Crossed products for weak Hopf algebras with coalgebra splitting

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Abstract

In this paper we prove that if \( g : B \to H \) is a morphism of weak Hopf algebras which is split as a coalgebra morphism, then there exists a subalgebra \( B_H \) of \( B \), morphisms \( \varphi_{B_H} : H \otimes B_H \to B_H \), \( \sigma_{B_H} : H \otimes H \to B_H \) and an isomorphism of algebras and right \( H \)-comodules \( b_{B_H} : B \to B_H \times H \), being \( B_H \times H \) a subobject of \( B_H \otimes H \) with its algebra structure defined by a crossed product. Also, we obtain the dual results and as a consequence we prove the Radford’s theorem for weak Hopf algebras.

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Introduction

Weak Hopf algebras (or quantum groupoids in the terminology of Nikshych and Vainerman [6]) have been introduced by Böhm, Nill, and Szlachányi [3] as a new generalization of Hopf algebras and groupoid algebras. Roughly speaking, a weak Hopf algebra \( H \) in a
symmetric monoidal category is an object that has both algebra and coalgebra structures with some relations between them and that possesses an antipode \( \lambda_H \) which not necessarily verify \( \lambda_H \ast \text{id}_H = \text{id}_H \ast \lambda_H = \varepsilon_H \otimes \eta_H \) where \( \varepsilon_H, \eta_H \) are the counity and unity morphisms, respectively. The main differences with other Hopf algebraic constructions, such as quasi-Hopf algebras and rational Hopf algebras, are the following: weak Hopf algebras are coassociative but the coproduct is not required to preserve the unity \( \eta_H \) or equivalently the counity is not an algebra morphism. Some motivations to study weak Hopf algebras come from their connection with the theory of algebra extensions, the important applications in the study of dynamical twists of Hopf algebras and their link with quantum field theories and operator algebras.

The main result of this paper is a generalization, for weak Hopf algebras living in a symmetric monoidal category with split idempotents, of the well-know result, due to Blattner, Cohen, and Montgomery, which shows that if \( H \rightarrowtail H ightarrow 0 \) is an exact sequence of Hopf algebras with coalgebra splitting, then \( H \cong A^{\mu_H} \), where \( A \) is the left Hopf kernel of \( \pi \) and \( \sigma \) is a suitable cocycle (see [2, Theorem (4.14)]). In Section 2, we prove that if \( g : B \rightarrow H \) is a morphism of weak Hopf algebras and there exists a morphism of coalgebras \( f : H \rightarrow B \) such that \( g \circ f = \text{id}_H \) and \( f \circ \eta_H = \eta_B \), using the idempotent morphism \( q_B^\mu_H = \mu_B \circ (B \otimes (\lambda_B \circ f \circ g)) \circ \delta_B : B \rightarrow B \) and an equalizer diagram it is possible to construct an algebra \( BH \) and morphisms \( \varphi_{BH} : H \otimes BH \rightarrow BH \) and \( \sigma_{BH} : H \otimes H \rightarrow BH \) such that there exists a subobject \( BH \times H \) of \( BH \otimes H \) isomorphic with \( B \) as algebras and with algebra structure (the crossed product) defined by

\[
\eta_{BH \times H} = r_B \circ (\eta_{BH} \otimes \eta_H),
\]

\[
\mu_{BH \times H} = r_B \circ (\mu_{BH} \otimes H) \circ (\mu_{BH} \otimes \sigma_{BH} \otimes \mu_H) \circ (B_H \otimes \psi_{BH} \otimes \delta_H \otimes H)
\]

\[
\circ (B_H \otimes H \otimes c_{H,BH} \otimes H) \circ (B_H \otimes \delta_H \otimes B_H \otimes H) \circ (s_B \otimes s_B),
\]

where \( s_B \) is the inclusion of \( BH \times H \) in \( BH \otimes H \) and \( r_B \) the projection of \( BH \otimes H \) on \( BH \times H \). Of course, when \( H, B \) are Hopf algebras we recover the result of Blattner, Cohen, and Montgomery. For this reason, we denote the algebra \( BH \times H \) by \( BH^{\varphi_{BH} \otimes \sigma_{BH} H} \).

In the third section we prove the dual results using arguments similar to the ones developed in Section 2, but passing to the opposite category. Finally, linking the information of these two sections with the results of [1], we obtain our version of Radford’s theorem for weak Hopf algebras with projection, that is: if \( H, B \) are weak Hopf algebras in \( C \) and \( g : B \rightarrow H, f : H \rightarrow B \) are morphisms of weak Hopf algebras such that \( g \circ f = \text{id}_H \), the object \( BH \) belongs to the category \( W \mathcal{W} \mathcal{Y} \mathcal{D} \) (defined in [1]) and verifies that \( B \) is isomorphic to \( BH \times H \) as weak Hopf algebras, being the (co)algebra structure in \( BH \times H \) the smash (co)product. Also, we obtain the expression for the antipode of \( BH \times H \).

1. Preliminaries

In what follows \( C \) denotes a symmetric monoidal category with tensor product \( \otimes \), symmetry isomorphism \( c \) and base object \( K \). We will suppose too that \( C \) admits split idem-
potents, i.e., for every morphism \( q: Y \to Y \) such that \( q = q \circ q \) exists and object \( Z \) and morphisms \( i: Z \to Y \) and \( p: Y \to Z \) such that \( q = i \circ p \) and \( p \circ i = \text{id}_Z \).

An algebra in \( \mathcal{C} \) is a triple \( A = (A, \eta_A, \mu_A) \) where \( A \) is an object in \( \mathcal{C} \) and \( \eta_A: K \to A \) (unit), \( \mu_A: A \otimes A \to A \) (product) are morphisms in \( \mathcal{C} \) such that \( \mu_A \circ (A \otimes \eta_A) = \text{id}_A = \mu_A \circ (\eta_A \otimes A) \), \( \mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A) \). Given two algebras \( A = (A, \eta_A, \mu_A) \) and \( B = (B, \eta_B, \mu_B) \), \( f: A \to B \) is an algebra morphism if \( \mu_B \circ (f \otimes f) = f \circ \mu_A \), \( f \circ \eta_A = \eta_B \). Also, if \( A, B \) are algebras in \( \mathcal{C} \), the object \( A \otimes B \) is also an algebra in \( \mathcal{C} \) where \( A \otimes B = \eta_A \otimes \eta_B \) and \( \mu_{A \otimes B} = \mu_A \otimes \mu_B \circ (A \otimes c_{B,A} \otimes B) \).

A coalgebra in \( \mathcal{C} \) is a triple \( D = (D, \varepsilon_D, \delta_D) \) where \( D \) is an object in \( \mathcal{C} \) and \( \varepsilon_D: D \to K \) (counit), \( \delta_D: D \to D \otimes D \) (coproduct) are morphisms in \( \mathcal{C} \) such that \( \varepsilon_D \circ \delta_D = \text{id}_D = \varepsilon_D \circ (\varepsilon_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D \circ \delta_D \). If \( D = (D, \varepsilon_D, \delta_D) \) and \( E = (E, \varepsilon_E, \delta_E) \) are coalgebras, \( f: D \to E \) is a coalgebra morphism if \( f \circ \delta_D = \delta_E \circ f \), \( \varepsilon_E \circ f = \varepsilon_D \). When \( D, E \) are coalgebras in \( \mathcal{C} \), \( D \otimes E \) is a coalgebra in \( \mathcal{C} \) where \( \varepsilon_{D \otimes E} = \varepsilon_D \otimes \varepsilon_E \) and \( \delta_{D \otimes E} = (D \otimes c_{E,D} \otimes E) \circ (\delta_D \otimes \delta_E) \).

Weak Hopf algebras are generalizations of Hopf algebras. The axioms are the same as the ones for a Hopf algebra, except that the coproduct of the unit, the product of the counit and the antipode condition are replaced by weaker properties. Below we collect the definition of an basic properties of weak Hopf algebras.

**Definition 1.1.** A weak Hopf algebra \( H \) in \( \mathcal{C} \) is by definition an algebra \( (H, \eta_H, \mu_H) \) and coalgebra \( (H, \varepsilon_H, \delta_H) \) such that the following axioms hold:

(a1) \( \delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_H \circ H \).

(a2) \( \varepsilon_H \circ \mu_H = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \circ H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (c_{H,H} \otimes \delta_H) \circ H) \).

(a3) \( \varepsilon_H \circ \delta_H \circ \eta_H = (H \otimes \mu_H \circ H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H) \).

(a4) There exists a morphism \( \lambda_H: H \to H \) in \( \mathcal{C} \) (called antipode of \( H \)) verifying:

(a4-1) \( \mu_H \circ (H \otimes \lambda_H) \circ \delta_H = ((\varepsilon_H \circ \mu_H) \circ H) \circ (D \otimes c_{H,H} \circ H) \circ (\delta_H \circ \eta_H) \).

(a4-2) \( \mu_H \circ (\lambda_H \otimes H) \circ \delta_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)). \)

(a4-3) \( \mu_H \circ (\mu_H \circ H) \circ (\lambda_H \otimes H) \circ \delta_H = \lambda_H \).

Axioms (a2) and (a3) above are the weaker version to the usual bialgebra axioms of \( \delta_H \) being a unit preserving map and \( \varepsilon_H \) being an algebra homomorphism. Axioms (a4-1), (a4-2), and (a4-3) generalize the properties of the antipode in a Hopf algebra with respect to the counit \( \varepsilon_H \). Observe that in the definition of Hopf algebra, (a2)–(a4) are replaced by the conditions

(a2′) \( \varepsilon_H \circ \mu_H = \varepsilon_H \otimes \varepsilon_H \).

(a3′) \( \delta_H \circ \eta_H = \eta_H \otimes \eta_H \).

