James orthogonality and orthogonal decompositions of Banach spaces

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Received 2 December 2004
Available online 18 April 2005
Submitted by T.D. Benavides

Abstract

We establish decompositions of a uniformly convex and uniformly smooth Banach space $B$ and dual space $B^*$ in the form $B = M \oplus J^*M^\perp$ and $B^* = M^\perp \oplus JM$, where $M$ is an arbitrary subspace in $B$, $M^\perp$ is its annihilator (subspace) in $B^*$, $J : B \to B^*$ and $J^* : B^* \to B$ are normalized duality mappings. The sign $\oplus$ denotes the James orthogonal summation (in fact, it is the direct sums of the corresponding subspaces and manifolds). In a Hilbert space $H$, these representations coincide with the classical decomposition in a shape of direct sum of the subspace $M$ and its orthogonal complement $M^\perp$: $H = M \oplus M^\perp$.

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Keywords: Banach spaces; Modulus of convexity; Modulus of smoothness; Generalized projection operator; James orthogonality; g-Orthogonality; J-co-ordinate sum; Decompositions; Subspaces; Nonlinear manifolds

1. Preliminaries

Let $B$ be a real uniformly convex and uniformly smooth (hence reflexive) Banach space with the norm $\| \cdot \|$, $B^*$ its dual space with the norm $\| \cdot \|_*$, $\theta_B$ and $\theta_{B^*}$ be origins of $B$
and $B^*$, respectively [1,3,9]. If we denote by $\rho_B(\tau)$ modulus of smoothness of $B$ and by $\delta_B(\varepsilon)$ its modulus of convexity, then a uniform smoothness means that
\[
h_B(\tau) = \frac{\rho_B(\tau)}{\tau} \to 0 \quad \text{as} \quad \tau \to 0,
\]
and uniform convexity means that
\[
\delta_B(\varepsilon) > 0 \quad \text{as} \quad \varepsilon > 0.
\]
As usually, we introduce a dual product in $B^* \times B$ by $\langle \phi, x \rangle$, where $\phi \in B^*$ and $x \in B$. Let $J : B \to B^*$ be the normalized duality mappings in $B$ defined as
\[
\langle Jx, x \rangle = \|Jx\|_* \|x\| = \|x\|^2.
\]
It is known that in our conditions the operator $J$ is well defined, strictly monotone, continuous, coercive, bounded and homogeneous. Besides, $J$ is uniformly continuous and uniformly monotone mapping on each bounded set of $B$ along with the following estimates (see [1,2]): if $\|x\| \leq R$ and $\|y\| \leq R$,
\[
(2L)^{-1} R^2 \delta_B\left(\|x - y\|/2R\right) \leq \langle Jx - Jy, x - y \rangle \leq 2LR^2 \rho_B(4\|x - y\|/R) \tag{1.1}
\]
and
\[
\|Jx - Jy\|_* \leq 8Rh_B\left(16L\|x - y\|/R\right), \tag{1.2}
\]
where $1 < L < 1.7$. In general, $J$ is nonlinear and multiple-valued in $B$. However, in any smooth Banach space $J$ is single-valued and demicontinuous, moreover, $Jx = \text{grad}\|x\|^2$. If the norm in $B$ is Fréchet differentiable, then $J$ is continuous. In a Hilbert space, $J$ is the identity (i.e., linear) operator $I$. The normalized duality mapping $J^* : B^* \to B$ in $B^*$ has the same properties and $J^* = J^{-1}$, where $J^{-1}$ is the inverse operator to $J$. Below we give analytical representations of $J$ in the uniformly convex and uniformly smooth Banach spaces $l^p$, $L^p$ and $W^p_m$ for $p \in (1, \infty)$:

(i) $l^p$:

\[
Jx = \|x\|_p^{2-p} y \in l^q, \quad p^{-1} + q^{-1} = 1, \quad x = \{x_1, x_2, \ldots\}, \quad y = \{x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, \ldots\}, \quad p^{-1} + q^{-1} = 1;
\]

(ii) $L^p$:

\[
Jx = \|x\|_L^p x \in L^q;
\]

(iii) $W^p_m$:

\[
Jx = \|x\|_{W^p_m}^2 \sum (-1)^{|\alpha|} D^\alpha (|D^\alpha x|^{p-2} D^\alpha x) \in W^q_{-m}.
\]

Let $M$ be an arbitrary closed subspace of $B$ and $M^\perp \subset B^*$ be its annihilator [4,10]. Let $P_M$ and $\Pi_{M^\perp}$ be the metric and generalized projection operators onto $M$ and $M^\perp$, respectively. Let us recall the definitions of the metric and generalized projection operators for an arbitrary closed convex subset $\Omega$ of $B$.

