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A Remark on the Minimal Index of Subfactors

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Let $M \supsetneqq N$ be a factor-subfactor pair with a faithful normal conditional expectation $E: M \to N$ satisfying Ind $E < +\infty$. Let $N \subsetneqq M \subsetneqq M_1 \subsetneqq M_2 \subsetneqq \cdots$ be the Jones tower of basic extensions, and $E_k: M_k \to M_{k-1}$ be the dual expectation constructed canonically from the given E. We prove that, if $E: M \to N$ is minimal, then so is the composition $E \circ E_1 \circ \cdots \circ E_k: M_k \to N$. Some consequences of this result are also presented.

1. INTRODUCTION

Motivated by Jones' index theory [10] for II_1-factors we have developed index theory for general factors in [12, 15, 16]. Let us point out that this framework naturally appears in the Quantum Field Theory context. Here, for a given normal conditional expectation E from a factor onto its subfactor the index value Ind E is defined (so that Ind E does depend on the choice of E).

In the Jones theory the normal conditional expectation E_N determined by the unique II_1-trace plays an important role. For general factors it was shown in [7, 8, 15], that one can choose a unique normal conditional expectation E_0 satisfying Ind $E_0 = \mathrm{Min}_E(\mathrm{Ind}\, E)$ (when Ind $E < +\infty$ and this property does not depend on the choice of E). Various characterizations of this minimal E_0 are known (see 2.3).

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The notion of basic extension is crucial in index theory. Starting from a factor-subfactor pair $M \supseteqq N$, one considers the basic extension $M_1 = J_M N' J_M = \langle M, e_N \rangle$, where e_N is the Jones projection and J_M is the modular conjugation associated with M. Repeating this construction, one obtains the Jones tower $N \subseteqq M \subseteqq M_1 \subseteqq M_2 \cdots$ of factors. By setting $E_1 = (\text{Ind } E)^{-1} J_M E^{-1}(J_M \cdot J_M) J_M$, one obtains the dual expectation from M_1 onto M. Repeating this procedure again, one obtains $E_2: M_2 \to M_1$, $E_3: M_3 \to M_2$, and so on.

The purpose of the present article is to show that, if $E: M \to N$ is minimal, then so is the composition $E \circ E_1 \circ \cdots \circ E_k: M_k \to N$. After some preliminary results in Section 2, we will prove this main result in Section 3. By the simple trick of tensoring by a factor of type III_1, we will reduce this problem to the II_1 case so that we will be able to use the result in [22]. In Section 4 some consequences will be explained. Other consequences will be contained in [17].

Basic facts on index theory can be found in [3, 10, 12, 15, 16, 19] while standard results on the modular theory and type III factors that will be needed in the article are treated in [23].

2. Preliminaries

Let $M \supseteqq N$ be a factor-subfactor pair with a faithful normal conditional expectation $E: M \to N$. Throughout we will assume $\text{Ind}\, E < +\infty$ in the sense of [12, 15, 16].

2.1. Let φ be a faithful normal state on N, and we set $\psi = \varphi \circ E \in M_*^+$. Since $\sigma_t^\psi | N = \sigma_t^\varphi$, we obtain the following inclusion of von Neumann algebras of type II_∞:

$$\tilde{N} = N \rtimes_{\sigma^\varphi} \mathbb{R} \subseteqq \tilde{M} = M \rtimes_{\sigma^\psi} \mathbb{R}.$$

This inclusion does not depend on the choice of $\varphi \in N_*^+$ thanks to Connes' Radon–Nikodym theorem. Note that the dual action $\{\theta_t^{\tilde{N}}\}_{t \in \mathbb{R}}$ on $\tilde{N}$ is exactly the restriction of that $\{\theta_t^{\tilde{M}}\}_{t \in \mathbb{R}}$ on $\tilde{M}$. Hence, in what follows, we shall simply denote them by $\{\theta_t\}_{t \in \mathbb{R}}$. From the construction, the dual weight $\hat{\varphi}$ on $\tilde{N}$ is just the restriction of $\hat{\psi}$. In particular, $\hat{\psi}$ is semi-finite on $\tilde{N}$. Because of $\sigma_t^{\hat{\psi}}|_{\tilde{N}} = \sigma_t^{\hat{\varphi}}$ $(= \text{Ad}\, \lambda(t))$ Takesaki's theorem guarantees that there exists a unique normal conditional expectation $\hat{E}: \tilde{M} \to \tilde{N}$ conditioned by $\hat{\psi}$. Notice that $\theta_t \circ \hat{E} = \hat{E} \circ \theta_t$ (which follows from the uniqueness of $\hat{E}$ and $\hat{\psi} \cdot \theta_t = \hat{\psi}$). As was pointed out in [14, 15], $\hat{E}$ also comes from the canonical trace $\text{tr}_{\tilde{M}}$ (scaled in the usual way under the dual action), and we have $tr_{\tilde{N}} \circ \hat{E} = tr_{\tilde{M}}$.

