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Iwasawa theory of totally real fields for certain non-commutative p-extensions

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ABSTRACT

In this paper, we will prove the non-commutative Iwasawa main conjecture—formulated by John Coates, Takako Fukaya, Kazuya Kato, Ramdorai Sujatha and Otmar Venjakob (2005)—for certain specific non-commutative p-adic Lie extensions of totally real fields by using theory on integral logarithms introduced by Robert Oliver and Laurence R. Taylor, theory on Hilbert modular forms introduced by Pierre Deligne and Kenneth A. Ribet, and so on. Our results give certain generalization of the recent work of Kazuya Kato on the proof of the main conjecture for Galois extensions of Heisenberg type.

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0. Introduction

The Iwasawa main conjecture, which predicts the mysterious relation between "arithmetic" characteristic elements and "analytic" p-adic zeta functions, has been proven under many situations for abelian extensions of number fields. However, it took many years even to formulate the main conjecture for non-commutative extensions. In 2005, John Coates et al. [5] formulated the main conjecture for elliptic curves without complex multiplication by using algebraic K-theory. Then Kazuya Kato [13] has constructed the p-adic zeta functions and proven the main conjecture for certain specific p-adic Lie extensions called "Heisenberg-type extensions" of totally real number fields. Mahesh Kakde [11] has recently generalized Kato's method and proven the main conjecture for another type of extension. On the other hand, Jürgen Ritter and Alfred Weiss had also formulated the main conjecture for 1-dimensional p-adic Lie extensions called "equivariant Iwasawa main conjecture" [19] in a little different way from the formulation of John Coates et al., and proven it for certain special cases [20–22]. In this paper, we will prove the Iwasawa main conjecture—the formulation of John Coates et al.—for another type of non-commutative p-extensions of totally real number fields by generalizing the method of Kazuya Kato [13].

Let p be a prime number, F a totally real number field and F^{∞} a totally real p-adic Lie extension of F containing the cyclotomic $\mathbb{Z}_p$-extension of F. Assume that only finitely many primes of F ramify in F^{∞}. For simplicity, also assume that Iwasawa's $\mu = 0$ conjecture is valid (see condition (♯) in Section 1.2 for more general condition). The aim of this paper is to prove the following theorem.

Theorem 0.1 *(= Theorem 2.1). Assume the following two conditions*:

(1) *the Galois group G of F^{∞}/F is isomorphic to* $\begin{pmatrix} 1 & \mathbb{F}_p & \mathbb{F}_p & \mathbb{F}_p \\ 0 & 1 & \mathbb{F}_p & \mathbb{F}_p \\ 0 & 0 & 1 & \mathbb{F}_p \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \Gamma$ *where Γ is a commutative p-adic Lie group isomorphic to $\mathbb{Z}_p$*;
(2) *the prime number p is not equal to either* 2 *or* 3.

Then the p-adic zeta function $\xi_{F^{\infty}/F}$ for F^{∞}/F exists and the Iwasawa main conjecture is true for F^{∞}/F.

We will review the formulation of the non-commutative Iwasawa main conjecture in Section 1.2. Now let us summarize how to prove the theorem. The basic strategy to construct the p-adic zeta functions and prove the main conjecture for non-commutative extensions has been proposed by David Burns, and Kazuya Kato applied it in his results [13]. We will also use their technique to prove our main theorem (Theorem 0.1); precisely we first construct the family $\mathfrak{F}$ consisting of pairs $\{(U_i, V_i)\}_{i=0}^4$ of subgroups of G such that each V_i is normal in U_i and all Artin representations of G are obtained as $\mathbb{Z}$-linear combinations of representations induced by characters of the abelian groups $\{U_i/V_i\}_{i=0}^4$ (see Section 3.1 for details). Let $\Lambda(G)$ (resp. $\Lambda(U_i)$, $\Lambda(U_i/V_i)$) denote the Iwasawa algebra of G (resp. U_i, U_i/V_i). Consider the composition

$$\theta_{S,i} : K_1\bigl(\Lambda(G)_S\bigr) \xrightarrow{\mathrm{Nr}_i} K_1\bigl(\Lambda(U_i)_S\bigr) \xrightarrow{\text{canonical}} K_1\bigl(\Lambda(U_i/V_i)_S\bigr) = \Lambda(U_i/V_i)_S^{\times}$$

for each i where $\Lambda(G)_S$ (resp. $\Lambda(U_i)_S$, $\Lambda(U_i/V_i)_S$) denotes *the canonical Ore localization of* $\Lambda(G)$ (*resp.* $\Lambda(U_i)$, $\Lambda(U_i/V_i)$) introduced by John Coates et al. in [5, Section 2] (also refer to Section 1.2 of this article) and Nr_i denotes the norm map. Set $\theta_S = (\theta_{S,i})_{i=0}^4$. Then each $\Lambda(U_i/V_i)_S^{\times}$ contains *the p-adic zeta pseudomeasure ξ_i for F_{V_i}/F_{U_i}* constructed by Pierre Deligne, Kenneth A. Ribet and Jean-Pierre Serre [6,24] (F_{U_i} and F_{V_i} denote the maximal intermediate fields of F^{∞}/F fixed by U_i and V_i respectively). Now suppose that there exists such an element ξ in $K_1(\Lambda(G)_S)$ as $\theta_S(\xi)$ coincides with $(\xi_i)_{i=0}^4$. Then we may check that ξ satisfies *the interpolation property* which characterizes the p-adic zeta function for F^{∞}/F (see Definition 1.5) by using Brauer induction, hence the element ξ is nothing but the p-adic zeta function for F^{∞}/F. Moreover ξ satisfies "the main conjecture" $\partial(\xi) = -[C_{F^{\infty}/F}]$ because the main conjecture for each abelian extension F_{V_i}/F_{U_i} holds by virtue of the deep results of Andrew Wiles [29] (see Section 2.2 for details). Therefore we have only to verify that $(\xi_i)_{i=0}^4$ is contained in the image of the map θ_S. However it is difficult to characterize the image of θ_S. On the other hand, we may characterize the image Ψ of $\theta = (\theta_i)_{i=0}^4$ by using theory on integral logarithms introduced by Robert Oliver and Laurence R. Taylor [17,18] (see Section 3 and Section 4), where θ_i is the composition

$$\theta_i : K_1\big(\Lambda(G)\big) \xrightarrow{\mathrm{Nr}_i} K_1\big(\Lambda(U_i)\big) \xrightarrow{\text{canonical}} K_1\big(\Lambda(U_i/V_i)\big) = \Lambda(U_i/V_i)^\times.$$

By this fact and easy diagram chasing, we may conclude that $(\xi_i)_{i=0}^4$ is contained in the image of θ_S if it is contained in a certain subgroup Ψ_S of $\prod_i \Lambda(U_i/V_i)_S^\times$ (which contains the image of θ_S) characterized by certain norm relations and congruences (see Theorem 2.4 and Section 5 for details).

Now the proof of Theorem 0.1 reduces to the verification of norm relations and congruences for $(\xi_i)_{i=0}^4$. We may easily verify the norm relations by formal calculation using the interpolation properties for $\{\xi_i\}_{i=0}^4$. To study congruences among $\{\xi_i\}_{i=0}^4$, we will use the q-expansion principle for Hilbert modular forms proven by Pierre Deligne and Kenneth A. Ribet [6] (see Section 6). Kazuya Kato, Jürgen Ritter and Alfred Weiss also used the q-expansion principle and successfully derived congruences among abelian p-adic zeta pseudomeasures which were necessary to verify their results [13,20–22]. Unfortunately it is difficult to derive all the desired congruences in our case by only using the technique of Kato, Ritter and Weiss. Therefore we will use the existence of the p-adic zeta function for a certain non-commutative subextension F_N/F of F^∞/F which is of Heisenberg type (note that the existence of the p-adic zeta functions for Heisenberg-type extensions has already been proven by Kazuya Kato [13]), and complete the proof of our Theorem 0.1 by using certain induction (see Section 7 for details).

Notation

In this paper, p always denotes a positive prime number. We denote by $\mathbb{N}$ the set of natural numbers (that is, the set of *strictly* positive integers), denote by $\mathbb{Z}$ (resp. $\mathbb{Z}_p$) the ring of integers (resp. p-adic integers), and denote by $\mathbb{Q}$ (resp. $\mathbb{Q}_p$) the rational number field (resp. the p-adic number field). For an arbitrary group G, we denote by $\mathrm{Conj}(G)$ the set of all conjugacy classes of G. For a pro-finite group P, $\Lambda(P)$ denotes its Iwasawa algebra (that is, its completed group ring over $\mathbb{Z}_p$). We denote by Γ a commutative p-adic Lie group isomorphic to $\mathbb{Z}_p$. Throughout this paper, we fix a topological generator t of Γ (in other words, we fix an isomorphism $\Lambda(\Gamma) \xrightarrow{\simeq} \mathbb{Z}_p[\![T]\!]$; $t \mapsto 1+T$ where $\mathbb{Z}_p[\![T]\!]$ is the formal power series ring over $\mathbb{Z}_p$). For an arbitrary finite group Δ, $\mathbb{Z}_p[\mathrm{Conj}(\Delta)]$ denotes the free $\mathbb{Z}_p$-module of finite rank with the free basis $\mathrm{Conj}(\Delta)$. For an arbitrary pro-finite group P, $\mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]$ denotes the projective limit of the free $\mathbb{Z}_p$-modules $\mathbb{Z}_p[\mathrm{Conj}(P_\lambda)]$ over finite quotient groups P_λ of P. We always assume that all associative rings have unity. For an associative ring R, we denote by $\mathrm{M}_n(R)$ the ring of $n \times n$-matrices with entries in R and denote by $\mathrm{GL}_n(R)$ the multiplicative group of $\mathrm{M}_n(R)$. For a module M over a commutative ring R and a finite subset S of M, $[S]_R$ denotes the R-submodule of M generated by S. In this article all Grothendieck groups are regarded as *additive* abelian groups, whereas all Whitehead groups are regarded as *multiplicative* abelian groups. Finally we fix embeddings of the algebraic closure of the rational number field $\overline{\mathbb{Q}}$ into the complex number field $\mathbb{C}$ and the algebraic closure of the p-adic number field $\overline{\mathbb{Q}}_p$.

1. Preliminaries

1.1. Brief review on theory of integral logarithms

Integral logarithmic homomorphisms were first introduced by Robert Oliver and Laurence R. Taylor [17,18] to study the structure of Whitehead groups of group rings of fundamental groups. We will use these homomorphisms to translate "the additive theta map" into "the (multiplicative) theta map" (see Section 4 for details). Jürgen Ritter and Alfred Weiss also used them to formulate their "equivariant Iwasawa theory" [19]. We refer to [1,15,26] for basic results on (lower) algebraic K-theory.

Let R be an absolutely unramified complete discrete valuation ring with mixed characteristics $(0, p)$ and K its fractional field. In the following, we assume that p is odd for simplicity. Fix the Frobenius endomorphism $\tilde{\varphi}: K \to K$ on K if its residue field is not perfect. Let Δ be a finite p-group and $R[\Delta]$ its group ring over R. Note that $R[\Delta]$ is a local ring whose maximal ideal $\mathfrak{m}_{R[\Delta]}$

is the kernel of the canonical surjection $R[\Delta] \to R/pR$. We define $R[\mathrm{Conj}(\Delta)]$ (resp. $K[\mathrm{Conj}(\Delta)]$) as the free R-module (resp. the K-vector space) with the free basis $\mathrm{Conj}(\Delta)$. Then the logarithm $\log_p(1+y) = \sum_{j=1}^{\infty}(-1)^{j-1}(y^j/j)$ converges p-adically on the multiplicative group $1+\mathfrak{m}_{R[\Delta]}$ and induces a group homomorphism $\log_p : K_1(R[\Delta], \mathfrak{m}_{R[\Delta]}) \to K[\mathrm{Conj}(\Delta)]$. It is known that we may extend this homomorphism to $\log_p : K_1(R[\Delta]) \to K[\mathrm{Conj}(\Delta)]$ if R is the integer ring of a finite unramified extension of $\mathbb{Q}_p$ (see [17, Theorem 2.8]). Finally, we define the "Frobenius correspondence" φ on $K[\mathrm{Conj}(\Delta)]$ by the relation $\varphi(\sum_{[g]\in\mathrm{Conj}(\Delta)} a_{[g]}[g]) = \sum_{[g]\in\mathrm{Conj}(\Delta)} \tilde{\varphi}(a_{[g]})[g^p]$.[2]

Proposition–Definition 1.1 *(Integral logarithm, Oliver–Taylor). For an element x in $K_1(R[\Delta], \mathfrak{m}_{R[\Delta]})$, let $\Gamma_{\Delta,\mathfrak{m}}(x)$ denote the element in $K[\mathrm{Conj}(\Delta)]$ defined as $\log_p(x) - p^{-1}\varphi(\log_p(x))$. Then $\Gamma_{\Delta,\mathfrak{m}}$ induces a homomorphism of abelian groups $\Gamma_{\Delta,\mathfrak{m}} : K_1(R[\Delta], \mathfrak{m}_{R[\Delta]}) \longrightarrow R[\mathrm{Conj}(\Delta)]$ which we call the integral logarithmic homomorphism for $R[\Delta]$. Moreover $\Gamma_{\Delta,\mathfrak{m}}$ is extended uniquely to the group homomorphism $\Gamma_\Delta : K_1(R[\Delta]) \longrightarrow R[\mathrm{Conj}(\Delta)]$ if R is the integer ring of a finite unramified extension K of $\mathbb{Q}_p$.*

Proof. Apply the proof of [17, Theorem 6.2] replacing $\mathbb{Z}_p$ and $\mathbb{Q}_p$ by R and K respectively. □

When R is the p-adic integer ring $\mathbb{Z}_p$, we may derive further information about the kernel and the cokernel of the integral logarithms [17, Theorem 6.6]:

Theorem 1.2. *Let Δ be a finite p-group and Δ^{ab} its abelization. Then the following sequence is exact*:

$$1 \to K_1(\mathbb{Z}_p[\Delta])/K_1(\mathbb{Z}_p[\Delta])_{\mathrm{tors}} \xrightarrow{\Gamma_\Delta} \mathbb{Z}_p[\mathrm{Conj}(\Delta)] \xrightarrow{\omega_\Delta} \Delta^{\mathrm{ab}} \to 1$$

where $K_1(\mathbb{Z}_p[\Delta])_{\mathrm{tors}}$ is the torsion part of $K_1(\mathbb{Z}_p[\Delta])$ and ω_Δ is the homomorphism defined by $\omega_\Delta(\sum_{[g]\in\mathrm{Conj}(\Delta)} a_{[g]}[g]) = \prod_{[g]\in\mathrm{Conj}(\Delta)} \bar{g}^{a_{[g]}}$ ($\bar{g}$ denotes the image of $[g]$ in Δ^{ab}).

The torsion part $K_1(R[\Delta])_{\mathrm{tors}}$ was well studied by Graham Higman [10] (for abelian Δ) and Charles Terence Clegg Wall [28] (for general Δ):

Theorem 1.3. *Let Δ be a finite p-group.*

(1) *The torsion part of $K_1(R[\Delta])$ is isomorphic to the multiplicative abelian group $\mu(R) \times \Delta^{\mathrm{ab}} \times SK_1(R[\Delta])$ where $\mu(R)$ is the multiplicative group generated by all roots of unity contained in R.*
(2) *The group $SK_1(R[\Delta])$ is finite if R is the integer ring of a number field K.*
(3) *The group $SK_1(R[\Delta])$ is trivial if Δ is abelian.*

Proof. See [28, Theorem 4.1] for (1) and [28, Theorem 2.5] for (2) respectively. The claim of (3) obviously follows from the definition of $SK_1(R[\Delta])$. □

1.2. Non-commutative Iwasawa theory for totally real fields

Now we review the formulation of the non-commutative Iwasawa main conjecture for totally real p-adic Lie extensions of totally real number fields. Let p be an odd prime number, F a totally real number field and F^∞/F a Galois extension of infinite degree satisfying the following three conditions:

1. the Galois group of F^∞/F is a compact p-adic Lie group;
2. only finitely many primes of F ramify in F^∞;
3. F^∞ is totally real and contains the cyclotomic $\mathbb{Z}_p$-extension F^{cyc} of F.

