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# Identities of symmetry for $q$-Bernoulli polynomials

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ABSTRACT

In this paper, we derive eight basic identities of symmetry in three variables related to $q$-Bernoulli polynomials and the $q$-analogue of power sums. These and most of their corollaries are new, since there have been results only concerning identities of symmetry in two variables. These abundant symmetries shed new light even on the existing identities so as to yield some further interesting ones. The derivations of the identities are based on the $p$-adic integral expression of the generating function for the $q$-Bernoulli polynomials and the quotient of integrals that can be expressed as the exponential generating function for the $q$-analogue of power sums.

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## 1. Introduction and preliminaries

Let $p$ be a fixed prime. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}_p$ will respectively denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. For a uniformly differentiable (also called continuously differentiable) function $f : \mathbb{Z}_p \longrightarrow \mathbb{C}_p$ (cf. [1]), the Volkenborn integral of $f$ is defined by

$$\int_{\mathbb{Z}_p} f(z)\mathrm{d}\mu(z) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{j=0}^{p^N-1} f(j).$$

Then it is easy to see that

$$\int_{\mathbb{Z}_p} f(z+1)\mathrm{d}\mu(z) = \int_{\mathbb{Z}_p} f(z)\mathrm{d}\mu(z) + f'(0). \tag{1.1}$$

Let $|\ |_p$ be the normalized absolute value of $\mathbb{C}_p$, such that $|p|_p = \frac{1}{p}$, and let

$$E = \left\{ t \in \mathbb{C}_p | |t|_p < p^{\frac{-1}{p-1}} \right\}. \tag{1.2}$$

Assume that $q, t \in \mathbb{C}_p$, with $q-1, t \in E$, and so $q^z = \exp(z \log q)$ and $\mathrm{e}^{zt}$ are, as functions of $z$, analytic functions on $\mathbb{Z}_p$. By applying (1.1) to $f$, with $f(z) = q^z \mathrm{e}^{tz}$, we get the $p$-adic integral expression for the generating function for $q$-Bernoulli numbers $B_{n,q}$ (cf. [2,3]):

$$\int_{\mathbb{Z}_p} q^z \mathrm{e}^{zt} \mathrm{d}\mu(z) = \frac{\log q + t}{q\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} B_{n,q} \frac{t^n}{n!} (q-1, t \in E). \tag{1.3}$$

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So we have the following $p$-adic integral expression for the generating function for the $q$-Bernoulli polynomials $B_{n,q}(x)$:

$$\int_{\mathbb{Z}_p} q^z e^{(x+z)t} d\mu(z) = \frac{\log q + t}{qe^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!} (q-1, t \in E, x \in \mathbb{Z}_p). \tag{1.4}$$

Here and throughout this paper, we will have many instances where we are able to interchange the integral and infinite sum. That is justified by Proposition 55.4 in [1]. Let $S_{k,q}(n)$ denote the $q$-analogue of the $k$th power sum of the first $n+1$ nonnegative integers, namely

$$S_{k,q}(n) = \sum_{i=0}^{n} i^k q^i = 0^k q^0 + 1^k q^1 + \cdots + n^k q^n. \tag{1.5}$$

In particular,

$$S_{0,q}(n) = \sum_{i=0}^{n} q^i = \frac{q^{n+1}-1}{q-1} = [n+1]_q, \qquad S_{k,q}(0) = \begin{cases} 1, & \text{for } k = 0, \\ 0, & \text{for } k > 0. \end{cases} \tag{1.6}$$

From (1.3) and (1.5), one easily derives the following identities: for $w \in \mathbb{Z}_{>0}$,

$$\frac{w \int_{\mathbb{Z}_p} q^x e^{xt} d\mu(x)}{\int_{\mathbb{Z}_p} q^{wy} e^{wyt} d\mu(y)} = \sum_{i=0}^{w-1} q^i e^{it} = \sum_{k=0}^{\infty} S_{k,q}(w-1) \frac{t^k}{k!} (q-1, t \in E). \tag{1.7}$$

In what follows, we will always assume that the Volkenborn integrals of the various exponential functions on $\mathbb{Z}_p$ are defined for $q-1, t \in E$ (cf. (1.2)), and therefore this will not be mentioned.

[4–8] are some of the previous works on identities of symmetry in two variables involving Bernoulli polynomials and power sums. For a brief history, one is referred to those papers. In [9], the idea of [6] was adopted to produce many new identities for symmetry in three variables involving Bernoulli polynomials and power sums.

In this paper, we will produce eight basic identities of symmetry in three variables $w_1, w_2, w_3$ related to $q$-Bernoulli polynomials and the $q$-analogue of power sums (cf. (4.8), (4.9), (4.12), (4.15), (4.19), (4.21), (4.23), (4.24)). These and most of their corollaries seem to be new, since there have been results only concerning identities of symmetry in two variables in the literature. These abundant symmetries shed new light even on the existing identities. For instance, it is known that (1.8) and (1.9) are equal (cf. [2, (2.20)]) and (1.10) and (1.11) are too (cf. [2, (2.25)]). In fact, (1.8)–(1.11) are all equal, as they can be derived from one and the same $p$-adic integral. Perhaps [2] neglected to mention this. Also, we have a bunch of new identities in (1.12)–(1.15). All of these were obtained as corollaries (cf. Corollaries 4.9, 4.12 and 4.15) to some of the basic identities by specializing the variable $w_3$ as 1. They would not be unearthed if more symmetries had not been available.

$$\sum_{k=0}^{n} \binom{n}{k} B_{k,q^{w_2}}(w_1 y_1) S_{n-k,q^{w_1}}(w_2 - 1) w_1^{n-k} w_2^{k-1} \tag{1.8}$$

$$= \sum_{k=0}^{n} \binom{n}{k} B_{k,q^{w_1}}(w_2 y_1) S_{n-k,q^{w_2}}(w_1 - 1) w_2^{n-k} w_1^{k-1} \tag{1.9}$$

$$= w_1^{n-1} \sum_{i=0}^{w_1-1} q^{w_2 i} B_{n,q^{w_1}} \left( w_2 y_1 + \frac{w_2}{w_1} i \right) \tag{1.10}$$

$$= w_2^{n-1} \sum_{i=0}^{w_2-1} q^{w_1 i} B_{n,q^{w_2}} \left( w_1 y_1 + \frac{w_1}{w_2} i \right) \tag{1.11}$$

$$= \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k,q^{w_1 w_2}}(y_1) S_{l,q^{w_2}}(w_1 - 1) S_{m,q^{w_1}}(w_2 - 1) w_1^{k+m-1} w_2^{k+l-1} \tag{1.12}$$

$$= w_1^{n-1} \sum_{k=0}^{n} \binom{n}{k} S_{n-k,q^{w_1}}(w_2 - 1) w_2^{k-1} \sum_{i=0}^{w_1-1} q^{w_2 i} B_{k,q^{w_1 w_2}} \left( y_1 + \frac{i}{w_1} \right) \tag{1.13}$$

$$= w_2^{n-1} \sum_{k=0}^{n} \binom{n}{k} S_{n-k,q^{w_2}}(w_1 - 1) w_1^{k-1} \sum_{i=0}^{w_2-1} q^{w_1 i} B_{k,q^{w_1 w_2}} \left( y_1 + \frac{i}{w_2} \right) \tag{1.14}$$

$$= (w_1 w_2)^{n-1} \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} q^{w_2 i + w_1 j} B_{n,q^{w_1 w_2}} \left( y_1 + \frac{i}{w_1} + \frac{j}{w_2} \right). \tag{1.15}$$

The derivations of the identities are based on the $p$-adic integral expression for the generating function for the $q$-Bernoulli polynomials in (1.4) and the quotient of integrals in (1.7) that can be expressed as the exponential generating function for the $q$-analogue of power sums. We are indebted for this idea to the paper [2].