As usual we regard $N(\cong \pi(N))$, $M(\cong \pi(M))$ as subalgebras of $\tilde{N}$ and $\tilde{M}$, respectively. We claim that the restriction of $\hat{E}\colon \tilde{M} \to \tilde{N}$ to M is precisely $E\colon M \to N$. In fact, for $x \in M$ we compute

$$\begin{aligned}\hat{\varphi}\left(\hat{E}(x)\int_{-\infty}^{\infty} y(t)\,\lambda(t)\,dt\right) &= \hat{\psi}\left(\int_{-\infty}^{\infty} xy(t)\,\lambda(t)\,dt\right)\\ &= \psi(xy(0)) = \varphi(E(x)\,y(0))\\ &= \hat{\varphi}\left(E(x)\int_{-\infty}^{\infty} y(t)\,\lambda(t)\,dt\right),\end{aligned}$$

where $\int_{-\infty}^{\infty} y(t)\,\lambda(t)\,dt$ is a "smooth" element in $\tilde{N}$. We thus get

$$\hat{E}\left(\int_{-\infty}^{\infty} x(t)\,\lambda(t)\,dt\right) = \int_{-\infty}^{\infty} E(x(t))\,\lambda(t)\,dt \qquad (x(t) \in M). \tag{1}$$

Let M_1 $(= J_M N' J_M)$ be the basic extension of $M \supseteq N$, and E_1 $(= (\operatorname{Ind} E)^{-1}\; J_M E^{-1}(J_M \cdot J_M)\, J_M)\colon M_1 \to M$ be the dual expectation. Repeating the argument so far for $E_1\colon M_1 \to M$, we also obtain $(M_1)^{\sim} = M_1 \rtimes_\sigma \mathbb{R}$ relative to the modular automorphisms associated with $\psi \circ E_1$ and $\hat{E}_1\colon (M_1)^{\sim} \to \tilde{M}$.

To identify $(M_1)^{\sim}$ with the basic extension of $\tilde{M} \supseteq \tilde{N}$, we use the next lemma.

LEMMA 1. *Let F be a normal conditional expectation from a von Neumann algebra B onto its subalgebras C. Assume that $A(\supseteq B)$ is a von Neumann algebra with a projection e whose central support is 1 and that there exists a bounded faithful normal operator-valued weight $G\colon A \to B$ such that*

$$G(e) = 1$$

$$G(xe)e = xe,\ x \in A,$$

$$exe = F(x)e,\ x \in B.$$

Then there exists an isomorphism $\pi\colon A \to \langle B, e_c\rangle$, the basic extension of $B \supseteq C$, such that

$$\pi(e) = e_c,$$

$$\pi(x) = x,\ x \in B,$$

$$\pi \circ G \circ \pi^{-1} = J_B F^{-1}(J_B \cdot J_B)\, J_B.$$

Proof. This can be proved by the obvious modification of arguments in [6, proof of Theorem 8]. We just point out that the π constructed there is injective even if the involved algebras are not necessarily factors. In fact, let us assume $\pi(x)=0$ $(x\in A)$. Recall that one can find a family $\{x_i\}_{i=1,2,\ldots}$ $(\subseteqq B)$ satisfying

$$0=\pi(x)\,\Lambda(x_i)=\Lambda(G(xx_i e))$$

and $G(xx_i e)=0$ since Λ is injective. We thus get

$$xx_i ex_i^*=G(xx_i e)\,ex_i^*=0.$$

Summing over i, we conclude

$$0=\sum_{i=1}^{\infty} xx_i ex_i^*=x. \quad \blacksquare$$

We apply Lemma 1 for $(M_1)^{\sim}$ $(\supseteqq \tilde{M})$ and $(\operatorname{Ind} E)\,\hat{E}_1$. Let e be the Jones projection of $M\supseteqq N$ $(e\in M_1\subseteqq (M_1)^{\sim})$. The central support of e in M_1 (hence in $(M_1)^{\sim}$) is oviously 1.