Fix a finite set Σ of primes of F containing all primes which ramify in F^∞. Let G be the Galois group of F^∞/F, H that of $F^\infty/F^{\mathrm{cyc}}$ and Γ that of F^{cyc}/F respectively. Note that H is a normal closed subgroup of G and the quotient group Γ is isomorphic to $\mathbb{Z}_p$ by definition. Let S denote a subset of $\Lambda(G)$ consisting of an element f such that the quotient module $\Lambda(G)/\Lambda(G)f$ is finitely generated

[2] φ is *not* necessarily induced by a group endomorphism because $g \mapsto g^p$ is *not* a group endomorphism in general.

as a left $\Lambda(H)$-module. John Coates et al. showed that S was a left and right Ore set without zero divisors [5, Theorem 2.4], which they called *the canonical Ore set for the group G* (see [14,25] for details of Ore localization). Let $\Lambda(G)_S$ denote the left Ore localization of $\Lambda(G)$ with respect to S (which is canonically isomorphic to the right Ore localization). It is well known that $\Lambda(G)_S$ is a semi-local ring [5, Theorem 4.2]. Now consider the Berrick–Keating's localization exact sequence [2] for the Ore localization $\Lambda(G) \to \Lambda(G)_S$:

$$K_1\big(\Lambda(G)\big) \to K_1\big(\Lambda(G)_S\big) \xrightarrow{\partial} K_0\big(\Lambda(G), \Lambda(G)_S\big) \to 0. \tag{1.1}$$

The connecting homomorphism ∂ is surjective by [5, Proposition 3.4]. Let $\mathscr{C}^{\mathrm{Perf}}(\Lambda(G))$ denote the category of complexes of finitely generated left $\Lambda(G)$-modules quasi-isomorphic to bounded complexes of finitely generated projective left $\Lambda(G)$-modules, and $\mathscr{C}_S^{\mathrm{Perf}}(\Lambda(G))$ the full subcategory of $\mathscr{C}^{\mathrm{Perf}}(\Lambda(G))$ generated by all objects of $\mathscr{C}^{\mathrm{Perf}}(\Lambda(G))$ with S-torsion cohomology groups. Then the relative Grothendieck group $K_0(\Lambda(G), \Lambda(G)_S)$ is canonically identified with the Grothendieck group $K_0(\mathscr{C}_S^{\mathrm{Perf}}(\Lambda(G)), \mathrm{qis})$ of the Waldhausen category $\mathscr{C}_S^{\mathrm{Perf}}(\Lambda(G))$ with quasi-isomorphisms as weak equivalences. Therefore for an arbitrary object $K^{\cdot}$ of the category $\mathscr{C}_S^{\mathrm{Perf}}(\Lambda(G))$, there exists an element $f_{K^{\cdot}}$ in $K_1(\Lambda(G)_S)$ satisfying $\partial(f_{K^{\cdot}}) = -[K^{\cdot}]$. We call such $f_{K^{\cdot}}$ *the characteristic element of* $K^{\cdot}$. By the localization exact sequence (1.1), the characteristic element of $K^{\cdot}$ is determined up to multiplication by an element in $K_1(\Lambda(G))$. Especially consider the complex

$$C = C_{F^\infty/F} = R\mathrm{Hom}\big(R\Gamma_{\text{ét}}\big(\operatorname{Spec}\mathcal{O}_{F^\infty}[1/\Sigma], \mathbb{Q}_p/\mathbb{Z}_p\big), \mathbb{Q}_p/\mathbb{Z}_p\big) \tag{1.2}$$

where $\Gamma_{\text{ét}}$ is the global section functor for étale topology. The cohomology groups of C are calculated as $H^0(C) = \mathbb{Z}_p$, $H^{-1}(C) = \mathrm{Gal}(M_\Sigma/F^\infty)$ where M_Σ is the maximal abelian pro-p extension of F^∞ unramified outside Σ, and $H^q(C) = 0$ for the other q. We denote $\mathrm{Gal}(M_\Sigma/F^\infty)$ by $X_\Sigma(F^\infty/F)$. Yoshitaka Hachimori and Romyar T. Sharifi proved the following proposition [9, Lemma 3.4]:

Proposition 1.4. *Let G' be a pro-p open subgroup of G and F' the maximal intermediate field of F^∞/F fixed by G'. Then the followings are equivalent*:

(1) *the Galois group $X_\Sigma(F^\infty/F)$ is S-torsion as a $\Lambda(G)$-module*;
(2) *the Iwasawa μ-invariant of the cyclotomic $\mathbb{Z}_p$-extension $(F')^{\mathrm{cyc}}/F'$ is zero.*

In particular $X_\Sigma(F^\infty/F)$ is S-torsion if the following condition is satisfied:

($\sharp$) there exists a finite subextension F' of F^∞ such that the Galois group of F^∞/F' is pro-p and $\mu((F')^{\mathrm{cyc}}/F') = 0$.

Kenkichi Iwasawa conjectured that the Iwasawa μ-invariant of the cyclotomic $\mathbb{Z}_p$-extension of an arbitrary number field is zero—Iwasawa's $\mu = 0$ conjecture—which implies that condition ($\sharp$) should be always satisfied. Famous Ferrero–Washington's theorem [7] asserts that condition ($\sharp$) is satisfied if $F/\mathbb{Q}$ is a finite abelian extension. We always assume condition ($\sharp$) in the rest of this article. This assumption implies that C is an object of $\mathscr{C}_S^{\mathrm{Perf}}(\Lambda(G))$, hence we may define *the characteristic element $f_{F^\infty/F}$ for F^∞/F* as that of the complex C.

Next we will define the "p-adic zeta function" as the element in $K_1(\Lambda(G)_S)$ satisfying certain interpolation properties. Let $\rho : G \to \mathrm{GL}_d(\overline{\mathbb{Q}})$ denote an Artin representation (that is, the kernel of ρ is an open subgroup) and let $\kappa : \mathrm{Gal}(F(\mu_{p^\infty})/F) \to \mathbb{Z}_p^\times$ denote the p-adic cyclotomic character. Since κ^r factors through the group Γ for an arbitrary natural number r divisible by $p-1$, $\rho\kappa^r$ induces a ring homomorphism $\Lambda(G) \to \mathrm{M}_d(E)$ where E is a certain finite extension of $\mathbb{Q}_p$. This also induces a homomorphism of Whitehead groups:

$$\mathrm{ev}_{\rho\kappa^r} : K_1\big(\Lambda(G)\big) \to K_1\big(\mathrm{M}_d(E)\big) \xrightarrow{\simeq} K_1(E) \cong E^\times$$

(the isomorphism $K_1(\mathrm{M}_d(E)) \xrightarrow{\simeq} K_1(E)$ is induced by the Morita equivalence between $\mathrm{M}_d(E)$ and E). Composing this with the natural inclusion $E^\times \to \overline{\mathbb{Q}}_p^\times$, we obtain the map $\mathrm{ev}_{\rho\kappa^r} : K_1(\Lambda(G)) \to \overline{\mathbb{Q}}_p^\times$. As is discussed in [5, Section 2], this map can be extended (in a non-trivial way) to $\mathrm{ev}_{\rho\kappa^r} : K_1(\Lambda(G)_S) \to \overline{\mathbb{Q}}_p \cup \{\infty\}$, which is called *the evaluation map at* $\rho\kappa^r$. We denote by $f(\rho\kappa^r)$ the element $\mathrm{ev}_{\rho\kappa^r}(f)$ for f in $K_1(\Lambda(G)_S)$.

Definition 1.5 *(p-adic zeta function).* If an element $\xi_{F^\infty/F}$ in $K_1(\Lambda(G)_S)$ satisfies the following *interpolation property*

$$\xi_{F^\infty/F}\big(\rho\kappa^r\big) = L_\Sigma\big(1-r; F^\infty/F, \rho\big) \tag{1.3}$$

for an arbitrary Artin representation ρ of G and an arbitrary natural number r divisible by $p-1$, we call it *the p-adic zeta function for* F^∞/F (here $L_\Sigma(s; F^\infty/F, \rho)$ denotes the complex Artin L-function of ρ in which the Euler factors at Σ are removed).

The Iwasawa main conjecture is formulated as follows:

Conjecture 1.6. *Let F and F^∞ be as above.*

(1) (*The existence of the p-adic zeta function*)
The p-adic zeta function $\xi_{F^\infty/F}$ for F^∞/F exists.
(2) (*The non-commutative Iwasawa main conjecture*)
The p-adic zeta function $\xi_{F^\infty/F}$ satisfies $\partial(\xi_{F^\infty/F}) = -[C_{F^\infty/F}]$.

Remark 1.7 *(The abelian case).* Assume that $G = \mathrm{Gal}(F^\infty/F)$ is an *abelian* p-adic Lie group. In this case, John Coates observed that if certain congruences among the special values of the partial zeta functions were proven, we could construct the p-adic L-function for F^∞/F [4, Hypotheses (H_n) and (C_0)]. These congruences were proven by Pierre Deligne and Kenneth A. Ribet [6] due to their deep results on Hilbert–Blumenthal modular varieties. Then by using Deligne–Ribet's congruences, Jean-Pierre Serre [24] constructed the element $\xi_{F^\infty/F}$ in the totally quotient ring of $\Lambda(G)$—*Serre's p-adic zeta pseudomeasure for* F^∞/F—which satisfied the following two properties:

(1) the element $(1-g)\xi_{F^\infty/F}$ is contained in $\Lambda(G)$ for arbitrary g in G;
(2) the element $\xi_{F^\infty/F}$ satisfies the interpolation property (1.3).

Remark 1.8. The non-commutative Iwasawa main conjecture which we introduced here was first formulated by John Coates, Takako Fukaya, Kazuya Kato, Ramdorai Sujatha and Otmar Venjakob [5] for elliptic curves without complex multiplication (the GL_2-conjecture). Then Takako Fukaya and Kazuya Kato [8] formulated the main conjecture for rather general cases and showed the compatibility of the main conjecture with the equivariant Tamagawa number conjecture [3].

2. The main theorem and Burns' technique

2.1. The main theorem

Consider the 1-dimensional pro-p-adic Lie group G which is the direct product of the finite p-group G^f defined as

$$G^f = \begin{pmatrix} 1 & \mathbb{F}_p & \mathbb{F}_p & \mathbb{F}_p \\ 0 & 1 & \mathbb{F}_p & \mathbb{F}_p \\ 0 & 0 & 1 & \mathbb{F}_p \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the commutative p-adic Lie group Γ isomorphic to $\mathbb{Z}_p$. In the following, we fix generators of G^f and denote them by

$$\alpha = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\delta = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \zeta = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The center of G^f is the cyclic subgroup generated by ζ and there exist four non-trivial fundamental relations

$$[\alpha, \beta] = \delta, \qquad [\beta, \gamma] = \varepsilon, \qquad [\alpha, \varepsilon] = \zeta, \qquad [\delta, \gamma] = \zeta$$

where $[x, y]$ denotes the commutator $xyx^{-1}y^{-1}$ of x and y. From now on we will assume that p is *not* equal to either 2 or 3, then the exponent of the finite p-group G^f coincides with p.

Theorem 2.1. *Let p be a positive prime number not equal to either* 2 *or* 3*, F a totally real number field and F^∞ a totally real Galois extension of F whose Galois group is isomorphic to $G = G^f \times \Gamma$. Assume that F^∞/F satisfies conditions* 2, 3 *and condition* (♯) *in Section* 1.2*. Then the p-adic zeta function $\xi_{F^\infty/F}$ for F^∞/F exists and the main conjecture* (*Conjecture* 1.6(2)) *is true for F^∞/F.*

Remark 2.2. On the uniqueness of the p-adic zeta function, we may conclude by Theorem 2.1 and Proposition 4.5 that the p-adic zeta function $\xi_{F^\infty/F}$ exists uniquely *up to multiplication by an element in $SK_1(\mathbb{Z}_p[G^f])$*. It is well known that $SK_1(\mathbb{Z}_p[\Delta])$ vanishes for many finite p-groups Δ. For the p-group G^f, Otmar Venjakob announced to the author that he and Peter Schneider have recently proven the vanishing of $SK_1(\mathbb{Z}_p[G^f])$, hence the p-adic zeta function $\xi_{F^\infty/F}$ in Theorem 2.1 is determined uniquely if we admit their results.

2.2. Burns' technique

There exists a certain strategy to construct the p-adic zeta functions for non-commutative extensions of totally real fields, which was first observed by David Burns. We will summarize his outstanding idea in this subsection.

Let F be a totally real number field and F^∞ a totally real p-adic Lie extension of F satisfying conditions 1–3 and condition (♯) in Section 1.2. Set $G = \mathrm{Gal}(F^\infty/F)$ (we do not have to assume that G is isomorphic to $G^f \times \Gamma$ in this subsection). Let $\mathfrak{F}$ be a family consisting of a pair (U, V) where U is an open subgroup of G and V is an open subgroup of H respectively such that V is normal in U and the quotient group U/V is commutative. Assume that the family $\mathfrak{F}$ satisfies the following hypothesis:

(♭) an arbitrary Artin representation of G is isomorphic to a $\mathbb{Z}$-linear combination of induced representations $\mathrm{Ind}_U^G(\chi_{U,V})$ as a virtual representation, where each (U, V) is an element in $\mathfrak{F}$ and $\chi_{U,V}$ is a character of the abelian group U/V of finite order.

In the following, we assume that there exists a family $\mathfrak{F}$ satisfying hypothesis (♭) and fix such $\mathfrak{F}$. For each pair (U, V) in $\mathfrak{F}$, let $\theta_{U,V} : K_1(\Lambda(G)) \to \Lambda(U/V)^\times$ denote the homomorphism defined as the composition of the norm map $\mathrm{Nr}_{\Lambda(G)/\Lambda(U)}$ and the canonical map $K_1(\Lambda(U)) \to \Lambda(U/V)^\times$.

Similarly define the homomorphism $\theta_{S,U,V}: K_1(\Lambda(G)_S) \to \Lambda(U/V)_S^\times$ for the canonical Ore localization $\Lambda(G)_S$ (we use the same symbol S for the canonical Ore set for U/V by abuse of notation). Set $\theta = (\theta_{U,V})_{(U,V)\in\mathfrak{F}}$ and $\theta_S = (\theta_{S,U,V})_{(U,V)\in\mathfrak{F}}$ respectively. Let Ψ_S be a certain subgroup of $\prod_{(U,V)\in\mathfrak{F}} \Lambda(U/V)_S^\times$ and let Ψ be the intersection of Ψ_S and $\prod_{(U,V)\in\mathfrak{F}} \Lambda(U/V)^\times$.

Definition 2.3 *(The theta map).* (See [13, Section 2.4].) Let G, $\mathfrak{F}$, θ, θ_S, Ψ and Ψ_S be as above. Suppose that θ, θ_S, Ψ and Ψ_S satisfy the following conditions:

(θ-1) the group Ψ coincides with the image of the map θ;
(θ-2) the group Ψ_S contains the image of the map θ_S.

Then we call the induced surjection $\theta: K_1(\Lambda(G)) \to \Psi$ *the theta map for the family* $\mathfrak{F}$, and call the induced homomorphism $\theta_S: K_1(\Lambda(G)_S) \to \Psi_S$ *the localized theta map for the family* $\mathfrak{F}$.

For each pair (U,V) in $\mathfrak{F}$, let F_U (resp. F_V) be the maximal subfield of F^∞ fixed by U (resp. V). Since the Galois group of F_V/F_U is abelian by assumption, the p-adic zeta pseudomeasure $\xi_{U,V}$ for F_V/F_U exists uniquely as an invertible element in the totally quotient ring of $\Lambda(U/V)$ (see Remark 1.7).

Theorem 2.4 *(David Burns). (See [13, Proposition 2.5].) Let F, F^∞ and G be as above. Assume that there exist a family $\mathfrak{F}$ of pairs (U,V) satisfying hypothesis ($\flat$), the theta map θ and the localized theta map θ_S for $\mathfrak{F}$. Also assume that $(\xi_{U,V})_{(U,V)\in\mathfrak{F}}$ is contained in Ψ_S. Then the p-adic zeta function $\xi_{F^\infty/F}$ for F^∞/F exists uniquely up to multiplication by the element in the kernel of θ. Moreover, the Iwasawa main conjecture is true for F^∞/F.*

Proof. Let f be an arbitrary characteristic element for F^∞/F (that is, an element in $K_1(\Lambda(G)_S)$ satisfying $\partial(f) = -[C_{F^\infty/F}]$). Let $u_{U,V}$ be the element defined as $\xi_{U,V}\theta_{S,U,V}(f)^{-1}$ for each pair (U,V) in $\mathfrak{F}$. Then $\partial(u_{U,V}) = 0$ holds by the functoriality of the connected homomorphism ∂ and the main conjecture $\partial(\xi_{U,V}) = -[C_{F_V/F_U}]$ for the abelian extension F_V/F_U proven by Wiles [29]. Hence $u_{U,V}$ is contained in $\Lambda(U/V)^\times$ by the exact sequence (1.1). On the other hand $(u_{U,V})_{(U,V)\in\mathfrak{F}}$ is contained in Ψ_S by (θ-2) and the assumption on $(\xi_{U,V})_{(U,V)\in\mathfrak{F}}$. Therefore $(u_{U,V})_{(U,V)\in\mathfrak{F}}$ is contained in the intersection of $\prod_{(U,V)\in\mathfrak{F}} \Lambda(U/V)^\times$ and Ψ_S, which coincides with Ψ by definition. Then there exists an element u in $K_1(\Lambda(G))$ satisfying $\theta(u) = (u_{U,V})_{(U,V)\in\mathfrak{F}}$ by condition (θ-1). Put $\xi_{F^\infty/F} = uf$ (we denote the image of u in $K_1(\Lambda(G)_S)$ by the same symbol u). By construction $\xi_{F^\infty/F}$ satisfies $\partial(\xi_{F^\infty/F}) = -[C_{F^\infty/F}]$ and $\theta_S(\xi_{F^\infty/F}) = (\xi_{U,V})_{(U,V)\in\mathfrak{F}}$. We may verify the interpolation property (1.3) by formal calculation, using the equation $\theta_S(\xi_{F^\infty/F}) = (\xi_{U,V})_{(U,V)\in\mathfrak{F}}$, hypothesis ($\flat$) and the inductivity of Artin L-functions. Finally suppose that $\xi^{(i)}_{F^\infty/F}$ is an element in $K_1(\Lambda(G)_S)$ which satisfies equations $\partial(\xi^{(i)}_{F^\infty/F}) = -[C_{F^\infty/F}]$ and $\theta_S(\xi^{(i)}_{F^\infty/F}) = (\xi_{U,V})_{(U,V)\in\mathfrak{F}}$ for $i = 1, 2$. Then the element w defined as $\xi^{(1)}_{F^\infty/F}(\xi^{(2)}_{F^\infty/F})^{-1}$ is identified with an element in $K_1(\Lambda(G))$ by the localization exact sequence (1.1), and the equation $\theta(w) = 1$ holds by construction. □

We remark that Kazuya Kato constructed the theta maps for p-adic Lie groups of Heisenberg type [13] and for certain open subgroups of $\mathbb{Z}_p^\times \ltimes \mathbb{Z}_p$ [12].

3. The additive theta map

The following three sections are devoted to the construction of the theta map θ and the localized theta map θ_S under the settings of Theorem 2.1. In this section, we construct a family $\mathfrak{F} = \{(U_i, V_i)\}_{i=0}^4$ satisfying hypothesis ($\flat$), define a $\mathbb{Z}_p$-module homomorphism $\theta^+: \mathbb{Z}_p[\![\mathrm{Conj}(G)]\!] \to \prod_{i=0}^4 \mathbb{Z}_p[\![U_i/V_i]\!]$ and characterize its image Ω. We will show that θ^+ induces an isomorphism between $\mathbb{Z}_p[\![\mathrm{Conj}(G)]\!]$ and Ω, which we call *the additive theta map for the family* $\mathfrak{F}$.