**Definition 1.1.** The operator $P_\Omega : B \to \Omega$ is called the metric projection operator onto $\Omega$ if it assigns to each $x \in B$ its nearest point $\bar{x} \in \Omega$, i.e., a solution of the following minimization problem:
\[
P_\Omega x = \bar{x}; \quad \bar{x}: \|x - \bar{x}\|^2 = \inf_{\xi \in \Omega} \|x - \xi\|^2. \tag{1.3}
\]
Under our conditions, the metric projection operator $P_{\Omega}$ is well defined, that is, there exists the unique projection $\bar{x}$ for each $x \in B$ called the best approximation.

The main property of $P_{\Omega}$ can be expressed as follows: the point $\bar{x}$ is the metric projection of $x \in B$ on $\Omega \subset B$ if and only if the inequality

$$\langle J(x - \bar{x}), \bar{x} - \xi \rangle \geq 0, \quad \forall \xi \in \Omega,$$

(1.4)

is satisfied [5,6,8,14,18]. In a Hilbert space $H$, (1.4) has the form

$$(x - \bar{x}, \bar{x} - \xi) \geq 0, \quad \forall \xi \in \Omega.$$

(1.5)

The construction of the generalized projection operator $\Pi_{\Omega} : B \to \Omega$ is based on the Lyapunov functional [1]

$$W(x, \xi) = \|x\|^2 - 2\langle Jx, \xi \rangle + \|\xi\|^2,$$

(1.6)

which is closely related with the Young–Fenchel transformation for conjugate functions [1,10]. It has been shown in [1] that $W(x, \xi)$ is positive, differentiable, coercive and finite on each bounded set of $B$.

**Definition 1.2.** Operator $\Pi_{\Omega} : B \to \Omega$ is called the generalized projection operator onto $\Omega$ if it assigns to each $x \in B$ a minimum point $\hat{x} \in \Omega$ of the functional $W(x, \xi)$, i.e., a solution of the following minimization problem:

$$\Pi_{\Omega}x = \hat{x}; \quad \hat{x}: W(x, \hat{x}) = \inf_{\xi \in \Omega} W(x, \xi).$$

Generalized projection operator $\Pi_{\Omega}$ is also well defined in uniformly convex and uniformly smooth Banach spaces. The property (1.4) for $\Pi_{\Omega}$ is written as follows (see [1]):

$$\langle Jx - J\hat{x}, \hat{x} - \xi \rangle \geq 0, \quad \forall \xi \in \Omega.$$

(1.7)

The inequalities (1.4) and (1.7) are essential to the proof of the following result:

**Theorem 1.3** (Alber [4]). Every element $x \in B$ has one and only one decomposition

$$x = PMx + J^* \Pi_{M^\perp} Jx$$

(1.8)

with the orthogonality relation

$$\langle \Pi_{M^\perp} Jx, v \rangle = 0, \quad \forall v \in M.$$

(1.9)

It has been also shown that the similar decomposition takes place for an element $\psi \in B^*$, namely:

$$\psi = PM^\perp \psi + J \Pi_M J^* \psi$$

(1.10)

and

$$\langle \zeta, \Pi_M J^* \psi \rangle = 0, \quad \forall \zeta \in M^\perp.$$

(1.11)

If $\Omega = B$, then $P_{\Omega}$ and $\Pi_{\Omega}$ are the identity operator $I$. It is obvious that (1.4), (1.5) and (1.7) coincide in a Hilbert space $H$ and $W(x, \xi) = \|x - \xi\|^2$. Therefore, $P_{\Omega} = \Pi_{\Omega}$. Since
\( H = H^* \), the decomposition formulas (1.8) and (1.10) are transformed into the classical Beppo Levi decomposition (see, for instance, [16]):

\[
x = P_M x + P_{M\perp} x, \quad \forall x \in H.
\]

In this case, \( M \perp \) is the orthogonal complement to a subspace \( M \). Furthermore, \( P_M \) is a linear operator in \( H \) and orthogonal in the sense that for each \( x \in H \),

\[
(x - P_M x, v) = 0, \quad \forall v \in M. \tag{1.12}
\]

Therefore,

\[
(x, P_M x) = \|P_M x\|^2. \tag{1.13}
\]

Emphasize also that, in general, the operators \( P_{\Omega} \) and \( \Pi_{\Omega} \) are nonlinear in Banach (not Hilbert) spaces even if \( \Omega = M \). At the same time, there holds the following proposition which describes “the conditional linearity” of \( P_M \):

**Proposition 1.4.** Let \( x \) be an arbitrary element of the Banach space \( B \) and \( y \) an arbitrary element of a subspace \( M \subset B \). Then

\[
P_M(\alpha x + \beta y) = \alpha P_M x + \beta P_M y, \quad \forall \alpha, \beta: -\infty < \alpha, \beta < +\infty. \tag{1.14}
\]

The equality (1.14) plays important role when we prove the main theorem below. Unfortunately, we have to state that the generalized projection \( \Pi_M \) does not possess this property. Moreover, we later show that if the generalized projection operator \( \Pi_M \) is conditionally linear then it is a metric projection operator (see Corollary 2.19).

In the sequel, we will use the analytical representation of the generalized projection onto one-dimensional subspaces.

