Journal of the Egyptian Mathematical Society (2012) **20**, 179–182

Egyptian Mathematical Society
Journal of the Egyptian Mathematical Society
www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Estimate for initial Maclaurin coefficients of bi-univalent functions for a class defined by fractional derivatives

S.P. Goyal [a], **Pranay Goswami** [b,*]

[a] *Department of Mathematics, University of Rajasthan, Jaipur 302055, India*
[b] *Department of Mathematics, School of Liberal Studies, Bharat Ratna Dr B.R. Ambedkar University, New Delhi 110006, India*

Received 10 April 2012; revised 15 August 2012; accepted 26 August 2012
Available online 4 October 2012

KEYWORDS
Univalent functions;
Bi-univalent functions;
Subordination;
Coefficient estimates;
Convolution (Hadamard Product)

Abstract Estimates for second and third Maclaurin coefficients of certain bi-univalent functions in the open unit disk defined by convolution are determined. Certain special cases are also indicated.

1. Introduction and definitions

Let $\mathscr{A}$ be the class of functions f which are analytic univalent functions in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ with normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ and having form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \quad (z \in \mathbb{D}). \tag{1.1}$$

Familiar subclasses of starlike and convex functions for which either of the quantity $Re\ \{zf'(z)/f(z)\} > 0$ or $\{1 + zf''(z)/f'(z)\} > 0$. The class consisting these two functions are given by $\mathscr{S}^*$ and $\mathscr{C}$, respectively. For a constant $\beta \in (-\pi/2, \pi/2)$, a function f is univalent on $\mathbb{D}$ and satisfies the condition that $Re\ \{e^{(i\beta)} zf'(z)/f(z)\} > 0$ in $\mathbb{D}$. We denote this class by $\mathscr{SP}$.

The Koebe one-quarter theorem [2] ensures that the image of $\mathbb{D}$ under every univalent function $f \in \mathscr{A}$ contains a disk of radius 1/4. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z,\ (z \in \mathbb{D})$ and

$$f^{-1}(f(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geqslant \frac{1}{4}\right).$$

A function $f \in \mathscr{A}$ is said to be bi-univalent in $\mathbb{D}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{D}$. We denote the class of bi-univalent functions by σ. Lewin [4] investigated the class σ of bi-univalent functions and obtained the bound for the second coefficient. Brannan and Taha [1] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike functions and bi-convex functions and obtained estimates on the initial coefficients. Recently, Srivastava et al. [8] and Frasin and Aouf [3] introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients.

Let f and g be analytic functions in $\mathbb{D}$, we say that f is subordinate to g, written as $f \prec g$, if there exists a Schwarz function $w(z)$ in $\mathbb{D}$, with $w(0) = 0$ and $|w(z)| < 1\ (z \in \mathbb{D})$, such that $f(z) = g(w(z))$. In particular, when g is univalent, then the above subordination is equivalent to $f(0) = 0$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. For functions $f,\ g \in \mathscr{A}$ given by

* Corresponding author.
E-mail addresses: somprg@mail.com (S.P. Goyal), pranaygoswami83@gmail.com (P. Goswami).
Peer review under responsibility of Egyptian Mathematical Society.

ELSEVIER Production and hosting by Elsevier

1110-256X © 2012 Egyptian Mathematical Society. Production and hosting by Elsevier B.V. Open access under CC BY-NC-ND license.
http://dx.doi.org/10.1016/j.joems.2012.08.020

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad z \in \mathbb{D},$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{D}.$$

Definition 1.1 (*cf. [6,7], see also [9]*). Let the function f be analytic in a simple connected region of the z-plane containing the origin. The fractional derivative or order 'λ' is defined by

$$\left(D_z^{\lambda} f\right)(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \quad (0 \leqslant \lambda < 1), \tag{1.2}$$

where multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Using Definition 1.1 and its known extension involving fractional derivative and fractional integrals, Owa and Srivastava [6] introduced the fractional differ-integral operator $\Omega_z^{\lambda} : \mathcal{A} \to \mathcal{A}$ defined by

$$\left(\Omega_z^{\lambda} f\right)(z) = \Gamma(2-\lambda) z^{\lambda} \left(D_z^{\lambda} f\right)(z) \quad (\lambda \neq 2, 3, 4, \ldots;\ z \in \mathbb{D}). \tag{1.3}$$

Note that $\left(\Omega_z^0 f\right)(z) = z f'(z)$ and $\left(\Omega_z^1 f\right)(z) = f(z)$.