(a4) There exists a morphism \( \lambda_H: H \to H \) in \( \mathcal{C} \) verifying:

\[
\mu_H \circ (H \otimes \lambda_H) \circ \delta_H = \mu_H \circ (\lambda_H \otimes H) \circ \delta_H = \varepsilon_H \otimes \eta_H.
\]
Therefore, a Hopf algebra is always a weak Hopf algebra. Then, a weak Hopf algebra is a Hopf algebra if and only if the morphism $\delta_H$ (comultiplication) is unit-preserving and if and only if the counit is a homomorphism of algebras.

If $H$ is a weak Hopf algebra, the antipode $\lambda_H$ is unique, antimultiplicative, anticomultiplicative and leaves the unit $\eta_H$ and the counit $\varepsilon_H$ invariant:

$$
\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ \eta_{H,H}, \quad \delta_H \circ \lambda_H = \varepsilon_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H,
$$

$$
\lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H.
$$

If we define the morphisms $\Pi^L_H, \Pi^R_H, \Pi^L_H$, and $\Pi^R_H$ by

$$
\Pi^L_H = \left( (\varepsilon_H \circ \mu_H) \otimes H \right) \circ (H \otimes c_{H,H}) \circ \left( (\delta_H \circ \eta_H) \otimes H \right): H \rightarrow H,
$$

$$
\Pi^R_H = \left( H \otimes (\varepsilon_H \circ \mu_H) \right) \circ (c_{H,H} \otimes \delta_H \circ \eta_H): H \rightarrow H,
$$

$$
\Pi^L_H = \left( (H \otimes (\varepsilon_H \circ \mu_H)) \circ \left( (\delta_H \circ \eta_H) \otimes H \right) \right): H \rightarrow H,
$$

$$
\Pi^R_H = \left( (\varepsilon_H \circ \mu_H) \otimes H \right) \circ \left( H \otimes (\delta_H \circ \eta_H) \right): H \rightarrow H,
$$

it is straightforward to show (see [3]) that they are idempotent and $\Pi^L_H, \Pi^R_H$ verify the equalities:

$$
\Pi^L_H = \mu_H \circ (H \otimes \lambda_H) \circ \delta_H, \quad \Pi^R_H = \mu_H \circ (\lambda_H \otimes H) \circ \delta_H.
$$

Moreover, we have that (see [5])

$$
\Pi^R_H \circ \Pi^L_H = \Pi^L_H, \quad \Pi^L_H \circ \Pi^R_H = \Pi^R_H, \quad \Pi^L_H \circ \Pi^L_H = \Pi^R_H, \quad \Pi^R_H \circ \Pi^R_H = \Pi^L_H,
$$

it is easy to show the formulas:

$$
\Pi^L_H = \Pi^R_H \circ \lambda_H = \lambda_H \circ \Pi^L_H, \quad \Pi^R_H = \Pi^L_H \circ \lambda_H = \lambda_H \circ \Pi^R_H,
$$

$$
\Pi^L_H \circ \lambda_H = \Pi^L_H \circ \Pi^R_H = \lambda_H \circ \Pi^R_H, \quad \Pi^R_H \circ \lambda_H = \Pi^R_H \circ \Pi^L_H = \lambda_H \circ \Pi^L_H.
$$

Finally, if $\lambda_H$ is bijective (for example, when $H$ is finite), in [9] we can find the equalities:

$$
\Pi^L_H = \mu_H \circ (H \otimes \lambda_H^{-1}) \circ c_{H,H} \circ \delta_H, \quad \Pi^R_H = \mu_H \circ (\lambda_H^{-1} \otimes H) \circ c_{H,H} \circ \delta_H.
$$

A morphism between weak Hopf algebras $H$ and $B$ is a morphism $f: H \rightarrow B$ which is both algebra and coalgebra morphism. If $f: H \rightarrow B$ is a weak Hopf algebra morphism, then $\lambda_B \circ f = f \circ \lambda_H$ (see [1, 1.4]).
2. Morphisms of weak Hopf algebras with coalgebra splitting and crossed products

In this section we obtain the main result of this paper. In Theorem 2.8 we will prove that if \( H, B \) are weak Hopf algebras in \( C \) and \( g : B \to H \) is a weak Hopf algebra morphism such that there exist a coalgebra morphism \( f : H \to B \) verifying \( g \circ f = \text{id}_H \) and \( f \circ \eta_H = \eta_B \), then it is possible to find an object \( B_H \), defined by an equalizer diagram, morphisms \( \varphi_{B_H} : H \otimes B_H \to B_H \), \( \sigma_{B_H} : H \otimes H \to B_H \) and an isomorphism of algebras and comodules \( b_H : B \to B_H \times H \) being \( B_H \times H \) a subobject of \( B_H \otimes H \) with its algebra structure twisted by the morphism \( \sigma_{B_H} \). Of course, the multiplication in \( B_H \times H \) is a generalization of the crossed product and in the Hopf algebra case Theorem 2.8 is the classical and well-know result obtained by Blattner, Cohen, and Montgomery in [2].

**Proposition 2.1.** Let \( H, B \) be weak Hopf algebras in \( C \). Let \( g : B \to H \) be a morphism of weak Hopf algebras and \( f : H \to B \) be a morphism of coalgebras such that \( g \circ f = \text{id}_H \). Then the following morphism is an idempotent in \( C \):

\[
q_B^H = \mu_B \circ (B \otimes (\lambda_B \circ f \circ g)) \circ \delta_B : B \to B.
\]

**Proof.** We have

\[
q_B^H \circ q_B^H = \mu_B \circ (\mu_B \otimes (\lambda_B \circ f \circ g \circ \mu_B)) \circ \delta_B \circ (B \otimes (\lambda_B \circ f \circ g) \circ \delta_B) \\
= \mu_B \circ (\mu_B \otimes (\lambda_B \circ f \circ \mu_H)) \circ (B \otimes c_{H,B} \otimes H) \circ (B \otimes g \otimes \lambda_B \otimes (g \circ \lambda_B)) \circ (B \otimes B \otimes \delta_B) \circ (\delta_B \otimes (\delta_B \circ f \circ g)) \circ \delta_B \\
= \mu_B \circ (B \otimes (\lambda_B \circ f \circ \Pi_{\mu_B}^H) \otimes (\lambda_B \circ f \circ g)) \circ \delta_B \\
= \mu_B \circ (B \otimes (\lambda_B \circ f \circ \Pi_{\mu_B}^H) \otimes (\lambda_B \circ f \circ g)) \circ \delta_B \\
= \mu_B \circ (B \otimes (\lambda_B \circ f \circ \Pi_{\mu_B}^H \otimes B) \otimes \delta_B) \circ (\delta_B \circ f \circ g)) \circ \delta_B = q_B^H.
\]

Note that the first equality follows from (a1), the second and the third ones from the associativity, the coassociativity, the naturality of \( c \), the condition of morphism of weak Hopf algebras for \( g \) and the anticomultiplicative nature of the antipode. In the fourth one we use the equality \( f \circ \Pi_{\mu_B}^H = \Pi_{\mu_B}^H \circ f \). The fifth one follows from the condition of morphism of coalgebras for \( f \) and the antimultiplicative nature of the antipode. Finally, in the fifth one we apply the equality \( \mu_B \circ (\Pi_{\mu_B}^H \otimes B) \circ \delta_B = \text{id}_B \).

As a consequence of Proposition 2.1, we obtain that there exist an epimorphism \( p_B^H \), a monomorphism \( i_B^H \), and an object \( B_H \) such that the diagram

\[
\begin{array}{ccc}
B & \xrightarrow{q_B^H} & B \\
\downarrow{p_B^H} & & \uparrow{i_B^H} \\
B_H & & \\
\end{array}
\]

commutes and \( p_B^H \circ i_B^H = \text{id}_{B_H} \).
Proposition 2.2. Let $H$, $B$ be weak Hopf algebras in $C$. Let $g : B \to H$ be a morphism of weak Hopf algebras and $f : H \to B$ be a morphism of coalgebras such that $g \circ f = \text{id}_H$ and $f \circ \eta_H = \eta_B$. Then the following diagram is an equalizer diagram in $C$:

$$
\begin{array}{c}
B_H & \xrightarrow{i^B_H} & B & \xrightarrow{(B \otimes g) \circ \delta_B} & B \otimes H.
\end{array}
$$

(D1)

Proof. Let $h : B_H \to B \otimes H$ be the morphism defined by

$$
h = (\mu_B \otimes H) \circ (B \otimes c_{H, B}) \circ \left( (\Pi^L_H \otimes (\lambda_B \circ f)) \circ \delta_H \circ g \right) \circ \delta_B \circ i^B_H.
$$

We have that $(B \otimes g) \circ \delta_B \circ i^B_H = h$. Indeed, using repeatedly the associativity, the coassociativity, the naturality of $c$, the condition of morphism of weak Hopf algebras for $g$, the condition of morphism of coalgebras for $f$ and the anti(co)multiplicative nature of the antipode, we obtain

$$
(B \otimes g) \circ \delta_B \circ i^B_H = (B \otimes g) \circ \delta_B \circ q^B_H \circ i^B_H = (B \otimes g) \circ \mu_B \otimes B \circ (\delta_B \circ (\delta_B \circ \lambda_B \circ f \circ g)) \circ \delta_B \circ i^B_H = (B \otimes g) \circ \mu_B \otimes (B \otimes c_{H, B} \circ (\lambda_B \otimes \lambda_B) \circ \delta_B \circ f \circ g))
$$

$$
\circ (B \otimes \delta_B) \circ \delta_B \circ i^B_H = \mu_B \otimes (B \otimes \delta_H \circ (\lambda_H \circ f \circ \lambda_H) \circ c_{H, H} \circ \delta_H)) \circ (B \otimes (\delta_H \circ g))
$$

$$
\circ \delta_B \circ i^B_H = h.
$$

Thus, $(B \otimes g) \circ \delta_B \circ i^B_H = (B \otimes (\Pi^L_H \circ g)) \circ \delta_B \circ i^B_H$ because, by the idempotent character of $\Pi^L_H$, we have $h = (B \otimes \Pi^L_H) \circ h$.