3.1. Construction of the family $\mathfrak{F}$

Consider the family $\mathfrak{F}$ consisting of the following five pairs of subgroups $\{(U_i, V_i)\}_{i=0}^4$ of G:

$$U_0 = G, \qquad V_0 = \begin{pmatrix} 1 & 0 & \mathbb{F}_p & \mathbb{F}_p \\ 0 & 1 & 0 & \mathbb{F}_p \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \{1\},$$

$$U_1 = \begin{pmatrix} 1 & \mathbb{F}_p & \mathbb{F}_p & \mathbb{F}_p \\ 0 & 1 & 0 & \mathbb{F}_p \\ 0 & 0 & 1 & \mathbb{F}_p \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \Gamma, \qquad V_1 = \begin{pmatrix} 1 & 0 & 0 & \mathbb{F}_p \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \{1\},$$

$$U_2 = \begin{pmatrix} 1 & 0 & \mathbb{F}_p & \mathbb{F}_p \\ 0 & 1 & \mathbb{F}_p & \mathbb{F}_p \\ 0 & 0 & 1 & \mathbb{F}_p \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \Gamma, \qquad V_2 = \begin{pmatrix} 1 & 0 & 0 & \mathbb{F}_p \\ 0 & 1 & 0 & \mathbb{F}_p \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \{1\},$$

$$U_3 = \begin{pmatrix} 1 & 0 & 0 & \mathbb{F}_p \\ 0 & 1 & \mathbb{F}_p & \mathbb{F}_p \\ 0 & 0 & 1 & \mathbb{F}_p \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \Gamma, \qquad V_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \mathbb{F}_p \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \{1\},$$

$$U_4 = \begin{pmatrix} 1 & 0 & 0 & \mathbb{F}_p \\ 0 & 1 & 0 & \mathbb{F}_p \\ 0 & 0 & 1 & \mathbb{F}_p \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \Gamma, \qquad V_4 = \{\mathrm{id}\} \times \{1\}.$$

Note that V_i is the commutator subgroup of U_i for each i (thus the quotient group U_i/V_i is abelian).

Proposition 3.1. *The family $\mathfrak{F}$ satisfies hypothesis* ($\flat$) *in Section* 2.2.

Before the proof, note that G^f is regarded as the semi-direct product

$$\begin{pmatrix} 1 & \mathbb{F}_p & \mathbb{F}_p \\ 0 & 1 & \mathbb{F}_p \\ 0 & 0 & 1 \end{pmatrix} \ltimes \begin{pmatrix} \mathbb{F}_p \\ \mathbb{F}_p \\ \mathbb{F}_p \end{pmatrix} \left(\cong \begin{pmatrix} 1 & \mathbb{F}_p & \mathbb{F}_p & 0 \\ 0 & 1 & \mathbb{F}_p & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ltimes \begin{pmatrix} 1 & 0 & 0 & \mathbb{F}_p \\ 0 & 1 & 0 & \mathbb{F}_p \\ 0 & 0 & 1 & \mathbb{F}_p \\ 0 & 0 & 0 & 1 \end{pmatrix} \right)$$

by the obvious manner which we denote by $H \ltimes N$. The subgroup H acts on N from the left as ordinary product of matrices. We will identify the group N with the 3-dimensional $\mathbb{F}_p$-vector space, and choose such a basis $\{v_\ell\}_{\ell=1}^3$ of N as $x_1 v_1 + x_2 v_2 + x_3 v_3$ corresponds to ${}^t(x_1, x_2, x_3)$. Set $N_\ell = \mathbb{F}_p v_\ell$ for each ℓ. To show Proposition 3.1, it suffices to check that $\mathfrak{F}^f = \{(U_i^f, V_i^f)\}_{i=0}^4$ satisfies hypothesis ($\flat$) for the finite p-group G^f where U_i^f and V_i^f denote the first factors ("finite parts") of U_i and V_i respectively. We use representation theory of semi-direct products of finite groups [23, Chapitre 9.2].

Proof of Proposition 3.1. Let $\mathfrak{X}(N)$ denote the character group of the abelian group N. It consists of a character $\chi_{i,j,k}$ for $0 \leqslant i, j, k \leqslant p-1$ which is defined by $\chi_{ijk}({}^t(x_1, x_2, x_3)) = \exp(2\pi\sqrt{-1}(x_1 i + x_2 j + x_3 k)/p)$. The left action of H on N naturally induces the right action of H on $\mathfrak{X}(N)$). It is easy to see that $\bigcup_{\ell=1}^3 \mathfrak{X}(N_\ell)$ forms a set of representatives of the orbital decomposition $\mathfrak{X}(N)/H$. Let H_ℓ denote the isotropic subgroup of H at a character in $\mathfrak{X}(N_\ell) \setminus \{\chi_{0,0,0}\}$ for each ℓ. Then H_ℓ has the following explicit description:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \mathbb{F}_p \\ 0 & 0 & 1 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 1 & \mathbb{F}_p & \mathbb{F}_p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad H_3 = H.$$

By using representation theory on semi-direct products, an arbitrary irreducible representation of G^f is isomorphic to one of the following representations [23, Chapitre 9.2, Théorème 17]:

- $\mathrm{Ind}_{H_1 \ltimes N}^{G^f}(\rho_1 \otimes \chi_1)$ where ρ_1 is a character of H_1 and χ_1 is an element in $\mathfrak{X}(N_1) \setminus \{\chi_{0,0,0}\}$ (namely χ_1 coincides with $\chi_{i,0,0}$ for certain i except for 0).
- $\mathrm{Ind}_{H_2 \ltimes N}^{G^f}(\rho_2 \otimes \chi_2)$ where ρ_2 is a character of H_2 and χ_2 is an element in $\mathfrak{X}(N_2) \setminus \{\chi_{0,0,0}\}$ (namely χ_2 coincides with $\chi_{0,j,0}$ for certain i except for 0).
- $\rho_3 \otimes \chi_3$ where ρ_3 is an irreducible representation of H_3 and χ_3 is an element in $\mathfrak{X}(N_3)$ (namely χ_3 coincides with $\chi_{0,0,k}$ for certain k).

Note that $\rho_1 \otimes \chi_1$ (resp. $\rho_2 \otimes \chi_2$) is regarded as a character of the abelian group U_3^f/V_3^f (resp. U_1^f/V_1^f). By applying the similar argument to the semi-direct product

$$H = \begin{pmatrix} 1 & \mathbb{F}_p & \mathbb{F}_p \\ 0 & 1 & \mathbb{F}_p \\ 0 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & \mathbb{F}_p & \mathbb{F}_p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rtimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \mathbb{F}_p \\ 0 & 0 & 1 \end{pmatrix}$$

(which we will denote by $N' \rtimes H'$), we may prove that an arbitrary irreducible representation of H $(= H_3)$ is isomorphic to either a character of the abelization of H_3 or the induced representation of a character of the abelian subgroup N'. Hence each $\rho_3 \otimes \chi_3$ is isomorphic to the induced representation of a character of either U_0^f/V_0^f or U_1^f/V_1^f. Consequently the family $\mathfrak{F}^f = \{(U_i^f, V_i^f)\}_{i=0,1,3}$ satisfies hypothesis (♭) for the group G^f. We will add (U_2^f, V_2^f) and (U_4^f, V_4^f) to our family $\mathfrak{F}^f$ for certain technical reasons. $\square$

3.2. *Construction of the isomorphism θ^+*

Take a system of representatives $\{a_1, a_2, \ldots, a_{r_i}\}$ of the left coset decomposition G/U_i for each i. Set $\mathrm{Tr}_i([g]) = \sum_{j=1}^{r_i} \tau_j([g])$ for each i and for a conjugacy class $[g]$ of G, where $\tau_j([g])$ is defined as the conjugacy class of $a_j^{-1} g a_j$ in U_i if $a_j^{-1} g a_j$ is contained in U_i and 0 otherwise. It is easy to see that $\mathrm{Tr}_i([g])$ is independent of the choice of representatives $\{a_j\}_{j=1}^{r_i}$, thus Tr_i induces a well-defined $\mathbb{Z}_p$-module homomorphism $\mathbb{Z}_p[\![\mathrm{Conj}(G)]\!] \to \mathbb{Z}_p[\![\mathrm{Conj}(U_i)]\!]$ which we call *the trace homomorphism from* $\mathbb{Z}_p[\![\mathrm{Conj}(G)]\!]$ *to* $\mathbb{Z}_p[\![\mathrm{Conj}(U_i)]\!]$. Now we define the homomorphism θ_i^+ as the composition of the trace map Tr_i and the natural surjection $\mathbb{Z}_p[\![\mathrm{Conj}(U_i)]\!] \to \mathbb{Z}_p[\![U_i/V_i]\!]$. Set $\theta^+ = (\theta_i^+)_{i=0}^4$. The value $\theta^+([g])$ for each conjugacy class $[g]$ of G^f is given in Table 1. There we use the notation $[a, b, c, d, e, f]$ for the conjugacy class containing $\begin{pmatrix} 1 & a & d & f \\ 0 & 1 & b & e \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and the notation h_{u_i} for the element $1 + u_i + \cdots + u_i^{p-1}$ where u_i is an element in U_i/V_i.

Let I_i^f be the image of the map $\theta_i^{+,f} : \mathbb{Z}_p[\mathrm{Conj}(G^f)] \to \mathbb{Z}_p[U^f/V^f]$ induced by θ_i^+ for each i except for 0. Then by Table 1, we easily see that each I_i^f is described as in Table 2. Note that generators of each I_i^f given in Table 2 are linearly independent over $\mathbb{Z}_p$ except for $i = 3$; those of I_3^f have one non-trivial relation $\sum_{f=0}^{p-1} p^2 \zeta^f = p \cdot p h_\zeta$. For each natural number n, let $G^{(n)}$ denote the finite p-group $G^f \times \Gamma/\Gamma^{p^n}$ and $U_i^{(n)}$ the finite p-group $U_i^f \times \Gamma/\Gamma^{p^n}$ respectively. Set $I_i^{(n)} = I_i^f \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Gamma/\Gamma^{p^n}]$ (regarded as a $\mathbb{Z}_p$-submodule of $\mathbb{Z}_p[U_i^{(n)}/V_i]$). Obviously the projective system $\{I_i^{(n)}\}_{n \in \mathbb{N}}$ has the limit I_i which coincides with the image of θ_i^+.

Table 1
Calculation of $\theta^+([g])$ for $[g]$ in $\mathrm{Conj}(G^f)$.

Conjugacy class	$\theta^+ = (\theta_0^+, \theta_1^+, \theta_2^+, \theta_3^+, \theta_4^+)$
$\mathfrak{c}_{\mathrm{I}}^{(a,b,c)} = [a,b,c,d,e,f]$ $(a \neq 0, b \neq 0)$	$(\alpha^a\beta^b\gamma^c, 0, 0, 0, 0)$
$\mathfrak{c}_{\mathrm{II}}^{(a,c,d,e)} = [a,0,c,d,e,f]$ $(a \neq 0, c \neq 0)$	$(\alpha^a\gamma^c,\ \alpha^a\gamma^c\delta^d\varepsilon^e h_{\varepsilon^c\delta^{-a}}, 0, 0, 0)$
$\mathfrak{c}_{\mathrm{III}}^{(a,e)} = [a,0,0,d,e,f]$ $(a \neq 0)$	$(\alpha^a,\ \alpha^a\varepsilon^e h_\delta, 0, 0, 0)$
$\mathfrak{c}_{\mathrm{IV}}^{(b,c)} = [0,b,c,d,e,f]$ $(b \neq 0, c \neq 0)$	$(\beta^b\gamma^c,\ 0,\ \beta^b\gamma^c h_\delta,\ \beta^b\gamma^c h_\zeta, 0)$
$\mathfrak{c}_{\mathrm{V}}^{(b,f)} = [0,b,0,d,e,f]$ $(b \neq 0)$	$(\beta^b,\ 0,\ \beta^b h_\delta,\ p\beta^b\zeta^{f-e\cdot\frac{d}{b}}, 0)$
$\mathfrak{c}_{\mathrm{VI}}^{(c,d)} = [0,0,c,d,e,f]$ $(c \neq 0, d \neq 0)$	$(\gamma^c,\ \gamma^c\delta^d h_\varepsilon,\ p\gamma^c\delta^d, 0, 0)$
$\mathfrak{c}_{\mathrm{VII}}^{(c)} = [0,0,c,0,e,f]$ $(c \neq 0)$	$(\gamma^c,\ \gamma^c h_\varepsilon,\ p\gamma^c,\ p\gamma^c h_\zeta, p\gamma^c h_\varepsilon h_\zeta)$
$\mathfrak{c}_{\mathrm{VIII}}^{(d,e)} = [0,0,0,d,e,f]$ $(d \neq 0)$	$(1,\ p\delta^d\varepsilon^e,\ p\delta^d, 0, 0)$
$\mathfrak{c}_{\mathrm{IX}}^{(e)} = [0,0,0,0,e,f]$ $(e \neq 0)$	$(1,\ p\varepsilon^e,\ p,\ ph_\zeta, p^2\varepsilon^e h_\zeta)$
$\mathfrak{c}_{\mathrm{X}}^{(f)} = [0,0,0,0,0,f]$	$(1, p, p, p^2\zeta^f, p^3\zeta^f)$

Table 2
Calculation of I_i^f for each i.

$I_1^f = [\alpha^a\gamma^c\delta^d\varepsilon^e h_{\varepsilon^c\delta^{-a}}]_{\mathbb{Z}_p}$, $\quad I_2^f = [\beta^b\gamma^c h_\delta\ (b \neq 0),\ p\gamma^c\delta^d]_{\mathbb{Z}_p}$,
$I_3^f = [p^2\zeta^f, p\gamma^c h_\zeta, \beta^b\gamma^c h_\zeta\ (b \neq 0, c \neq 0),\ p\beta^b\zeta^f\ (b \neq 0)]_{\mathbb{Z}_p}$,
$I_4^f = [p^3\zeta^f, p^2\varepsilon^e h_\zeta\ (e \neq 0),\ p\gamma^c h_\varepsilon h_\zeta\ (c \neq 0)]_{\mathbb{Z}_p}$.

Definition 3.2. We define Ω to be the $\mathbb{Z}_p$-submodule of $\prod_{i=0}^4 \mathbb{Z}_p[\![U_i/V_i]\!]$ consisting of an element $(y_i)_{i=0}^4$ such that

1. (trace relations) the following equations hold:
   (rel-1) $\mathrm{Tr}_{\mathbb{Z}_p[\![U_0/V_0]\!]/\mathbb{Z}_p[\![U_1/V_0]\!]}(y_0) \equiv y_1$,
   (rel-2) $\mathrm{Tr}_{\mathbb{Z}_p[\![U_0/V_0]\!]/\mathbb{Z}_p[\![U_2/V_0]\!]}(y_0) \equiv y_2$,
   (rel-3) $\mathrm{Tr}_{\mathbb{Z}_p[\![U_2/V_2]\!]/\mathbb{Z}_p[\![U_3/V_2]\!]}(y_2) \equiv y_3$,
   (rel-4) $\mathrm{Tr}_{\mathbb{Z}_p[\![U_1/V_2]\!]/\mathbb{Z}_p[\![U_1\cap U_2/V_2]\!]}(y_1) \equiv \mathrm{Tr}_{\mathbb{Z}_p[\![U_2/V_2]\!]/\mathbb{Z}_p[\![U_1\cap U_2/V_2]\!]}(y_2)$,
   (rel-5) $\mathrm{Tr}_{\mathbb{Z}_p[\![U_1/V_1]\!]/\mathbb{Z}_p[\![U_4/V_1]\!]}(y_1) \equiv y_4$,
   (rel-6) $\mathrm{Tr}_{\mathbb{Z}_p[\![U_3/V_3]\!]/\mathbb{Z}_p[\![U_4/V_3]\!]}(y_3) \equiv y_4$
   (see Fig. 1);
2. the element y_i is contained in I_i for each i except for 0.

Proposition–Definition 3.3. *The homomorphism θ^+ induces an isomorphism between $\mathbb{Z}_p[\![\mathrm{Conj}(G)]\!]$ and Ω, which we call the additive theta map for $\mathfrak{F}$.*

Proof. It is clear by construction that Ω contains the image of θ^+, hence we will prove the injectivity and surjectivity of the induced map $\theta^+ : \mathbb{Z}_p[\![\mathrm{Conj}(G)]\!] \to \Omega$. It suffices to show that the homomorphism $\theta^{+,(n)} : \mathbb{Z}_p[\mathrm{Conj}(G^{(n)})] \to \Omega^{(n)}$ induced by θ^+ is isomorphic for each natural number n, where $\Omega^{(n)}$ is defined to be the $\mathbb{Z}_p$-submodule of $\prod_{i=0}^4 \mathbb{Z}_p[U_i^{(n)}/V_i]$ satisfying the conditions corresponding to those in Definition 3.2.

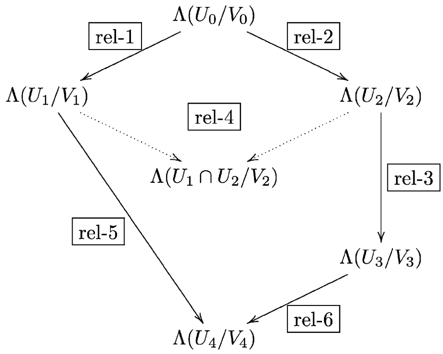

Fig. 1. Trace and norm relations.

Injectivity. Let $y=\sum_{[g]\in \mathrm{Conj}(G^{(n)})} m_{[g]}[g]$ be an element contained in the kernel of $\theta^{+,(n)}$. To prove that y is equal to zero, we will show that $\tilde{\rho}(y)$ vanishes for an arbitrary class function $\tilde{\rho}$ on the group $G^{(n)}$. By hypothesis ($\flat$), it suffices to prove that $\tilde{\chi}_i(y)$ vanishes for each i where $\tilde{\chi}_i$ is the associated character to the representation $\mathrm{Ind}_{U_i^{(n)}}^{G^{(n)}}(\chi_i)$ induced by an arbitrary character χ_i of the abelian group $U_i^{(n)}/V_i$. Then we have

$$\tilde{\chi}_i(y)=\sum_{[g]\in \mathrm{Conj}(G^{(n)})} m_{[g]} \sum_{j, a_j^{-1}ga_j\in U_i^{(n)}} \chi_i\bigl(a_j^{-1}ga_j\bigr)=\chi_i\circ\theta_i^{+,(n)}(y)$$

by the definition of $\tilde{\chi}_i$ (where $\{a_1,\dots,a_{r_i}\}$ is a system of representatives of the left coset decomposition $G^{(n)}/U_i^{(n)}$). Therefore the proof is done because $\theta_i^{+,(n)}(y)$ vanishes for each i by assumption.