We easily check

$$(\operatorname{Ind} E)\,\hat{E}_1(e)=1,$$
$$(\operatorname{Ind} E)\,\hat{E}_1(xe)e=xe, \qquad x\in (M_1)^{\sim},$$
$$exe=\hat{E}(x)e, \qquad x\in \tilde{M}.$$

In fact, since E_1 is known to satisfy similar properties, the above formulas can be checked by "component-wise calculation" thanks to (1) (for $\hat{E}_1$). Therefore, $(M_1)^{\sim}$ can be identified with the basic extension of $\tilde{M}\supseteqq\tilde{N}$, and $e=\pi(e)$ is precisely the Jones projection for $\tilde{M}\supseteqq\tilde{N}$.

We also have

$$J_{\tilde{M}}(\hat{E})^{-1}(J_{\tilde{M}}\cdot J_{\tilde{M}})\,J_{\tilde{M}}=(\operatorname{Ind} E)\,\hat{E}_1.$$

In particular, we get

$$(\hat{E})^{-1}(1)=(\operatorname{Ind} E)\,1$$

(even if $\tilde{M}$ is not necessarily a factor).

Finally we remark that the above argument works for general crossed products $M\rtimes_\alpha G\supseteqq N\rtimes_\alpha G$ as long as α_g leaves N invariant and $\alpha_g\circ E=E\circ\alpha_g (g\in G)$.

2.2. It is well known that $\sigma_t^{\varphi\circ E}$ leaves the relative commutant $M\cap N'$ invariant and $\sigma_t^{\varphi\circ E}|_{M\cap N'}$ does not depend on the choice of φ (so that the restriction can be denoted by σ_t^E). Using the facts

$$\sigma_t^{\hat{\psi}} = \text{Ad}\,\lambda(t),$$
$$\sigma_t^{\hat{\psi}}|_M = \sigma_t^{\psi} \qquad (=\sigma_t^E \text{ on } M \cap N'),$$
$$(\tilde{M})_\theta = M,$$

one sees that

$$(\tilde{M} \cap \tilde{N}')_\theta = (M \cap N')_E, \qquad \text{the fixed point algebra of } \{\sigma_t^E\}.$$

Let p be a projection in this algebra. Then the index (i.e., the local index) of $E_p: x \in pMp \to E(p)^{-1} E(x)\, p \in N_p$ is given by

$$\text{Ind}\, E_p = E(p)\, E^{-1}(p). \tag{2}$$

(see Proposition 4.2 in [12]) We consider the (local) index of $(\hat{E})_{p=\pi(p)}$: $p\tilde{M}p \to \tilde{N}_p$. Notice that $\psi = \varphi \circ E$ satisfies

$$\begin{aligned}\psi(E_p(x)) &= E(p)^{-1}\, \psi(E(x)\, p)\\ &= E(p)^{-1}\, \varphi(E(x)\, E(p))\\ &= \psi(x)\end{aligned}$$

for each $x \in pMp$. This E_p is determined by $\varphi|_{pMp}$ (i.e., $\psi|_{N_p} \circ E_p = \psi|_{pMp}$). Since $\sigma_t^{\psi}(p) = p$, the modular actions of $\psi|_{pMp}$ and $\psi|_{N_p}$ are just the restrictions of σ_t^{ψ} to the relevant algebras. Consequently, the associated inclusion of II_∞- algebras (see the first part of 2.1) is simply

$$(N_p)^{\sim} = \tilde{N}_p \subseteqq (pMp)^{\sim} = p\tilde{M}p.$$

Since E is just the restriction of $\hat{E}$ to E, we have $\hat{E}(p) = E(p)$ and from (1) one easily checks $(E_p)^{\wedge} = (\hat{E})_p$. Therefore, we have

$$\text{Ind}(\hat{E})_p = \text{Ind}(E_p)^{\wedge} = \text{Ind}\, E_p \tag{3}$$

(when $\tilde{M}$ is a factor).