## 2. Several quotient types for Volkenborn integrals

Here we will introduce several quotient types for Volkenborn integrals on $\mathbb{Z}_p$ or $\mathbb{Z}_p^3$ from which some interesting identities follow owing to the built-in symmetries in $w_1, w_2, w_3$. In the following, $w_1, w_2, w_3$ are all positive integers and all of the explicit expressions for integrals in (2.2), (2.4), (2.6) and (2.8) are obtained from the identity in (1.3).

(a) Type $\Lambda_{23}^i$ (for $i = 0, 1, 2, 3$):

$$I(\Lambda_{23}^i) = \frac{\int_{\mathbb{Z}_p^3} q^{w_2w_3x_1+w_1w_3x_2+w_1w_2x_3} e^{\left(w_2w_3x_1+w_1w_3x_2+w_1w_2x_3+w_1w_2w_3\left(\sum_{j=1}^{3-i} y_j\right)\right)t} d\mu(X)}{\left(\int_{\mathbb{Z}_p} q^{w_1w_2w_3x_4} e^{w_1w_2w_3x_4t} d\mu(x_4)\right)^i}, \tag{2.1}$$

with $d\mu(X) = d\mu(x_1)d\mu(x_2)d\mu(x_3)$,

$$= \frac{(w_1w_2w_3)^{2-i}(\log q + t)^{3-i} e^{w_1w_2w_3\left(\sum_{j=1}^{3-i} y_j\right)t} (q^{w_1w_2w_3} e^{w_1w_2w_3t} - 1)^i}{(q^{w_2w_3} e^{w_2w_3t} - 1)(q^{w_1w_3} e^{w_1w_3t} - 1)(q^{w_1w_2} e^{w_1w_2t} - 1)}. \tag{2.2}$$

(b) Type $\Lambda_{13}^i$ (for $i = 0, 1, 2, 3$):

$$I(\Lambda_{13}^i) = \frac{\int_{\mathbb{Z}_p^3} q^{w_1x_1+w_2x_2+w_3x_3} e^{\left(w_1x_1+w_2x_2+w_3x_3+w_1w_2w_3\left(\sum_{j=1}^{3-i} y_j\right)\right)t} d\mu(x_1)d\mu(x_2)d\mu(x_3)}{\left(\int_{\mathbb{Z}_p} q^{w_1w_2w_3} e^{w_1w_2w_3x_4t} d\mu(x_4)\right)^i} \tag{2.3}$$

$$= \frac{(w_1w_2w_3)^{1-i}(\log q + t)^{3-i} e^{w_1w_2w_3\left(\sum_{j=1}^{3-i} y_j\right)t} (q^{w_1w_2w_3} e^{w_1w_2w_3t} - 1)^i}{(q^{w_1} e^{w_1t} - 1)(q^{w_2} e^{w_2t} - 1)(q^{w_3} e^{w_3t} - 1)}. \tag{2.4}$$

(c-0) Type $\Lambda_{12}^0$:

$$I(\Lambda_{12}^0) = \int_{\mathbb{Z}_p^3} q^{w_1x_1+w_2x_2+w_3x_3} e^{(w_1x_1+w_2x_2+w_3x_3+w_2w_3y+w_1w_3y+w_1w_2y)t} d\mu(x_1)d\mu(x_2)d\mu(x_3) \tag{2.5}$$

$$= \frac{w_1w_2w_3(\log q + t)^3 e^{(w_2w_3+w_1w_3+w_1w_2)yt}}{(q^{w_1} e^{w_1t} - 1)(q^{w_2} e^{w_2t} - 1)(q^{w_3} e^{w_3t} - 1)}. \tag{2.6}$$

(c-1) Type $\Lambda_{12}^1$:

$$I(\Lambda_{12}^1) = \frac{\int_{\mathbb{Z}_p^3} q^{w_1x_1+w_2x_2+w_3x_3} e^{(w_1x_1+w_2x_2+w_3x_3)t} d\mu(x_1)d\mu(x_2)d\mu(x_3)}{\int_{\mathbb{Z}_p^3} q^{w_2w_3z_1+w_1w_3z_2+w_1w_2z_3} e^{(w_2w_3z_1+w_1w_3z_2+w_1w_2z_3)t} d\mu(z_1)d\mu(z_2)d\mu(z_3)} \tag{2.7}$$

$$= \frac{(w_1w_2w_3)^{-1}(q^{w_2w_3} e^{w_2w_3t} - 1)(q^{w_1w_3} e^{w_1w_3t} - 1)(q^{w_1w_2} e^{w_1w_2t} - 1)}{(q^{w_1} e^{w_1t} - 1)(q^{w_2} e^{w_2t} - 1)(q^{w_3} e^{w_3t} - 1)}. \tag{2.8}$$

All of the above $p$-adic integrals of various types are invariant under all permutations of $w_1, w_2, w_3$, as one can see either from $p$-adic integral representations in (2.1), (2.3), (2.5) and (2.7) or from their explicit evaluations in (2.2), (2.4), (2.6) and (2.8).

## 3. Identities for $q$-Bernoulli polynomials

First, let's consider Type $\Lambda_{23}^i$, for each $i = 0, 1, 2, 3$.