Now, let $t : D \to B$ be a morphism such that $(B \otimes (\Pi^L_H \circ g)) \circ \delta_B \circ t = (B \otimes g) \circ \delta_B \circ t$. If $v = p^B_H \circ t$, since $\lambda_B \otimes \Pi_B^L = \Pi_B^L \otimes p^B_H$ and $\mu_B \circ (B \otimes (f \circ \Pi^L_B \circ g)) \circ \delta_B = \text{id}_B$ (in this equality we use $f \circ \eta_H = \eta_B$), we obtain $i^B_H \circ v = t$. Therefore, (D1) is an equalizer diagram.

Proposition 2.3. Let $H$, $B$ be weak Hopf algebras in $C$. Let $g : B \to H$ be a morphism of weak Hopf algebras and $f : H \to B$ be a morphism of coalgebras such that $g \circ f = \text{id}_H$ and $f \circ \eta_H = \eta_B$. Then $(B_H, \eta_{B_H} = p^B_H \circ \eta_B, (\mu_B \circ (i^B_H \otimes i^B_H)))$ is an algebra in $C$.

Proof. Note that the morphisms $\eta_{B_H}$ and $\mu_B \circ (i^B_H \otimes i^B_H)$ are the factorizations, through the equalizer $i^B_H$, of the morphisms $\eta_B$ and $\mu_B \circ (i^B_H \otimes i^B_H)$. It is an easy exercise to show that $(B_H, \eta_{B_H}, \mu_{B_H})$ is an algebra in $C$. □
**Proposition 2.4.** Let $H$, $B$ be weak Hopf algebras in $C$. Let $g : B \to H$ be a morphism of weak Hopf algebras and $f : H \to B$ be a morphism of coalgebras such that $g \circ f = \text{id}_H$ and $f \circ \eta_H = \eta_B$. There exists an unique morphism $\varphi_{BH} : H \otimes B_H \to B_H$ such that $i_H^B \circ \varphi_{BH} = y_B$ where $y_B : H \otimes B_H \to B$ is the morphism defined by

$$y_B = \mu_B \circ (B \otimes (\mu_B \circ c_{B,B})) \circ (f \otimes (\lambda_B \circ f) \otimes B) \circ (\delta_H \otimes i_H^B).$$

Moreover, the morphism $\varphi_{BH}$ verifies

1. $\varphi_{BH} = p_H^B \circ \mu_B \circ (f \otimes i_H^B)$.
2. $\varphi_{BH} \circ (\eta_H \otimes B_H) = \text{id}_{B_H}$.
3. $\varphi_{BH} \circ (H \otimes \eta_B) = \varphi_{BH} \circ (\Pi^L_H \otimes \eta_B)$.
4. $\mu_{BH} \circ (\varphi_{BH} \otimes B_H) \circ (H \otimes \eta_B \otimes B_H) = \varphi_{BH} \circ (\Pi^L_H \otimes B_H)$.
5. $\varphi_{BH} \circ (H \otimes \mu_B) = \mu_B \circ (\varphi_{BH} \otimes \varphi_{BH}) \circ (H \otimes c_{H,B} \otimes B_H) \circ (\delta_H \otimes B_H \otimes B_H)$.
6. $\mu_{BH} \circ c_{B,H,H} \circ ((\varphi_{BH} \circ (H \otimes \eta_B)) \otimes B_H) = \varphi_{BH} \circ (\Pi^L_H \otimes B_H)$.

**Proof.** Let $h' : H \otimes B_H \to B \otimes H$ be the morphism given by

$$h' = (\mu_B \otimes H) \circ (\mu_B \otimes (\epsilon_H \circ \mu_B) \otimes c_{H,B}) \circ (f \otimes c_{H,B} \otimes B \otimes \Pi^L_H \otimes (\lambda_B \circ f))
\circ (\delta_H \otimes \delta_B \otimes \delta_B) \circ (H \otimes c_{H,B}) \circ (\delta_H \otimes i_H^B).$$

This morphism verifies that $(B \otimes g) \circ \delta_B \circ y_B = h'$. Indeed, using the equality $\mu_H \circ (H \otimes \Pi^L_H) = ((\epsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$, the naturality of the braiding, the anticomultiplicative nature of the antipode, the condition of morphism of weak Hopf algebras for $g$ and the condition of morphism of coalgebras for $f$, we obtain

$$(B \otimes g) \circ \delta_H \circ y_B
= (B \otimes g) \circ \mu_B \circ (B \otimes B \otimes \mu_B) \circ (\delta_B \otimes \delta_B \otimes \delta_B) \circ (B \otimes c_{B,B})
\circ (f \otimes (\lambda_B \circ f) \otimes B) \circ (B \otimes i_H^B)
= \mu_B \circ (B \otimes H \otimes \mu_B \otimes H) \circ (f \otimes H \otimes B \otimes c_{H,B} \otimes H)
\circ (\delta_H \otimes ((B \otimes g) \otimes \delta_B) \otimes (\lambda_B \otimes (\lambda_B \otimes \lambda_H) \circ c_{H,H} \circ \delta_H))
\circ (H \otimes c_{H,B}) \circ (\delta_H \otimes i_H^B)
= (\mu_B \otimes H) \circ (B \otimes c_{H,B}) \circ (\mu_B \otimes \mu_B \otimes B)
\circ (\delta_H \otimes (\lambda_B \otimes (\lambda_B \otimes f) \otimes \delta_H))
\circ (B \otimes c_{H,B} \otimes H \otimes H)
\circ (((f \otimes H) \otimes \delta_H) \otimes ((B \otimes g) \otimes \delta_B) \otimes H) \circ (H \otimes c_{H,B}) \circ (\delta_H \otimes i_H^B)
= \mu_B \circ (B \otimes (\epsilon_B \circ \mu_B) \otimes c_{H,B}) \circ (B \otimes H \otimes H \otimes \mu_B \otimes B)
\circ (B \otimes H \otimes c_{H,B} \otimes H \otimes B)
it follows that

\[ \circ (\mu_B \otimes \delta_B \otimes H \otimes ((\lambda_H \otimes f) \circ \delta_H)) \circ (f \otimes c_{H,B} \otimes g \otimes H) \circ (\delta_H \otimes \delta_B \otimes H) \]

\[ \circ (H \otimes c_{H,B}) \circ (\delta_H \otimes i^B_H) = h'. \]

Then \((B \otimes \Pi^L_H) \circ h' = h' \) because \(\Pi^L_H\) is an idempotent morphism. Thus, \((B \otimes g) \circ \delta_B \circ \eta_B = (B \otimes (\Pi^L_H \circ g)) \circ \delta_B \circ \eta_B \) and, as a consequence, there exists an unique morphism \(\varphi_B : H \otimes B_H \rightarrow B_H\) verifying the equality \(i^B_H \circ \varphi_B = \eta_B\).

Next we will prove the assertions (1) to (6).

1. Since \(i^B_H \circ p^B_H \circ \mu_B \circ (f \otimes i^B_H) = \eta_B\), we have \(\varphi_B = p^B_H \circ \mu_B \circ (f \otimes i^B_H)\). Indeed, using the equality

\[ f \circ \mu_H \circ (H \otimes \Pi^L_H) = \mu_B \circ (B \otimes \Pi^L_B) \circ (f \otimes f) \quad (E1) \]

it follows that

\[
i^B_H \circ p^B_H \circ \mu_B \circ (f \otimes i^B_H)
= q^B_H \circ \mu_B \circ (f \otimes i^B_H)
= \mu_B \circ (B \otimes (\lambda_B \circ f)) \circ \mu_B \circ ((f \otimes H) \circ \delta_H) \circ ((B \otimes g) \circ \delta_B \circ i^B_H)
= \mu_B \circ (B \otimes (\lambda_B \circ f)) \circ \mu_B \circ ((f \otimes H) \circ \delta_H) \circ ((B \otimes (\Pi^L_H \circ g) \circ \delta_B \circ i^B_H))
= \mu_B \circ (B \otimes (\lambda_B \circ \mu_B \circ (f \otimes (\Pi^L_B \circ f \circ g)))) \circ (f \otimes c_{H,B} \otimes B) \circ (\delta_B \otimes (\delta_B \circ i^B_H))
= \mu_B \circ (B \otimes (\lambda_B \circ (\mu_B \otimes (\lambda_B \circ (f \otimes \Pi^L_B \circ g)))) \circ (f \otimes i^B_H)
= \mu_B \circ (B \otimes (\lambda_B \otimes (\Pi^L_B \circ \eta_B))) \circ ((\delta_B \circ f) \otimes B_H) = \eta_B.

2. Trivially \(\varphi_B \circ (\eta_H \otimes B_H) = \text{id}_{B_H}\) because \(f \circ \eta_H = \eta_B\).

3. Composing with \(i^B_H\) and using the equality \(q^B_H \circ \eta_B = \eta_B\), we obtain

\[
i^B_H \circ \varphi_B \circ (H \otimes \eta_B) = i^B_H \circ p^B_H \circ \mu_B \circ (f \otimes (q^B_H \circ \eta_B)) = i^B_H \circ p^B_H \circ \mu_B \circ (f \otimes \eta_B)
= q^B_H \circ f = \Pi^L_H \circ f = \Pi^L_B \circ \Pi^L_H \circ f = \Pi^L_B \circ f \circ \Pi^L_H
= i^B_H \circ \varphi_B \circ (\Pi^L_H \otimes \eta_B).