2.3. When $M \cap N' \neq \mathbb{C}1$, there are many normal conditional expectations $E: M \to N$ and the index $\text{Ind}\, E$ does depend on the choice of E. However, the canonical choice of E is possible. In fact, it was shown in [7, 8, 15] that there exists a unique normal conditional expectation $E_0: M \to N$ satisfying $\text{Ind}\, E_0 = \text{Min}_E(\text{Ind}\, E)$. In what follows, this unique E_0 will be called the minimal conditional expectation. (For II_1- factors $M \supseteqq N$, the minimal E_0 may not be the same as the expectation E_N determined by the II_1- trace.) In the non-factor case see [9]. As was shown in [8], this E_0 is tracial on $M \cap N'$ and characterized by the property

$$E_0^{-1} = (\text{Ind}\, E_0)\, E_0 \qquad \text{on} \quad M \cap N'. \tag{4}$$

When $E = E_0$, for a projection p in $M \cap N'$ $(= M \cap N')_E)$ we get

$$\text{Ind } E_p = (\text{Ind } E)\, E(p)^2 \tag{5}$$

because of (2) and (4). Hence we obtain "additivity of the square roots of indices" (see [15]),

$$\sum_{i=1}^{n} (\text{ind } E_{p_i})^{1/2} = \sum_{i=1}^{n} (\text{Ind } E)^{1/2}\, E(p_i) = (\text{Ind } E)^{1/2}$$

for any partition $\{p_i\}_{i=1,2,\ldots,n}$ $(\subseteqq (M \cap N')_{proj})$ of the unit. This additivity turns out to be a very useful characterization of the minimal expectation E_0.

THEOREM 2 [7, 15]. *A normal conditional expectation E is minimal (i.e., $E = E_0$) if and only if there is a partition $\{p_i\}_{i=1,2,\ldots,n}$ of the unit consisting of minimal projections in $M \cap N'$ such that*

$$p_i \in (M \cap N')_E, \quad \text{Ind } E_{p_i} = (\text{Ind } E)\, E(p_i)^2,$$

for $i = 1, 2, \ldots, n$.

In the next section, this characterization will be used to reduce the proof of our main result to the corresponding result for II_1- factors.

3. MAIN RESULT

Starting from $E: M \to N$, Ind $E < +\infty$, as usual we obtain a sequence of the dual expectations $E_1: M_1 \to M$, $E_2: M_2 \to M_1$, and so on, where $N \subseteqq M \subseteqq M_1 \subseteqq M_2 \subseteqq \cdots$ is the Jones tower of factors.

THEOREM 3. *If $E: M \to N$ is minimal, then for each $k = 1, 2, \ldots$ the composition $E \circ E_1 \circ \cdots \circ E_k: M_k \to N$ is also minimal.*

Proof. Let A be a factor of type III_1. We consider

$$E \otimes Id_A: M \otimes A \to N \otimes A.$$

The basic extensions are obviously $M_k \otimes A$ with the dual expectations $E_k \otimes Id_A$. The characterization (4) (as well as the condition in Theorem 2) is not effected by this "trivial tensoring procedure." Since $M \otimes A$ (and $N \otimes A$) is of type III_1, we may and do assume that $M \supseteqq N$ are factors of type III_1 (so that the $\tilde{M}_k$'s are factors of type II_∞).

We can choose a partition $\{p_i\}_{i=1,2,\ldots,n}$ of the unit consisting of minimal projections in $\tilde{M} \cap \tilde{N}'$ and satisfying $\theta_t(p_i) = p_i$. In fact, being finite dimensional, $\tilde{M} \cap \tilde{N}'$ is of the form $\sum_{j=1}^{k} \oplus M_{n_j}(\mathbb{C})$. The connectedness of $\mathbb{R}$ shows that each central minimal projection in $\tilde{M} \cap \tilde{N}'$ is fixed by θ_t. Hence, θ_t sends each $M_{n_j}(\mathbb{C})$ into itself, and $\theta_t|_{M_{n_j}(\mathbb{C})} = \mathrm{Ad}\, H_j^{it}$ for some positive $n_j \times n_j$-matrix H_j. By using eigenvectors for H_j, one can choose n_j rank-1 projections fixed by $\mathrm{Ad}\, H_j^{it}$.

First we claim that $\hat{E}: \tilde{M} \to \tilde{N}$ is minimal. The above minimal projections $\{p_i\}$ satisfy

$$p_i \in (\tilde{M} \cap \tilde{N}')_\theta = (M \cap N')_E = M \cap N' \qquad \text{(since } E \text{ is minimal)},$$

$$\sigma_t^{\hat{E}}(p_i) = p_i \qquad \text{(since } \hat{E} \text{ comes from the trace)}.$$

Then (3) and (5) guarantee

$$\begin{aligned} \mathrm{Ind}\, (\hat{E})_{p_i} &= \mathrm{Ind}\, E_{p_i} \\ &= (\mathrm{Ind}\, E)\, E(p_i)^2 \\ &= (\mathrm{Ind}\, \hat{E})\, \hat{E}(p_i)^2, \end{aligned}$$

and $\hat{E}$ is minimal thanks to Theorem 2.