Surjectivity. Let $(y_i)_{i=0}^4$ be an arbitrary element in $\Omega^{(n)}$. Then each y_i is described as a $\mathbb{Z}_p[\Gamma/\Gamma^{p^n}]$-linear combination of generators of $I_i^{(n)}$:

$$\begin{aligned}
y_0&=\sum_{a,b,c}\kappa_{abc}\alpha^a\beta^b\gamma^c,\\
y_1&=\sum_{d,e}\lambda_{de}^{(1)}\delta^d\varepsilon^e+\sum_{a\neq 0,e}\lambda_{ae}^{(2)}\alpha^a\varepsilon^e h_\delta\\
&\quad+\sum_{c\neq 0,d}\lambda_{cd}^{(3)}\gamma^c\delta^d h_\varepsilon+\sum_{a\neq 0,c\neq 0,d,e}\lambda_{acde}^{(4)}\alpha^a\gamma^c\delta^d\varepsilon^e h_{\varepsilon^c\delta^{-a}},\\
y_2&=\sum_{b\neq 0,c}\mu_{bc}^{(1)}\beta^b\gamma^c h_\delta+\sum_{c,d}p\mu_{cd}^{(2)}\gamma^c\delta^d,\\
y_3&=\sum_f p^2\nu_f^{(1)}\zeta^f+\sum_c p\nu_c^{(2)}\gamma^c h_\zeta+\sum_{b\neq 0,c\neq 0}\nu_{bc}^{(3)}\beta^b\gamma^c h_\zeta+\sum_{b\neq 0,f}p\nu_{bf}^{(4)}\beta^b\zeta^f,\\
y_4&=\sum_f p^3\sigma_f^{(1)}\zeta^f+\sum_{e\neq 0}p^2\sigma_e^{(2)}\varepsilon^e h_\zeta+\sum_{c\neq 0}p\sigma_c^{(3)}\gamma^c h_\varepsilon h_\zeta.
\end{aligned}$$

Note that $\nu_f^{(1)}$ $(0\leqslant f\leqslant p-1)$ and $\nu_0^{(2)}$ are not determined uniquely because of the relation $\sum_{f=0}^{p-1}p^2\zeta^f=p\cdot ph_\zeta$. The trace relations in Definition 3.2 give constraints among coefficients above, which are described explicitly as follows:

(rel-1) $\kappa_{000} = \sum_{d,e} \lambda_{de}^{(1)}, \qquad \kappa_{a00} = \sum_{e} \lambda_{ae}^{(2)} \quad (a \neq 0),$

$$\kappa_{00c} = \sum_{d} \lambda_{cd}^{(3)} \quad (c \neq 0), \qquad \kappa_{a0c} = \sum_{d,e} \lambda_{acde}^{(4)} \quad (a \neq 0, c \neq 0),$$

(rel-2) $\kappa_{00c} = \sum_{d} \mu_{cd}^{(2)}, \qquad \kappa_{0bc} = \mu_{bc}^{(1)} \quad (b \neq 0),$

(rel-3) $\mu_{bc}^{(1)} = \nu_{bc}^{(3)} \quad (b \neq 0, c \neq 0), \qquad \mu_{b0}^{(1)} = \sum_{f} \nu_{bf}^{(4)} \quad (b \neq 0),$

$$\mu_{c0}^{(2)} = \nu_{c}^{(2)} \quad (c \neq 0), \qquad \mu_{00}^{(2)} = \sum_{f} \nu_{f}^{(1)} + \nu_{0}^{(2)},$$

(rel-4) $\sum_{e} \lambda_{de}^{(1)} = \mu_{0d}^{(2)}, \qquad \lambda_{cd}^{(3)} = \mu_{cd}^{(2)} \quad (c \neq 0),$

(rel-5) $\lambda_{00}^{(1)} = \sum_{f} \sigma_{f}^{(1)}, \qquad \lambda_{0e}^{(1)} = \sigma_{e}^{(2)} \quad (e \neq 0), \qquad \lambda_{c0}^{(3)} = \sigma_{c}^{(3)} \quad (c \neq 0),$

(rel-6) $\nu_{f}^{(1)} = \sigma_{f}^{(1)}, \qquad \nu_{0}^{(2)} = \sum_{e \neq 0} \sigma_{e}^{(2)}, \qquad \nu_{c}^{(2)} = \sigma_{c}^{(3)} \quad (c \neq 0).$

We remark that in order to derive the relations $\nu_f^{(1)} = \sigma_f^{(1)}$ and $\nu_0^{(2)} = \sum_{e \neq 0} \sigma_e^{(2)}$ in (rel-6), we have to replace $\nu_f^{(1)}$ and $\nu_0^{(2)}$ appropriately by using the relation $\sum_{f=0}^{p-1} p^2 \zeta^f = p \cdot ph_\zeta$. Then we may calculate directly that the element y in $\mathbb{Z}_p[\mathrm{Conj}(G^{(n)})]$ defined as

$$\begin{aligned} y = & \sum_{a \neq 0, b \neq 0, c} \kappa_{abc} \mathfrak{c}_{\mathrm{I}}^{(a,b,c)} + \sum_{a \neq 0, c \neq 0, d, e} \lambda_{acde}^{(4)} \mathfrak{c}_{\mathrm{II}}^{(a,c,d,e)} + \sum_{a \neq 0, e} \lambda_{ae}^{(2)} \mathfrak{c}_{\mathrm{III}}^{(a,e)} \\ & + \sum_{b \neq 0, c \neq 0} \mu_{bc}^{(1)} \mathfrak{c}_{\mathrm{IV}}^{(b,c)} + \sum_{b \neq 0, f} \nu_{bf}^{(4)} \mathfrak{c}_{\mathrm{V}}^{(b,f)} + \sum_{c \neq 0, d \neq 0} \mu_{cd}^{(2)} \mathfrak{c}_{\mathrm{VI}}^{(c,d)} + \sum_{c \neq 0} \nu_{c}^{(2)} \mathfrak{c}_{\mathrm{VII}}^{(c)} \\ & + \sum_{d \neq 0, e} \lambda_{de}^{(1)} \mathfrak{c}_{\mathrm{VIII}}^{(d,e)} + \sum_{e \neq 0} \sigma_{e}^{(2)} \mathfrak{c}_{\mathrm{IX}}^{(e)} + \sum_{f} \nu_{f}^{(1)} \mathfrak{c}_{\mathrm{X}}^{(f)} \end{aligned}$$

satisfies $\theta^{+,(n)}(y) = (y_i)_{i=0}^4$ by using the explicit trace relations above. □

4. Translation into the multiplicative theta map

In this section we will translate the additive theta map θ^+ constructed in the previous subsection into *the multiplicative theta map* θ. The main tool is *the integral logarithmic homomorphism* introduced in Section 1.1. Let θ_i denote the composition of the norm map Nr_i and the canonical homomorphism $K_1(\Lambda(U_i)) \to \Lambda(U_i/V_i)^\times$ for each i, as in Section 0. Set $\theta = (\theta_i)_{i=0}^4$.

Proposition–Definition 4.1 *(Frobenius homomorphism). Set $\varphi(g) = g^p$ for arbitrary g in G. Then φ induces a group homomorphism $\varphi : G \to \Gamma$, which we call the Frobenius homomorphism. We denote the induced ring homomorphism $\Lambda(G) \to \Lambda(\Gamma)$ by the same symbol φ, and also call it the Frobenius homomorphism.*

Proof. Since the exponent of G^f is p, φ coincides with $\varphi_\Gamma \circ \pi_G$ where $\varphi_\Gamma : \Gamma \to \Gamma$ is the Frobenius endomorphism on Γ induced by $t \mapsto t^p$ and π_G is the canonical projection $G \to \Gamma$, hence φ is clearly a group homomorphism. □

Remark 4.2. The correspondence $g \mapsto g^p$ does not induce a group endomorphism $\varphi : G \to G$ in general.

In the following, we also denote by the same symbol φ the induced ring homomorphism $\Lambda(U_i/V_i) \to \Lambda(\Gamma)$.

Lemma 4.3. *Let G_1^f and G_2^f be arbitrary subgroups of G^f. For $\ell = 1, 2$, set $G_\ell = G_\ell^f \times \Gamma$. Suppose that G_1 contains G_2. Then the Verlagerung homomorphism $\mathrm{Ver}_{G_2}^{G_1} : G_1^{\mathrm{ab}} \to G_2^{\mathrm{ab}}$ coincides with the composition of the e-th power of the Frobenius homomorphism $\varphi^e : G_1^{\mathrm{ab}} \to \Gamma$ and the canonical injection $\Gamma \to G_2^{\mathrm{ab}}$ where we denote the index of G_2 in G_1 by p^e.*

Proof. By the transitivity of the Verlagerung homomorphism, it suffices to prove the claim when the index of G_2 in G_1 is p. In this case it is well known that G_2 is normal in G_1. Let σ be an element in G_1^f which generates G_1/G_2. Then $\{\sigma^j\}_{j=0}^{p-1}$ is a set of representatives of G_1/G_2. Suppose that an element g in G_1 satisfies $gG_2 = \sigma^{j(g)} G_2$. Then by the definition of the Verlagerung homomorphisms, we have $\mathrm{Ver}_{G_2}^{G_1}(g) = \prod_{j=0}^{p-1} \sigma^{-(j+j(g))} g \sigma^j = g^p = \varphi(g)$. □

Hence in our special case (more generally, in the case when the exponent of the finite part G^f is p), we may identify every Verlagerung homomorphism with the several power of the Frobenius homomorphism.

Definition 4.4. We define Ψ to be the subgroup of $\prod_{i=0}^4 \Lambda(U_i/V_i)^\times$ consisting of an element $(\eta_i)_{i=0}^4$ such that

1. (norm relations) the following equations hold:
   (rel-1) $\mathrm{Nr}_{\Lambda(U_0/V_0)/\Lambda(U_1/V_0)}(\eta_0) \equiv \eta_1$,
   (rel-2) $\mathrm{Nr}_{\Lambda(U_0/V_0)/\Lambda(U_2/V_0)}(\eta_0) \equiv \eta_2$,
   (rel-3) $\mathrm{Nr}_{\Lambda(U_2/V_2)/\Lambda(U_3/V_2)}(\eta_2) \equiv \eta_3$,
   (rel-4) $\mathrm{Nr}_{\Lambda(U_1/V_2)/\Lambda(U_1 \cap U_2/V_2)}(\eta_1) \equiv \mathrm{Nr}_{\Lambda(U_2/V_2)/\Lambda(U_1 \cap U_2/V_2)}(\eta_2)$,
   (rel-5) $\mathrm{Nr}_{\Lambda(U_1/V_1)/\Lambda(U_4/V_1)}(\eta_1) \equiv \eta_4$,
   (rel-6) $\mathrm{Nr}_{\Lambda(U_3/V_3)/\Lambda(U_4/V_3)}(\eta_3) \equiv \eta_4$
   (see Fig. 1);
2. (congruences) the congruence $\eta_i \equiv \varphi(\eta_0)^{(G:U_i)/p} \bmod I_i$ holds for each i except for 0.

Proposition 4.5. *The homomorphism θ induces a surjection from $K_1(\Lambda(G))$ onto Ψ with kernel $SK_1(\mathbb{Z}_p[G^f])$. In other words, θ induces the theta map for the family $\mathfrak{F}$ (in the sense of Definition 2.3).*

In the rest of this section, we prove Proposition 4.5 by using the additive theta map θ^+ and the integral logarithmic homomorphisms. Since the integral logarithmic homomorphisms are defined only for group rings of finite groups, we fix a natural number n and prove the isomorphy for finite quotients $\theta^{(n)} : K_1(\mathbb{Z}_p[G^{(n)}])/SK_1(\mathbb{Z}_p[G^f]) \xrightarrow{\simeq} \Psi^{(n)} (\subseteq \prod_i \mathbb{Z}_p[U_i^{(n)}/V_i]^\times)$ where $\Psi^{(n)}$ is the subgroup of $\prod_i \mathbb{Z}_p[U_i^{(n)}/V_i]^\times$ defined by the same conditions as in Definition 4.4. Then we may obtain the desired surjection by taking the projective limit.

4.1. Logarithmic isomorphisms

In the following three subsections, we fix a natural number n.

Lemma 4.6. *For each i except for 0, $1+I_i^{(n)}$ is a multiplicative subgroup of $\mathbb{Z}_p[U_i^{(n)}/V_i]^\times$ and the p-adic logarithmic homomorphism induces an isomorphism between $1+I_i^{(n)}$ and $I_i^{(n)}$.*

Proof. By direct calculation, we have

$$\begin{aligned}
\big(I_1^{(n)}\big)^2 &= \big[p\alpha^a\gamma^c\delta^d\varepsilon^e h_{\varepsilon^c\delta^{-a}}, p\delta^d\varepsilon^e h_{\varepsilon^c\delta^{-a}}, \alpha^a\gamma^c h_{\varepsilon,\delta}\big((a,c)\neq(0,0)\big)\big]_{\mathbb{Z}_p[\Gamma/\Gamma^{p^n}]},\\
\big(I_2^{(n)}\big)^2 &= \big[p\beta^b\gamma^c h_\delta, p^2\gamma^c\delta^d\big]_{\mathbb{Z}_p[\Gamma/\Gamma^{p^n}]}, \qquad \big(I_3^{(n)}\big)^2 = \big[p\beta^b\gamma^c h_\zeta, p^2\beta^b\zeta^f\big]_{\mathbb{Z}_p[\Gamma/\Gamma^{p^n}]},\\
\big(I_4^{(n)}\big)^2 &= \big[p^6\zeta^f, p^5\varepsilon^e h_\zeta, p^4\gamma^c h_\varepsilon h_\zeta\big]_{\mathbb{Z}_p[\Gamma/\Gamma^{p^n}]}
\end{aligned}$$

where $h_{\delta,\varepsilon}$ is defined as $\sum_{0\leqslant d,e\leqslant p-1}\delta^d\varepsilon^e$. This calculation implies that $(I_i^{(n)})^2$ is contained in $I_i^{(n)}$, thus each $1+I_i^{(n)}$ is stable under multiplication. Moreover we may calculate directly that $(I_i^{(n)})^4$ coincides with $p(I_i^{(n)})^3$ for $i=1$ and $(I_i^{(n)})^3$ coincides with $p(I_i^{(n)})^2$ for other i. Set $N=3$ if i is equal to 1 and $N=2$ otherwise. Note that the topology on $(I_i^{(n)})^N$ induced by the filtration $\{(I_i^{(n)})^{N+m}\}_{m\in\mathbb{N}}$ coincides with the p-adic topology by the calculation above.

The existence of inverse elements. By the argument above, y^m is contained in $p^{m-N}(I_i^{(n)})^N$ for an arbitrary element y in $I_i^{(n)}$ and for an arbitrary natural number m larger than N. Hence $(1+y)^{-1}=\sum_{m\geqslant 0}(-1)^m y^m$ converges in $I_i^{(n)}$ with respect to the p-adic topology.

Convergence of logarithms. Take an arbitrary element y in $I_i^{(n)}$. Then by simple calculation y^m/m is contained in $I_i^{(n)}$ for m less than N (use the assumption $p\neq 2,3$) and in $(p^{m-N}/m)I_i^{(n)}$ for m strictly larger than N. Therefore the logarithm $\log(1+y)=\sum_{m\geqslant 1}(-1)^{m-1}(y^m/m)$ converges p-adically in $I_i^{(n)}$.

Logarithmic isomorphisms. Let m be a natural number. Since $\{(I_i^{(n)})^m\}^{N+1}$ is contained in $p^m(I_i^{(n)})^{mN}$, we may show that $1+(I_i^{(n)})^m$ is a subgroup of $1+I_i^{(n)}$ and the logarithm on $1+(I_i^{(n)})^m$ converges p-adically in $(I_i^{(n)})^m$ by the same argument as above. Since $\{(I_i^{(n)})^{N+m}\}_{m\in\mathbb{N}}$ gives the p-adic topology on $(I_i^{(n)})^N$, it is sufficient to show that the logarithm induces an isomorphism

$$\big(1+\big(I_i^{(n)}\big)^m\big)/\big(1+\big(I_i^{(n)}\big)^{m+1}\big)\xrightarrow{\simeq}\big(I_i^{(n)}\big)^m/\big(I_i^{(n)}\big)^{m+1}; \qquad 1+y\mapsto y.$$

To prove it, we have only to verify that y^{p^k}/p^k is contained in $(I_i^{(n)})^{m+1}$ for an arbitrary element $1+y$ in $1+(I_i^{(n)})^m$ and for an arbitrary natural number k. By direct calculation, y^{p^k} is contained in $p^{p^k-3}(I_1^{(n)})^2$ if both i and m are equal to 1 and contained in $p^{mp^k-m-1}(I_i^{(n)})^{m+1}$ otherwise. Then it is easy to see that both p^{p^k-3}/p^k and p^{mp^k-m-1}/p^k are integers (use the assumption $p\neq 2,3$). $\square$

For the later use, we now introduce the $\mathbb{Z}_p$-submodule J_3 of $\mathbb{Z}_p[\![U_3/V_3]\!]$, which is defined as the image of the composition of $\mathrm{Tr}_{\mathbb{Z}_p[\![\mathrm{Conj}(U_2)]\!]/\mathbb{Z}_p[\![\mathrm{Conj}(U_3)]\!]}$ and the canonical surjection $\mathbb{Z}_p[\![\mathrm{Conj}(U_3)]\!]\to\mathbb{Z}_p[\![U_3/V_3]\!]$. Then J_3 has the explicit description $[\beta^b\gamma^c h_\zeta(c\neq 0), p\beta^b\zeta^f]_{\mathbb{Z}_p}\otimes_{\mathbb{Z}_p}\Lambda(\Gamma)$, which we denote by $J_3^f\otimes_{\mathbb{Z}_p}\Lambda(\Gamma)$. Set $J_3^{(n)}=J_3^f\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[\Gamma/\Gamma^{p^n}]$. Since $(J_3^{(n)})^3=p(J_3^{(n)})^2$ holds by simple calculation, we may prove that $1+J_3^{(n)}$ is a multiplicative subgroup of $\mathbb{Z}_p[U_3^{(n)}/V_3]^\times$ and the p-adic logarithm induces an isomorphism between $1+J_3^{(n)}$ and $J_3^{(n)}$ by the same argument as Lemma 4.6. Note that $J_3^{(n)}$ contains $I_3^{(n)}$ by construction.