Thus, \(\varphi_B \circ (H \otimes \eta_B) = \varphi_B \circ (\Pi^L_H \otimes \eta_B)\). (4) This equality is a consequence of (3). Indeed,

\[
\mu_B \circ (\varphi_B \otimes B_H) \circ (H \otimes \eta_B \otimes B_H)
= p^B_H \circ \mu_B \circ ((q^B_H \circ f) \otimes i^B_H)
= p^B_H \circ \mu_B \circ ((\Pi^L_B \circ f) \otimes i^B_H)
= \varphi_B \circ (\Pi^L_H \otimes B_H).
\]

4. We have
\[\mu_B \circ (\varphi_B \otimes \varphi_B) \circ (H \otimes c_{B,B} \otimes B_H) \circ (\delta_H \otimes B_H \otimes B_H)\]
\[= p_B^L \circ \mu_B \circ ((i_H^R \circ \varphi_B) \otimes (i_H^R \circ \varphi_B)) \circ (H \otimes c_{H,B} \otimes B_H) \circ (\delta_H \otimes B_H \otimes B_H)\]
\[= p_B^L \circ \mu_B \circ (B \otimes \mu_B) \circ (\mu_{B \otimes B} \otimes \mu_B) \circ (B \otimes B \otimes c_{B,B} \otimes c_{B,B})\]
\[= (B \otimes B \otimes B \otimes c_{B,B} \otimes B)\]
\[\circ (\delta_H \otimes B_H \otimes B_H)\]
\[= p_B^L \circ \mu_B \circ (\mu_B \otimes (\mu_B \circ (\Pi_R^B \otimes B))) \circ (B \otimes B \otimes B \otimes \mu_B) \circ (B \otimes c_{B,B} \otimes c_{B,B})\]
\[= (B \otimes B \otimes c_{B,B} \otimes B) \circ (\delta_H \otimes i_H^B \otimes i_H^B)\]
\[= (p_B^L \otimes \varepsilon_B) \circ \mu_B \otimes (\mu_B \otimes \mu_B) \circ (B \otimes B \otimes \delta_B \otimes B \otimes B)\]
\[= (B \otimes B \otimes c_{B,B} \otimes B \otimes B)\]
\[\circ (\delta_H \otimes \varepsilon_B) \circ (\delta_H) \otimes B_H\]
\[= (p_H^L \otimes \varepsilon_B) \circ \mu_B \otimes (\varepsilon_B \otimes \mu_B) \circ (B \otimes B \otimes \delta_B \otimes B \otimes B)\]
\[= (B \otimes c_{B,B} \otimes B \otimes B)\]
\[\circ (\delta_H \otimes \varepsilon_B) \circ (\delta_H) \otimes B_H\]
\[= (p_H^L \otimes \varepsilon_B) \circ \mu_B \otimes (\varepsilon_B \otimes \mu_B) \circ (B \otimes B \otimes \delta_B \otimes B \otimes B)\]
\[= (B \otimes c_{B,B} \otimes B \otimes B)\]
\[\circ (\delta_H \otimes \varepsilon_B) \circ (\delta_H) \otimes B_H\]
\( \circ (\mu_B \otimes c_{B,B} \otimes g) \circ (f \otimes c_{H,R} \otimes \delta_H) \circ (\delta_H \otimes i_H^R \otimes i_H^R) \)

\[ = p_H^R \circ \mu_B \circ (\mu_B \otimes \lambda_B \circ f \circ \mu_H \circ (H \otimes \Pi_H^L)) \circ (\mu_B \otimes c_{B,B} \otimes g) \]

\[ \circ (f \otimes c_{H,R} \otimes \delta_B) \circ (\delta_H \otimes i_H^R \otimes i_H^R) \]

\[ = p_H^R \circ \mu_B \circ (\mu_B \otimes \lambda_B \otimes \mu_B \circ (B \otimes (f \circ g))) \circ (\mu_B \otimes c_{B,B} \otimes B) \]

\[ \circ (B \otimes c_{B,B} \otimes \delta_B) \circ (f \otimes f \otimes B \otimes B) \circ (\delta_H \otimes i_H^R \otimes i_H^R) \]

\[ = p_H^R \circ \mu_B \circ (B \otimes (\mu_B \circ c_{B,B})) \circ ((f \otimes (\lambda_B \circ f)) \otimes \delta_H) \otimes (\mu_B \circ (i_H^R \otimes (q_H^B \circ i_H^R))) \]

\[ = p_H^R \circ y_B \circ (H \otimes \mu_B) \]

\[ = \varphi_{B_H} \circ (H \otimes \mu_B) \]

In the last equalities we use repeatedly the condition of weak Hopf algebra morphism for \( g \), the condition of coalgebra morphism for \( f \), the equality \( g \circ f = id_H \), the antipode, the naturality of the symmetry morphism \( c \) and \( (B \otimes g) \circ \delta_B \circ i_H^R = (B \otimes (\Pi_H^L \circ g)) \circ \delta_B \circ i_H^R \). Also, in the fourth equality we use \( \mu_B \circ (\Pi_H^R \otimes B) = (B \otimes (\Pi_H^R \circ \mu_B)) \circ (c_{B,B} \otimes B) \circ (B \otimes \delta_B) \). The sixth and the twelfth ones follow from \( \mu_H \circ (H \otimes \Pi_H^L) = ((\epsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H) \). In the ninth one we apply the equality \( \epsilon_B \circ \Pi_B^L = \epsilon_B \). Finally, the thirteenth one follows from (E1).

(6) First, note that

\[ i_H^R \circ \mu_B \circ c_{B_{H},B_H} \circ (\varphi_{B_H} \circ (H \otimes \eta_B)) \circ B_H = \mu_B \circ c_{B,B} \circ ((\Pi_B^L \circ f) \otimes i_H^R). \]

On the other hand, using the equalities \( \Pi_B^R \circ \Pi_B^L = \lambda_B \circ \Pi_B^L \) and \( \Pi_B^L = \lambda_B \circ \Pi_B^L \), we obtain

\[ i_H^R \circ \varphi_{B_H} \circ (\Pi_B^L \otimes B_H) = y_B \circ (\Pi_B^L \otimes B_H) \]

\[ = \mu_B \circ (B \otimes (\mu_B \circ c_{B,B} \circ (\lambda_B \otimes i_H^R))) \circ ((\delta_B \circ \Pi_B^L \circ f) \otimes B_H) \]

\[ = \mu_B \circ (B \otimes (\mu_B \circ c_{B,B} \circ (\lambda_B \otimes (\epsilon_B \circ \mu_B) \otimes i_H^R))) \circ (\delta_B \circ B \otimes f \otimes B_H) \]

\[ \circ (\delta_B \otimes \eta_B) \otimes H \otimes B_H \]

\[ = \mu_B \circ (B \otimes (\mu_B \circ c_{B,B} \circ (((\lambda_B \otimes \mu_B) \otimes (\epsilon_B \circ \mu_B) \otimes B))) \circ (\delta_B \otimes \eta_B) \circ f \otimes i_H^R) \]

\[ = \mu_B \circ ((\mu_B \circ (B \otimes (\lambda_B \otimes (\Pi_B^L \circ f) \otimes B) \circ (\lambda_B \otimes (\epsilon_B \circ \mu_B) \otimes B) \circ (\delta_B \otimes \eta_B) \circ f \otimes i_H^R) \]

\[ = \mu_B \circ c_{B,B} \circ (B \otimes (\mu_B \circ (B \otimes (\Pi_B^R \circ \Pi_B^L)) \circ \delta_B) \circ (\lambda_B \otimes (\epsilon_B \circ \mu_B) \otimes B) \]

\[ \circ (\delta_B \otimes \eta_B) \otimes f \otimes i_H^R) \]

\[ = \mu_B \circ c_{B,B} \circ (B \otimes (\mu_B \circ (B \otimes (\Pi_B^R \circ \Pi_B^L))) \circ \delta_B) \circ (\lambda_B \otimes (\epsilon_B \circ \mu_B) \otimes B) \]

\[ \circ (\delta_B \otimes \eta_B) \otimes f \otimes i_H^R) \]

\[ = \mu_B \circ c_{B,B} \circ (B \otimes (\mu_B \circ (B \otimes (\Pi_B^R \circ \Pi_B^L))) \circ \delta_B) \circ (\lambda_B \otimes (\epsilon_B \circ \mu_B) \otimes B) \]

\[ \circ (\delta_B \otimes \eta_B) \otimes f \otimes i_H^R) \]
\[
\circ (\delta_B \circ \eta_B) \otimes f \otimes i_H^B = \\
= \mu_B \circ c_{B,B} \circ (B \otimes (\mu_B \circ (f \circ \Pi^R_B \circ \Pi^L_B \circ \eta_B) \circ \delta_B) \otimes (\lambda_B \otimes (\epsilon_B \circ \mu_B) \otimes B) \circ (\delta_B \circ \eta_B) \otimes f \otimes i_H^B) = \\
= \mu_B \circ c_{B,B} \circ ((\lambda_B \circ \Pi^L_B \circ f) \otimes i_H^B) = \\
= \mu_B \circ c_{B,B} \circ ((\Pi^L_B \circ f) \otimes i_H^B).
\]

Therefore, \( \mu_{B_H} \circ ((\varphi_{B_H} \circ (H \otimes \eta_{B_H})) \otimes B_H) = \varphi_{B_H} \circ (\Pi^L_H \otimes B_H). \) \( \square \)

**Remark 2.5.** In Proposition 2.4 we use the equality (E1). For to prove it, we only need a morphism of weak Hopf algebras \( g : B \to H \) and a morphism of coalgebras \( f : H \to B \) such that \( g \circ f = \text{id}_H \). Indeed,

\[
f \circ \mu_H \circ (H \otimes \Pi^L_H) = ((\epsilon_H \circ \mu_H) \otimes f) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H) = \\
= ((\epsilon_H \circ \mu_H) \otimes f) \circ ((g \circ f) \otimes c_{H,H}) \circ (\delta_H \otimes (g \circ f)) = \\
= ((\epsilon_B \circ \mu_B) \otimes B) \circ (B \otimes c_{B,B}) \circ (\delta_B \otimes B) \circ (f \otimes f) = \\
= \mu_B \circ (B \otimes \Pi^L_B) \circ (f \otimes f).
\]

Also, by an analogous calculus, it is possible to prove

\[
f \circ \mu_H \circ (\Pi^R_H \otimes H) = \mu_B \circ (\Pi^R_B \otimes B) \circ (f \otimes f).
\]  

(E2)

**Proposition 2.6.** Let \( H, B \) be weak Hopf algebras in \( \mathcal{C} \). Let \( g : B \to H \) be a morphism of weak Hopf algebras and \( f : H \to B \) be a morphism of coalgebras such that \( g \circ f = \text{id}_H \) and \( f \circ \eta_H = \eta_B \). There exists an unique morphism \( \sigma_{B_H} : H \otimes H \to B_H \) such that \( i_H^B \circ \sigma_{B_H} = \sigma_B \) where \( \sigma_B : H \otimes H \to B \) is the morphism defined by

\[
\sigma_B = \mu_B \circ ((\mu_B \circ (f \otimes f) \circ (\lambda_B \circ f \circ \mu_H))) \circ \delta_{H \otimes H}.
\]

As a consequence, we have the equality \( \sigma_{B_H} = \pi_H^B \circ \sigma_B \).

