The present paper is a contribution to the problem when the Hewitt realcompactification commutes with products. The results published below have a sense only in the case measurable cardinals exist. But in spite of this fact the results are of some interest; they indicate certain connections with the compact case $\beta(P \times Q)$ and complete in a way papers [2], [3] by Comfort and Negrepontis.

In the first section properties are proved being equivalent to pseudo-m-compactness and generalizing those known for pseudocompactness (m stands for the first measurable cardinal and also for the first corresponding ordinal). A space playing the same role in pseudo-m-compactness as real numbers in pseudocompactness is described.

The second section deals with the equality $v(P \times Q) = vP \times vQ$ in special cases. We find out in Theorem 5 that partial analogy of our realcompact case with the Glicksberg theorem on $\beta(P \times Q)$ (see [5], [7]) holds. Some results of this section were published without proofs in the preliminary communication [10]. Further results of this sort concerning function spaces are prepared.

All the spaces under consideration are supposed to be uniformizable Hausdorff. The terminology of [1] and [6] is used.

§ 1.

We shall need some facts about pseudo-m-compactness which was introduced by Isbell in [11]). We shall use the following equivalent definition—see [14] (a family of sets in a space is called discrete if each point of the space has a neighborhood intersecting at most one member of the family):

Definition 1. A space $P$ is said to be *pseudo-m-compact* if each discrete family of open sets in $P$ is of nonmeasurable cardinal.

This equivalent definition (and further ones—see e.g. [8]) is, in fact,
of covering character. What we need is an analogy with defining of pseudocompactness by means of the space $R$ of real numbers. We shall try to give such characterizations. First we shall describe a space playing similar role in pseudo-$m$-compactness as $R$ in pseudocompactness (it is a star space from [12], p. 190).

Denote by $S$ the following metric space: The underlying set equals to a quotient $(M \times I)/r$, where $I$ is the closed unit interval $[0, 1]$, $M$ is a discrete space of cardinality $m$ and the equivalence $r$ is a union of the identity on $M \times I$ and of $M \times \{0\}$; the equivalence class $M \times \{0\}$ will be denoted by $\langle m, 0 \rangle$, $m \in M$, the remaining points are pairs $\langle m, x \rangle$ with $m \in M$ and $x \in [0, 1]$. The metric $d$ on $S$ is defined by

\[
\begin{align*}
  d(\langle 0, \langle m, x \rangle \rangle) &= d(\langle m, x \rangle, 0) = x \\
  d(\langle m, x \rangle, \langle n, y \rangle) &= |x - y| & \text{if } m = n \\
  x + y & & \text{if } m \neq n.
\end{align*}
\]

Properties of the space $S$: $S$ is a complete metric space with the density character $m$; it is connected and locally connected but it is not locally compact, realcompact and pseudo-$m$-compact; realcompact and pseudo-$m$-compact subspaces of $S$ are just subspaces of nonmeasurables cardinals.

**Proposition 1.** Let $\{U_m | m \in M\}$ be a discrete family of nonvoid open subsets of a space $P$ and let $y_m \in U_m$ for each $m \in M$. Then every mapping $f: \{y_m | m \in M\} \rightarrow S$ can be continuously extended on $P$ into $S$.

**Proof.** (See similar assertion in [6], 3 L. 1.) Let $f y_m = \langle n_m, x_m \rangle$ where $x_m \in [0, 1]$. For each $m \in M$ there is a continuous function $f_m: P \rightarrow [0, 1]$ such that $f_m y_m = x_m$, $f_m[P - U_m] = \{0\}$. Define $g: P \rightarrow S$ in the following way:

\[
\begin{align*}
  gy = 0 & & \text{if } y \notin \cup \{U_m | m \in M\} \\
  gy = \langle n_m, f_m y \rangle & & \text{if } y \in U_m.
\end{align*}
\]

Since the restriction of $g$ to $P - \cup \{U_n | n \neq m\}$ is equal to the composition of $f_m$ and the embedding of the copy of $[0, 1]$ onto its $n_m$-copy in $S$ and, hence, is continuous, the continuity of $g$ follows from the fact that the family $\{P - \cup \{U_n | n \neq m\} | m \in M\}$ is an interior covering of $P$.