4.2. *$\Psi^{(n)}$ contains the image of $\theta^{(n)}$*

Lemma 4.7. *The following diagram commutes for each i:*

$$\begin{array}{ccc}
K_1(\mathbb{Z}_p[G^{(n)}]) & \xrightarrow{\log} & \mathbb{Q}_p[\mathrm{Conj}(G^{(n)})] \\
{\scriptstyle \mathrm{Nr}_{\mathbb{Z}_p[G^{(n)}]/\mathbb{Z}_p[U_i^{(n)}]}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_{\mathbb{Q}_p[\mathrm{Conj}(G^{(n)})]/\mathbb{Q}_p[\mathrm{Conj}(U_i^{(n)})]}} \\
K_1(\mathbb{Z}_p[U_i^{(n)}]) & \xrightarrow[\log]{} & \mathbb{Q}_p[\mathrm{Conj}(U_i^{(n)})]
\end{array}$$

Proof. By the proof of [18, Theorem 1.4], the following diagram commutes

$$\begin{array}{ccc}
K_1(\mathbb{Z}_p[G^{(n)}]) & \xrightarrow{\log} & \mathbb{Q}_p[\mathrm{Conj}(G^{(n)})] \\
{\scriptstyle \mathrm{Nr}_{\mathbb{Z}_p[G^{(n)}]/\mathbb{Z}_p[U_i^{(n)}]}}\downarrow & & \downarrow{\scriptstyle R'} \\
K_1(\mathbb{Z}_p[U_i^{(n)}]) & \xrightarrow[\log]{} & \mathbb{Q}_p[\mathrm{Conj}(U_i^{(n)})]
\end{array}$$

where $R' : \mathbb{Z}_p[\mathrm{Conj}(G^{(n)})] \to \mathbb{Z}_p[\mathrm{Conj}(U_i^{(n)})]$ is defined as follows: for an arbitrary element x in $G^{(n)}$, take a set of representatives $\{a'_1, \ldots, a'_{s_i}\}$ of the double coset decomposition $\langle x\rangle\backslash G^{(n)}/U_i^{(n)}$, and let $\mathfrak{J}$ be the finite set consisting of such j as $a'^{-1}_j x a'_j$ is contained in $U_i^{(n)}$. Then $R'(x)$ is defined as an element $\sum_{j\in\mathfrak{J}} a'^{-1}_j x a'_j$. Obviously it suffices to show that R' coincides with the trace.

(Case-1). Assume that $\{a'_j\}_{1\leqslant j\leqslant s_i}$ represents the left coset decomposition $G^{(n)}/U_i^{(n)}$. Then R' coincides with the trace map by definition.

(Case-2). Assume that $\{x^k a'_j\}_{0\leqslant k\leqslant p-1, 1\leqslant j\leqslant s_i}$ represents the left coset decomposition $G^{(n)}/U_i^{(n)}$. In this case $R'(x)$ is equal to $p^{-1}\,\mathrm{Tr}(x)$ by definition. Note that j is contained in $\mathfrak{J}$ if and only if xa'_j is contained in $a'_j U_i^{(n)}$. This is impossible because $\{x^k a'_j\}_{0\leqslant k\leqslant p-1, 1\leqslant j\leqslant s_i}$ represents the left coset decomposition $G^{(n)}/U_i^{(n)}$. Therefore $\mathfrak{J}$ is empty and $\mathrm{Tr}(x) = R'(x) = 0$ holds. □

Proposition 4.8. *The image of an element η in $K_1(\mathbb{Z}_p[G^{(n)}])$ under the composite map $\theta_i^{+,(n)} \circ \Gamma_{G^{(n)}}$ is described as $\log(\varphi(\theta_0^{(n)}(\eta))^{-(G:U_i)/p}\theta_i^{(n)}(\eta))$ for each i except for 0, where $\Gamma_{G^{(n)}}$ is the integral logarithm for $G^{(n)}$ (see Proposition–Definition 1.1).*

Proof. The claim follows from Lemma 4.7 and the relation

$$\frac{(G:U_i)}{p}\varphi \circ \theta_0^{+,(n)} = \theta_i^{+,(n)} \circ \frac{1}{p}\varphi,$$

which is easy to verify for each g in $G^{(n)}$. □

Proposition 4.9. *The following congruences hold for arbitrary η in $K_1(\mathbb{Z}_p[G^{(n)}])$:*

(1) $\theta_i^{(n)}(\eta) \equiv \varphi(\theta_0^{(n)}(\eta)) \bmod I_i^{(n)}$ *for $i = 1, 2$;*
(2) $\theta_3^{(n)}(\eta) \equiv \varphi(\theta_0^{(n)}(\eta))^p \bmod J_3^{(n)}$;
(3) $\theta_4^{(n)}(\eta) \equiv \varphi(\theta_0^{(n)}(\eta))^{p^2} \bmod p\mathbb{Z}_p[U_4^{(n)}/V_4]$.

Proof. We may verify the congruence (1) by the calculation similar to the proof of [27, Lemma 1.7], hence we will give the proof only for the congruences (2) and (3). Note that $p\mathbb{Z}_p[U_4^{(n)}/V_4]$ obviously contains $I_4^{(n)}$ (see Table 2).

The congruence (2). Let $\eta = \sum_{0\leqslant i,j,k\leqslant p-1}(\eta_{i,j}^{(k)}\beta^k)\alpha^i\delta^j$ be an arbitrary element in $\mathbb{Z}_p[G^{(n)}]$ where each ciefficient $\eta_{i,j}^{(k)}$ is an element in $\mathbb{Z}_p[U_4^{(n)}]$. Then we may easily calculate that $\alpha^\ell\delta^m\eta$ is equal to $\sum_{0\leqslant i,j\leqslant p-1}(\sum_{k=0}^{p-1}\nu_{\ell,m}(\eta_{i-\ell,j-m-\ell k}^{(k)})\beta^k)\alpha^i\delta^j$ where $\nu_{\ell,m}(x)$ is the element defined as $(\alpha^\ell\delta^m)x(\alpha^\ell\delta^m)^{-1}$ for each $0\leqslant \ell,m\leqslant p-1$ (we regard the sub-indices i,j of $\eta_{i,j}^{(k)}$ as an element in $(\mathbb{Z}/p\mathbb{Z})^{\oplus 2}$ and the upper-index (k) as an element in $\mathbb{Z}/p\mathbb{Z}$). Then we may calculate $\theta_3^{(n)}(\eta)$ as follows (we use the same notation $\nu_{\ell,m}(\eta_{i-\ell,j-m-\ell k}^{(k)})$ for its image in $\mathbb{Z}_p[U_3^{(n)}/V_3]$):

$$\begin{aligned}\theta_3^{(n)}(\eta) &= \det\left(\sum_{k=0}^{p-1}\nu_{\ell,m}\big(\eta_{(i,j)-(\ell,m)-(0,\ell k)}^{(k)}\big)\beta^k\right)_{(i,j),(\ell,m)}\\ &= \sum_{\sigma\in\mathfrak{S}^{(p^2)}}\operatorname{sgn}(\sigma)\prod_{(\ell,m)\in(\mathbb{Z}/p\mathbb{Z})^{\oplus 2}}\left(\sum_{k=0}^{p-1}\nu_{\ell,m}\big(\eta_{\sigma(\ell,m)-(\ell,m)-(0,\ell k)}^{(k)}\big)\beta^k\right)\end{aligned}$$

where $\mathfrak{S}^{(p^2)}$ is the permutation group of the finite set $\{(\ell,m)\mid 0\leqslant \ell,m\leqslant p-1\}$. We denote by P_σ the σ-component of $\theta_3^{(n)}(\eta)$. Since the commutator $[\beta,\delta]$ is trivial, the equation

$$\begin{aligned}\nu_{0,\mu}(P_\sigma) &= \operatorname{sgn}(\sigma)\prod_{0\leqslant \ell,m\leqslant p-1}\left(\sum_{k=0}^{p-1}\nu_{\ell,m+\mu}\big(\eta_{\sigma(\ell,m)-(\ell,m)-(0,\ell k)}^{(k)}\big)\beta^k\right)\\ &= \operatorname{sgn}(\sigma)\prod_{0\leqslant \ell,m\leqslant p-1}\left(\sum_{k=0}^{p-1}\nu_{\ell,m}\big(\eta_{(\sigma(\ell,m-\mu)+(0,\mu))-(\ell,m)-(0,\ell k)}^{(k)}\big)\beta^k\right) \qquad (4.1)\end{aligned}$$

holds for each μ in $\mathbb{Z}/p\mathbb{Z}$. First suppose that σ does not satisfy

$$\sigma(\ell,m) = \sigma(\ell,m-\mu)+(0,\mu) \qquad (4.2)$$

for certain μ in $\mathbb{Z}/p\mathbb{Z}$, then the right-hand side of (4.1) coincides with P_{τ_μ} where τ_μ is the permutation defined by $\tau_\mu(\ell,m)=\sigma(\ell,m-\mu)+(0,\mu)$.[3] Each τ_μ is distinct by assumption, hence $\sum_{\mu=0}^{p-1}P_{\tau_\mu}$ is contained in $\sum_{\mu=0}^{p-1}\nu_{0,\mu}(\mathbb{Z}_p[U_3^{(n)}/V_3])$ which is no other than the image of the trace map from $\mathbb{Z}_p[\mathrm{Conj}(U_2^{(n)})]$ to $\mathbb{Z}_p[U_3^{(n)}/V_3]$. In other words, $\sum_{\mu=0}^{p-1}P_{\tau_\mu}$ is contained in $J_3^{(n)}$ by definition.

Next suppose that σ satisfies the relation (4.2) for an arbitrary μ in $\mathbb{Z}_p/p\mathbb{Z}_p$. Set $\sigma(\ell,0)=(a_\ell,b_\ell)$, then $\sigma(\ell,-\mu)=(a_\ell,b_\ell-\mu)$ holds for each μ in $\mathbb{Z}/p\mathbb{Z}$ by (4.2). This implies that all permutations satisfying (4.2) are described as $c_{s,h}(\ell,m)=(s(\ell),h_\ell+m)$, where s is a permutation of the set $\{0,1,\ldots,p-1\}$ and $h=(h_\ell)_\ell$ is an element in $(\mathbb{Z}/p\mathbb{Z})^{\oplus p}$. We may calculate $P_{c_{s,h}}$ as

[3] Note that $\operatorname{sgn}(\sigma)$ is equal to $\operatorname{sgn}(\tau_\mu)$ because τ_μ coincides with a composition $c_\mu\circ\sigma\circ c_{-\mu}$, where c_μ is a cyclic permutation of degree p defined as $(\ell,m)\mapsto(\ell,m+\mu)$.

$$P_{c_{s,h}} = \operatorname{sgn}(c_{s,h}) \prod_{0\leqslant \ell,m\leqslant p-1} \left(\sum_{k=0}^{p-1} \nu_{\ell,m}\bigl(\eta^{(k)}_{c_{s,h}(\ell,m)-(\ell,m)-(0,\ell k)}\bigr)\beta^k \right)$$
$$= \operatorname{sgn}(s) \prod_{0\leqslant \ell,m\leqslant p-1} \left(\sum_{k=0}^{p-1} \nu_{\ell,m}\bigl(\eta^{(k)}_{(s(\ell)-\ell,h_\ell-\ell k)}\bigr)\beta^k \right)$$

(we use the relation $\operatorname{sgn}(c_{s,h}) = \operatorname{sgn}(s)^p$). Set $Q_{s,h;\ell} = \sum_{k=0}^{p-1} \eta^{(k)}_{(s(\ell)-\ell,h_\ell-\ell k)}\beta^k$. We claim that the congruence $\prod_{m=0}^{p-1} \nu_{0,m}(Q_{s,h;\ell}) \equiv \varphi(Q_{s,h;\ell}) \bmod J_3^{(n)}$ holds; in general, let $x = \sum_{u\in U_3^f/V_3^f} x_u u$ be an element in $\mathbb{Z}_p[U_3^{(n)}/V_3]$ (where each x_u is in $\mathbb{Z}_p[\Gamma/\Gamma^{p^n}]$). Then

$$\prod_{m=0}^{p-1} \nu_{0,m}(x) \equiv \sum_{u\in U_3^f/V_3^f} \prod_{m=0}^{p-1} \nu_{0,m}(x_u)\nu_{0,m}(u) \mod J_3^{(n)}$$
$$\equiv \sum_{u\in U_3^f/V_3^f} x_u^p \operatorname{Ver}^{U_2^{(n)}}_{U_3^{(n)}}(u) \mod J_3^{(n)}$$

holds by [27, Chapter 5, Lemma 1.9]. Recall that $\operatorname{Ver}^{U_2^{(n)}}_{U_3^{(n)}}(u)$ is equal to $\varphi(u) = 1$ by Lemma 4.3, thus the claim follows because x_u^p is congruent to $\varphi(x_u)$ modulo $p\mathbb{Z}_p[\Gamma/\Gamma^{p^n}]$ (recall that $J_3^{(n)}$ contains $p\mathbb{Z}_p[\Gamma/\Gamma^{p^n}]$). Now we obtain

$$\sum_{s,h} P_{c_{s,h}} \equiv \sum_{s,h} \operatorname{sgn}(s)\left(\prod_{\ell=0}^{p-1} \nu_{\ell,0}(\varphi(Q_{s,h;\ell})) \right) \mod J_3^{(n)}$$
$$\equiv \sum_{s,h} \operatorname{sgn}(s)\left(\sum_{0\leqslant k_0,\dots,k_{p-1}\leqslant p-1} \prod_{\ell=0}^{p-1} \varphi\bigl(\eta^{(k_\ell)}_{(s(\ell)-\ell,h_\ell-\ell k_\ell)}\bigr) \right) \mod J_3^{(n)} \tag{4.3}$$

(we use the obvious relation $\nu_{\ell,0}\circ\varphi = \varphi$ for the second congruence). We denote the right-hand side of (4.3) by $\sum_{s,h,(k_\ell)_\ell} R_{s,h;k_0,\dots,k_{p-1}}$. First suppose that $s(\ell)-\ell$ (resp. k_ℓ, $h_\ell-\ell k_\ell$) is equal to a certain constant λ (resp. κ, μ) determined independent of ℓ. Such s and h are described as $s_\lambda; \ell \mapsto \ell+\lambda$ and $(h_{\mu,\kappa})_\ell = \ell\kappa+\mu$ for each λ,κ and μ respectively, and the equation $R_{s_\lambda,h_{\mu,\kappa};\kappa,\dots,\kappa} = \varphi(\eta^{(\kappa)}_{\lambda,\mu})^p$ holds (note that $\operatorname{sgn}(s_\lambda)$ is 1 because s_λ is a cyclic permutation of degree p). Otherwise set $k_\ell^{(w)} = k_{\ell+w}$, $s^{(w)}(\ell) = s(\ell+w)-w$ and $h_\ell^{(w)} = h_{\ell+w}-wk_{\ell+w}$ for each w in $\mathbb{Z}/p\mathbb{Z}$ respectively. Then each $R_{s^{(w)},h^{(w)};k_0^{(w)},\dots,k_{p-1}^{(w)}}$ is a distinct term in the expansion (4.3) but has the same value by construction. Hence the element $\sum_{w=0}^{p-1} R_{s^{(w)},h^{(w)};k_0^{(w)},\dots,k_{p-1}^{(w)}}$ is contained in $J_3^{(n)}$ because $J_3^{(n)}$ contains $p\mathbb{Z}_p[\Gamma/\Gamma^{p^n}]$. Consequently the element $\theta_3^{(n)}(\eta)$ is congruent to

$$\sum_{0\leqslant\lambda,\mu,\kappa\leqslant p-1} R_{s_\lambda,h_{\mu,\kappa};\kappa,\kappa,\dots,\kappa} = \sum_{0\leqslant\lambda,\mu,\kappa\leqslant p-1} \varphi\bigl(\eta^{(\kappa)}_{\lambda,\mu}\bigr)^p \equiv \varphi(\eta)^p \mod J_3^{(n)}.$$

The congruence (3). Similarly to the argument in the proof of (2), we may calculate $\theta_4(\eta)$ for an element $\eta = \sum_{k,\ell,m} \eta_{k,\ell,m}\alpha^k\beta^\ell\delta^m$ in $\mathbb{Z}_p[G^{(n)}]$ (each $\eta_{k,\ell,m}$ is in $\mathbb{Z}_p[U_4^{(n)}]$) as

$$\theta_4(\eta) = \sum_{\sigma \in \mathfrak{S}^{(p^3)}} \mathrm{sgn}(\sigma) \prod_{0 \leqslant k,\ell,m \leqslant p-1} \nu_{k,\ell,m}(\eta_{\sigma(k,\ell,m)-(k,\ell,m)})$$

where $\mathfrak{S}^{(p^3)}$ is the permutation group of $(\mathbb{Z}/p\mathbb{Z})^{\oplus 3}$. Let P_σ denote the σ-component of the right-hand side of the equation above. First suppose that the permutation $\tau_{\kappa,\lambda,\mu}$ defined by $(k,\ell,m) \mapsto \sigma(k-\kappa, \ell-\lambda, m-\mu) + (\kappa,\lambda,\mu)$ is distinct for each (κ,λ,μ) in $(\mathbb{Z}_p/p\mathbb{Z}_p)^{\oplus 3}$. Then $\sum_{\kappa,\lambda,\mu} P_{\tau_{\kappa,\lambda,\mu}}$ is contained in $I_4^{(n)}$ (hence also contained in $p\mathbb{Z}_p[U_4^{(n)}/V_4]$) by the similar argument to the proof of (2). On the other hand, suppose that σ is the permutation $c_{\kappa,\lambda,\mu}$ for (κ,λ,μ) in $(\mathbb{Z}_p/p\mathbb{Z}_p)^{\oplus 3}$ defined by $(k,\ell,m) \mapsto (k+\kappa, \ell+\lambda, m+\mu)$. Then we obtain the congruence $P_{c_{\kappa,\lambda,\eta}} \equiv \varphi(\eta_{\kappa,\lambda,\mu})^{p^2} \bmod p\mathbb{Z}_p[U_4^{(n)}/V_4]$ similarly to the proof of (2).[4] □

Since each $\theta_i^{(n)}$ is essentially a norm map, it is clear that $(\theta_i^{(n)}(\eta))_{i=0}^4$ satisfies the desired norm relations for $\Psi^{(n)}$. Note that the congruences in Proposition 4.9(1) are the desired ones for $i = 1, 2$. Now set $J_4^{(n)} = p\mathbb{Z}_p[U_4^{(n)}/V_4]$. Then we may define the element $y_i = \log(\varphi(\theta_0^{(n)}(\eta))^{-(G:U_i)/p}\theta_i^{(n)}(\eta))$ in $J_i^{(n)}$ for $i = 3, 4$ by the congruences (2) and (3) in Proposition 4.9. On the other hand, y_i is an element in $I_i^{(n)}$ by Proposition 4.8. Hence $\varphi(\theta_0^{(n)}(\eta))^{-(G:U_i)/p}\theta_i^{(n)}(\eta)$ is contained in $1 + I_i^{(n)}$ because the p-adic logarithm induces an injection on $1 + J_i^{(n)}$ and an isomorphism between $1 + I_i^{(n)}$ and $I_i^{(n)}$.[5] This implies the desired congruence for $i = 3, 4$, thus $(\theta_i^{(n)}(\eta))_{i=0}^4$ is contained in $\Psi^{(n)}$.