(a-0) The following results can be easily obtained from (1.4) and (1.7):

$$I(\Lambda_{23}^0) = \int_{\mathbb{Z}_p} q^{w_2w_3x_1} e^{w_2w_3(x_1+w_1y_1)t} d\mu(x_1) \int_{\mathbb{Z}_p} q^{w_1w_3x_2} e^{w_1w_3(x_2+w_2y_2)t} d\mu(x_2) \int_{\mathbb{Z}_p} q^{w_1w_2x_3} e^{w_1w_2(x_3+w_3y_3)t} d\mu(x_3)$$

$$= \left(\sum_{k=0}^{\infty} \frac{B_{k,q^{w_2w_3}}(w_1y_1)}{k!}(w_2w_3t)^k\right)\left(\sum_{l=0}^{\infty} \frac{B_{l,q^{w_1w_3}}(w_2y_2)}{l!}(w_1w_3t)^l\right)\left(\sum_{m=0}^{\infty} \frac{B_{m,q^{w_1w_2}}(w_3y_3)}{m!}(w_1w_2t)^m\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,q^{w_2w_3}}(w_1y_1)B_{l,q^{w_1w_3}}(w_2y_2)B_{m,q^{w_1w_2}}(w_3y_3) w_1^{l+m} w_2^{k+m} w_3^{k+l}\right)\frac{t^n}{n!}, \tag{3.1}$$

where the inner sum is over all nonnegative integers $k$, $l$, $m$, with $k+l+m=n$, and

$$\binom{n}{k,l,m}=\frac{n!}{k!l!m!}. \tag{3.2}$$

(a-1) Here we write $I(\Lambda_{23}^1)$ in two different ways:
(1)

$$\begin{aligned} I(\Lambda_{23}^1) &= \frac{1}{w_3}\int_{\mathbb{Z}_p} q^{w_2w_3x_1}e^{w_2w_3(x_1+w_1y_1)t}d\mu(x_1)\int_{\mathbb{Z}_p} q^{w_1w_3x_2}e^{w_1w_3(x_2+w_2y_2)t}d\mu(x_2)\frac{w_3\int_{\mathbb{Z}_p} q^{w_1w_2x_3}e^{w_1w_2x_3t}d\mu(x_3)}{\int_{\mathbb{Z}_p} q^{w_1w_2w_3x_4}e^{w_1w_2w_3x_4t}d\mu(x_4)} \\ &= \frac{1}{w_3}\left(\sum_{k=0}^{\infty}B_{k,q^{w_2w_3}}(w_1y_1)\frac{(w_2w_3t)^k}{k!}\right)\sum_{l=0}^{\infty}B_{l,q^{w_1w_3}}(w_2y_2)\frac{(w_1w_3t)^l}{l!}\left(\sum_{m=0}^{\infty}S_{m,q^{w_1w_2}}(w_3-1)\frac{(w_1w_2t)^m}{m!}\right) \quad (3.3) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k+l+m=n}\binom{n}{k,l,m}B_{k,q^{w_2w_3}}(w_1y_1)B_{l,q^{w_1w_3}}(w_2y_2)S_{m,q^{w_1w_2}}(w_3-1)w_1^{l+m}w_2^{k+m}w_3^{k+l-1}\right)\frac{t^n}{n!}. \quad (3.4) \end{aligned}$$

(2) Invoking (1.7), (3.3) can also be written as

$$\begin{aligned} I(\Lambda_{23}^1) &= \frac{1}{w_3}\sum_{i=0}^{w_3-1}q^{w_1w_2i}\int_{\mathbb{Z}_p} q^{w_2w_3x_1}e^{w_2w_3(x_1+w_1y_1)t}d\mu(x_1)\int_{\mathbb{Z}_p} q^{w_1w_3x_2}e^{w_1w_3\left(x_2+w_2y_2+\frac{w_2}{w_3}i\right)t}d\mu(x_2) \\ &= \frac{1}{w_3}\sum_{i=0}^{w_3-1}q^{w_1w_2i}\left(\sum_{k=0}^{\infty}B_{k,q^{w_2w_3}}(w_1y_1)\frac{(w_2w_3t)^k}{k!}\right)\left(\sum_{l=0}^{\infty}B_{l,q^{w_1w_3}}\left(w_2y_2+\frac{w_2}{w_3}i\right)\frac{(w_1w_3t)^l}{l!}\right) \\ &= \sum_{n=0}^{\infty}\left(w_3^{n-1}\sum_{k=0}^{n}\binom{n}{k}B_{k,q^{w_2w_3}}(w_1y_1)w_1^{n-k}w_2^k\sum_{i=0}^{w_3-1}q^{w_1w_2i}B_{n-k,q^{w_1w_3}}\left(w_2y_2+\frac{w_2}{w_3}i\right)\right)\frac{t^n}{n!}. \quad (3.5) \end{aligned}$$

(a-2) Here we write $I(\Lambda_{23}^2)$ in three different ways:
(1)

$$\begin{aligned} I(\Lambda_{23}^2) &= \frac{1}{w_2w_3}\int_{\mathbb{Z}_p} q^{w_2w_3x_1}e^{w_2w_3(x_1+w_1y_1)t}d\mu(x_1)\frac{w_2\int_{\mathbb{Z}_p} q^{w_1w_3x_2}e^{w_1w_3x_2t}d\mu(x_2)}{\int_{\mathbb{Z}_p} q^{w_1w_2w_3x_4}e^{w_1w_2w_3x_4t}d\mu(x_4)}\frac{w_3\int_{\mathbb{Z}_p} q^{w_1w_2x_3}e^{w_1w_2x_3t}d\mu(x_3)}{\int_{\mathbb{Z}_p} q^{w_1w_2w_3x_4}e^{w_1w_2w_3x_4t}d\mu(x_4)} \\ &= \frac{1}{w_2w_3}\left(\sum_{k=0}^{\infty}B_{k,q^{w_2w_3}}(w_1y_1)\frac{(w_2w_3t)^k}{k!}\right)\left(\sum_{l=0}^{\infty}S_{l,q^{w_1w_3}}(w_2-1)\frac{(w_1w_3t)^l}{l!}\right) \\ &\quad\times\left(\sum_{m=0}^{\infty}S_{m,q^{w_1w_2}}(w_3-1)\frac{(w_1w_2t)^m}{m!}\right) \quad (3.6) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k+l+m=n}\binom{n}{k,l,m}B_{k,q^{w_2w_3}}(w_1y_1)S_{l,q^{w_1w_3}}(w_2-1)S_{m,q^{w_1w_2}}(w_3-1)w_1^{l+m}w_2^{k+m-1}w_3^{k+l-1}\right)\frac{t^n}{n!}. \quad (3.7) \end{aligned}$$

(2) Invoking (1.7), (3.6) can also be written as

$$\begin{aligned} I(\Lambda_{23}^2) &= \frac{1}{w_2w_3}\sum_{i=0}^{w_2-1}q^{w_1w_3i}\int_{\mathbb{Z}_p} q^{w_2w_3x_1}e^{w_2w_3\left(x_1+w_1y_1+\frac{w_1}{w_2}i\right)t}d\mu(x_1)\frac{w_2\int_{\mathbb{Z}_p} q^{w_1w_2x_3}e^{w_1w_2x_3t}d\mu(x_3)}{\int_{\mathbb{Z}_p} q^{w_1w_2w_3x_4}e^{w_1w_2w_3x_4t}d\mu(x_4)} \\ &= \frac{1}{w_2w_3}\sum_{i=0}^{w_2-1}q^{w_1w_3i}\left(\sum_{k=0}^{\infty}B_{k,q^{w_2w_3}}\left(w_1y_1+\frac{w_1}{w_2}i\right)\frac{(w_2w_3t)^k}{k!}\right)\left(\sum_{l=0}^{\infty}S_{l,q^{w_1w_2}}(w_3-1)\frac{(w_1w_2t)^l}{l!}\right) \quad (3.8) \\ &= \sum_{n=0}^{\infty}\left(w_2^{n-1}\sum_{k=0}^{n}\binom{n}{k}S_{n-k,q^{w_1w_2}}(w_3-1)w_1^{n-k}w_3^{k-1}\sum_{i=0}^{w_2-1}q^{w_1w_3i}B_{k,q^{w_2w_3}}\left(w_1y_1+\frac{w_1}{w_2}i\right)\right)\frac{t^n}{n!}. \quad (3.9) \end{aligned}$$