In the following two theorems we shall state equivalent properties of pseudo-$m$-compactness which are analogous to those of pseudocompactness for $R$ instead of $S$, $N$ instead of $M$ and compactness instead of realcompactness. By $\beta S P$ we mean an $S$-compactification of $P$ in the sense of [4], i.e., a reflection of $P$ in the full subcategory of closed subspaces of powers $S^4$ and their homeomorphs. One of the copies of $\beta S P$ can be constructed by the Čech's method (a closure of $P$ in $S^C(r, S)$).

**Theorem 1.** The following properties of a space $P$ are equivalent:

1. $P$ is pseudo-$m$-compact;
(2) There is no copy $M'$ of $M$ in $P$ such that each continuous mapping on $M'$ into $S$ can be continuously extended on $P$ into $S$;

(3) If $f$ is a continuous mapping on $P$ into $S$ then $f[P]$ is realcompact (i.e., $f[P]$ is of nonmeasurable cardinal);

(4) $vP = \beta S P$.

Proof. Let $M'$ be a copy of $M$ in $P$ with the property from (2). Then there is a continuous mapping $f$ on $P$ into $S$ such that $f[M'] = M \times (1)$. Thus a continuous image of $P$ is not pseudo-m-compact and hence $P$ is not pseudo-m-compact, too. Therefore (1) implies (2). Now we shall prove that (2) implies (3). Suppose that there is a continuous mapping $f$ on $P$ into $S$ such that $f[P]$ is of measurable cardinal. We may suppose that $f[P]$ contains $M \times (1)$. By Proposition 1, $f^{-1}[M \times (1)]$ contains a copy of $M$ with the property required in (2). The implication (3) to (4) follows at once from a construction of $\beta S P$. Indeed, we can regard $\beta S P$ as a closure of $P$ in $SC(P,S)$ and, hence by (3), $\beta S P$ is realcompact; since always $\beta S P \subseteq v P$, the condition (4) is fulfilled. It remains to prove that (4) implies (1). Let $P$ be not pseudo-m-compact. Then there is a discrete family $\{U_m | m \in M\}$ of nonvoid open sets in $P$. If $D$ is constructed so that $D \subseteq \bigcup \{U_m | m \in M\}$, $D \cap U_m$ is a one-point set for each $m$, then by Proposition 1, $\beta S D$ is equal to the closure of $D$ in $\beta S P$ and $v D$ is equal to the closure of $D$ in $v P$. Since always $\beta S D = D$, $v D \neq D$ the equality $\beta S P = v P$ cannot hold in this case.

For an additional characterization of pseudo-m-compactness we need a concept related to that of $G_\delta$-set. The following definition is sufficient to our purposes but, unfortunately, very complicated; we believe there exists a more convenient form of it.

Definition 2. A subset $A$ of a space $P$ is said to be a $G_\delta(m)$-set in $P$ if there exists a family $\{G_\alpha \mid \alpha < m\}$ of open subsets in $P$ such that

(a) $\bigcap \{G_\alpha \mid \alpha < m\} = A$;

(b) if $\alpha < \beta$ then $G_\alpha \supset G_\beta$ and if $\alpha$ is limit then $G_\alpha = \bigcap \{G_\beta \mid \beta < \alpha\}$;

(c) $\{G_\alpha \mid \alpha < m\}$ is an open collection of measurable cardinal which is discrete in $P - A$.

Theorem 2. A space $P$ is pseudo-m-compact if and only if every $G_\delta(m)$-set in $vP$ meets $P$.