Choose a projection p in $\tilde{N}$ satisfying $tr_{\tilde{N}}(p) = 1$. Then one can choose a partition $\{p_i\}_{i=1,2,\ldots}$ ($\subseteqq \tilde{N}_{proj}$) of the unit such that $p_1 = p$, each p_i is equivalent to p (in $\tilde{N}$), and the p_i's are mutually orthogonal.

Using partial isometries (in $\tilde{N}$) realizing equivalence between the p_i's and p, we can construct the conjugacy between

$$\tilde{M} \supseteqq \tilde{N} \qquad \text{and} \qquad p\tilde{M}p \otimes B(H) \supseteqq \tilde{N}p \otimes B(H)$$

(in the usual way). Under this conjugacy, $\hat{E}$ corresponds to

$$\hat{E}|_{pMp} \otimes Id_{B(H)}.$$

Note that $\hat{E}|_{pMp}: p\tilde{M}p \to \tilde{N}p$ arises from the unique II_1- trace $tr_{\tilde{M}}|_{p\tilde{M}p}$ on $p\tilde{M}p$ (since $\hat{E}$ arises from $tr_{\tilde{M}}$, see 2.1). Obviously, $\hat{E}|_{p\tilde{M}p}$ is minimal (recall the characterization (4)). Hence, by the Pimsner–Popa theorem [22] (see the final remark in [7]), $\hat{E} \circ \hat{E}_1 \circ \cdots \circ \hat{E}_k: \tilde{M}_k \to \tilde{N}$ is minimal.

Choose a partition $\{q_j\}_{j=1,2,\ldots,m}$ of the unit consisting of minimal projections in $M_k \cap N'$ and satisfying $q_j \in (M_k \cap N')_{E \circ E_1 \circ \cdots \circ E_k}$ (by repeating similar arguments as above).

We now compute

$$\begin{aligned} \mathrm{Ind}(E \circ E_1 \circ \cdots \circ E_k)_{q_j} &= \mathrm{Ind}(\hat{E} \circ \hat{E}_1 \circ \cdots \circ \hat{E}_k)_{q_j} \qquad \text{(by (3))} \\ &= \mathrm{Ind}(\hat{E} \circ \hat{E}_1 \circ \cdots \circ \hat{E}_k)(\hat{E} \circ \hat{E}_1 \circ \cdots \circ \hat{E}_k)(q_j)^2 \\ &\qquad \text{(by (5): } \hat{E} \circ \hat{E}_1 \circ \cdots \circ \hat{E}_k \text{ is minimal)} \\ &= \mathrm{Ind}(E \circ E_1 \circ \cdots \circ E_k)(E \circ E_1 \circ \cdots \circ E_k)(q_j)^2. \end{aligned}$$

Therefore, the composition $E \circ E_1 \circ \cdots \circ E_k$ is minimal thanks to Theorem 2 again. ∎

It is an interesting problem to characterize the minimal expectation by a condition similar to that in Corollary 4.5(iii) in [21], which might give us a much more natural proof of Theorem 3.

Remark 4 (Converse of the Pimsner–Popa Inequality). Let $M \supseteq N$ be II_1-factors. Assume that the expection E_N arising from the II_1-trace satisfies $E(x) \geq \varepsilon x$, $x \in M_+$, for some $\varepsilon > 0$. It was shown in [21] that we get $[M:N]$ ($= \mathrm{Ind}\, E_N) < +\infty$. (If the above $\varepsilon > 0$ is the best constant, then we get $\varepsilon^{-1} = [M:N] \cdots$ which is easily seen by considering the Jones projection associated with a downward basic construction.) The converse of the Pimsner–Popa inequality for general factors has been considered by several authors (see [1] and also see [18] for a partial result). We would like to point out that the "tensoring trick" in the above proof gives us a slick proof of the converse of the Pimsner–Popa inequality. Assume that $E: M \to N$ satisfies

$$E(x) \geq \varepsilon x, \qquad x \in M_+ (\varepsilon > 0). \tag{6}$$

For simplicity, let us assume that $M \supseteq N$ are properly infinite factors. (For example, in the finite dimensional case one has to assume the complete positivity of the map $x \in M \to E(x) - \varepsilon x \in M$.) Properly infiniteness guarantees that

$$(E \otimes Id)(x) \geq \varepsilon x, \qquad x \in (M \otimes B(H))_+. \tag{7}$$