**Proof.** We only need to show that \( (B \otimes g) \circ \delta_B \circ \sigma_B = (B \otimes (\Pi^L_H \circ g)) \circ \delta_B \circ \sigma_B \). First, note that \( (B \otimes g) \circ \delta_B \circ \sigma_B = \Delta \) being \( \Delta : H \otimes H \to B \otimes H \) the morphism defined by

\[
\Delta = (\mu_B \circ \Pi^L_H) \circ (\mu_B \circ c_{H,B}) \circ (f \otimes f \otimes ((H \otimes (\lambda_B \circ f)) \circ \delta_H \circ \mu_H)) \circ \delta_{H \otimes H}.
\]

Indeed,
\[(B \otimes g) \circ \delta_B \circ \sigma_B \]
\[= (B \otimes g) \circ \mu_B \otimes B \circ (\mu_B \otimes B \otimes \delta_B) \circ (\delta_B \otimes \delta_B \otimes (\lambda_B \circ f)) \circ (f \otimes f \otimes \mu_H) \circ \delta_H \otimes H \]
\[= \mu_B \otimes H \circ \left( (\mu_B \otimes H \otimes (\lambda_B \circ \lambda_B) \circ (f \otimes H) \circ \delta_H) \otimes (f \otimes H) \circ \delta_H \otimes (\delta_H \circ \mu_H) \right) \circ \delta_H \otimes H \]
\[= (\mu_B \otimes H) \circ (B \otimes c_{H,B}) \circ (B \otimes \mu_H \otimes (\lambda_B \circ f)) \circ (\mu_B \otimes H \otimes \lambda_H \otimes H) \]
\[\circ (f \otimes f \otimes \delta_H \otimes \delta_H) \circ \delta_H \otimes H \]
\[= (\mu_B \otimes H \circ (B \otimes c_{H,B}) \circ (B \otimes \mu_H \otimes \lambda_B) \circ (B \otimes H \otimes (\lambda_H \circ f) \circ \delta_H) \]
\[= (\mu_B \otimes H \circ (B \otimes c_{H,B}) \circ (B \otimes \mu_H \otimes \lambda_H \otimes H) \circ (f \otimes f \otimes \delta_H \otimes \delta_H) \circ \delta_H \otimes H \]
\[= \Delta. \]

Thus, \((B \otimes g) \circ \delta_B \circ \sigma_B = (B \otimes (\Pi_H^L \circ g)) \circ \delta_B \circ \sigma_B\), because \(\Pi^L_H\) is idempotent. \(\square\)

2.7. Let \(H, B\) be weak Hopf algebras in \(\mathcal{C}\). Let \(g : B \to H\) be a morphism of weak Hopf algebras and \(f : H \to B\) be a morphism of coalgebras such that \(g \circ f = \text{id}_H\) and \(f \circ \eta_H = \eta_B\). Let \(\omega_B : B_H \otimes H \to B\) be the morphism defined by

\[\omega_B = \mu_B \circ (i^B_H \otimes f).\]

If we define \(\omega'_B : B \to B_H \otimes H\) by

\[\omega'_B = (p^B_H \otimes g) \circ \delta_B,\]

we have \(\omega_B \circ \omega'_B = \text{id}_B\). As a consequence, the morphism \(\Omega_B = \omega'_B \circ \omega_B : B_H \otimes H \to B_H \otimes H\) is idempotent and there exists a diagram

\[
\begin{array}{ccc}
B_H \otimes H & \xrightarrow{\omega_B} & B_H \\
\downarrow{\omega'_B} & & \downarrow{\mu_B} \\
B_H \times H & \xrightarrow{s_B} & B_H \otimes H \\
\end{array}
\]

where

\[s_B \circ r_B = \Omega_B, \quad r_B \circ s_B = \text{id}_{B_H \times H}, \quad b_B = r_B \circ \omega'_B.\]
It is easy to prove that the morphism $b_B$ is an isomorphism with inverse $b_B^{-1} = \omega_B \circ s_B$. Therefore, the object $B_H \times H$ is an algebra with unit and product defined by

$$
\eta_{B_H \times H} = b_B \circ \eta_B, \quad \mu_{B_H \times H} = b_B \circ \mu_B \circ (b_B^{-1} \otimes b_B^{-1}).
$$

respectively.

Also, $B_H \times H$ is a right $H$-comodule where

$$
\rho_{B_H \times H} = (b_B \otimes H) \circ (B \otimes g) \circ \delta_B \circ b_B^{-1}.
$$

Of course, with these structures $b_B$ is an isomorphism of algebras and right $H$-comodules being $\rho_B = (B \otimes g) \circ \delta_B$.

On the other hand, we can define the following morphisms:

$$
\eta_{B_H \otimes B_H}^*: K \to B_H \times H, \quad \mu_{B_H \otimes B_H}^*: B_H \times H \otimes B_H \times H \to B_H \times H,
$$

$$
\rho_{B_H \otimes B_H}^*: B_H \to B_H \times H \otimes H,
$$

where

$$
\eta_{B_H \otimes B_H}^* = r_B \circ (\eta_B \otimes \eta_H),
$$

$$
\mu_{B_H \otimes B_H}^* = r_B \circ (\mu_B \otimes H) \circ (\mu_B \otimes \sigma_B \otimes \mu_H) \circ (B_H \otimes \varphi_{B_H} \otimes \delta_H \otimes H) \circ (B_H \otimes \delta_H \otimes B_H \otimes H) \circ (s_B \otimes s_B),
$$

$$
\rho_{B_H \otimes B_H}^* = (r_B \otimes H) \circ (B_H \otimes \delta_H) \circ s_B.
$$

Finally, we denote by $B_H \otimes_{\sigma_B}^* H$ (the crossed product of $B_H$ and $H$) the triple

$$
(B_H \times H, \eta_{B_H \otimes_{\sigma_B}^* H}, \mu_{B_H \otimes_{\sigma_B}^* H}).
$$

**Theorem 2.8.** Let $H$, $B$ be weak Hopf algebras in $\mathcal{C}$. Let $g: B \to H$ be a morphism of weak Hopf algebras and $f: H \to B$ be a morphism of coalgebras such that $g \circ f = \id_H$ and $f \circ \eta_H = \eta_B$. Then, $B_H \otimes_{\sigma_B}^* H$ is an algebra, $(B_H \times H, \rho_{B_H \otimes_{\sigma_B}^* H})$ is a right $H$-comodule and $b_B: B \to B_H \otimes_{\sigma_B}^* H$ is an isomorphism of algebras and right $H$-comodules.

**Proof.** For to prove the theorem we only need to show that

$$
\eta_{B_H \otimes_{\sigma_B}^* H} = \eta_{B_H \times H}, \quad \mu_{B_H \otimes_{\sigma_B}^* H} = \mu_{B_H \times H}, \quad \rho_{B_H \otimes_{\sigma_B}^* H} = \rho_{B_H \times H}.
$$

Firstly, using the equality $f \circ \eta_H = \eta_B$, we have

$$
s_B \circ \eta_{B_H \otimes_{\sigma_B}^* H} = s_B \circ r_B \circ (\eta_B \otimes \eta_B) = \omega_B' \circ \omega_B \circ (\eta_B \otimes \eta_B) = s_B \circ b_B \circ \eta_B
$$

$$
= s_B \circ \eta_{B_H \times H}.
$$
Thus, $\eta_B H \circ \omega_B H = \eta_B H \times H$.