Proof. If there is a $G_\delta(m)$-set in $vP$ disjoint with $P$ then $P$ is not pseudo-m-compact by (c). Thus the condition is necessary. Now, let $P$ be not pseudo-m-compact. Then there is a discrete family $\{U_\alpha \mid \alpha < m\}$ of nonvoid open sets in $P$. Choose a nonvoid open $V_\alpha$ for each $\alpha$ such that $V_\alpha$ and $P - U_\alpha$ are functionally separated in $P$. Let $V'_\alpha$ be an open set in $vP$ the trace of which in $P$ is $V_\alpha$ and denote by $G_\alpha$ the open set $\bigcup \{V'_\beta \mid \alpha \leq \beta < m\}$. We shall prove that $A = \bigcap \{G_\alpha \mid \alpha < m\}$ is a $G_\delta(m)$-set in $vP$ disjoint with $P$. We need verify that the system $\{G_\alpha\}$ and $A$ satisfy (c) and that $A$ is disjoint with $P$. For the rest of the proof we identify
As it was seen in the proof we may request for $\{G_{\alpha+1} - G_\alpha\}$ to be uniformly discrete in $P$. First we shall prove (c). Let $x \in vP - A$; then there is a $\beta$ such that $x \not\in G_\beta$. As in the proof of Proposition 1 we can construct a continuous mapping $f: P \to S$ such that $f[V_{\alpha,1}] = \langle x, 1 \rangle$ for $\alpha < \beta$ and $f[P - \cup \{U_\alpha | x < \beta\}] = \{0\}$. Extending this mapping continuously on $vP$ we obtain a mapping $g$ with the same image as $f$. Thus $gx$ has a neighborhood in $S$ intersecting at most one $\langle x, 1 \rangle$, $\alpha < \beta$. It follows that $x$ has a neighborhood intersecting at most one $V_\alpha = G_\alpha - G_{\alpha+1}$, $\alpha < \beta$ and, consequently, has a neighborhood intersecting at most one $G_\alpha - G_{\alpha+1}$, $\alpha < m$. The remaining property of (c) is clear. Finally, since $P \cap \bar{G}_\alpha = \cup \{V_\beta [\beta \geq \alpha]\}$, the set $P \cap A$ is empty. The proof is complete.

**Remark.** As it was seen in the proof we may request for $\{(G_\alpha - G_{\alpha+1}) \cap P\}$ to be uniformly discrete in $P$.

§ 2.

In this section we shall be interested in the equality $v(P \times Q) = vP \times vQ$ (by this equality we mean that the continuous mapping from $v(P \times Q)$ into $vP \times vQ$, leaving all the points of $P \times Q$ fixed, is a homeomorphism onto).

The following theorem is not so important but it is of some interest. It generalizes the example 4.8 from [3] and the proposition 4.6 from [13] and entails that all members of $\mathcal{F}$ from [13] are of nonmeasurable cardinal.

**Theorem 3.** Let $Q$ be discrete. Then $v(P \times Q) = vP \times vQ$ if and only if either $P$ or $Q$ is of nonmeasurable cardinal.

**Proof.** The case when card $Q$ is nonmeasurable is trivial. If card $Q$ is measurable and card $P$ is nonmeasurable, then each continuous bounded function on $P \times Q$ can be continuously extended on $\beta P \times Q$ and, hence, on $\beta P \times vQ$ by Theorem 2.8 in [3]. Thus it remains to prove that if both card $P$ and card $Q$ are measurable then there is a continuous function $f$ on $P \times Q$ which cannot be continuously extended on $vP \times vQ$. We may and shall assume that card $Q \leq$ card $P$. It is easy to construct an injective mapping $h: Q \to P$ such that there is a point $q_0 \in vQ - Q$ with $\bar{h}q_0 \not\in h[Q]$, where $\bar{h}: vQ \to vP$ is a continuous extension of $h$. Indeed, there is an injective mapping $k: Q \to P$ such that $k[Q] \neq P$; if $k$ has not the property required for $h$ we pick out $p_1 \in P - k[Q]$, $p_2 \in \bar{k}[vQ - Q]$ and put $h = h_{Q \times k^{-1}[p_2]} \cup \langle k^{-1}[p_2], p_1 \rangle$; then $q_0 \in \bar{k}^{-1}[p_2] - Q$. Now, define $f: vP \times Q \to [0, 1]$ as follows: for every $q \in Q$, $f(\langle \cdot, q \rangle)$ is a continuous function on $vP$ into $[0, 1]$ having the value 1 in $hq$ and the value 0 in $\bar{h}q_0$. Of course, this function $f$ cannot be continuously extended on $vP \times vQ$ since it has the value 0 on the set $\{\langle \bar{h}q_0, q\rangle | q \in Q\}$, the value 1 on $\{\langle hq, q \rangle | q \in Q\}$ and closures of both these sets meet in $vP \times vQ$ (they contain $\langle \bar{h}q_0, q_0 \rangle$).
Corollary. If card P is measurable and Q is not pseudo-m-compact then \( v(P \times Q) \neq vP \times vQ \).

Proof. The space Q contains a discrete subspace M of measurable cardinal such that \( P \times M \) is C-embedded in \( P \times Q \). Therefore, by the foregoing theorem, \( v(P \times Q) \neq vP \times vQ \).

Our following assertion completes the theorem 2.2 from [2] for the case of measurable cardinals. The theorem 2.2 from [2] asserts that if Q is a locally compact realcompact space of nonmeasurable cardinal, then \( v(P \times Q) = vP \times vQ \) for each space P. In the proof we shall make use of the following formal modification of SHIROMA theorem ([6], [15]): P is realcompact if and only if it is pseudo-m-compact and has a complete uniformity.