In fact, $M \supseteq N$ is conjugate to $M \otimes B(H) \supseteq N \otimes B(H)$ by the standard argument, and via this conjugacy E corresponds to $E \otimes Id$. Now let A be a factor of type III_1. Then $E \otimes Id_A : M \otimes A \to N \otimes A$ satisfies (6) thanks to (7) and we have $\mathrm{Ind}(E \otimes Id_A) = \mathrm{Ind}\, E$. Therefore, as in the proof of Theorem 3, we may and do assume that $M \supseteq N$ are factors of type III_1. We then consider the "second dual" $\hat{\hat{E}}: \tilde{M} \rtimes_\theta \mathbb{R} \to \tilde{N} \rtimes_\theta \mathbb{R}$ determined by the bidual weight $\hat{\hat{\psi}}$. By the Takesaki duality $\tilde{M} \rtimes_\theta \mathbb{R} \supseteq \tilde{N} \rtimes_\theta \mathbb{R}$ can be identified with $M \otimes B(H) \supseteq N \otimes B(H)$. Through this identification $\hat{\hat{E}}$ obviously corresponds to $E \otimes Id$. (Hence $\mathrm{Ind}\, \hat{\hat{E}} = \mathrm{Ind}\, E$.) Thus, $\hat{\hat{E}}$ satisfies (6) due to (7). The expectation $\hat{E}$ being the restriction of $\hat{\hat{E}}$, $\hat{E}$ also satisfies (6). As in the proof of Theorem 3 one can reduce the situation to the II_1-factor case and use the abovementioned II_1-result due to Pimsner and Popa. We thus conclude $\mathrm{Ind}\, \hat{E} < +\infty$ and

$$\begin{aligned} \mathrm{Ind}\, E &= \mathrm{Ind}\, \hat{\hat{E}} \\ &= \mathrm{Ind}\, \hat{E} \qquad \text{(see the last paragraph of 2.1)} \\ &< +\infty. \end{aligned}$$

4. Consequences of the Main Result

In the next two corollaries we assume $\operatorname{Ind} E < +\infty$.

COROLLARY 5. *If $M \cap N' = \mathbb{C}1$ (or more generally if E is minimal), then the modular action on $M_k \cap N'$ associated with $E \circ E_1 \circ \cdots \circ E_k$ is trivial.*

Proof. Since $E \circ E_1 \circ \cdots \circ E_k$ is minimal (Theorem 3), it is tracial on $M_k \cap N'$. ∎

According to [20], having an irreducible ($M \cap N' = \mathbb{C}1$) factor-subfactor pair means that one has a "group-like" object (quantized group). Thus the above corollary means that a quantized group of finite "order" is always "unimodular". Is a discrete quantized group ($E: M \to N$ exists and $M \cap N' = 1$) "unimodular"? For a non-discrete inclusion the result is false [2].

COROLLARY 6. *Under the same assumptions as in Corollary 5, the relative commutant $M_k \cap N'$ is the fixed point algebra of "II_∞-relative commutant" $\tilde{M}_k \cap \tilde{N}'$ under the dual action (i.e., $M_k \cap N' = (\tilde{M}_k \cap \tilde{N}')_\theta$).*

Proof. As remarked in 2.2 we have $(\tilde{M}_k \cap \tilde{N}')_\theta = (M_k \cap N')_{E \circ E_1 \circ \cdots \circ E_k}$. Thus the result follow Corollary 5. ∎

This result gives us a practical method for computing the higher relative commutant $M_k \cap N'$ of an inclusion $M \supseteq N$ of type III factors.

COROLLARY 7. *If $M \supseteq N$ are factors of type III_1 and* $\operatorname{Ind} E < 4$, *then we have $M_k \cap N' = \tilde{M}_k \cap \tilde{N}'$, $k = 1, 2, \ldots$.*