4.3. Surjectivity for finite quotients

Let $(\eta_i)_{i=0}^4$ be an arbitrary element in $\Psi^{(n)}$. Then for each i except for 0, the multiplicative group $1 + I_i$ contains an element $\varphi(\eta_0)^{-(G:U_i)/p}\eta_i$ by the congruences in Definition 4.4. Hence the element y_i defined as $\log(\varphi(\eta_0)^{-(G:U_i)/p}\eta_i)$ is contained in $I_i^{(n)}$ by Lemma 4.6. For $i = 0$, set $y_0 = \Gamma_{U_0^{(n)}/V_0}(\eta_0)$. Then $(y_i)_{i=0}^4$ satisfies the trace relations for $\Omega^{(n)}$ by using Lemma 4.7, thus $(y_i)_{i=0}^4$ is an element in $\Omega^{(n)}$. By the additive theta isomorphism (Proposition–Definition 3.3), there exists a unique element y in $\mathbb{Z}_p[\mathrm{Conj}(G^{(n)})]$ corresponding to $(y_i)_{i=0}^4$.

Proposition 4.10. *Let y be as above. Then $\omega_{G^{(n)}}(y)$ vanishes where $\omega_{G^{(n)}}$ is the group homomorphism introduced in Theorem* 1.2. *In other words, y is contained in the image of the integral logarithm $\Gamma_{G^{(n)}}$.*

Proof. The homomorphism $\omega_{G^{(n)}}$ coincides with $\omega_{U_0^{(n)}/V_0} \circ \theta_0^{+,(n)}$ by definition where $\theta_0^{+,(n)}$ is the canonical abelization $G^{(n)} \to U_0^{(n)}/V_0$, thus we obtain the equation $\omega_{G^{(n)}}(y) = \omega_{U_0^{(n)}/V_0}(y_0) = 1$ because y_0 is contained in the image of $\Gamma_{U_0^{(n)}/V_0}$ (see Theorem 1.2). □

Let η' be an element in $K_1(\mathbb{Z}_p[G^{(n)}])$ satisfying $\Gamma_{G^{(n)}}(\eta') = y$. Then η' is determined up to multiplication by a torsion element in $K_1(\mathbb{Z}_p[G^{(n)}])$ by Theorem 1.2. Note that $\theta^{+,(n)} \circ \Gamma_{G^{(n)}}(\eta')$ coincides with $(y_i)_{i=0}^4$ by construction. Combining with Proposition 4.8, we obtain the following equations:

---

[4] Here we use the congruence $\eta_{\kappa,\lambda,\mu}^p \equiv \varphi(\eta_{\kappa,\lambda,\mu}) \bmod p\mathbb{Z}_p[U_4^{(n)}/V_4]$. This is the reason why we have to replace $I_4^{(n)}$ by $p\mathbb{Z}_p[U_4^{(n)}/V_4]$.

[5] For $i = 4$, the p-adic logarithm induces an isomorphism between $1 + p\mathbb{Z}_p[U_4^{(n)}/V_4]$ and $p\mathbb{Z}_p[U_4^{(n)}/V_4]$ since p is an odd prime number. For $i = 3$, see the last paragraph of Section 4.1.

$$y_0 = \Gamma_{U_0^{(n)}/V_0}(\eta_0) = \Gamma_{U_0^{(n)}/V_0}\big(\theta_0^{(n)}(\eta')\big), \tag{4.4}$$

$$y_i = \log \frac{\eta_i}{\varphi(\eta_0)^{(G:U_i)/p}} = \log \frac{\theta_i^{(n)}(\eta')}{\varphi(\theta_0^{(n)}(\eta'))^{(G:U_i)/p}} \quad \text{for } 1 \leqslant i \leqslant 4. \tag{4.5}$$

From Eq. (4.4) and the fact that the integral logarithmic homomorphism is injective on $K_1(\mathbb{Z}_p[U_0^{(n)}/V_0])$ modulo torsion elements, there exists a unique element τ in $K_1(\mathbb{Z}_p[U_0^{(n)}/V_0])_{\mathrm{tors}}$ satisfying $\eta_0 = \tau\theta_0^{(n)}(\eta')$. The element τ is identified with an element in $\mu_{p-1}(\mathbb{Z}_p) \times (U_0^{(n)}/V_0)$ by Higman–Wall's Theorem 1.3, thus the equation $\theta_0^{(n)}(\eta'\tau) = \theta_0^{(n)}(\eta')\tau = \eta_0$ holds. Set $\eta = \eta'\tau$. Then Eq. (4.5) holds when we replace η' by η (recall that τ is contained in the kernel of $\Gamma_{G^{(n)}}$). Therefore we obtain $\theta_i^{(n)}(\eta) = \eta_i$ for each i by Lemma 4.6, which implies that $\theta^{(n)}$ induces an surjection onto $\Psi^{(n)}$.

4.4. *Taking the projective limit*

Consider the following diagram with exact rows and an injective left vertical arrow:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(\theta^{(n)}) & \longrightarrow & K_1(\mathbb{Z}_p[G^{(n)}]) & \xrightarrow{\theta^{(n)}} & \Psi & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow{\scriptstyle \mathrm{Log}} & & \\
0 & \longrightarrow & \ker(\Gamma_{G^{(n)}}) & \longrightarrow & K_1(\mathbb{Z}_p[G^{(n)}]) & \xrightarrow[\theta^{+,(n)}\circ\Gamma_{G^{(n)}}]{} & \mathrm{Im}(\theta^{+,(n)}\circ\Gamma_{G^{(n)}}) & \longrightarrow & 0,
\end{array}$$

where Log is defined by the formula $\mathrm{Log}((\eta_i)_{i=0}^4) = (\log(\varphi(\eta_0)^{-(G:U_i)/p}\eta_i))_{i=0}^4$ (we denote $\Gamma_{U_0^{(n)}/V_0}(\eta)$ by $\log(\varphi(\eta_0)^{-1/p}\eta_0)$ by abuse of notation). It is easy to see that the kernel of Log is coincides with the image of $\theta^{(n)}|_{\ker(\Gamma_{G^{(n)}})}$, which is isomorphic to $\mu_{p-1}(\mathbb{Z}_p) \times U_0^{(n)}/V_0$ by Higman–Wall's Theorem 1.3. Hence the kernel of $\theta^{(n)}$ is isomorphic $SK_1(\mathbb{Z}_p[G^{(n)}])$ by the snake lemma. We remark that $SK_1(\mathbb{Z}_p[G^{(n)}])$ is decomposed as the direct sum of $SK_1(\mathbb{Z}_p[G^f])$ and $SK_1(\mathbb{Z}_p[\Gamma/\Gamma^{p^n}])$ by Oliver's theorem [16, Proposition 25], and $SK_1(\mathbb{Z}_p[\Gamma/\Gamma^{p^n}])$ is trivial since Γ/Γ^{p^n} is abelian. Therefore we obtain the following exact sequence of projective systems of abelian groups:

$$1 \longrightarrow SK_1(\mathbb{Z}_p[G^f]) \longrightarrow K_1(\mathbb{Z}_p[G^{(n)}]) \xrightarrow{\theta^{(n)}} \Psi^{(n)} \longrightarrow 1. \tag{4.6}$$

The projective system $\{SK_1(\mathbb{Z}_p[G^f])\}_{n\in\mathbb{N}}$ obviously satisfies the Mittag–Leffler condition, hence we obtain the exact sequence

$$1 \to SK_1(\mathbb{Z}_p[G^f]) \to \varprojlim_{n\in\mathbb{N}} K_1(\mathbb{Z}_p[G^{(n)}]) \xrightarrow{(\theta^{(n)})_n} \Psi \to 1 \tag{4.7}$$

by taking the projective limit of (4.6). The kernel of $\Lambda(G) \to \mathbb{Z}_p[G^{(n)}]$ satisfies the condition $(*)$ in [8, Section 1.4.1], hence by the same argument as the proof of [8, Proposition 1.5.1], we may prove that $\varprojlim_{n\in\mathbb{N}} K_1(\mathbb{Z}_p[G^{(n)}])$ is isomorphic to $K_1(\Lambda(G))$. The proof of Proposition 4.5 is now finished.

5. Localized theta map

First note that the canonical Ore set S for a 1-dimensional p-adic Lie group G coincides with the multiplicative closed set $\Lambda(G) \setminus \mathfrak{m}_G$ where $\mathfrak{m}_G$ is the kernel of the augmentation map $\Lambda(G) \to \Lambda(\Gamma)/p\Lambda(\Gamma)$ (use [5, Lemma 2.1 (3) or (4)] for $H = G^f$). Especially for $G = G^f \times \Gamma$,

$\Lambda(G)_S$ (resp. $\Lambda(U_i/V_i)_S$) is isomorphic to $\Lambda(\Gamma)_{(p)}[G^f]$ (resp. $\Lambda(\Gamma)_{(p)}[U^f/V^f]$) where $\Lambda(\Gamma)_{(p)}$ is the localization of $\Lambda(\Gamma)$ with respect to the prime ideal $p\Lambda(\Gamma)$. Let $R=\widehat{\Lambda(\Gamma)_{(p)}}$ denote the p-adic completion of $\Lambda(\Gamma)_{(p)}$ and let Ψ_S be the subgroup of $\prod_i \Lambda(U_i/V_i)_S^\times$ consisting of an element $(\eta_{S,i})_{i=0}^4$ such that

1. (norm relations) the following equations hold:
 (rel-1) $\mathrm{Nr}_{\Lambda(U_0/V_0)_S/\Lambda(U_1/V_0)_S}(\eta_{S,0}) \equiv \eta_{S,1}$,
 (rel-2) $\mathrm{Nr}_{\Lambda(U_0/V_0)_S/\Lambda(U_2/V_0)_S}(\eta_{S,0}) \equiv \eta_{S,2}$,
 (rel-3) $\mathrm{Nr}_{\Lambda(U_2/V_2)_S/\Lambda(U_3/V_2)_S}(\eta_{S,2}) \equiv \eta_{S,3}$,
 (rel-4) $\mathrm{Nr}_{\Lambda(U_1/V_2)_S/\Lambda(U_1\cap U_2/V_2)_S}(\eta_{S,1}) \equiv \mathrm{Nr}_{\Lambda(U_2/V_2)_S/\Lambda(U_1\cap U_2/V_2)_S}(\eta_{S,2})$,
 (rel-5) $\mathrm{Nr}_{\Lambda(U_1/V_1)_S/\Lambda(U_4/V_1)_S}(\eta_{S,1}) \equiv \eta_{S,4}$,
 (rel-6) $\mathrm{Nr}_{\Lambda(U_3/V_3)_S/\Lambda(U_4/V_3)_S}(\eta_{S,3}) \equiv \eta_{S,4}$
 (See Fig. 1);
2. (congruences) the congruence $\eta_{S,i} \equiv \varphi(\eta_{S,0})^{(G:U_i)/p} \bmod I_{S,i}$ holds for each i except for 0 where $I_{S,i}$ is defined as $I_i^f \otimes_{\mathbb{Z}_p} \Lambda(\Gamma)_{(p)}$.

Let $\theta_S=(\theta_{S,i})_{i=0}^4$ be the family of the homomorphisms $\theta_{S,i}$ which is defined as the composition of the norm map $\mathrm{Nr}_{\Lambda(G)_S/\Lambda(U_i)_S}$ and the canonical homomorphism $K_1(\Lambda(U_i)_S) \to \Lambda(U_i/V_i)_S^\times$ (see Section 0).

Proposition 5.1. *The image of θ_S is contained in Ψ_S.*

Proof. It suffices to show that $(\theta_{S,i}(\eta_S))_{i=0}^4$ satisfies the norm relations and congruences above for η_S in $K_1(\Lambda(G)_S)$. Norm relations are obviously satisfied by the definition of θ_S. The congruences for $i=1,2$ are obtained by the argument similar to the proof of [27, Lemma 1.7]. For $i=3,4$, we obtain the congruence $\theta_{S,i}(\eta_S) \equiv \theta_{S,0}(\eta_S)^{(G:U_i)/p} \bmod J_{S,i}$ by the same argument as Proposition 4.9 where $J_{S,i}$ is defined as $J_i^f \otimes_{\mathbb{Z}_p} \Lambda(\Gamma)_{(p)}$. Set $\widehat{I}_i = I_i^f \otimes_{\mathbb{Z}_p} R$ and $\widehat{J}_i = J_i^f \otimes_{\mathbb{Z}_p} R$ respectively. Then we obtain the logarithmic isomorphism $1+\widehat{I}_i \xrightarrow{\sim} \widehat{I}_i$ and $1+\widehat{J}_i \xrightarrow{\sim} \widehat{J}_i$ for $i=3,4$ by the argument similar to the proof of Lemma 4.6. Therefore we obtain the congruence $\theta_{S,i}(\eta_S) \equiv \theta_{S,0}(\eta_S)^{(G:U_i)/p} \bmod \widehat{I}_i$ similarly to the argument following the proof of Proposition 4.9 by using the generalized integral logarithm $\Gamma_{R,G^{(n)}}$ (see the following Remark 5.2). The proof is now finished since the intersection of $\widehat{I}_i$ and $\Lambda(U_i/V_i)_S$ coincides with $I_{S,i}$. (We may easily check that for an arbitrary element in $\widehat{I}_i \cap \Lambda(U_i/V_i)_S$, its coefficient of each generator of I_i^f in Table 2 has to be contained in $\Lambda(\Gamma)[p^{-1}] \cap R = \Lambda(\Gamma)_{(p)}$. Also refer the argument in the proof of Proposition 5.3.) □

Remark 5.2. The integral logarithm $\Gamma_{\Delta,\mathfrak{m}}$ for $R(\Delta)$ is also extended to a group homomorphism $\Gamma_{R,\Delta}$: $K_1(R[\Delta]) \to R[\mathrm{Conj}(\Delta)]$ if R is the completion of the localized Iwasawa algebra $\widehat{\Lambda(\Gamma)_{(p)}}$; let x be an element in $K_1(R[\Delta])$ and $\tilde{x}$ its lift to $R[\Delta]^\times$. There exist an element $\tilde{r}$ in $R^\times$ and $\tilde{y}$ in $1+\mathfrak{m}_{R[\Delta]}$ such that $\tilde{x}=\tilde{r}\tilde{y}$ holds. Then put $\Gamma_{R,\Delta}(x)=\Gamma_R(\tilde{r})+\Gamma_{\Delta,\mathfrak{m}}(y)$ where $\Gamma_R : R^\times \to R$ is the integral logarithm for R (regarded as a group ring of the trivial group) and y is the image of $\tilde{y}$ in $K_1(R[\Delta], \mathfrak{m}_{R[\Delta]})$.

Proposition 5.3. *The intersection of Ψ_S and $\prod_i \Lambda(U_i/V_i)^\times$ coincides with Ψ.*

Proof. It suffices to prove that $I_{S,i} \cap \Lambda(U_i/V_i)$ coincides with I_i for each i except for 0. The $\mathbb{Z}_p$-module I_i is clearly contained in $I_{S,i} \cap \Lambda(U_i/V_i)$. Note that all monomials which appear in the generators of I_i^f in Table 2 are of the form of $p^k u_i$ for u_i in U_i^f/V_i^f and they are distinct except for $i=3$. First suppose that i is not equal to 3. Then for an arbitrary element in $I_{S,i} \cap \Lambda(U_i/V_i)$, the coefficient of each generator in I_i^f has to be contained in $\Lambda(\Gamma)[p^{-1}] \cap \Lambda(\Gamma)_{(p)}$, which coincides with $\Lambda(\Gamma)$. Hence the intersection $I_{S,i} \cap \Lambda(U_i/V_i)$ is contained in I_i. Next consider the case $i=3$. Take an arbitrary element in $I_{S,3} \cap \Lambda(U_3/V_3)$ and let $\nu_f^{(1)}$ (resp. $\nu_0^{(2)}$) denote its coefficient of $p^2\zeta^f$ (resp. ph_ζ), which is contained in $\Lambda(\Gamma)_{(p)}$. Then the coefficient of ζ^f is $p^2\nu_f^{(1)} + p\nu_0^{(2)}$ for each f, which is contained in $\Lambda(\Gamma)$ by definition. Replace $\nu_0^{(2)}$ by $\tilde{\nu}_0^{(2)} = \nu_0^{(2)} + p\nu_0^{(1)}$ and $\nu_f^{(1)}$ by $\tilde{\nu}_f^{(1)} = \nu_f^{(1)} - \nu_0^{(1)}$ using the relation $\sum_{f=0}^{p-1} p^2\zeta^f = p \cdot ph_\zeta$. Then $\tilde{\nu}_0^{(2)}$ is contained in $\Lambda(\Gamma)[p^{-1}] \cap \Lambda(\Gamma)_{(p)} (= \Lambda(\Gamma))$ by construction, thus each $\tilde{\nu}_f^{(2)}$ is also contained in $\Lambda(\Gamma)$ automatically. This implies that the intersection of $I_{S,3}$ and $\Lambda(U_3/V_3)$ is also contained in I_3. □

Hence we obtain *the localized theta map* $\theta_S : K_1(\Lambda(G)_S) \to \Psi_S$ for $\mathfrak{F}$.