Motivated by the work of Srivastava et al. [8] and Mishra and Gochhayat [5], we introduce a new subclass of bi univalent functions $\mathcal{SB}_{\sigma}^{\lambda,\beta}(h)$.

Definition 1.2. Let

$h : \mathbb{D} \to \mathbb{C}$,

be a convex univalent function such that

$h(0) = 1 \quad \text{and} \quad h(\bar{z}) = \overline{h(z)}, \quad (z \in \mathbb{D};\ Re(h(z)) > 0).$

A function $f(z)$ is said to be in the class $\mathcal{SB}_{\sigma}^{\lambda,\beta}(h)$, if the following conditions are satisfied:

$$f \in \sigma \quad \text{and} \quad e^{i\beta} \frac{\left(\Omega_z^{\lambda} f\right)(z)}{z} \prec h(z) \cos\beta + i \sin\beta, \quad (z \in \mathbb{D}), \tag{1.4}$$

and

$$e^{i\beta} \frac{\left(\Omega_z^{\lambda} g\right)(w)}{w} \prec h(w) \cos\beta + i \sin\beta, \quad (w \in \mathbb{D}); \tag{1.5}$$

where $\beta \in (-\pi/2, \pi/2)$, $\lambda \neq 2, 3, \ldots$ and $g = f^{-1}$.

Remark 1.1. If we set $h(z) = 1 + Az/1 + Bz, -1 \leqslant B < A \leqslant 1$, then the class $\mathcal{SB}_{\sigma}^{\lambda,\beta}(h)$ reduces to $\mathcal{SB}_{\sigma}^{\lambda,\beta}(A, B)$ which is define as

$$f \in \sigma \quad \text{and} \quad e^{i\beta} \frac{\left(\Omega_z^{\lambda} f\right)(z)}{z} \prec \frac{1 + Az}{1 + Bz} \cos\beta + i \sin\beta, \quad (z \in \mathbb{D}), \tag{1.6}$$

and

$$e^{i\beta} \frac{\left(\Omega_z^{\lambda} g\right)(w)}{w} \prec \frac{1 + Aw}{1 + Bw} \cos\beta + i \sin\beta, \quad (w \in \mathbb{D}); \tag{1.7}$$

where $\beta \in (-\pi/2, \pi/2)$ and $g = f^{-1}$ and $\lambda \neq 2, 3, \ldots$.

Remark 1.2. Taking $\lambda = 0$in above class, then we have $\mathcal{SB}_{\sigma}^{0}(A, B)$ and if $f \in \mathcal{SB}_{\sigma}^{0}(A, B)$

$$f \in \sigma \quad \text{and} \quad e^{i\beta} f'(z) \prec \frac{1 + Az}{1 + Bz} \cos\beta + i \sin\beta, \quad (z \in \mathbb{D}), \tag{1.8}$$

and

$$e^{i\beta} g'(w) \prec \frac{1 + Aw}{1 + Bw} \cos\beta + i \sin\beta, \quad (w \in \mathbb{D}); \tag{1.9}$$

where $\beta \in (-\pi/2, \pi/2)$ and $g = f^{-1}$.

Now substituting $A = 1 - 2\alpha,\ 0 \leqslant \alpha < 1,\ B = -1$ and $\beta = 0$, we get known class $\mathcal{B}_{\sigma}(\beta)$ which is studied by Srivastva et al. [8].

Remark 1.3. Taking $\lambda = 1$ in the class $\mathcal{SB}_{\sigma}^{\lambda,\beta}(A, B)$, we have $\mathcal{SB}_{\sigma}^{1,\beta}(A, B)$ and if $f \in \mathcal{SB}_{\sigma}^{\lambda,\beta}(A, B)$, then

$$f \in \sigma \quad \text{and} \quad e^{i\beta} \frac{f(z)}{z} \prec \frac{1 + Az}{1 + Bz} \cos\beta + i \sin\beta, \quad (z \in \mathbb{D}), \tag{1.10}$$

and

$$e^{i\beta} \frac{g(w)}{w} \prec \frac{1 + Aw}{1 + Bw} \cos\beta + i \sin\beta, \quad (w \in \mathbb{D}); \tag{1.11}$$

where $\beta \in (-\pi/2, \pi/2)$ and $g = f^{-1}$.