Theorem 4. Let Q be locally compact realcompact. Then \( v(P \times Q) = vP \times vQ \) if and only if either card Q is nonmeasurable or P is pseudo-m-compact.

Proof. Since \( (P \times Q) = C(P, C(Q)) \) and \( (vP \times Q) = C(vP, C(Q)) \) (these equalities stand for the canonical bijections), where \( C(Q) \) has the compact-open topology, the equality \( v(P \times Q) = vP \times vQ \) holds if and only if each continuous mapping \( f: P \to C(Q) \) can be continuously extended to a mapping on \( vP \) into \( C(Q) \). If Q is of nonmeasurable cardinal, then \( C(Q) \) is realcompact (it has a complete uniformity). If P is pseudo-m-compact then \( f[P] \) has the same property and so \( f[P] \) is realcompact. In both cases \( f \) can be continuously extended on \( vP \) into \( C(Q) \). We have proved that our condition is sufficient. Its necessity follows immediately from Corollary of Theorem 3.

Assume for a while that P and Q are of measurable cardinals. If either Q is discrete or locally compact realcompact, then \( v(P \times Q) = vP \times vQ \) if and only if \( P \times Q \) is pseudo-m-compact. This assertion is analogous to Glicksberg theorem saying that if P and Q are infinite spaces then \( \beta(P \times Q) = \beta P \times \beta Q \) if and only if \( P \times Q \) is pseudocompact. As we find out in Theorem 5 and the example following it this analogy holds in one direction only.

Theorem 5. Let P and Q be of measurable cardinals. If \( v(P \times Q) = vP \times vQ \) then \( P \times Q \) is pseudo-m-compact.

Proof. We shall proceed similarly as Glicksberg in [7]. Let P, Q be spaces of measurable cardinals such that \( v(P \times Q) = vP \times vQ \). Then, by Corollary of Theorem 3, P and Q are pseudo-m-compact spaces. Assume that \( P \times Q \) is not pseudo-m-compact. Then there is a \( G_m \)-set A in \( vP \times vQ \) disjoint with \( P \times Q \) (Theorem 2) and a corresponding family \( \{G_n\} \) with properties stated in Definition 2 and in addition such that the collection \( \{(G_n-G_{n+1}) \cap (P \times Q)\} \) is uniformly discrete in \( P \times Q \) (the remark following Theorem 2). First we shall prove that \( A \cap (P \times vQ) = \emptyset \). Let \( \langle p, q \rangle \in A \cap (P \times vQ) \) and let \( \{U_n\} \) be a discrete system of open sets in
$P \times Q$ such that for each $\alpha$ the sets $(P \times Q) - U_\alpha$ and $(G_\alpha - G_{\alpha+1}) \cap (P \times Q)$ are functionally separated in $P \times Q$. Then $\{U_\alpha \cap ((p) \times Q)\}$ is a discrete system of open sets in $(p) \times Q$ of cardinality $m$. Indeed, if $\{U_\alpha \cap ((p) \times Q)\}$ had smaller cardinality than $m$, then there would be a $\beta$ such that $U_\alpha \cap ((p) \times Q) = \emptyset$ for $\alpha \geq \beta$; but in this case no continuous function on $P \times Q$ being equal to 1 on $G_\beta \cap (P \times Q)$ and 0 on $P \times Q - \cup \{U_\alpha | \alpha \geq \beta\}$ could be continuously extended to $(p, q) \cap (P \times Q)$. This fact (discreteness of $\{U_\alpha \cap ((p) \times Q)\}$) contradicts to pseudo-$m$-compactness of $Q$. Now, take a subset $X$ of $G_\beta \cap (P \times Q)$ such that $X \cap (G_\alpha - G_{\alpha+1})$ is a one-point set for each $\alpha$. Pick out an $x = (p, q) \in X - (P \times Q)$, the closure being in $vP \times vQ$. Then for each $\alpha$ there is a neighborhood $U_\alpha$ of $x$ such that $U_\alpha \cap (P \times Q) \subseteq G_\alpha$ and hence $x \in A$. By the first part of the proof $p \not\in P$. We shall prove that the collection $\{(G_\alpha - G_{\alpha+1}) \cap (P \times (q))\}$ is an open discrete collection in $P \times (q)$ of measurable cardinality, which contradicts to pseudo-$m$-compactness of $P$. It is clear that the collection is open and discrete in $P \times (q)$. Assume it has a nonmeasurable cardinal. Since $A \cap (P \times (q)) = \emptyset$ there is an $\alpha < m$ such that $G_\alpha \cap (P \times (q)) = \emptyset$. It follows there exists a continuous function defined on $P \times vQ$ with a value 1 on $X$ and 0 on $P \times (q)$; evidently this function cannot be continuously extended to $(p, q)$. The proof is complete.