6. Congruences among abelian *p*-adic zeta pseudomeasures

Let F_{U_i} (resp. F_{V_i}) be the maximal subfield of F^∞ fixed by U_i (resp. V_i). In the previous sections we have constructed the theta map θ and the localized theta map θ_S for the family $\mathfrak{F}$. If we may apply Theorem 2.4 to our case $G = G^f \times \Gamma$, the element $(\xi_i)_{i=0}^4$ is expected to be contained in Ψ_S where ξ_i is the p-adic zeta pseudomeasure for F_{V_i}/F_{U_i}. In order to show it, we have to verify that ξ_i's satisfy the norm relations and the congruences in the definition of Ψ_S. The norm relations among ξ_i's are easily verified by formal calculation using their interpolation properties, hence what is the most difficult is to derive the desired congruences among abelian p-adic zeta pseudomeasures.

6.1. Congruences

Now let us study the congruences among ξ_i's.

Proposition 6.1 *(Congruences among ξ_i's). The p-adic zeta pseudomeasures $\{\xi_i\}_{i=0}^4$ satisfy the following congruences*:

(1) $\xi_i \equiv \varphi(\xi_0) \bmod I_{S,i}$ *for* $i = 1, 2$;
(2) $\xi_3 \equiv c_3 \bmod J_{S,3}$ *where c_3 is a certain element in* $\Lambda(\Gamma)_{(p)}$;
(3) $\xi_4 \equiv c_4 \bmod I_{S,4}$ *where c_4 is a certain element in* $\Lambda(\Gamma)_{(p)}$.

Remark 6.2. These congruences are *not* sufficient to prove that $(\xi_i)_{i=0}^4$ is contained in Ψ_S. Hence we have to modify Burns' technique 2.4 to prove our main theorem (Theorem 2.1), which will be discussed in the next section.

Remark 6.3. We may replace c_i by an element in $\Lambda(\Gamma)$ for $i = 3$ and 4 because c_3 (resp. c_4) is determined modulo $p\Lambda(\Gamma)_{(p)}$ (resp. $p^3\Lambda(\Gamma)_{(p)}$). Note that the intersection of $J_{S,3}$ (resp. $I_{S,4}$) and $\Lambda(\Gamma)_{(p)}$ coincides with $p\Lambda(\Gamma)_{(p)}$ (resp. $p^3\Lambda(\Gamma)_{(p)}$).

The congruence (1) is just the Ritter–Weiss' congruence [21] for the extensions $F_{V_i}/F_{U_i}/F$. They derived such a kind of congruences by using Deligne–Ribet's theory on Hilbert modular forms [6]. Kazuya Kato also obtained similar kinds of congruences in [13, Section 4]. In the following, we will prove Proposition 6.1 (2) and (3) by mimicking the method of Ritter–Weiss and Kato.

6.2. Approximation of abelian *p*-adic zeta pseudomeasures

Suppose that i is equal to either 3 or 4 in this subsection. To simplify the notation, we denote U_i/V_i by W_i and U_i^f/V_i^f by W_i^f respectively. For an arbitrary open subgroup $\mathcal{U}$ of W_i, let $m(\mathcal{U})$ be the non-negative integer defined by the relation $\kappa^{p-1}(\mathcal{U}) = 1 + p^{m(\mathcal{U})}\mathbb{Z}_p$ where κ is the p-adic cyclotomic character. Then we obtain the canonical isomorphism between $\mathbb{Z}_p[\![W_i]\!]$ and $\varprojlim_{\mathcal{U} \subseteq W_i:\text{open}} \mathbb{Z}_p[W_i/\mathcal{U}]/p^{m(\mathcal{U})}\mathbb{Z}_p[W_i/\mathcal{U}]$ (see [21, Lemma 1]). Let ε denote a $\mathbb{C}$-valued locally constant function on W_i. Then there exists an open subgroup $\mathcal{U}$ of W_i such that ε is constant on each coset of $W_i/\mathcal{U}$, hence we may describe ε as $\sum_{x \in W_i/\mathcal{U}} \varepsilon(x)\delta^{(x)}$ where $\delta^{(x)}$ is the characteristic function with respect to a coset x, that is, $\delta^{(x)}(w)$ is equal to 1 if w is contained in x and 0 otherwise.

Definition 6.4 *(Partial zeta function).* Let x be an arbitrary coset of $W_i/\mathcal{U}$. Then we define *the partial zeta function for* $\delta^{(x)}$ as the function

$$\zeta_{F_{V_i}/F_{U_i}}\big(s, \delta^{(x)}\big) = \sum_{0 \neq \mathfrak{a} \subseteq \mathcal{O}_{F_{U_i}}} \frac{\delta^{(x)}(((F_{V_i}/F_{U_i}), \mathfrak{a}))}{(\mathcal{N}\mathfrak{a})^s}$$

where $((F_{V_i}/F_{U_i}), -)$ is the Artin symbol for F_{V_i}/F_{U_i} and $\mathcal{N}\mathfrak{a}$ is the absolute norm of an ideal $\mathfrak{a}$. This function has analytic continuation to the whole complex plane except for a simple pole at 1, and it is known that $\zeta_{F_{V_i}/F_{U_i}}(1-k, \delta^{(x)})$ is a rational number for an arbitrary natural number k. We also define *the partial zeta function* $\zeta_{F_{V_i}/F_{U_i}}(s, \varepsilon)$ *for a locally constant function* ε *on* W_i as the function $\sum_{x \in W_i/\mathcal{U}} \varepsilon(x)\zeta_{F_{V_i}/F_{U_i}}(s, \delta^{(x)})$, where $\varepsilon = \sum_{x \in W_i/\mathcal{U}} \varepsilon(x)\delta^{(x)}$ is the decomposition as above.

For an arbitrary element w in W_i, an arbitrary $\mathbb{Q}_p$-valued locally constant function ε on W_i and an arbitrary natural number k divisible by $p-1$, we define a p-adic rational number $\Delta_i^w(1-k, \varepsilon)$ by

$$\Delta_i^w(1-k, \varepsilon) = \zeta_{F_{V_i}/F_{U_i}}(1-k, \varepsilon) - \kappa(w)^k \zeta_{F_{V_i}/F_{U_i}}(1-k, \varepsilon_w)$$

where ε_w is the locally constant function on W_i defined by $\varepsilon_w(w') = \varepsilon(ww')$ for w' in W_i. Pierre Deligne and Kenneth A. Ribet showed [6, Théorème (0.4)] that $\Delta_i^w(1-k, \delta^{(x)})$ is a p-adic integer for an arbitrary element w in W_i and an arbitrary coset x of $W_i/\mathcal{U}$ (also refer to [4, Hypothesis (H_{n-1})]). Jürgen Ritter and Alfred Weiss showed the following proposition [21, Proposition 2] which approximated the p-adic zeta pseudomeasure by the value $\Delta_i^w(1-k, \delta^{(x)})$:

Proposition 6.5 *(Approximation lemma, Ritter–Weiss). Let $\mathcal{U}$ be an arbitrary open subgroup of W_i. Then for an arbitrary natural number k divisible by $p-1$ and an arbitrary element w in W_i, $(1-w)\xi_i$ maps to*

$$\sum_{x \in W_i/\mathcal{U}} \Delta_i^w\big(1-k, \delta^{(x)}\big)\kappa(x)^{-k} x \mod p^{m(\mathcal{U})}\mathbb{Z}_p[W_i/\mathcal{U}] \tag{6.1}$$

under the canonical surjection $\mathbb{Z}_p[\![W_i]\!] \to \mathbb{Z}_p[W_i/\mathcal{U}]/p^{m(\mathcal{U})}\mathbb{Z}_p[W_i/\mathcal{U}]$ (note that $\kappa(x)^{-k}$ is well defined by the definition of $m(\mathcal{U})$).

6.3. Sufficient conditions

In this subsection, we will reduce the congruences among pseudomeasures to those among special values of partial zeta functions by using the approximation lemma (Proposition 6.5), and derive the sufficient conditions for Proposition 6.1 (2) and (3). Let NU_i denote the normalizer of U_i in G. Obviously NU_i coincides with U_2 if i is equal to 3 and with G otherwise. The quotient group NU_i/U_i acts on the set of locally constant functions on W_i by $\varepsilon^\sigma(w) = \varepsilon(\sigma^{-1}w\sigma)$ where ε is a locally constant function on W_i and σ is an element in NU_i/U_i.

Proposition 6.6. *The following* (2)′ *and* (3)′ *are the sufficient conditions for the congruences in Proposition* 6.1 (2) *and* (3) *to hold respectively*:

(2)′ *the congruence $\Delta_3^w(1-k, \delta^{(y)}) \equiv 0 \mod p\mathbb{Z}_p$ holds for an arbitrary element w in Γ and for each coset y of W_3/Γ^{p^j} (j is a natural number) which is not contained in Γ and fixed by NU_3/U_3 $(= U_2/U_3)$;*

(3)′ *the congruence $\Delta_4^w(1-k, \delta^{(y)}) \equiv 0 \mod p^{m_y}\mathbb{Z}_p$ holds for an arbitrary element w in Γ and for each coset y of W_4/Γ^{p^j} (j is a natural number) not contained in Γ. Here p^{m_y} is the order of $(NU_4/U_4)_y = (G/U_4)_y$, the isotropic subgroup of G/U_4 at y.*

Proof. Apply the approximation lemma (Proposition 6.5) to $(1-w)\xi_i$:

$$(1-w)\xi_i \equiv \sum_{y\in W_i/\Gamma^{p^j}} \Delta_i^w\bigl(1-k,\delta^{(y)}\bigr)\kappa(y)^{-k}y \mod p^{m(\Gamma^{p^j})}. \tag{6.2}$$

Then by the condition (2)′ and (3)′, we may calculate as

$$\begin{aligned}
&\sum_{\sigma\in(NU_i/U_i)/(NU_i/U_i)_y} \Delta_i^w\bigl(1-k,\delta^{(\sigma^{-1}y\sigma)}\bigr)\kappa\bigl(\sigma^{-1}y\sigma\bigr)^{-k}\sigma^{-1}y\sigma \\
&\quad= \Delta_i^w\bigl(1-k,\delta^{(y)}\bigr)\kappa(y)^{-k}\sum_{\sigma\in(NU_i/U_i)/(NU_i/U_i)_y}\sigma^{-1}y\sigma \\
&\quad\equiv p^{m_y}\sum_{\sigma\in(NU_i/U_i)/(NU_i/U_i)_y}\sigma^{-1}y\sigma \mod p^{m(\Gamma^{p^j})}\mathbb{Z}_p\bigl[W_i/\Gamma^{p^j}\bigr]
\end{aligned}$$

for each coset y of W_i/Γ^{p^j} not contained in Γ. The right-hand side of the equation above is no other than the image of y under the trace map from $\mathbb{Z}_p[\![\mathrm{Conj}(NU_i)]\!]$ to $\mathbb{Z}_p[\![U_i/V_i]\!]$, hence the right-hand side of the congruence (6.2) is contained in $(\mathbb{Z}_p[\Gamma/\Gamma^{p^j}] + J_3^{(j)})/p^{m(\Gamma^{p^j})}$ (resp. $(\mathbb{Z}_p[\Gamma/\Gamma^{p^j}] + I_4^{(j)})/p^{m(\Gamma^{p^j})}$). By taking the projective limit, we obtain congruences $(1-w)\xi_3 \equiv c_{w,3} \bmod J_3$ and $(1-w)\xi_4 \equiv c_{w,4} \bmod I_4$ where both $c_{w,3}$ and $c_{w,4}$ are certain elements in $\Lambda(\Gamma)$. Since $1-w$ is invertible in $\Lambda(\Gamma)_{(p)}$, we obtain the congruences (2) and (3) in Proposition 6.1 by setting $c_i = (1-w)^{-1}c_{w,i}$. □

6.4. Hilbert modular forms and Hilbert–Eisenstein series

In this subsection, we will review Deligne–Ribet's theory on Hilbert modular forms (especially the q-expansion principle). See [6] and [21, Section 3] for details. Let K be a totally real number field of degree r, K^∞/K an abelian totally real p-adic Lie extension and Σ a fixed finite set of primes of K containing all primes which ramify in K^∞. Let $\mathfrak{h}_K$ denote the Hilbert upper-half plane associated to K defined as $\{\tau\in K\otimes\mathbb{C} \mid \mathrm{Im}(\tau)\gg 0\}$. For an even non-negative integer k, we define the action of $\mathrm{GL}_2(K\otimes\mathbb{R})^+$ (the group generated by matrices with totally positive determinants) on the set of all $\mathbb{C}$-valued functions on $\mathfrak{h}_K$ by

$$\left(F|_k\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right)(\tau) = \mathcal{N}(ad-bc)^{k/2}\mathcal{N}(c\tau+d)^{-k}F\left(\frac{a\tau+b}{c\tau+d}\right)$$

where $\mathcal{N}: K\otimes\mathbb{C}\to\mathbb{C}$ is the usual norm map.

Definition 6.7 *(Hilbert modular forms).* Take a non-zero integral ideal $\mathfrak{f}$ of $\mathcal{O}_K$ with all prime factors in Σ and set

$$\Gamma_{00}(\mathfrak{f}) = \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(K) \;\middle|\; a,d\in 1+\mathfrak{f},\ b\in\mathfrak{D}^{-1},\ c\in\mathfrak{f}\mathfrak{D}\right\}$$

where $\mathfrak{D}$ is the differential of K. *A Hilbert modular form F of weight k on $\Gamma_{00}(\mathfrak{f})$* is a holomorphic function[6] $F:\mathfrak{h}_K\to\mathbb{C}$ satisfying $F|_kM = F$ for an arbitrary element M in $\Gamma_{00}(\mathfrak{f})$.

[6] If K is the rational number field $\mathbb{Q}$, we also assume that F is holomorphic at ∞.

A Hilbert modular form F has a Fourier series expansion called *the standard q-expansion*: $c(0)+\sum_{\mu\in\mathcal{O}_K,\ \mu\gg 0} c(\mu)q_K^\mu$ (q_K^μ is a complex number defined as $\exp(2\pi\sqrt{-1}\mathrm{Tr}_{K/\mathbb{Q}}(\mu\tau))$). Deligne and Ribet [6, Theorem (6.1)] constructed the Hilbert–Eisenstein series attached to a locally constant function ε.

Theorem–Definition 6.8 *(Hilbert–Eisenstein series). Let ε be a locally constant function on* $\mathrm{Gal}(K^\infty/K)$ *and k an even non-negative integer. Then there exist an integral ideal $\mathfrak{f}$ of $\mathcal{O}_K$ with its all prime factors in Σ and a Hilbert modular form $G_{k,\varepsilon}$ of weight k on $\Gamma_{00}(\mathfrak{f})$ characterized by the following standard q-expansion*:

$$2^{-r}\zeta_{K^\infty/K}(1-k,\varepsilon)+\sum_{\substack{\mu\in\mathcal{O}_K\\ \mu\gg 0}}\left(\sum_{\substack{\mu\in\mathfrak{a}\subseteq\mathcal{O}_K\\ \text{prime to }\Sigma}}\varepsilon(\mathfrak{a})\kappa(\mathfrak{a})^{k-1}\right)q_K^\mu$$

where $\zeta_{K^\infty/K}(s,\varepsilon)$ is the partial zeta function for ε defined in the same manner as Definition 6.4 ($\varepsilon(\mathfrak{a})$ and $\kappa(\mathfrak{a})$ denote $\varepsilon(((K^\infty/K),\mathfrak{a}))$ and $\kappa(((K^\infty/K),\mathfrak{a}))$ respectively, where $((K^\infty/K),-)$ is the Artin symbol for K^∞/K). The Hilbert modular form $G_{k,\varepsilon}$ is called the Hilbert–Eisenstein series of weight k attached to ε.

Next let us discuss the q-expansion of Hilbert modular forms at cusps. Let $\mathbb{A}_K^{\mathrm{fin}}$ denote the finite adèle ring of K. By the strong approximation theorem $\mathrm{SL}_2(\mathbb{A}_K^{\mathrm{fin}})=\hat{\Gamma}_{00}(\mathfrak{f})\cdot\mathrm{SL}_2(K)$, each element M in $\mathrm{SL}_2(\mathbb{A}_K^{\mathrm{fin}})$ is decomposed as $M=M_1M_2$ where M_1 is an element in $\hat{\Gamma}_{00}(\mathfrak{f})$—the topological closure of $\Gamma_{00}(\mathfrak{f})$ in $\mathrm{SL}_2(\mathbb{A}_K^{\mathrm{fin}})$—and M_2 is that in $\mathrm{SL}_2(K)$. For such M, we define $F|_kM$ as $F|_kM_2$. Let F_α denote

$$F|_k\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}$$

for a finite idèle α. The Fourier series expansion of F_α is called *the q-expansion of F at the cusp determined by α*. Deligne and Ribet [6, Theorem (6.1)] determined the q-expansion of the Hilbert–Eisenstein series $G_{k,\varepsilon}$ at cusps:

Proposition 6.9. *Let k be an even non-negative integer and ε a locally constant function on* $\mathrm{Gal}(K^\infty/K)$. *Then the q-expansion of $G_{k,\varepsilon}$ at the cusp determined by a finite idèle α is given by*

$$\mathcal{N}((\alpha))^k\left\{2^{-r}\zeta_{K^\infty/K}(1-k,\varepsilon_a)+\sum_{\substack{\mu\in\mathcal{O}_K\\ \mu\gg 0}}\left(\sum_{\substack{\mu\in\mathfrak{a}\subseteq\mathcal{O}_K\\ \text{prime to }\Sigma}}\varepsilon_a(\mathfrak{a})\kappa(\mathfrak{a})^{k-1}\right)q_K^\mu\right\}$$

where (α) is the ideal generated by α and a is defined as $((K^\infty/K),(\alpha)\alpha^{-1})$.