(3) Invoking (1.7) once again, (3.8) can be written as

$$I(\Lambda_{23}^2) = \frac{1}{w_2w_3}\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_3-1}q^{w_1w_3i+w_1w_2j}\int_{\mathbb{Z}_p} q^{w_2w_3x_1}e^{w_2w_3\left(x_1+w_1y_1+\frac{w_1}{w_2}i+\frac{w_1}{w_3}j\right)t}d\mu(x_1)$$

$$= \frac{1}{w_2 w_3} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} \left( q^{w_1 w_3 i + w_1 w_2 j} \sum_{n=0}^{\infty} B_{n,q^{w_2 w_3}} \left( w_1 y_1 + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j \right) \frac{(w_2 w_3 t)^n}{n!} \right)$$

$$= \sum_{n=0}^{\infty} \left( (w_2 w_3)^{n-1} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} q^{w_1 w_3 i + w_1 w_2 j} B_{n,q^{w_2 w_3}} \left( w_1 y_1 + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j \right) \right) \frac{t^n}{n!}. \tag{3.10}$$

(a-3):

$$I(\Lambda_{23}^3) = \frac{1}{w_1 w_2 w_3} \frac{w_1 \int_{\mathbb{Z}_p} q^{w_2 w_3 x_1} e^{w_2 w_3 x_1 t} d\mu(x_1)}{\int_{\mathbb{Z}_p} q^{w_1 w_2 w_3 x_4} e^{w_1 w_2 w_3 x_4 t} d\mu(x_4)} \frac{w_2 \int_{\mathbb{Z}_p} q^{w_1 w_3 x_2} e^{w_1 w_3 x_2 t} d\mu(x_2)}{\int_{\mathbb{Z}_p} q^{w_1 w_2 w_3 x_4} e^{w_1 w_2 w_3 x_4 t} d\mu(x_4)}$$
$$\times \frac{w_3 \int_{\mathbb{Z}_p} q^{w_1 w_2 x_3} e^{w_1 w_2 x_3 t} d\mu(x_3)}{\int_{\mathbb{Z}_p} q^{w_1 w_2 w_3 x_4} e^{w_1 w_2 w_3 x_4 t} d\mu(x_4)}$$
$$= \frac{1}{w_1 w_2 w_3} \left( \sum_{k=0}^{\infty} S_{k,q^{w_2 w_3}}(w_1 - 1) \frac{(w_2 w_3 t)^k}{k!} \right) \left( \sum_{l=0}^{\infty} S_{l,q^{w_1 w_3}}(w_2 - 1) \frac{(w_1 w_3 t)^l}{l!} \right)$$
$$\times \left( \sum_{m=0}^{\infty} S_{m,q^{w_1 w_2}}(w_3 - 1) \frac{(w_1 w_2 t)^m}{m!} \right)$$
$$= \sum_{n=0}^{\infty} \left( \sum_{k+l+m=n} \binom{n}{k,l,m} S_{k,q^{w_2 w_3}}(w_1 - 1) S_{l,q^{w_1 w_3}}(w_2 - 1) \right.$$
$$\left. \times S_{m,q^{w_1 w_2}}(w_3 - 1) w_1^{l+m-1} w_2^{k+m-1} w_3^{k+l-1} \right) \frac{t^n}{n!}. \tag{3.11}$$

(b) For Type $\Lambda_{13}^i$ ($i = 0, 1, 2, 3$), we may consider analogous things to the ones in (a-0), (a-1), (a-2), and (a-3). However, these do not lead us to new identities. Indeed, if we substitute $w_2 w_3, w_1 w_3, w_1 w_2$ respectively for $w_1, w_2, w_3$ in (2.1), this amounts to replacing $t$ by $w_1 w_2 w_3 t$ and $q$ by $q^{w_1 w_2 w_3}$ in (2.3). So, upon replacing $w_1, w_2, w_3$ respectively by $w_2 w_3, w_1 w_3, w_1 w_2$ and dividing by $(w_1 w_2 w_3)^n$, in each of the expressions (3.1), (3.4), (3.5), (3.7), (3.9)–(3.11), we will get the corresponding symmetric identities for Type $\Lambda_{13}^i$ ($i = 0, 1, 2, 3$).

(c-0):

$$I(\Lambda_{12}^0) = \int_{\mathbb{Z}_p} q^{w_1 x_1} e^{w_1 (x_1 + w_2 y) t} d\mu(x_1) \int_{\mathbb{Z}_p} q^{w_2 x_2} e^{w_2 (x_2 + w_3 y) t} d\mu(x_2) \int_{\mathbb{Z}_p} q^{w_3 x_3} e^{w_3 (x_3 + w_1 y) t} d\mu(x_3)$$
$$= \left( \sum_{n=0}^{\infty} \frac{B_{k,q^{w_1}}(w_2 y)}{k!} (w_1 t)^k \right) \left( \sum_{l=0}^{\infty} \frac{B_{l,q^{w_2}}(w_3 y)}{l!} (w_2 t)^l \right) \left( \sum_{m=0}^{\infty} \frac{B_{m,q^{w_3}}(w_1 y)}{m!} (w_3 t)^m \right)$$
$$= \sum_{n=0}^{\infty} \left( \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,q^{w_1}}(w_2 y) B_{l,q^{w_2}}(w_3 y) B_{m,q^{w_3}}(w_1 y) w_1^k w_2^l w_3^m \right) \frac{t^n}{n!}. \tag{3.12}$$