On the other hand,

$$b^{-1}_B \circ \mu_B H \circ \omega_B H H$$

$$= \mu_B \circ \left((i^B_H \circ \mu_B \circ (B_H \otimes \mu_B H)) \otimes f \right) \circ (B_H \otimes \varphi_{B_H} \otimes \sigma_{B_H} \otimes \mu_H)$$

$$\circ (B_H \otimes H \otimes B_H \otimes \delta_{H \otimes H}) \circ (B_H \otimes H \otimes c_{H,B_H} \otimes H) \circ \left( ((B_H \otimes \delta_H) \otimes s_B) \otimes s_B \right)$$

$$= \mu_B \circ \left((i^B_H \circ \mu_B \circ (B \otimes \mu_B) \circ (\lambda_B \otimes f) \otimes \mu_B \otimes (\lambda_B \otimes f) \circ (H \otimes f \otimes f \otimes \mu_H) \right)$$

$$\circ \left((B \otimes B \otimes H \otimes (\delta_H \otimes \mu_H)) \circ (B \otimes c_{B,H} \otimes H) \right) \circ \left( ((B \otimes g) \otimes \delta_B) \otimes i^B_H \otimes H \right) \circ \left( (\omega_B \circ \sigma_B) \otimes s_B \right)$$

$$= \mu_B \circ \left((i^B_H \circ \mu_B) \otimes f \right) \circ \left((B \circ \sigma_B) \otimes (\lambda_B \otimes f) \circ (\mu_B \circ (\Pi^B_{H \otimes H} \otimes f)) \otimes (\lambda_B \otimes f) \otimes H \right)$$

$$\circ \left((B \otimes B \otimes H \otimes (\delta_H \otimes \mu_H)) \circ (B \otimes c_{B,H} \otimes H) \right) \circ \left( ((B \otimes g) \otimes \delta_B) \otimes i^B_H \otimes H \right) \circ \left( (\omega_B \circ \sigma_B) \otimes s_B \right)$$

$$= \mu_B \circ \left((i^B_H \circ \mu_B) \otimes f \right) \circ \left((B \otimes \sigma_B) \otimes (\lambda_B \otimes f) \circ (\mu_B \circ (\Pi^B_{H \otimes H} \otimes f)) \otimes (\lambda_B \otimes f) \otimes H \right)$$

$$\circ \left((B \otimes B \otimes H \otimes (\delta_H \otimes \mu_H)) \circ (B \otimes c_{B,H} \otimes H) \right) \circ \left( ((B \otimes g) \otimes \delta_B) \otimes i^B_H \otimes H \right) \circ \left( (\omega_B \circ \sigma_B) \otimes s_B \right)$$
\[ \mu_B \circ (b_B^{-1} \otimes b_B^{-1}). \]

Therefore, \( \mu_{B_H \sigma_B H} = \mu_{B_H \times H} \).

As in Proposition 2.4, in the last calculus, we use the condition of weak Hopf algebra morphism for \( g \), the condition of coalgebra morphism for \( f \), the equality \( g \circ f = \text{id}_H \), the antimultiplicative nature of the antipode and the naturality of the symmetry morphism \( \sigma \). In the second an third equalities we use the definitions of \( \mu_{B_H} \), \( \varphi_{B_H} \) and \( \sigma_{B_H} \). The fifth one follows from \( (E2) \) and the sixth one follows from \( \mu_H \circ (\Pi_B^B \otimes H) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_H \otimes H) \circ (H \otimes \delta_H) \).

Remark 2.9. We point out that if \( H \) and \( B \) are Hopf algebras, Theorem 2.8 is the result obtained by Blattner, Cohen, and Montgomery in [2]. Moreover, if \( f \) is an algebra morphism, we have \( \sigma_{B_H} = \varepsilon_H \otimes \eta_B \) and then \( B_H \otimes_{\sigma_{B_H}} H \) is the smash product of \( B_H \) and \( H \), denoted by \( B_H \sharp H \). Observe that the product of \( B_H \sharp H \) is

\[ \mu_{B_H \sharp H} = (\mu_{B_H} \otimes \mu_H) \circ (B_H \otimes (\varphi_{B_H} \otimes (H \otimes \delta_B \circ B_I) \circ (\delta_B \otimes B_H) \otimes H)). \]

In the following paragraph we study the expression of \( \mu_{B_H \sigma_{B_H} H} \) when \( f \) is a morphism of weak Hopf algebras.

2.10. Let \( H, B \) be weak Hopf algebras in \( C \). Let \( g : B \rightarrow H \), \( f : H \rightarrow B \) be morphisms of weak Hopf algebras such that \( g \circ f = \text{id}_H \). In this case \( \sigma_B = \Pi_B^B \circ f \circ \mu_H \) and then, using \( \mu_B \circ (\Pi_B^B \otimes B) \circ \delta_B = \text{id}_B \), we obtain

\[ \mu_{B_H \sigma_{B_H} H} = r_B \circ (\mu_{B_H} \otimes H) \circ (\mu_{B_H} \otimes \sigma_{B_H} \otimes \mu_H) \circ (B_H \otimes \varphi_{B_H} \otimes \delta_B \circ H). \]

\[ = r_B \circ (\mu_{B_H} \otimes (\varphi_{B_H} \otimes (H \otimes \delta_B \circ B_I) \circ (\delta_B \otimes B_H) \otimes H) \circ (s_B \otimes s_B)). \]

\[ = r_B \circ (\varphi_{B_H} \otimes (B_H \otimes (\varphi_{B_H} \otimes (H \otimes \delta_B \circ B_I) \circ (\delta_B \otimes B_H) \otimes H) \circ (s_B \otimes s_B)). \]

\[ = r_B \circ (\varphi_{B_H} \otimes (B_H \otimes (\varphi_{B_H} \otimes (H \circ \delta_B \circ B_I) \circ (f \circ \mu_H) \circ (B_H \otimes (H \otimes \delta_B \circ B_H) \otimes H) \circ (s_B \otimes s_B)). \]

\[ = r_B \circ (\varphi_{B_H} \otimes (B_H \otimes (\varphi_{B_H} \otimes (H \otimes \delta_B \circ B_H) \otimes H) \circ (s_B \otimes s_B)). \]

\[ = r_B \circ (\varphi_{B_H} \otimes (B_H \otimes (H \otimes \delta_B \circ B_H) \otimes H) \circ (s_B \otimes s_B)). \]

\[ = r_B \circ (\varphi_{B_H} \otimes (B_H \otimes (H \otimes \delta_B \circ B_H) \otimes H) \circ (s_B \otimes s_B)). \]

\[ = r_B \circ (\varphi_{B_H} \otimes (B_H \otimes (H \otimes \delta_B \circ B_H) \otimes H) \circ (s_B \otimes s_B)). \]
\[ \circ (B_H \otimes \delta_H \otimes B_H \otimes H) \circ (s_B \otimes s_B) \]
\[ = r_B \circ (\mu_B \otimes \mu_H) \circ (B_H \otimes ((\varphi_{B_H} \otimes H) \circ (H \otimes c_{H,B_H}) \circ (\delta_H \otimes B_H)) \otimes H) \]
\[ \circ (s_B \otimes s_B). \]

As a consequence, for analogy with the Hopf algebra case, when \( \sigma_B = \Pi^B_L \circ f \circ \mu_H, \) we will denote the triple \( B_H \sharp_{\sigma_{B_H}} H \) by \( B_H \sharp H \) (the smash product of \( B_H \) and \( H \)).

Therefore, if \( f \) and \( g \) are morphisms of weak Hopf algebras, we have the following particular case of Theorem 2.8.

**Corollary 2.11.** Let \( H, B \) be weak Hopf algebras in \( \mathcal{C} \). Let \( g : B \to H, f : H \to B \) be morphisms of weak Hopf algebras such that \( g \circ f = \text{id}_H \). Then \( B_H \sharp H \) is an algebra, \( (B_H \times H, \rho_{B_H \sharp H}) \) is a right \( H \)-comodule and \( b_B : B \to B_H \sharp H \) is an isomorphism of algebras and right \( H \)-comodules.

### 3. The dual version

In this section we develop the dual results of Section 2. The arguments are similar to the ones used in Section 2, but passing to the opposite category, and then we leave the details to the reader.

Let \( H, B \) be weak Hopf algebras in \( \mathcal{C} \). Let \( h : H \to B \) be a morphism of weak Hopf algebras and \( t : B \to H \) be a morphism of algebras such that \( t \circ h = \text{id}_H \) and \( \varepsilon_H \circ t = \varepsilon_B \). The morphism \( k_B^H : B \to B \) defined by

\[ k_B^H = \mu_B \circ (B \otimes (h \circ t \circ \lambda_B)) \circ \delta_B \]

is idempotent in \( \mathcal{C} \) and, as a consequence, we obtain that there exist an epimorphism \( l_B^H \), a monomorphism \( n_B^H \) and an object \( B^H \) such that the diagram

\[
\begin{array}{ccc}
B & \xrightarrow{k_B^H} & B \\
\downarrow{l_B^H} & & \downarrow{n_B^H} \\
B^H & & \\
\end{array}
\]

commutes and \( l_B^H \circ n_B^H = \text{id}_{B^H} \). Moreover, using the next coequalizer diagram in \( \mathcal{C} \)

\[
\begin{array}{ccc}
B \otimes H & \xrightarrow{\mu_B \circ (B \otimes h)} & B & \xrightarrow{l_B^H} & B^H \\
\mu_B \circ (B \otimes (h \circ \Pi^H_T)) & & & & \\
\end{array}
\]

(D2)

it is possible to obtain a coalgebra structure for \( B^H \). This structure is given by

\[ (B^H, \varepsilon_{B^H} = \varepsilon_B \circ n_B^H, \delta_{B^H} = (l_B^H \otimes l_B^H) \circ \delta_B \circ n_B^H). \]
Let \( y_B : B \to H \otimes B^H \) be the morphism defined by
\[
y_B = (\mu_H \otimes l_H^B) \circ (t \otimes (t \circ (t \circ \lambda_B) \otimes B) \circ (B \otimes (c_{B,B} \circ \delta_B)) \circ \delta_B.
\]

The morphism \( y_B \) verifies that \( y_B \circ \mu_B \circ (B \otimes h) = y_B \circ \mu_B \circ (B \otimes (\Pi_B^L \circ h)) \) and then, there exists an unique morphism \( r_{BH} : B^H \to H \otimes B^H \) such that \( r_{BH} \circ l_H^B = y_B \).