Now we introduce the Deligne–Ribet's deep q-expansion principle [6, Theorem (0.2) and Corollary (0.3)]:

Theorem 6.10 *(q-expansion principle, Deligne–Ribet). Let F_k denote a rational Hilbert modular form of weight k on $\Gamma_{00}(\mathfrak{f})$ (that is, all coefficients of the q-expansion of F_k at an arbitrary cusp are rational numbers), and suppose that F_k is equal to zero for all but finitely many k. Let α be a finite idèle of K and α_p the p-th component of α. Set $S(\alpha)=\sum_{k\geqslant 0}\mathcal{N}\alpha_pF_{k,\alpha}$.*

Assume that there exists a finite idèle γ such that all non-constant coefficients of $S(\gamma)$ are contained in $p^j\mathbb{Z}_{(p)}$ for a certain integer j. Then the difference between the constant terms of the q-expansions of $S(\alpha)$ and $S(\beta)$ is also contained in $p^j\mathbb{Z}_{(p)}$ for arbitrary two distinct finite idèles α and β.

6.5. *Proof of the sufficient conditions*

Now we prove the sufficient conditions $(2)'$ and $(3)'$ in Proposition 6.6.

Proof of Proposition 6.1. Suppose that i is equal to either 3 or 4 in the following. Let j be a sufficiently large integer and y a coset of W_i/Γ^{p^j} not contained in Γ. Let $G_{k,\delta^{(y)}}$ be the Hilbert–Eisenstein series of weight k attached to $\delta^{(y)}$ and let $E_{k,\delta^{(y)}}$ be the restriction of $G_{k,\delta^{(x)}}$ to F. Then the standard q-expansion of $E_{k,\delta^{(y)}}$ is given by

$$2^{-[F_{U_i}:\mathbb{Q}]}\zeta_{F_{V_i}/F_{U_i}}\big(1-k,\delta^{(y)}\big)+\sum_{\substack{\nu\in\mathcal{O}_{F_{U_i}}\\ \nu\gg 0}}\Bigg(\sum_{\substack{\nu\in\mathfrak{b}\subseteq\mathcal{O}_{F_{U_i}}\\ \text{prime to }\Sigma}}\delta^{(y)}(\mathfrak{b})\kappa(\mathfrak{b})^{k-1}\Bigg)q_F^{\mathrm{Tr}_{F_{U_i}/F}(\nu)} \tag{6.3}$$

where $q_{F_{U_i}}^{\nu}$ is defined as $\exp(2\pi\sqrt{-1}\mathrm{Tr}_{F_{U_i}/\mathbb{Q}}(\nu\tau))$. Let P_i denote the set of all pairs $(\mathfrak{b},\nu)$ where $\mathfrak{b}$ is a non-zero integral ideal of $\mathcal{O}_{F_{U_i}}$ prime to Σ and ν is a totally real element in $\mathfrak{b}$. Then P_i enjoys the natural action of NU_i and NU_i/U_i.

(Case-1). Suppose that the isotropic subgroup $(NU_i/U_i)_{(\mathfrak{b},\nu)}$ at an element $(\mathfrak{b},\nu)$ in P_i is trivial. Then we may easily calculate the sum of non-constant terms in (6.3) containing (NU_i/U_i)-orbit of $(\mathfrak{b},\nu)$-part:

$$\begin{aligned}\sum_{\sigma\in NU_i/U_i}\delta^{(y)}\big(\mathfrak{b}^{\sigma}\big)\kappa\big(\mathfrak{b}^{\sigma}\big)^{k-1}q_F^{\mathrm{Tr}_{F_{U_i}/F}(\nu^{\sigma})}&=\sum_{\sigma\in NU_i/U_i}\delta^{(\sigma y\sigma^{-1})}(\mathfrak{b})\kappa(\mathfrak{b})^{k-1}q_F^{\mathrm{Tr}_{F_{U_i}/F}(\nu)}\\&=p^{m_y}\sum_{\sigma\in(NU_i/U_i)/(NU_i/U_i)_y}\delta^{(\sigma y\sigma^{-1})}(\mathfrak{b})\kappa(\mathfrak{b})^{k-1}q_F^{\mathrm{Tr}_{F_{U_i}/F}(\nu)}.\end{aligned}$$

(Case-2). Suppose that the isotropy subgroup $(NU_i/U_i)_{(\mathfrak{b},\nu)}$ is not trivial. Let $F_{(\mathfrak{b},\nu)}$ denote the maximal intermediate field of F_{U_i}/F fixed by $(NU_i/U_i)_{(\mathfrak{b},\nu)}$ and $F'_{(\mathfrak{b},\nu)}$ that of F^{∞}/F fixed by the commutator subgroup of $(NU_i)_{(\mathfrak{b},\nu)}$. Then there exist a unique integral ideal $\mathfrak{a}$ of $\mathcal{O}_{F_{(\mathfrak{b},\nu)}}$ and a unique element μ in $\mathfrak{a}$ which satisfy $(\mathfrak{b},\nu)=(\mathfrak{a}\mathcal{O}_{F_{U_i}},\mu)$. For such $\mathfrak{a}$ and μ, the equation

$$\delta^{(y)}(\mathfrak{b})=\delta^{(y)}\big(\big((F_{V_i}/F_{U_i}),\mathfrak{a}\mathcal{O}_{F_{U_i}}\big)\big)=\delta^{(y)}\circ\mathrm{Ver}\big(\big((F'_{(\mathfrak{b},\nu)}/F_{(\mathfrak{b},\nu)}),\mathfrak{a}\big)\big)=0$$

holds since the image of the Verlagerung homomorphism is contained in Γ by Lemma 4.3 but y is not contained in Γ by assumption.

Therefore $E_{k,\delta^{(y)}}$ has all non-constant coefficients in $p^{m_y}\mathbb{Z}_{(p)}$ by (Case-1) and (Case-2). Take a finite idèle γ satisfying $((F_{V_i}/F_{U_i}),(\gamma)\gamma^{-1})=w$. Then the constant term of $E_{k,\delta^{(y)}}-E_{k,\delta^{(y)}}(\gamma)$ is also contained in $p^{m_y}\mathbb{Z}_{(p)}$ by q-expansion principle of Deligne and Ribet (Theorem 6.10), which is nothing but $2^{-[F_{U_i}:\mathbb{Q}]}\Delta_i^{w}(1-k,\delta^{(y)})$. Note that 2 is invertible since p is an odd prime number. □

7. Proof of the main theorem

In the previous section, we obtained congruences among the abelian p-adic zeta peudomeasures (Proposition 6.1). Unfortunately, these congruences are not sufficient to prove that $(\xi_i)_{i=0}^4$ is contained in Ψ_S, therefore we may not apply Burns' technique 2.4 directly to $(\xi_i)_{i=0}^4$. In this section we will modify the proof of Theorem 2.4 and prove our main theorem (Theorem 2.1) by using certain induction.

7.1. Kato's p-adic zeta function for F_N/F

Let N be a (closed) normal subgroup of G as was defined in Section 3.1. Set $\overline{G} = G/N, \overline{U}_i = U_i/N$ and $\overline{V}_i = V_iN/N$ respectively. Note that the p-adic Lie group

$$\overline{G} = \begin{pmatrix} 1 & \mathbb{F}_p & \mathbb{F}_p \\ 0 & 1 & \mathbb{F}_p \\ 0 & 0 & 1 \end{pmatrix} \times \Gamma$$

is a group "of Heisenberg type." Kazuya Kato [13, Theorem 4.1] has already proven the existence and the uniqueness of the p-adic zeta function for every totally real Galois extension of Heisenberg type, hence we obtain the following:

Theorem 7.1 *(K. Kato). The p-adic zeta function $\bar{\xi}$ for F_N/F exists uniquely and the Iwasawa main conjecture (Conjecture 1.6(2)) is true for F_N/F.*

Note that there exists a splitting exact sequence

$$1 \longrightarrow N \longrightarrow G \xrightarrow{\pi} \overline{G} \longrightarrow 1$$

with a section s defined as

$$s : \overline{G} \to G; \qquad \left(\begin{pmatrix} 1 & a & d \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, t^z \right) \mapsto \left(\begin{pmatrix} 1 & a & d & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, t^z \right).$$

Let $C = C_{F^\infty/F}$ denote the complex defined as (1.2), which is an object of $\mathscr{C}_S^{\mathrm{Perf}}(\Lambda(G))$ by hypothesis ($\sharp$). Let $[\overline{C}]$ denote the norm image of $[C]$ in $K_0(\mathscr{C}_S^{\mathrm{Perf}}(\Lambda(\overline{G})), \mathrm{qis})$. Note that Kato's p-adic zeta function $\bar{\xi}$ satisfies the main conjecture $\partial(\bar{\xi}) = -[\overline{C}]$ by Theorem 7.1.

Proposition 7.2. *There exists a characteristic element f for F^∞/F whose image in $K_1(\Lambda(\overline{G})_S)$ coincides with the Kato's p-adic zeta function $\bar{\xi}$.*

Proof. For an arbitrary element x in $K_1(\Lambda(G)_S)$, let $\bar{x}$ denote its image in $K_1(\Lambda(\overline{G})_S)$. Take an arbitrary characteristic element f' for F^∞/F. Then $\partial(\bar{\xi}(\bar{f}')^{-1})$ vanishes by the functoriality of the connecting homomorphism ∂ and the main conjecture for F_N/F. The localization exact sequence (1.1) implies that $\bar{\xi}(\bar{f}')^{-1}$ is the image of an element in $K_1(\Lambda(\overline{G}))$, which we denote by $\bar{u}$. Then the element f in $K_1(\Lambda(\overline{G})_S)$ defined as $f's(\bar{u})$ satisfies the assertion of the proposition where s denotes the homomorphism $K_1(\Lambda(\overline{G})) \to K_1(\Lambda(G))$ induced by s. $\square$

7.2. Completion of the proof

Let f be a characteristic element for F^∞/F whose image in $K_1(\Lambda(\overline{G})_S)$ coincides with $\bar{\xi}$ and f_i its image under the map $\theta_{S,i}$. Set $u_i = \xi_i f_i^{-1}$. Then $\partial(u_i)$ vanishes by construction, thus u_i is contained in $\Lambda(U_i/V_i)^\times$ by the localization sequence (1.1). To prove Theorem 2.1, it suffices to show that $(u_i)_{i=0}^4$ is contained in Ψ by the proof of Burns' technique (Theorem 2.4). Note that $(u_i)_{i=0}^4$ obviously satisfies the norm relations for Ψ since both $(f_i)_{i=0}^4$ and $(\xi_i)_{i=0}^4$ satisfy them.

Proposition 7.3. *Each u_i satisfies the following congruences*:

$$u_1 \equiv \varphi(u_0) \mod I_1, \qquad u_2 \equiv \varphi(u_0) \mod I_2,$$
$$u_3 \equiv d_3 \mod J_3, \qquad u_4 \equiv d_4 \mod I_4$$

where both d_3 and d_4 are certain elements in $\Lambda(\Gamma)$.

Recall that J_3 contains I_3 (see the last paragraph of Section 4.1).

Proof. Each f_i satisfies the congruences in the definition of Ψ_S since $(f_i)_{i=0}^4$ is contained in Ψ_S. Hence $\varphi(f_0)^{-(G:U_i)/p} f_i$ is contained in $1 + I_{S,i}$ for each i except for 0. On the other hand, each ξ_i satisfies the congruences in Proposition 6.1, thus $\varphi(\xi_0)\xi_i^{-1}$ is an element in $1 + I_{S,i}$ for $i = 1, 2$, $c_3\xi_3^{-1}$ is in $1 + J_{S,3}$ and $c_4\xi_4^{-1}$ is in $1 + I_{S,4}$ respectively. Then we may obtain the desired congruences by multiplying them appropriately since we may replace $c_i\varphi(\xi_0)^{-(G:U_i)/p}$ by an element d_i in $\Lambda(\Gamma)$ for $i = 3$ and 4 (refer to Remark 6.3). □

Lemma 7.4. *The element $(u_i)_{i=0}^4$ is contained in the kernel of the canonical surjection $\pi^\times : \prod_i \Lambda(U_i/V_i)^\times \to \prod_i \Lambda(\overline{U}_i/\overline{V}_i)^\times$.*

Proof. The claim follows from the construction of f (Proposition 7.2) and the compatibility between $\pi^\times$ and θ_i. □

Lemma 7.5. *The $\mathbb{Z}_p$-module J_3 contains $s \circ \pi(J_3)$ and the $\mathbb{Z}_p$-module I_4 contains $s \circ \pi(I_4)$ respectively.*

Proof. Just simple calculation. □

Lemma 7.6. *The element $\varphi(u_i)$ is equal to 1 for each i.*

Proof. The Frobenius homomorphism $\varphi : \Lambda(U_i/V_i) \to \Lambda(\Gamma)$ factors as $s \circ \bar{\varphi} \circ \pi$ where $\bar{\varphi} : \Lambda(\overline{U}_i/\overline{V}_i) \to \Lambda(\Gamma)$ is the Frobenius homomorphism on $\Lambda(\overline{U}_i/\overline{V}_i)$ and π is the canonical surjection $\Lambda(U_i/V_i) \to \Lambda(\overline{U}_i/\overline{V}_i)$. Hence the claim holds since $(u_i)_{i=0}^4$ is contained in the kernel of $\pi^\times$ by Lemma 7.4. □

Proof of Theorem 2.1. It is sufficient to verify that $(u_i)_{i=0}^4$ satisfies the desired congruences for Ψ. The congruence for $i = 1$ and 2 has been already proven in Proposition 7.3, hence we will verify the congruence $u_i \equiv \varphi(u_0)^{(G:U_i)/p} \mod I_i$ for $i = 3, 4$ as follows. By operating $s \circ \pi$ to the congruences in Proposition 7.3, we obtain the following congruences:

$$s \circ \pi(u_3) \equiv d_3 \mod s \circ \pi(J_3), \qquad s \circ \pi(u_4) \equiv d_4 \mod s \circ \pi(I_4).$$

Both $s \circ \pi(u_3)$ and $s \circ \pi(u_4)$ are equal to 1 by Lemma 7.4, thus the congruences

$$u_3 \equiv d_3 \equiv 1 = \varphi(u_0)^p \mod J_3, \qquad u_4 \equiv d_4 \equiv 1 = \varphi(u_0)^{p^2} \mod I_4 \tag{7.1}$$

hold by Lemmas 7.5 and 7.6. Hence we have only to verify the congruence

$$u_3 \equiv 1\big(= \varphi(u_0)^p\big) \mod I_3. \tag{7.2}$$

We remark that u_3 is contained in $1 + J_3$ and u_4 is contained in $1 + I_4$ respectively by (7.1), thus $\log u_3$ is contained in J_3 and $\log u_4$ is contained in I_4 by logarithmic isomorphisms (Lemma 4.6). We may describe $\log u_3$ and $\log u_4$ as $\Lambda(\Gamma)$-linear combinations of generators of J_3 and I_4 which we denote by

$$\log u_3 = \sum_{b,c\neq 0} \tilde{\nu}^{(3)}_{bc}\beta^b\gamma^c h_\zeta + \sum_{b,f} p\tilde{\nu}^{(4)}_{bf}\beta^b\zeta^f,$$

$$\log u_4 = \sum_f p^3\sigma^{(1)}_f\zeta^f + \sum_{e\neq 0} p^2\sigma^{(2)}_e\varepsilon^e h_\zeta + \sum_{c\neq 0} p\sigma^{(3)}_c\gamma^c h_\varepsilon h_\zeta.$$

Then we may calculate that $\mathrm{Tr}_{\mathbb{Z}_p[\![U_3/V_3]\!]/\mathbb{Z}_p[\![U_4/V_3]\!]}(\log u_3)$ is equal to an element defined as $\sum_{c\neq 0} p\tilde{\nu}^{(3)}_{0c}\gamma^c h_\zeta + \sum_f p^2\tilde{\nu}^{(4)}_{0f}h_\zeta$ and the image of $\log u_4$ in $\mathbb{Z}_p[\![U_4/V_3]\!]$ is equal to the element defined as $\sum_f p^2(p\sigma^{(1)}_f + \sum_{e\neq 0}\sigma^{(2)}_e)\zeta^f + \sum_{c\neq 0} p^2\sigma^{(3)}_c\gamma^c h_\zeta$. Since $(u_i)_{i=0}^4$ satisfies the norm relations for Ψ, we obtain the following relations among coefficients by the corresponding trace relation (rel-6):

$$\tilde{\nu}^{(4)}_{0f} = p\sigma^{(1)}_f + \sum_{e\neq 0}\sigma^{(2)}_e, \qquad \tilde{\nu}^{(3)}_{0c} = p\sigma^{(3)}_c \quad (c\neq 0).$$

Therefore if we set

$$\nu^{(1)}_f = \sigma^{(1)}_f, \qquad \nu^{(2)}_c = \begin{cases} \sum_{e\neq 0}\sigma^{(2)}_e & \text{if } c=0,\\ \sigma^{(3)}_c & \text{if } c\neq 0,\end{cases}$$

$$\nu^{(3)}_{bc} = \tilde{\nu}^{(3)}_{bc} \quad (b\neq 0, c\neq 0), \qquad \nu^{(4)}_{bf} = \tilde{\nu}^{(4)}_{bf} \quad (b\neq 0),$$

the element $\log u_3$ is written as

$$\sum_f p^2\nu^{(1)}_f\zeta^f + \sum_c p\nu^{(2)}_c\gamma^c h_\zeta + \sum_{b\neq 0,c\neq 0}\nu^{(3)}_{bc}\beta^b\gamma^c h_\zeta + \sum_{b\neq 0,f} p\nu^{(4)}_{bf}\beta^b\zeta^f,$$

which implies that $\log u_3$ is contained in I_3. Since the logarithm is injective on $1+J_3$ and induces an isomorphism between $1+I_3$ and I_3, u_3 is contained in $1+I_3$. Therefore the congruence (7.2) holds. □

Remark 7.7. The element $\xi = uf$ constructed as above satisfies $\theta_S(\xi) = (\xi_i)_{i=0}^4$. Since Ψ_S contains the image of θ_S, we especially obtain the following non-trivial congruences among abelian p-adic zeta pseudomeasures:

$$\xi_3 \equiv \varphi(\xi_0)^p \mod I_{S,3}, \qquad \xi_4 \equiv \varphi(\xi_0)^{p^2} \mod I_{S,4}.$$

It seems to be almost impossible to derive such congruences by using only q-expansion principle.

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