Remark 1.4. Taking $\lambda = 0$ in above class $\mathcal{SB}_{\sigma}^{\lambda,\beta}(A, B)$, we have $\mathcal{SB}_{\sigma}^{0,\beta}(A, B)$ and if $f \in \mathcal{SB}_{\sigma}^{0,\beta}(A, B)$, then

$$f \in \sigma \quad \text{and} \quad Re\left(e^{i\beta} f'(z)\right) > \alpha \cos\beta, \quad (z \in \mathbb{D}), \tag{1.12}$$

and

$$Re(e^{i\beta} g'(w)) > \alpha \cos\beta, \quad (w \in \mathbb{D}); \tag{1.13}$$

where $\beta \in (-\pi/2, \pi/2)$ and $g = f^{-1}$.

Remark 1.5. Taking $A = 1 - 2\alpha,\ B = -1$ in above class in class $\mathcal{SB}_{\sigma}^{1,\beta}(A, B)$, it reduces to $\mathcal{SB}_{\sigma}^{1,\beta}(\alpha)$ and if $f \in \mathcal{SB}_{\sigma}^{1,\beta}(\alpha)$, then

$$f \in \sigma \quad \text{and} \quad Re\left(e^{i\beta} \frac{f(z)}{z}\right) > \alpha \cos\beta, \quad (z \in \mathbb{D}), \tag{1.14}$$

and

$$Re\left(e^{i\beta} \frac{g(w)}{w}\right) > \alpha \cos\beta \quad (w \in \mathbb{D}). \tag{1.15}$$

The object of the paper is to estimates for the coefficients a_2 and a_3 for functions in the class $\mathcal{SB}_{\sigma}^{\lambda,\beta}(h)$ are obtained by employing the techniques used earlier by Srivastava et al. [8].

2. Main result

In order to prove our main result for the functions class $f \in \mathcal{SB}_{\sigma}^{k}(\beta)$, we first recall the following lemma:

Lemma 2.1 (*Theorem 3.3, p. 11, 13*). *Let the function $\varphi(z)$ given $\varphi(z) = \sum_{n=1}^{\infty} B_n z^n$ be convex in $\mathbb{D}$. Suppose also that the function $h(z)$ given by*

$$h(z) = \sum_{n=1}^{\infty} h_n z^n,$$

is holomorphic in $\mathbb{D}$. *If* $h(z) \prec \varphi(z)(z \in \mathbb{D})$, *then*

$$|h_n| \leqslant |B_1| \quad (n \in \mathbb{N}). \tag{2.1}$$

Theorem 2.2. *If* $f \in \mathcal{A}$ *satisfies* (1.1), *is in the class* $\mathcal{SB}_\sigma^{\lambda,\beta}(h)$. *Then*

$$|a_2| \leqslant \sqrt{\frac{|B_1|\cos\beta(2-\lambda)(3-\lambda)}{12}}, \tag{2.2}$$

and

$$|a_3| \leqslant \left(\frac{|B_1|(2-\lambda)}{2}\right)^2 + \frac{|B_1|\cos\beta(2-\lambda)(3-\lambda)}{12}, \tag{2.3}$$

where $\beta \in (-\pi/2, \pi/2)$ *and* $\lambda \neq 2, 3, \ldots$

Proof. From (1.4) and (1.5)

$$e^{i\beta}\frac{(\Omega_z^\lambda f)(z)}{z} = p(z)\cos\beta + i\sin\beta, \quad (z \in \mathbb{D}), \tag{2.4}$$

and

$$e^{i\beta}\frac{(\Omega_z^\lambda f)(w)}{w} = q(w)\cos\beta + i\sin\beta, \quad (w \in \mathbb{D}), \tag{2.5}$$

where $p(z) \prec h(z)$ and $q(w) \prec h(w)$ and have following forms:

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \quad z \in \mathbb{D}, \tag{2.6}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots \quad w \in \mathbb{D}. \tag{2.7}$$