(c-1):

$$I(\Lambda_{12}^1) = \frac{1}{w_1 w_2 w_3} \frac{w_2 \int_{\mathbb{Z}_p} q^{w_1 x_1} e^{w_1 x_1 t} d\mu(x_1)}{\int_{\mathbb{Z}_p} q^{w_1 w_2 z_3} e^{w_1 w_2 z_3 t} d\mu(z_3)} \frac{w_3 \int_{\mathbb{Z}_p} q^{w_2 x_2} e^{w_2 x_2 t} d\mu(x_2)}{\int_{\mathbb{Z}_p} q^{w_2 w_3 z_1} e^{w_2 w_3 z_1 t} d\mu(z_1)} \frac{w_1 \int_{\mathbb{Z}_p} q^{w_3 x_3} e^{w_3 x_3 t} d\mu(x_3)}{\int_{\mathbb{Z}_p} q^{w_3 w_1 z_2} e^{w_3 w_1 z_2 t} d\mu(z_2)}$$
$$= \frac{1}{w_1 w_2 w_3} \left( \sum_{k=0}^{\infty} S_{k,q^{w_1}}(w_2 - 1) \frac{(w_1 t)^k}{k!} \right) \left( \sum_{l=0}^{\infty} S_{l,q^{w_2}}(w_3 - 1) \frac{(w_2 t)^l}{l!} \right) \left( \sum_{m=0}^{\infty} S_{m,q^{w_3}}(w_1 - 1) \frac{(w_3 t)^m}{m!} \right)$$
$$= \sum_{n=0}^{\infty} \left( \sum_{k+l+m=n} \binom{n}{k,l,m} S_{k,q^{w_1}}(w_2 - 1) S_{l,q^{w_2}}(w_3 - 1) S_{m,q^{w_3}}(w_1 - 1) w_1^{k-1} w_2^{l-1} w_3^{m-1} \right) \frac{t^n}{n!}. \tag{3.13}$$

## 4. Main theorems

As we noted earlier in the last paragraph of Section 2, the various quotient types for Volkenborn integrals are invariant under any permutation of $w_1, w_2, w_3$. So the corresponding expressions in Section 3 are also invariant under any permutation of $w_1, w_2, w_3$. Thus our results concerning identities of symmetry will be immediate consequences of this observation.

However, not all permutations of an expression in Section 3 yield distinct ones. In fact, as these expressions are obtained by permuting $w_1, w_2, w_3$ in a single case labeled by them, they can be viewed as a group in a natural manner and hence it is isomorphic to a quotient of $S_3$. In particular, the number of possible distinct expressions are 1, 2, 3, or 6. (a-0), (a-1(1)), (a-1(2)), and (a-2(2)) give the full six identities of symmetry, (a-2(1)) and (a-2(3)) yield three identities of symmetry, and (c-0) and (c-1) give two identities of symmetry, while the expression in (a-3) yields no identities of symmetry.

Here we will just consider the cases of Theorems 4.8 and 4.17, leaving the others as easy exercises for the reader. As for the case of Theorem 4.8, in addition to (4.14)–(4.16), we get the following three cases:

$$\sum_{k+l+m=n}\binom{n}{k,l,m}B_{k,q^{w_2w_3}}(w_1y_1)S_{l,q^{w_1w_2}}(w_3-1)S_{m,q^{w_1w_3}}(w_2-1)w_1^{l+m}w_3^{k+m-1}w_2^{k+l-1}, \tag{4.1}$$

$$\sum_{k+l+m=n}\binom{n}{k,l,m}B_{k,q^{w_1w_3}}(w_2y_1)S_{l,q^{w_2w_3}}(w_1-1)S_{m,q^{w_1w_2}}(w_3-1)w_2^{l+m}w_1^{k+m-1}w_3^{k+l-1}, \tag{4.2}$$

$$\sum_{k+l+m=n}\binom{n}{k,l,m}B_{k,q^{w_1w_2}}(w_3y_1)S_{l,q^{w_1w_3}}(w_2-1)S_{m,q^{w_2w_3}}(w_1-1)w_3^{l+m}w_2^{k+m-1}w_1^{k+l-1}. \tag{4.3}$$

But, by interchanging $l$ and $m$, we see that (4.1)–(4.3) are respectively equal to (4.14)–(4.16).

As regards Theorem 4.17, in addition to (4.24) and (4.25), we have

$$\sum_{k+l+m=n}\binom{n}{k,l,m}S_{k,q^{w_1}}(w_2-1)S_{l,q^{w_2}}(w_3-1)S_{m,q^{w_3}}(w_1-1)w_1^{k-1}w_2^{l-1}w_3^{m-1}, \tag{4.4}$$

$$\sum_{k+l+m=n}\binom{n}{k,l,m}S_{k,q^{w_2}}(w_3-1)S_{l,q^{w_3}}(w_1-1)S_{m,q^{w_1}}(w_2-1)w_2^{k-1}w_3^{l-1}w_1^{m-1}, \tag{4.5}$$

$$\sum_{k+l+m=n}\binom{n}{k,l,m}S_{k,q^{w_1}}(w_3-1)S_{l,q^{w_3}}(w_2-1)S_{m,q^{w_2}}(w_1-1)w_1^{k-1}w_3^{l-1}w_2^{m-1}, \tag{4.6}$$

$$\sum_{k+l+m=n}\binom{n}{k,l,m}S_{k,q^{w_3}}(w_2-1)S_{l,q^{w_2}}(w_1-1)S_{m,q^{w_1}}(w_3-1)w_3^{k-1}w_2^{l-1}w_1^{m-1}. \tag{4.7}$$

However, (4.4) and (4.5) are equal to (4.24), as we can see by applying the permutations $k\to l, l\to m, m\to k$ for (4.4) and $k\to m, l\to k, m\to l$ for (4.5). Similarly, we see that (4.6) and (4.7) are equal to (4.25), by applying permutations $k\to l, l\to m, m\to k$ for (4.6) and $k\to m, l\to k, m\to l$ for (4.7).

**Theorem 4.1.** *Let $w_1, w_2, w_3$ be any positive integers. Then the following expression is invariant under any permutation of $w_1, w_2, w_3$, so it gives us six symmetries:*

$$\begin{aligned}
&\sum_{k+l+m=n}\binom{n}{k,l,m}B_{k,q^{w_2w_3}}(w_1y_1)B_{l,q^{w_1w_3}}(w_2y_2)B_{m,q^{w_1w_2}}(w_3y_3)w_1^{l+m}w_2^{k+m}w_3^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}B_{k,q^{w_2w_3}}(w_1y_1)B_{l,q^{w_1w_2}}(w_3y_2)B_{m,q^{w_1w_3}}(w_2y_3)w_1^{l+m}w_3^{k+m}w_2^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}B_{k,q^{w_1w_3}}(w_2y_1)B_{l,q^{w_2w_3}}(w_1y_2)B_{m,q^{w_1w_2}}(w_3y_3)w_2^{l+m}w_1^{k+m}w_3^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}B_{k,q^{w_1w_3}}(w_2y_1)B_{l,q^{w_1w_2}}(w_3y_2)B_{m,q^{w_2w_3}}(w_1y_3)w_2^{l+m}w_3^{k+m}w_1^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}B_{k,q^{w_1w_2}}(w_3y_1)B_{l,q^{w_2w_3}}(w_1y_2)B_{m,q^{w_1w_3}}(w_2y_3)w_3^{l+m}w_1^{k+m}w_2^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}B_{k,q^{w_1w_2}}(w_3y_1)B_{l,q^{w_1w_3}}(w_2y_2)B_{m,q^{w_2w_3}}(w_1y_3)w_3^{l+m}w_2^{k+m}w_1^{k+l}.
\end{aligned} \tag{4.8}$$