**Corollary.** If $P$ is a locally compact realcompact space and $Q$ is pseudo-$m$-compact, then $P \times Q$ is pseudo-$m$-compact.

Proof follows from the preceding theorem and from Theorem 4 in the case card $P$ and card $Q$ are measurable. The remaining cases are trivial (if card $P$ is nonmeasurable and $Q$ is pseudo-$m$-compact then $P \times Q$ is pseudo-$m$-compact, too).

We do not know whether the assertion of Corollary remains true after replacing realcompact by pseudo-$m$-compact).

It follows from Theorem 4 that in the preceding theorem the assumption on cardinality of $P$ and $Q$ cannot be omitted.

Unlike the compact case the converse of Theorem 5 does not hold. It is easy to state many examples of spaces $P$, $Q$ of measurable cardinals such that $P \times Q$ is pseudo-$m$-compact and $v(P \times Q) \neq vP \times vQ$. But in all the examples we know, this nonequality is caused by factors of non-measurable cardinals. We believe that the converse of Theorem 5 is true provided these factors are excluded.

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2) Now we can answer in the negative the question whether realcompactness in Corollary of Theorem 5 can be replaced by pseudo-$m$-compactness. It suffices to put $P = T_m$ with the usual order-topology (i.e., $P$ is locally compact pseudo-compact) and $Q = T_{m+1}^+ = \{ \omega \}$, which is the set $T_{m+1}$ with a discrete topology on $T_m$ and the usual order-neighborhoods at $\omega$ (i.e., $Q$ is realcompact); the product $P \times Q$ is not pseudo-$m$-compact because the cover $\{A_\alpha | -1 \leq \alpha < m\}$, where $A_{-1} = \{ (\beta, \gamma) | \beta \leq \gamma \leq m \}$ and $A_\alpha = \{ (\beta, \alpha) | \beta > \alpha \}$ for $\alpha > 0$, is a disjoint open (hence uniformizable) cover of $P \times Q$. (Added in proof.)
Example. Let $X$ be a compact space of measurable cardinal and $T'_{\omega_{1}+1}$ be the set $T_{\omega_{1}+1}$ with a $T_1$-topology inducing a discrete topology on $T_{\omega_1}$ and having the same base of neighborhoods at $\omega_1$ as the order topology in $T_{\omega_{1}+1}$. The space $T'_{\omega_{1}+1}$ is realcompact and $v(T_{\omega_{1}} \times T'_{\omega_{1}+1}) \neq T_{\omega_{1}} \times T_{\omega_{1}+1}$. Put $P = T_{\omega_{1}} + X$, $Q = T'_{\omega_{1}+1} + X$ (the operation $+$ means a disjoint union). Then $P \times Q$ is pseudo-m-compact and $v(P \times Q) \neq vP \times vQ$.

There appeared an interesting question in connection with the last example: do minimal cardinals $\alpha$, $\beta$ exist such that there are spaces $P$, $Q$ of cardinalities $\alpha$, $\beta$, respectively, with $v(P \times Q) = vP \times vQ$? The example above produces spaces with $\alpha = \beta = \aleph_1$. A slight modification of Example 5.3 from [13] produces spaces with $\alpha = \aleph_0$, $\beta = \exp \aleph_0$ (here $P = \aleph_0 \cup \{x\}$, $x \in \beta \aleph_0 - \aleph_0$, and $Q = \aleph_0 \cup X$ is a pseudocompact subspace of $\beta \aleph_0$ not containing $x$—for existence of such a $Q$ see e.g. [1], p. 864, 3 (f)) and, hence, shows that under the hypotheses of continuum there are spaces $P$, $Q$ of cardinalities $\aleph_0$ and $\aleph_1$, respectively, such that $v(P \times Q) \neq P \times vQ$. We wonder if such spaces exist without the assumption of the hypothesis of continuum.

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