Now, equating the coefficients in (2.6) and (2.7), we get

$$e^{i\beta}\left(\frac{2}{2-\lambda}\right)a_2 = p_1, \tag{2.8}$$

$$e^{i\beta}\frac{6}{(2-\lambda)(3-\lambda)}a_3 = p_2, \tag{2.9}$$

$$-e^{i\beta}\left(\frac{2}{2-\lambda}\right)a_2 = q_1, \tag{2.10}$$

and

$$e^{i\beta}\frac{6}{(2-\lambda)(3-\lambda)}\left(2a_2^2 - a_3\right) = q_2. \tag{2.11}$$

From (2.8) and (2.10), we get

$$p_1 = -q_1, \tag{2.12}$$

and

$$e^{2i\beta}\left(\frac{8}{2-\lambda}\right)^2 a_2^2 = \left(p_1^2 + q_1^2\right)\cos^2\beta. \tag{2.13}$$

Adding (2.9) and (2.11), it follows:

$$a_2^2 = \frac{(2-\lambda)(3-\lambda)}{12}(p_2 + q_2)e^{-i\beta}\cos\beta. \tag{2.14}$$

Again from (2.9) and (2.11)

$$a_3 - a_2^2 = \frac{(2-\lambda)(3-\lambda)}{12}(p_2 - q_2)e^{-i\beta}\cos\beta. \tag{2.15}$$

Substituting value of a_2^2 from (2.13) in (2.14), we get

$$a_3 = \frac{(2-\lambda)(3-\lambda)}{12}(p_2 - q_2)e^{-i\beta}\cos\beta + \frac{(2-\lambda)^2}{8}\left(p_1^2 + q_1^2\right)e^{-2i\beta}\cos^2\beta. \tag{2.16}$$

Since $p(z),\ q(w) \in h(\mathbb{D})$. According to Lemma 2.1, we find that

$$|p_k| = \left|\frac{p^{(k)}(0)}{k!}\right| \leqslant |B_1| \quad (k \in \mathbb{N}), \tag{2.17}$$

and

$$|q_k| = \left|\frac{q^{(k)}(0)}{k!}\right| \leqslant |B_1| \quad (k \in \mathbb{N}). \tag{2.18}$$

Using above Eq. (2.12) and using (2.17) and (2.18), we have

$$|a_2|^2 \leqslant \frac{(2-\lambda)(3-\lambda)}{12}(|q_2| + |p_2|)\cos\beta \leqslant \frac{|B_1|\cos\beta(2-\lambda)(3-\lambda)}{6}, \tag{2.19}$$

which gives (2.2). Now using (2.13) and (2.15) and from (2.17) and (2.18), we can easily get (2.3). This is the end of the Theorem 2.2. □

By setting, $h(z) = \frac{1+Az}{1+Bz}$, $-1 \leqslant B < A \leqslant 1$ in Theorem 2.2, we get the following corollary:

Corollary 2.3. *Let* $f \in \mathcal{A}$ *be in the class* $\mathcal{SB}_\sigma^{\lambda,\beta}(A, B)$. *Then*

$$|a_2| \leqslant \sqrt{\frac{(2-\lambda)(3-\lambda)(A-B)\cos\beta}{6}}, \tag{2.20}$$

and

$$|a_3| \leqslant \frac{(2-\lambda)^2(A-B)^2\cos^2\beta}{4} + \frac{(2-\lambda)(3-\lambda)(A-B)\cos\beta}{6}, \tag{2.21}$$

where $\beta \in (-\pi/2, \pi/2)$ *and* $\lambda \neq 2, 3, \ldots$ *Again putting,* $h(z) = \frac{1+(1-2\alpha)z}{1-z}, 0 \leqslant \alpha < 1$ *in Theorem 2.2, we have*

Corollary 2.4. *Let* $f \in \mathcal{A}$ *be in the class* $\mathcal{SB}_\sigma^{\lambda,\beta}(\alpha)$. *Then*

$$|a_2| \leqslant \sqrt{\frac{(2-\lambda)(3-\lambda)(1-\alpha)\cos\beta}{3}} \tag{2.22}$$

and

$$|a_3| \leqslant (2-\lambda)^2(1-\alpha)^2\cos^2\beta + \frac{(2-\lambda)(3-\lambda)(1-\alpha)\cos\beta}{3}, \tag{2.23}$$

where $\beta \in (-\pi/2, \pi/2)$ *and* $\lambda \neq 2, 3, \ldots$

If we take $\lambda = 0$ in Corollary 2.3, it gives

Corollary 2.5. *Let* $f \in \mathcal{A}$ *be in the class* $\mathcal{SB}_\sigma^{0,\beta}(A, B)$. *Then*

$$|a_2| \leqslant \sqrt{(A-B)\cos\beta}, \tag{2.24}$$

and

$$|a_3| \leqslant (A-B)^2\cos^2\beta + (A-B)\cos\beta, \tag{2.25}$$

where $\beta \in (-\pi/2, \pi/2)$.