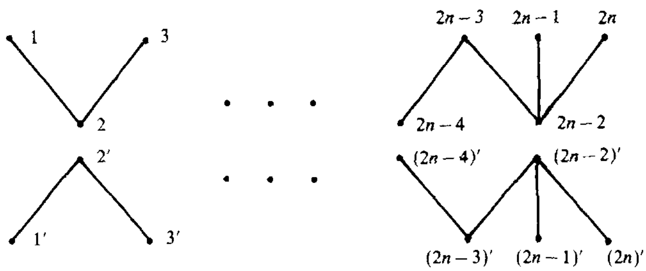
Proof. The intertwining property $\theta_t \circ E_k = E_k \circ \theta_t$ means that the Jones projections $\{e_k\}_{k=0,1,2,\ldots}$ are fixed under θ_t. The group $\mathbb{R}$ being connected, θ_t also acts trivially for central minimal projections (in each $\tilde{M}_k \cap \tilde{N}'$). The assumption $\operatorname{Ind} E < 4$ means (see [3]) that $\{\tilde{M}_k \cap \tilde{N}'\}_k$ is described by one of the Coxeter–Dynkin diagrams of types A, D, E. (More precisely, the principal graph (see [3]) of the Bratteli diagram describing $\{\tilde{M}_k \cap \tilde{N}'\}_k$ is one of the above Coxeter–Dynkin diagrams.) In each case one easily observes that $\tilde{M}_k \cap \tilde{N}'$ is the direct sum of the part generated by $e = e_0, e_1, \ldots, e_{k-1}$ (i.e., the "reflection of the previous step") and a certain subalgebra in $Z(\tilde{M}_k \cap \tilde{N}')$ (i.e., the "rest"). Therefore, we conclude that $\theta_t = Id$ on $\tilde{M}_k \cap \tilde{N}'$. ∎

For factors of type III_λ $(0 \leq \lambda < 1)$ one can consider the discrete decomposition so that we have a single automorphism θ_0 (action of $\mathbb{Z}$) instead of the one parameter automorphism group $\{\theta_t\}_{t \in \mathbb{R}}$. Assume that both of M

and N are AFD type III_λ $(0<\lambda<1)$. If the "type III relative commutants" coincide with "the type II relative commutants," then $M \supseteq N$ is a trivial inclusion (in the sense that $M \supseteq N$ comes from an inclusion of II_1-factors together the trivial tensoring) as was shown in [11, 19]. It is a very interesting problem to see if the same conclusion follows from Corollary 7 in the type III_1 setting.

Recall that the connectedness of $\mathbb{R}$ was crucial in the proof of Corollary 7. For type III_λ $(0 \leq \lambda < 1)$ factors, considering a $\mathbb{Z}$-action is relevant as was pointed out above. The situation completely changes and in fact, in the following, based on Corollary 6, we will construct an inclusion $M \supseteq N$ of factors of type III_λ $(0 \leq \lambda < 1)$ such that $\{M_k \cap N'\}_k$ and $\{\tilde{M}_k \cap \tilde{N}'\}_k$ are different. In particular, $M \supseteq N$ is not a trivial inclusion. A similar phenomenon for type III_λ $(0<\lambda<1)$ factors was explained in [19].

We start from the coupling system [20] corresponding to the Dynkin diagram D_{2n} $(n=2, 3, ...)$



Define the (period 2) graph automorphism π by

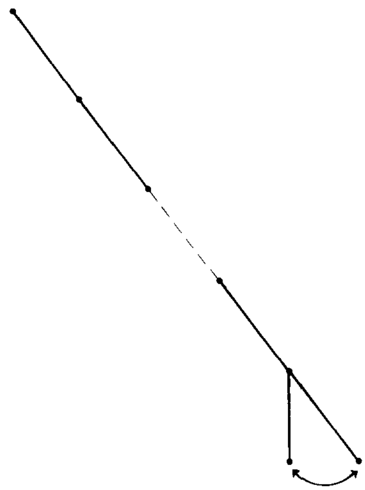
$$\pi(i)=i,\ \pi(i')=i',\ i=1, 2, ..., 2n-2,$$

$$\pi(2n-1)=2n,\ \pi(2n)=2n-1,$$

$$\pi((2n-1)')=(2n)',\ \pi((2n)')=(2n-1)'.$$

The map π is compatible with contragredient maps, and from the table in [20] we can easily check that π preserves connections (of cells). (Hence π is a period 2 automorphism in the sense of Ocneanu's coupling system.) From this coupling system we get the AFD II_1-factors $B \supseteq C$ such that the principal graph of $\{B_k \cap C'\}_k$ (associated with the Jones tower $C \subseteq B \subseteq B_1 \subseteq B_2 \subseteq \cdots$) is D_{2n}.

Then, π gives rise to the period 2 automorphism (still denoted by π) in $\mathrm{Aut}(B, C)$. The canonical extensions (still denoted by π) to B_k (uniquely determined by the property $\pi(e_k)=e_k$) act on $B_k \cap C'$ as in



Namely, π switches the "last two vertices" of D_{2n}.