Moreover, the morphism \( r_{BH} \) satisfies:

1. \( r_{BH} = (t \otimes l_H^B) \circ \delta_B \circ n_B^H. \)
2. \( (\varepsilon_H \otimes B^H) \circ r_{BH} = \text{id}_{B^H}. \)
3. \( (H \otimes (\varepsilon_H \otimes B^H)) \circ r_{BH} = (\Pi_H^L \otimes B^H) \circ r_{BH}. \)
4. \( (H \otimes \delta_B) \circ r_{BH} = (\mu_B \otimes B^H) \circ (H \otimes c_{B^H,H} \otimes B^H) \circ (r_{BH} \otimes r_{BH}) \circ \delta_B. \)
5. \( (H \otimes \varepsilon_B) \circ r_{BH} \otimes B^H \circ \delta_B = (\Pi_H^L \otimes B^H) \circ r_{BH}. \)
6. \( (((H \otimes \varepsilon_B) \circ r_{BH}) \otimes B^H) \circ c_{B^H,B^H} \circ \delta_B = \text{id}_{B^H}. \)

Let \( \gamma_B : B \to H \otimes H \) be the morphism defined by
\[
\gamma_B = \mu_{H \otimes H} \circ ((t \otimes t) \circ \delta_B) \circ (t \circ \lambda_B) \circ \delta_B.
\]

The morphism \( \gamma_B \) verifies that \( \gamma_B \circ \mu_B \circ (B \otimes h) = \gamma_B \circ \mu_B \circ (B \otimes (\Pi_B^L \circ h)) \) and then, there exists an unique morphism \( \gamma_{BH} : B^H \to H \otimes H \) such that \( \gamma_{BH} \circ l_H^B = \gamma_B \).

In a similar way to 2.7 it is not difficult to see that the morphism \( \Upsilon_B : B^H \otimes H \to B^H \otimes H \) defined by
\[
\Upsilon_B = \sigma_{B'} \circ \sigma_B,
\]

being \( \sigma_B = \mu_B \circ (n_B^H \otimes h) \) and \( \sigma_{B'} = (t_H^B \otimes t) \circ \delta_B \), is idempotent and there exists a diagram

\[
\begin{array}{ccc}
B^H \otimes H & \xrightarrow{\gamma_B} & B^H \otimes H \\
\sigma_B \downarrow & & \downarrow \sigma_B' \\
B \quad & & \quad B^H \boxdot H \\
\sigma_B' \uparrow & & \uparrow \sigma_B \\
B^H \otimes H & \xrightarrow{\gamma_B} & B^H \otimes H \\
\end{array}
\]

where
\[
v_B \circ u_B = \gamma_B, \quad u_B \circ v_B = \text{id}_{B^H \boxdot H}, \quad d_B = u_B \circ \sigma_{B'}.
\]
Moreover, $d_B$ is an isomorphism with inverse $d_B^{-1} = \sigma_B \circ v_B$ and the object $B^H \square H$ is a coalgebra with counit and coproduct defined by

$$
\varepsilon_B^H \square H = \varepsilon_B \circ d_B^{-1}, \quad \delta_B^H \square H = (d_B \otimes d_B) \circ \delta_B \circ d_B^{-1},
$$

respectively.

Also, $B^H \square H$ is a right $H$-module where

$$
\psi_B^H \square H = d_B \circ \mu_B \circ (d_B^{-1} \otimes h).
$$

With these structures $d_B$ is an isomorphism of coalgebras and right $H$-modules being $\psi_B = \mu_B \circ (B \otimes h)$. Finally, we define the morphisms

$$
\varepsilon_B^H \otimes \gamma_B^H : B^H \square H \rightarrow K, \quad \delta_B^H \otimes \gamma_B^H : B^H \square H \rightarrow B^H \square H \otimes B^H \square H, \\
\psi_B^H \otimes \gamma_B^H : B^H \square H \otimes H \rightarrow B^H \square H,
$$

where

$$
\varepsilon_B^H \otimes \gamma_B^H = (\varepsilon_B^H \otimes \varepsilon_H) \circ v_B, \\
\delta_B^H \otimes \gamma_B^H = (\mu_B \otimes \mu_H) \circ \left(B^H \otimes H \otimes c_{BH,H} \otimes H\right) \\
\circ \left(B^H \otimes r_{BH} \otimes \mu_H \otimes H\right) \circ \left(\delta_B^H \otimes \gamma_B^H \otimes \delta_H\right) \circ \left(\delta_B^H \otimes H\right) \circ v_B, \\
\psi_B^H \otimes \gamma_B^H = \mu_B \circ \left(B^H \otimes \mu_H\right) \circ (v_B \otimes H).
$$

If we denote by $B^H \otimes \gamma_B^H$ (the crossed coproduct of $B^H$ and $H$) the triple

$$
(B^H \square H, \varepsilon_B^H \otimes \gamma_B^H, \delta_B^H \otimes \gamma_B^H),
$$

we have the following theorem.

**Theorem 3.1.** Let $H$, $B$ be weak Hopf algebras in $C$. Let $h : H \rightarrow B$ be a morphism of weak Hopf algebras and $t : B \rightarrow H$ be a morphism of algebras such that $t \circ h = \varepsilon_H$ and $\varepsilon_H \circ t = \varepsilon_B$. Then, $B^H \otimes \gamma_B^H$ is a coalgebra, $(B^H \square H, \psi_B^H \otimes \gamma_B^H)$ is a right $H$-module and $d_B : B \rightarrow B^H \otimes \gamma_B^H$ is an isomorphism of coalgebras and right $H$-modules.

In the Hopf algebra case ($H$ and $B$ Hopf algebras) Theorem 3.1 is the dual of the result obtained by Blattner, Cohen, and Montgomery. In this case, if $t$ is a coalgebra morphism, we have $\gamma_B^H = \varepsilon_B^H \otimes \eta_H \otimes \eta_H$ and then $B^H \otimes \gamma_B^H$ is the smash coproduct of $B^H$ and $H$, denoted by $B^H \otimes H$. In $B^H \otimes H$ the coproduct is

$$
\delta_B^H \otimes H = \left(B^H \otimes \left((\mu_H \otimes B^H) \circ (H \otimes c_{BH,H}) \circ (r_{BH} \otimes H)\right) \otimes H\right) \circ \left(\delta_B^H \otimes H\right).
$$
Finally, when $t$ is a morphism of weak Hopf algebras we have $\gamma_B = \delta_H \circ \Pi^H_{\mathcal{L}} \circ t$ and then the expression of $\delta^H_B \circ \gamma_B^H_H$ is

$$\delta^H_B \circ \gamma_B^H_H = (u_B \otimes u_B) \circ (B^H \otimes ((\mu_H \otimes B^H) \circ (H \otimes c_{B,H}) \circ (r_{BH} \otimes H)) \otimes H)$$

$$\circ (\delta_B^H \otimes \delta_H) \circ v_B.$$

As a consequence, for analogy with the Hopf algebra case, when $\gamma_B = \delta_H \circ \Pi^H_{\mathcal{L}} \circ t$, we will denote the triple $B^H \ominus \gamma_B^H_H$ by $B^H \ominus H$ (the smash coproduct of $B^H$ and $H$).

Therefore, if $h$ and $t$ are morphisms of weak Hopf algebras, we have the following corollary.

**Corollary 3.2.** Let $H$, $B$ be weak Hopf algebras in $\mathcal{C}$. Let $t : B \rightarrow H$, $h : H \rightarrow B$ be morphisms of weak Hopf algebras such that $t \circ h = \text{id}_H$. Then, $B^H \circ H$ is a coalgebra, $(\theta_B^H \ominus H, \phi_B^H)$ is a right $H$-module and $\iota_B : B \rightarrow B^H \circ H$ is an isomorphism of coalgebras and right $H$-modules.

### 4. Radford’s theorem for weak Hopf algebras

In this section we give Radford’s theorem for weak Hopf algebras with projection. Suppose that $g : B \rightarrow H$ and $f : H \rightarrow B$ are morphisms of weak Hopf algebras such that $g \circ f = \text{id}_H$. Then $q_B^H = k_H^B$ and therefore $B_H = B^H$, $p_H^B = \iota_H^B$, and $\iota_H^B = n_H^B$. Thus

$$B_H \xrightarrow{\iota_H^B} B \xrightarrow{(B \otimes g) \circ \delta_B} B \otimes H$$

is an equalizer diagram and

$$B \otimes H \xrightarrow{\mu_B \circ (B \otimes f)} B \xrightarrow{p_H^B} B^H$$

is a coequalizer diagram.

Also, $\omega_B = \sigma_B$, $\omega_B' = \sigma_B'$ and then $B_H \times H = B^H \ominus H$.

The object $B_H$ is an algebra coalgebra and in [1, Proposition 2.8] we prove that the triple $(B_H, \varphi_B^H, r_B^H)$ belongs to $H^W \mathcal{YD}$ where $H^W \mathcal{YD}$ denotes the category of left weak Yetter–Drinfeld modules over $H$. That is, $M = (M, \varphi_M, r_M)$ is an object in $H^W \mathcal{YD}$ if $(M, \varphi_M)$ is a left $H$-module, $(M, r_M)$ is a left $H$-comodule and

(a) $$(\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ ((r_M \circ \varphi_M) \otimes H) \circ (H \otimes c_{H,M}) \circ (\delta_H \otimes M)$$

$$= (\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ (\mu_H \otimes \varphi_M \otimes H) \circ (H \otimes c_{H,M} \otimes M \otimes H)$$

$$\circ (\delta_H \otimes r_M \otimes \Pi^H_B) \circ (H \otimes c_{H,M} \circ (\delta_H \otimes M)).$$
(b) \((\mu_H \otimes \psi_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \circ \eta_H) \otimes r_M) = r_M.\)

A morphism in \(\mathcal{H}^{WYD}\) is a morphism of left \(H\)-modules and left \(H\)-comodules.

Also, using the morphism \(t_{BH,BH} = (\phi_{BH} \otimes BH) \circ (H \otimes c_{BH,BH}) \circ (r_{BH} \otimes BH)\):

\[BH \otimes BH \to BH \otimes BH\]

instead of \(c_{BH,BH}\), in [1, Proposition 2.9], we obtain that \(BH\) verifies similar conditions with the ones included in the definition of weak Hopf algebra, that is

(1) \(\delta_{BH} \circ \mu_{BH} = (\mu_{BH} \otimes \mu_{BH}) \circ (BH \otimes t_{BH,BH}) \circ BH \otimes \delta_{BH} \otimes \delta_{BH} \).