If we take $\lambda = 1$ in Corollary 2.3, we obtain

Corollary 2.6. *Let* $f \in \mathscr{A}$ *be in the class* $\mathscr{SB}_{\sigma}^{1,\beta}(A,B)$. *Then*

$$|a_2| \leqslant \sqrt{\frac{(A-B)\cos\beta}{3}}, \tag{2.26}$$

and

$$|a_3| \leqslant \frac{(A-B)^2}{4}\cos^2\beta + \frac{(A-B)\cos\beta}{3}, \tag{2.27}$$

where $\beta \in (-\pi/2, \pi/2)$.

Upon putting $A = 1-2\alpha$, $0 \leqslant \alpha < 1$ and $B = -1$, above Corollaries 2.4 and 2.5, we get following results

Corollary 2.7. *Let* $f \in \mathscr{A}$ *be in the class* $\mathscr{SB}_{\sigma}^{0,\beta}(\alpha)$. *Then*

$$|a_2| \leqslant \sqrt{2(1-\alpha)\cos\beta}, \tag{2.28}$$

and

$$|a_3| \leqslant 4(1-\alpha)^2\cos^2\beta + 2(1-\alpha)\cos\beta, \tag{2.29}$$

where $\beta \in (-\pi/2, \pi/2)$.

Put $\lambda = 1$ in Corollary 2.3, we obtain

Corollary 2.8. *Let* $f \in \mathscr{A}$ *be in the class* $\mathscr{SB}_{\sigma}^{1,\beta}(\alpha)$. *Then*

$$|a_2| \leqslant \sqrt{\frac{2(1-\alpha)\cos\beta}{3}}, \tag{2.30}$$

and

$$|a_3| \leqslant (1-\alpha)^2\cos^2\beta + \frac{2(1-\alpha)\cos\beta}{3}, \tag{2.31}$$

where $\beta \in (-\pi/2, \pi/2)$.

Remark 2.1. On taking $\beta = 0$ in Corollary 2.8, we obtain a known result due to Srivastava et al. [8].

Acknowledgments

The first author (S.P. Goyal) is thankful to CSIR, New Delhi, India for awarding Emeritus Scientistship, under scheme number 21(084)/10/EMR-II. The authors are also thankful to the worthy referee for his/her useful comments.

References

[1] D.A. Brannan, T.S. Taha, On some classes of bi-univalent functions, Studia Universitatis Babeş-Bolyai Mathematica 31 (2) (1986) 70–77.
[2] P.L. Duren, Univalent functions, Grundlehren der Mathematischen Wissenschaften, vol. 259, Springer, New York, 1983.
[3] B.A. Frasin, M.K. Aouf, New subclasses of bi-univalent functions, Applied Mathematics Letters 24 (9) (2011) 1569–1573.
[4] M. Lewin, On a coefficient problem for bi-univalent functions, Proceedings of the American Mathematical Society 18 (1967) 63–68.
[5] A.K. Mishra, P. Gochhayat, Second Hankel determinant for a class of analytic functions defined by fractional derivative, International Journal of Mathematics and Mathematical Sciences (2008). Art. ID 153280, p. 10.
[6] S. Owa, On the distortion theorems. I, Kyungpook Mathematical Journal 18 (1) (1978) 53–59.
[7] S. Owa, H.M. Srivastava, Univalent and starlike generalized hypergeometric functions, Canadian Journal of Mathematics 39 (5) (1987) 1057–1077.
[8] H.M. Srivastava, A.K. Mishra, P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Applied Mathematical Letters 23 (10) (2010) 1188–1192.
[9] H.M. Srivastava, S. Owa, An application of the fractional derivative, Mathematica Japonica 29 (3) (1984) 383–389.