Let $A \rtimes_{\theta_0} \mathbb{Z}$ be a discrete decomposition of a factor of type III_λ $(0 \leqq \lambda < 1)$ such that θ_0^2 is ergodic on the center $Z(A)$. (This assumption is automatic for a factor of type III_λ, $0 < \lambda < 1$, since A is a factor of type II_∞ in this case.) Let us consider the inclusion

$$N = (C \otimes A) \rtimes_{\pi \otimes \theta_0} \mathbb{Z} \subseteq M = (B \otimes A) \rtimes_{\pi \otimes \theta_0} \mathbb{Z}.$$

Let F be the unique expectation from B onto C (coming from the II_1 − trace τ_B). Then $F \otimes Id_A$ commutes with $\pi \otimes \theta_0$ so that it "extends" to the expectation $E: M \to N$ (Ind $E = [B: C] = 4\cos^2 \pi/(4n-2)$), see 2.1). We will show that this $M \supseteq N$ is a non-trivial inclusion.

Notice that $\pi \otimes \theta_0$ scales the traces $\tau_B \otimes \tau_A = (\tau_C \circ F) \otimes \tau_A$ and $\tau_C \otimes \tau_A$ in the same way (because of $\pi \circ F = F \circ \pi$). Therefore (recall the relationship between continuous crossed product decomposition and discrete crossed product decomposition for a type III factor) the continuous crossed products $\tilde{M}$, $\tilde{N}$ are given by the discrete systems $(B \otimes A, \pi \otimes \theta_0)$ and $(C \otimes A, \pi \otimes \theta_0)$ together with the common ceiling function (determined by how θ_0 scales τ_A). In particular, M and N (and $A \rtimes_{\theta_0} \mathbb{Z}$) have the same flow of weights (and $M \supseteq N$ is of the form $M = \mathfrak{A} \supseteq \mathfrak{B} = N$ in the sense of [14]). The basic extensions of $M \supseteq N$ are $M_k = (B_k \otimes A)_{\pi \otimes \theta_0} \mathbb{Z}$ by Lemma 1 (as in 2.1). Hence it is easy to see

$$\begin{aligned}(\tilde{M}_k \cap \tilde{N}')_\theta &= \{(B_k \otimes A) \cap (C \otimes A)'\}_{\pi \otimes \theta_0} \\ &= \{(B_k \cap C') \otimes Z(A)\}_{\pi \otimes \theta_0}.\end{aligned}$$

Due to $\pi^2 = Id$ and the ergodicity of θ_0^2 on $Z(A)$, this algebra is included in $B_k \cap C'$. Since $(\tilde{M}_k \cap \tilde{N}')_\theta = M_k \cap N'$ (Corollary 6), we conclude that

$$M_k \cap N' = (B_k \cap C')_\pi.$$

From the description of π (on $B_k \cap C'$) given before, one easily checks that the principal graph of $\{M_k \cap N'\}_k$ is the Dynkin diagram A_{4n-3} (note that the principal graph of $\{M_k \cap N'\}_k$ has to be one with index $= 4\cos^2(\pi/(4n-2))$).

On the other hand, $\tilde{M} \supseteqq \tilde{N}$ (having the same center) gives us a field of inclusions of II_∞-factors by looking at the common central decomposition. This gives us a field of "type II relative commutants" and hence that of principal graphs. Notice that all inclusions of II_∞-factors look like $B \otimes A(\omega) \supseteqq C \otimes A(\omega)$. Here, $A = \int_X^\oplus A(\omega)\, d\omega$ is the central decomposition and X is the base space of the flow of weights (represented with the ceiling function). Hence the field of principal graphs is constant, and we get the Dynkin diagram D_{2n}.

Therefore, we have seen that the "type II principal graph" D_{2n} shrinks to the "type III principal graph" A_{4n-3} since we have to look at the fixed point algebras under the symmetry π. Consequently, the inclusion $M \supseteqq N$ is not trivial.

Remark 8. The central ergodicity of θ_0^2 in the above discussion is essential. For example, assume that θ_0^2 is not ergodic on $Z(A)$ and $n = 2$. Then the principal graph of $\{M_k \cap N'\}_k$ is D_4 (not A_5) so that we get $M = N \rtimes_\alpha \mathbb{Z}_3$ by [13]. Since M and N have the same flow of weights, one easily sees that the α_g's are approximately pointwise inner and α_g, $g \neq e$, are not pointwise inner (in the sense of [4, 5]). Therefore, if the factors are further assumed to be AFD, then by [24] we conclude that $M \supseteqq N$ is conjugate to $(R \rtimes \mathbb{Z}_3) \otimes M \supseteqq R \otimes M$.

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