**Theorem 4.2.** *Let $w_1, w_2, w_3$ be any positive integers. Then the following expression is invariant under any permutation of $w_1, w_2, w_3$, so it gives us six symmetries:*

$$\sum_{k+l+m=n}\binom{n}{k,l,m}B_{k,q^{w_2w_3}}(w_1y_1)B_{l,q^{w_1w_3}}(w_2y_2)S_{m,q^{w_1w_2}}(w_3-1)w_1^{l+m}w_2^{k+m}w_3^{k+l-1}$$

$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,q^{w_2w_3}}(w_1y_1) B_{l,q^{w_1w_2}}(w_3y_2) S_{m,q^{w_1w_3}}(w_2-1) w_1^{l+m} w_3^{k+m} w_2^{k+l-1}$$
$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,q^{w_1w_3}}(w_2y_1) B_{l,q^{w_2w_3}}(w_1y_2) S_{m,q^{w_1w_2}}(w_3-1) w_2^{l+m} w_1^{k+m} w_3^{k+l-1}$$
$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,q^{w_1w_3}}(w_2y_1) B_{l,q^{w_1w_2}}(w_3y_2) S_{m,q^{w_2w_3}}(w_1-1) w_2^{l+m} w_3^{k+m} w_1^{k+l-1}$$
$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,q^{w_1w_2}}(w_3y_1) B_{l,q^{w_1w_3}}(w_2y_2) S_{m,q^{w_2w_3}}(w_1-1) w_3^{l+m} w_2^{k+m} w_1^{k+l-1}$$
$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,q^{w_1w_2}}(w_3y_1) B_{l,q^{w_2w_3}}(w_1y_2) S_{m,q^{w_1w_3}}(w_2-1) w_3^{l+m} w_1^{k+m} w_2^{k+l-1}. \quad (4.9)$$

Putting $w_3 = 1$ in (4.9), we get the following corollary.

**Corollary 4.3.** *Let $w_1, w_2$ be any positive integers.*

$$\sum_{k=0}^{n} \binom{n}{k} B_{k,q^{w_2}}(w_1y_1) B_{n-k,q^{w_1}}(w_2y_2) w_1^{n-k} w_2^{k} = \sum_{k=0}^{n} \binom{n}{k} B_{k,q^{w_1}}(w_2y_1) B_{n-k,q^{w_2}}(w_1y_2) w_2^{n-k} w_1^{k}$$
$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,q^{w_1w_2}}(y_1) B_{l,q^{w_1}}(w_2y_2) S_{m,q^{w_2}}(w_1-1) w_2^{k+m} w_1^{k+l-1}$$
$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,q^{w_1}}(w_2y_1) B_{l,q^{w_1w_2}}(y_2) S_{m,q^{w_2}}(w_1-1) w_2^{l+m} w_1^{k+l-1}$$
$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,q^{w_1w_2}}(y_1) B_{l,q^{w_2}}(w_1y_2) S_{m,q^{w_1}}(w_2-1) w_1^{k+m} w_2^{k+l-1}$$
$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,q^{w_2}}(w_1y_1) B_{l,q^{w_1w_2}}(y_2) S_{m,q^{w_1}}(w_2-1) w_1^{l+m} w_2^{k+l-1}. \quad (4.10)$$

Letting further $w_2 = 1$ in (4.10), we have the following corollary.

**Corollary 4.4.** *Let $w_1$ be any positive integer.*

$$\sum_{k=0}^{n} \binom{n}{k} B_{k,q}(w_1y_1) B_{n-k,q^{w_1}}(y_2) w_1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} B_{k,q^{w_1}}(y_1) B_{n-k,q}(w_1y_2) w_1^{k}$$
$$= \sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,q^{w_1}}(y_1) B_{l,q^{w_1}}(y_2) S_{m,q}(w_1-1) w_1^{k+l-1}. \quad (4.11)$$

**Theorem 4.5.** *Let $w_1, w_2, w_3$ be any positive integers. Then the following expression is invariant under any permutation of $w_1, w_2, w_3$, so it gives us six symmetries:*

$$w_1^{n-1} \sum_{k=0}^{n} \binom{n}{k} B_{k,q^{w_1w_2}}(w_3y_1) w_3^{n-k} w_2^{k} \sum_{i=0}^{w_1-1} q^{w_2w_3i} B_{n-k,q^{w_1w_3}}\left(w_2y_2 + \frac{w_2}{w_1}i\right)$$
$$= w_1^{n-1} \sum_{k=0}^{n} \binom{n}{k} B_{k,q^{w_1w_3}}(w_2y_1) w_2^{n-k} w_3^{k} \sum_{i=0}^{w_1-1} q^{w_2w_3i} B_{n-k,q^{w_1w_2}}\left(w_3y_2 + \frac{w_3}{w_1}i\right)$$
$$= w_2^{n-1} \sum_{k=0}^{n} \binom{n}{k} B_{k,q^{w_1w_2}}(w_3y_1) w_3^{n-k} w_1^{k} \sum_{i=0}^{w_2-1} q^{w_1w_3i} B_{n-k,q^{w_2w_3}}\left(w_1y_2 + \frac{w_1}{w_2}i\right)$$
$$= w_2^{n-1} \sum_{k=0}^{n} \binom{n}{k} B_{k,q^{w_2w_3}}(w_1y_1) w_1^{n-k} w_3^{k} \sum_{i=0}^{w_2-1} q^{w_1w_3i} B_{n-k,q^{w_1w_2}}\left(w_3y_2 + \frac{w_3}{w_2}i\right)$$

$$= w_3^{n-1}\sum_{k=0}^{n}\binom{n}{k}B_{k,q^{w_1w_3}}(w_2y_1)w_2^{n-k}w_1^k\sum_{i=0}^{w_3-1}q^{w_1w_2i}B_{n-k,q^{w_2w_3}}\left(w_1y_2+\frac{w_1}{w_3}i\right)$$

$$= w_3^{n-1}\sum_{k=0}^{n}\binom{n}{k}B_{k,q^{w_2w_3}}(w_1y_1)w_1^{n-k}w_2^k\sum_{i=0}^{w_3-1}q^{w_1w_2i}B_{n-k,q^{w_1w_3}}\left(w_2y_2+\frac{w_2}{w_3}i\right). \tag{4.12}$$

Letting $w_3 = 1$ in (4.12), we obtain alternative expressions for the identities in (4.10).