(2) \(\epsilon_{BH} \circ \mu_{BH} \circ (\mu_{BH} \otimes BH) = (\epsilon_{BH} \otimes \epsilon_{BH}) \circ (\mu_{BH} \otimes \mu_{BH}) \circ (BH \otimes \delta_{BH} \otimes BH) = (\epsilon_{BH} \otimes \epsilon_{BH}) \circ (\mu_{BH} \otimes \mu_{BH}) \circ (t_{BH,BH} \circ \delta_{BH}) \otimes BH\).

(3) \(\delta_{BH} \circ \delta_{BH} \circ \eta_{BH} = (BH \otimes \mu_{BH} \otimes BH) \circ (\delta_{BH} \otimes \delta_{BH}) \circ (\eta_{BH} \otimes \eta_{BH}) = (BH \otimes (\mu_{BH} \circ t_{BH,BH} \circ BH) \circ (\delta_{BH} \otimes \delta_{BH}) \circ (\eta_{BH} \otimes \eta_{BH})\).

(4) There exists an unique morphism \(\lambda_{BH} : BH \to BH\) in \(\mathcal{C}\) such that

\[i_H \circ \lambda_{BH} = \mu_B \circ ((f \circ g) \otimes \lambda_H) \circ \delta_B \circ i_H^{B}\]

and verifying:

(4-1) \(\mu_{BH} \circ (BH \otimes \lambda_{BH}) \circ \delta_{BH} = ((\epsilon_{BH} \circ \mu_{BH}) \otimes BH) \circ (BH \otimes t_{BH,BH}) \circ ((\delta_{BH} \circ \eta_{BH}) \otimes BH).\)

(4-2) \(\mu_{BH} \circ (\lambda_{BH} \otimes BH) \circ \delta_{BH} = (BH \otimes (\epsilon_{BH} \circ \mu_{BH})) \circ (BH \otimes (\delta_{BH} \circ \eta_{BH} \otimes BH))\).

(4-3) \(\mu_{BH} \circ (\mu_{BH} \otimes BH) \circ (\lambda_{BH} \otimes BH \otimes \lambda_{BH}) \circ (\delta_{BH} \otimes BH) \circ \delta_{BH} = \lambda_{BH}.\)

Finally, linking this information with the one obtained in the previous sections, we have Radford’s theorem (see [8]) for weak Hopf algebras:

**Theorem 4.1.** Let \(H, B\) be weak Hopf algebras in \(\mathcal{C}\). Let \(g : B \to H\) and \(f : H \to B\) be morphisms of weak Hopf algebras such that \(g \circ f = \text{id}_H\). Then there exists an object \(BH\) living in \(\mathcal{H}^{WYD}\) such that \(B\) is isomorphic to \(BH \times H\) as weak Hopf algebras, being the (co)algebra structure in \(BH \times H\) the smash (co)product. The expression for the antipode of \(BH \times H\) is

\[\lambda_{BH \times H} = r_B \circ (\varphi_{BH} \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \lambda_H) \otimes \lambda_{BH}) \circ (H \otimes c_{BH,H}) \circ \epsilon_B \circ s_B.\]

**Proof.** The central part of this theorem is a consequence of Corollaries 2.11 and 3.2. We only show the equality for the antipode of \(BH \times H\).

First, note that

\[\lambda_{BH \times H} = b_B \circ \lambda_B \circ b_B^{-1} = r_B \circ (\gamma_H \otimes g) \circ \delta_B \circ \lambda_B \circ \mu_B \circ (i_H \otimes f) \circ s_B\]
\[= r_B \circ c_{H,B_H} \circ ((\lambda_H \circ \mu_H) \otimes (p^B_H \circ \lambda_B \circ \mu_B)) \circ (g \otimes H \otimes B \otimes f) \circ \delta_{B \otimes H}\]
\[\circ (i_H^B \otimes H) \circ s_B\]
\[= r_B \circ c_{H,B_H} \circ (\mu_H \otimes (p^B_H \circ \mu_B \circ c_{B,H})) \circ (H \otimes c_{B,H} \otimes B)\]
\[\circ (((g \otimes \lambda_H) \otimes \delta_B) \otimes H \otimes (\lambda_B \circ f)) \circ (i_H^B \otimes \delta_H) \circ s_B.\]

On the other hand, using the antimultiplicative nature of $\lambda_H$ and the equality
\[\mu_B \circ ((f \circ \Pi^R_H \circ g) \otimes \lambda_B) \circ \delta_B = \lambda_B,\]
we have that
\[r_B \circ (\varphi_{B_H} \otimes H) \circ (H \otimes c_{H,B_H}) \circ ((\delta_H \circ \lambda_H \circ \mu_H) \otimes \lambda_{B_H}) \circ (H \otimes c_{B,H},H)\]
\[\circ (r_{B_H} \otimes H) \circ s_B\]
\[= r_B \circ ((p^B_H \circ \mu_B) \otimes H) \circ (f \otimes c_{H,B}) \circ (\lambda_H \otimes \lambda_H \otimes B)\]
\[\circ \left(((c_{H,H} \circ \mu_{H \otimes H} \circ (\delta_H \otimes \delta_H)) \otimes (\mu_B \circ ((f \circ g) \otimes \lambda_H) \circ \delta_B)) \circ (g \otimes c_{B,H})\right)\]
\[\circ (\delta_B \circ (i_H^B \otimes H) \circ s_B\]
\[= r_B \circ c_{H,B_H} \circ \left((H \otimes (p^B_H \circ \mu_B))\right)\]
\[\circ \left(((\delta_B \circ (i_H^B \otimes H) \circ (\mu_B \otimes ((f \circ \Pi^R_H \circ g) \otimes \lambda_B) \circ \delta_B))\right)\]
\[\circ (g \otimes \delta_H \otimes \delta_B) \circ (B \otimes c_{B,H}) \circ ((\delta_B \circ (i_H^B \otimes H) \circ s_B\]
\[= r_B \circ c_{H,B_H} \circ \left((\lambda_H \otimes \mu_H) \otimes (f \circ \lambda_H) \otimes (\mu_B \circ ((f \circ \Pi^R_H \circ g) \otimes \lambda_B) \circ \delta_B))\right)\]
\[\circ (g \otimes \delta_H \otimes B) \circ (B \otimes c_{B,H}) \circ ((\delta_B \circ (i_H^B \otimes H) \circ s_B\]
\[= r_B \circ c_{H,B_H} \circ \left((\mu_H \otimes (p^B_H \circ \mu_B \circ c_{B,H}))\right) \circ (H \otimes c_{B,H} \otimes B)\]
\[\circ \left(((g \otimes \lambda_H) \otimes \lambda_B) \circ (H \otimes (\lambda_B \circ f)) \circ (i_H^B \otimes \delta_H) \circ s_B.\right)\]

Therefore,
\[\lambda_{B_H \otimes H} = r_B \circ (\varphi_{B_H} \otimes H) \circ (H \otimes c_{H,B_H}) \circ ((\delta_H \circ \lambda_H \circ \mu_H) \otimes \lambda_{B_H}) \circ (H \otimes c_{B,H},H)\]
\[\circ (r_{B_H} \otimes H) \circ s_B.\]

**Remark 4.2.** If $g : B \rightarrow H$ and $f : H \rightarrow B$ are morphisms of weak Hopf algebras such that $g \circ f = \text{id}_H$, the morphism $\Omega_B = \omega_B \circ \omega_H$ admits a new formulation. Note that, using the equalities (6) of 2.4, $p^B_H \circ \mu_B \circ (B \otimes q^B_H) = p^B_H \circ \mu_B$ and the usual arguments in the weak Hopf algebra calculus, we have
\[ \Omega_B = (p_B^H \otimes \mu_H) \circ (\mu_B \otimes \gamma) \circ (B \otimes \pi_H) \circ ((B \otimes g) \circ \delta_B \circ i_B^H \otimes (\delta_B \circ f)) \]
\[ = (p_B^H \otimes \mu_H) \circ (\mu_B \otimes \gamma) \circ (B \otimes \pi_H) \circ ((B \otimes g) \circ \delta_B \circ i_B^H \otimes (f \otimes H) \circ \delta_B) \]
\[ = (p_B^H \otimes \epsilon_H \otimes H) \circ (\mu_B \otimes H) \circ ((B \otimes g) \circ \delta_B \circ i_B^H \otimes ((f \otimes \delta_B) \circ \delta_B)) \]
\[ = (p_B^H \otimes (\epsilon_H \circ g) \otimes H) \circ (\mu_B \otimes H) \circ (\delta_B \circ (f \otimes H) \circ \delta_B) \]
\[ = ((p_B^H \circ \mu_B) \otimes H) \circ (i_B^H \otimes ((f \otimes H) \circ \delta_B)) \]
\[ = (\pi_B^H \otimes H) \circ (\mu_B \circ (B \otimes (\Pi_B^H \circ f))) \circ H \circ (i_B^H \otimes \delta_B) \]
\[ = (\pi_B^H \otimes H) \circ (\mu_B \circ (\delta_B \circ (\Pi_B^H \circ f))) \circ (B_H \otimes \delta_B) \]
\[ = (\pi_B^H \otimes H) \circ (\delta_B \circ (\Pi_B^H \circ B_H)) \circ H \circ (\delta_B \circ (B_H \otimes \delta_B)) \]
\[ = (\varphi_H \otimes H) \circ (\delta_B \circ (B_H \otimes \delta_B)) \circ (B_H \otimes \delta_B) \]
\[ = (\varphi_H \otimes H) \circ (H \otimes \delta_B) \circ (\delta_B \circ (\gamma_H \otimes B_H)) \circ H. \]

Therefore, the object \( B_H \times H \) is the tensor product of \( B_H \) and \( H \) in the representation category of \( H \). This category is denoted by \( \text{Rep}(H) \) and were studied in [4] and [7].

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