**Corollary 4.6.** *Let $w_1, w_2$ be any positive integers.*

$$\sum_{k=0}^{n}\binom{n}{k}B_{k,q^{w_2}}(w_1y_1)B_{n-k,q^{w_1}}(w_2y_2)w_1^{n-k}w_2^k = \sum_{k=0}^{n}\binom{n}{k}B_{k,q^{w_1}}(w_2y_1)B_{n-k,q^{w_2}}(w_1y_2)w_2^{n-k}w_1^k$$

$$= w_1^{n-1}\sum_{k=0}^{n}\binom{n}{k}B_{k,q^{w_1w_2}}(y_1)w_2^k\sum_{i=0}^{w_1-1}q^{w_2i}B_{n-k,q^{w_1}}\left(w_2y_2+\frac{w_2}{w_1}i\right)$$

$$= w_1^{n-1}\sum_{k=0}^{n}\binom{n}{k}B_{k,q^{w_1}}(w_2y_1)w_2^{n-k}\sum_{i=0}^{w_1-1}q^{w_2i}B_{n-k,q^{w_1w_2}}\left(y_2+\frac{i}{w_1}\right)$$

$$= w_2^{n-1}\sum_{k=0}^{n}\binom{n}{k}B_{k,q^{w_1w_2}}(y_1)w_1^k\sum_{i=0}^{w_2-1}q^{w_1i}B_{n-k,q^{w_2}}\left(w_1y_2+\frac{w_1}{w_2}i\right)$$

$$= w_2^{n-1}\sum_{k=0}^{n}\binom{n}{k}B_{k,q^{w_2}}(w_1y_1)w_1^{n-k}\sum_{i=0}^{w_2-1}q^{w_1i}B_{n-k,q^{w_1w_2}}\left(y_2+\frac{i}{w_2}\right). \tag{4.13}$$

Putting further $w_2 = 1$ in (4.13), we have the alternative expressions for the identities for (4.11).

**Corollary 4.7.** *Let $w_1$ be any positive integer.*

$$\sum_{k=0}^{n}\binom{n}{k}B_{k,q^{w_1}}(y_1)B_{n-k,q}(w_1y_2)w_1^k = \sum_{k=0}^{n}\binom{n}{k}B_{k,q^{w_1}}(y_2)B_{n-k,q}(w_1y_1)w_1^k$$

$$= w_1^{n-1}\sum_{k=0}^{n}\binom{n}{k}B_{k,q^{w_1}}(y_1)\sum_{i=0}^{w_1-1}q^iB_{n-k,q^{w_1}}\left(y_2+\frac{i}{w_1}\right).$$

**Theorem 4.8.** *Let $w_1, w_2, w_3$ be any positive integers. Then we have the following three symmetries in $w_1, w_2, w_3$:*

$$\sum_{k+l+m=n}\binom{n}{k,l,m}B_{k,q^{w_2w_3}}(w_1y_1)S_{l,q^{w_1w_3}}(w_2-1)S_{m,q^{w_1w_2}}(w_3-1)w_1^{l+m}w_2^{k+m-1}w_3^{k+l-1} \tag{4.14}$$

$$= \sum_{k+l+m=n}\binom{n}{k,l,m}B_{k,q^{w_1w_3}}(w_2y_1)S_{l,q^{w_1w_2}}(w_3-1)S_{m,q^{w_2w_3}}(w_1-1)w_2^{l+m}w_3^{k+m-1}w_1^{k+l-1} \tag{4.15}$$

$$= \sum_{k+l+m=n}\binom{n}{k,l,m}B_{k,q^{w_1w_2}}(w_3y_1)S_{l,q^{w_2w_3}}(w_1-1)S_{m,q^{w_1w_3}}(w_2-1)w_3^{l+m}w_1^{k+m-1}w_2^{k+l-1}. \tag{4.16}$$

Putting $w_3 = 1$ in (4.14)–(4.16), we get the following corollary.

**Corollary 4.9.** *Let $w_1, w_2$ be any positive integers.*

$$\sum_{k=0}^{n}\binom{n}{k}B_{k,q^{w_2}}(w_1y_1)S_{n-k,q^{w_1}}(w_2-1)w_1^{n-k}w_2^{k-1} = \sum_{k=0}^{n}\binom{n}{k}B_{k,q^{w_1}}(w_2y_1)S_{n-k,q^{w_2}}(w_1-1)w_2^{n-k}w_1^{k-1}$$

$$= \sum_{k+l+m=n}\binom{n}{k,l,m}B_{k,q^{w_1w_2}}(y_1)S_{l,q^{w_2}}(w_1-1)S_{m,q^{w_1}}(w_2-1)w_1^{k+m-1}w_2^{k+l-1}. \tag{4.17}$$

Letting further $w_2 = 1$ in (4.17), we get the following corollary. This is also obtained in [2, (2.22)].

**Corollary 4.10.** *Let $w_1$ be any positive integer.*

$$B_{n,q}(w_1y_1) = \sum_{k=0}^{n}\binom{n}{k}B_{k,q^{w_1}}(y_1)S_{n-k,q}(w_1-1)w_1^{k-1}. \tag{4.18}$$

**Theorem 4.11.** *Let $w_1, w_2, w_3$ be any positive integers. Then the following expression is invariant under any permutation of $w_1, w_2, w_3$, so it gives us six symmetries:*

$$\begin{aligned}
&w_1^{n-1}\sum_{k=0}^{n}\binom{n}{k}S_{n-k,q^{w_1w_2}}(w_3-1)w_2^{n-k}w_3^{k-1}\sum_{i=0}^{w_1-1}q^{w_2w_3i}B_{k,q^{w_1w_3}}\left(w_2y_1+\frac{w_2}{w_1}i\right)\\
&= w_1^{n-1}\sum_{k=0}^{n}\binom{n}{k}S_{n-k,q^{w_1w_3}}(w_2-1)w_3^{n-k}w_2^{k-1}\sum_{i=0}^{w_1-1}q^{w_2w_3i}B_{k,q^{w_1w_2}}\left(w_3y_1+\frac{w_3}{w_1}i\right)\\
&= w_2^{n-1}\sum_{k=0}^{n}\binom{n}{k}S_{n-k,q^{w_1w_2}}(w_3-1)w_1^{n-k}w_3^{k-1}\sum_{i=0}^{w_2-1}q^{w_1w_3i}B_{k,q^{w_2w_3}}\left(w_1y_1+\frac{w_1}{w_2}i\right)\\
&= w_2^{n-1}\sum_{k=0}^{n}\binom{n}{k}S_{n-k,q^{w_2w_3}}(w_1-1)w_3^{n-k}w_1^{k-1}\sum_{i=0}^{w_2-1}q^{w_1w_3i}B_{k,q^{w_1w_2}}\left(w_3y_1+\frac{w_3}{w_1}i\right)\\
&= w_3^{n-1}\sum_{k=0}^{n}\binom{n}{k}S_{n-k,q^{w_1w_3}}(w_2-1)w_1^{n-k}w_2^{k-1}\sum_{i=0}^{w_3-1}q^{w_1w_2i}B_{k,q^{w_2w_3}}\left(w_1y_1+\frac{w_1}{w_3}i\right)\\
&= w_3^{n-1}\sum_{k=0}^{n}\binom{n}{k}S_{n-k,q^{w_2w_3}}(w_1-1)w_2^{n-k}w_1^{k-1}\sum_{i=0}^{w_3-1}q^{w_1w_2i}B_{k,q^{w_1w_3}}\left(w_2y_1+\frac{w_2}{w_3}i\right).
\end{aligned} \tag{4.19}$$

Putting $w_3 = 1$ in (4.19), we obtain the following corollary. In Section 1, the identities in (4.17), (4.20) and (4.22) are combined to give those in (1.8)–(1.15).

**Corollary 4.12.** *Let $w_1, w_2$ be any positive integers.*

$$\begin{aligned}
w_1^{n-1}\sum_{i=0}^{w_1-1}q^{w_2i}B_{n,q^{w_1}}\left(w_2y_1+\frac{w_2}{w_1}i\right) &= w_2^{n-1}\sum_{i=0}^{w_2-1}q^{w_1i}B_{n,q^{w_2}}\left(w_1y_1+\frac{w_1}{w_2}i\right)\\
&= \sum_{k=0}^{n}\binom{n}{k}B_{k,q^{w_1}}(w_2y_1)S_{n-k,q^{w_2}}(w_1-1)w_2^{n-k}w_1^{k-1}\\
&= \sum_{k=0}^{n}\binom{n}{k}B_{k,q^{w_2}}(w_1y_1)S_{n-k,q^{w_1}}(w_2-1)w_1^{n-k}w_2^{k-1}\\
&= w_1^{n-1}\sum_{k=0}^{n}\binom{n}{k}S_{n-k,q^{w_1}}(w_2-1)w_2^{k-1}\sum_{i=0}^{w_1-1}q^{w_2i}B_{k,q^{w_1w_2}}\left(y_1+\frac{i}{w_1}\right)\\
&= w_2^{n-1}\sum_{k=0}^{n}\binom{n}{k}S_{n-k,q^{w_2}}(w_1-1)w_1^{k-1}\sum_{i=0}^{w_2-1}q^{w_1i}B_{k,q^{w_1w_2}}\left(y_1+\frac{i}{w_2}\right).
\end{aligned} \tag{4.20}$$

Letting further $w_2 = 1$ in (4.20), we get the following corollary. This is the multiplication formula for $q$-Bernoulli polynomials (cf. [2, (2.26)]) together with the relatively new identity mentioned in (4.18).

**Corollary 4.13.** *Let $w_1$ be any positive integer.*

$$\begin{aligned}
B_{n,q}(w_1y_1) &= w_1^{n-1}\sum_{i=0}^{w_1-1}q^{i}B_{n,q^{w_1}}\left(y_1+\frac{i}{w_1}\right)\\
&= \sum_{k=0}^{n}\binom{n}{k}B_{k,q^{w_1}}(y_1)S_{n-k,q}(w_1-1)w_1^{k-1}.
\end{aligned}$$

**Theorem 4.14.** *Let $w_1, w_2, w_3$ be any positive integers. Then we have the following three symmetries in $w_1, w_2, w_3$:*

$$\begin{aligned}
&(w_1w_2)^{n-1}\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1} q^{w_2w_3i+w_1w_3j}B_{n,q^{w_1w_2}}\left(w_3y_1+\frac{w_3}{w_1}i+\frac{w_3}{w_2}j\right)\\
&=(w_2w_3)^{n-1}\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_3-1} q^{w_1w_3i+w_1w_2j}B_{n,q^{w_2w_3}}\left(w_1y_1+\frac{w_1}{w_2}i+\frac{w_1}{w_3}j\right)\\
&=(w_3w_1)^{n-1}\sum_{i=0}^{w_3-1}\sum_{j=0}^{w_1-1} q^{w_1w_2i+w_2w_3j}B_{n,q^{w_1w_3}}\left(w_2y_1+\frac{w_2}{w_3}i+\frac{w_2}{w_1}j\right).
\end{aligned} \tag{4.21}$$

Letting $w_3 = 1$ in (4.21), we have the following corollary.

**Corollary 4.15.** *Let $w_1, w_2$ be any positive integers.*

$$\begin{aligned}
w_1^{n-1}\sum_{j=0}^{w_1-1} q^{w_2j}B_{n,q^{w_1}}\left(w_2y_1+\frac{w_2}{w_1}j\right) &= w_2^{n-1}\sum_{i=0}^{w_2-1} q^{w_1i}B_{n,q^{w_2}}\left(w_1y_1+\frac{w_1}{w_2}i\right)\\
&=(w_1w_2)^{n-1}\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1} q^{w_2i+w_1j}B_{n,q^{w_1w_2}}\left(y_1+\frac{i}{w_1}+\frac{j}{w_2}\right).
\end{aligned} \tag{4.22}$$

**Theorem 4.16.** *Let $w_1, w_2, w_3$ be any positive integers. Then we have the following two symmetries in $w_1, w_2, w_3$:*

$$\begin{aligned}
&\sum_{k+l+m=n}\binom{n}{k,\,l,\,m} B_{k,q^{w_3}}(w_1y)B_{l,q^{w_1}}(w_2y)B_{m,q^{w_2}}(w_3y)w_3^kw_1^lw_2^m\\
&=\sum_{k+l+m=n}\binom{n}{k,\,l,\,m} B_{k,q^{w_2}}(w_1y)B_{l,q^{w_1}}(w_3y)B_{m,q^{w_3}}(w_2y)w_2^kw_1^lw_3^m.
\end{aligned} \tag{4.23}$$

**Theorem 4.17.** *Let $w_1, w_2, w_3$ be any positive integers. Then we have the following two symmetries in $w_1, w_2, w_3$:*

$$\sum_{k+l+m=n}\binom{n}{k,\,l,\,m} S_{k,q^{w_3}}(w_1-1)S_{l,q^{w_1}}(w_2-1)S_{m,q^{w_2}}(w_3-1)w_3^{k-1}w_1^{l-1}w_2^{m-1} \tag{4.24}$$

$$=\sum_{k+l+m=n}\binom{n}{k,\,l,\,m} S_{k,q^{w_2}}(w_1-1)S_{l,q^{w_1}}(w_3-1)S_{m,q^{w_3}}(w_2-1)w_2^{k-1}w_1^{l-1}w_3^{m-1}. \tag{4.25}$$

Putting $w_3 = 1$ in (4.24) and (4.25) and multiplying the resulting identity by $w_1w_2$, we get the following corollary.

**Corollary 4.18.** *Let $w_1, w_2$ be any positive integers.*

$$\sum_{k=0}^{n}\binom{n}{k} S_{k,q^{w_1}}(w_2-1)S_{n-k,q}(w_1-1)w_1^k = \sum_{k=0}^{n}\binom{n}{k} S_{k,q^{w_2}}(w_1-1)S_{n-k,q}(w_2-1)w_2^k.$$

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