**Proposition 1.5** [4]. Let \( M_\alpha \subset B \) be a one-dimensional subspace spanned upon the element \( e_\alpha \) with the unit norm, i.e., \( \|e_\alpha\| = 1 \). Then the generalized projection \( \Pi_{M_\alpha} x \) of an arbitrary element \( x \in B \) on \( M_\alpha \) is \( \langle Jx, e_\alpha \rangle e_\alpha \), where \( \langle Jx, e_\alpha \rangle \) is the generalized Fourier coefficient.

This result has no analogue in a Banach (not Hilbert) space for the metric projection \( P_{M_\alpha} x \).

### 2. Main results

First of all, we present some orthogonality concepts in Banach spaces which is used in this section [4,9,15].

**Definition 2.1.** We say that an element \( \phi \in B^* \) is \( d \)-orthogonal (orthogonal in the dual sense) to \( x \in B \) and write \( \phi \perp^d x \) if \( \langle \phi, x \rangle = 0 \). An element \( \phi \in B^* \) is \( d \)-orthogonal to a subset \( K \subset B \), i.e., \( \phi \perp^d K \), if it is \( d \)-orthogonal to each element of \( K \). A subset \( K_2 \subset B^* \) is \( d \)-orthogonal to subset \( K_1 \subset B \), i.e., \( K_2 \perp^d K_1 \), if each element \( \phi \in K_2 \) is \( d \)-orthogonal to \( K_1 \).
It is obvious that the \(d\)-orthogonality is a symmetric characterization of \(B\).

**Definition 2.2.** An element \(x \in B\) is said to be \(g\)-orthogonal to \(y \in B\) (orthogonal in the generalized sense) and written \(x \perp^g y\) if \(\langle Jx, y \rangle = 0\). An element \(x \in B\) is \(g\)-orthogonal to a subset \(K\), i.e., \(x \perp^g K\), if it is \(g\)-orthogonal to each element of \(K\). A subset \(K_2 \subset B\) is \(g\)-orthogonal to subset \(K_1 \subset B\), i.e., \(K_2 \perp^g K_1\), if each element \(x \in K_2\) is \(g\)-orthogonal to \(K_1\).

**Definition 2.3.** We say that an element \(x \in B\) is \(j\)-orthogonal to \(y \in B\) (orthogonal in the James’ sense) and write \(x \perp^j y\) if
\[
\|x\| \leq \|x + ty\|, \quad \forall t \in \mathbb{R}.
\]
(2.1)
An element \(x \in B\) is \(j\)-orthogonal to a subset \(K \subset B\), i.e., \(x \perp^j K\), if \(x\) is \(j\)-orthogonal to each \(y \in K\). A subset \(K_2 \subset B\) is \(j\)-orthogonal to subset \(K_1 \subset B\), i.e., \(K_2 \perp^j K_1\), if each element \(x \in K_2\) is \(j\)-orthogonal to \(K_1\).

The basic properties of the James orthogonality have been obtained in [9, Chapter 2, Theorems 3–5]. We establish some of them for the generalized orthogonality. First of all observe that if \(\phi \in B^*, x \in B\) and \(\phi \perp^d x\), then \(J^*\phi \perp^g x\). By Definition 2.2, if \(x\) is \(g\)-orthogonal to \(y\), then \(x\) is \(g\)-orthogonal to \(\lambda y\) for all \(\lambda \in \mathbb{R}\), and \(\lambda x\) is also \(g\)-orthogonal to \(y\). The generalized orthogonality is asymmetric characterization of a Banach (not Hilbert) space \(B\) because \(\langle Jx, y \rangle \neq \langle Jy, x \rangle\), in general. Therefore, if \(x \in B\) is \(g\)-orthogonal to \(y \in B\), then \(y\) is not necessary \(g\)-orthogonal to \(x\). As it was noted in [9], the same fact characterizes the James orthogonality too.

**Theorem 2.4.** Let \(x, y \in B\), \(x \neq \theta_B\). Then \(x \perp^g (\alpha x + y)\) if and only if
\[
\alpha = -\frac{\langle Jx, y \rangle}{\|x\|^2}.
\]
(2.2)
If \(B\) is a smooth space, then \(\alpha\) is unique.

**Proof.** Trivially,
\[
\langle Jx, \alpha x + y \rangle = \alpha \langle Jx, x \rangle + \langle Jx, y \rangle = \alpha \|x\|^2 + \langle Jx, y \rangle = 0.
\]
Therefore, (2.2) holds. If \(B\) is smooth, then the duality mapping \(J\) is single-valued and hence \(\alpha\) is unique. \(\square\)

**Remark 2.5.** It follows from (2.2) that \(|\alpha| \leq \|x\|^{-1} \|y\|\).

**Theorem 2.6.** If \(x, y, z \in B\) such that \(x \perp^g y\) and \(x \perp^g z\), then \(x \perp^g (y + z)\).

**Proof.** If \(x \perp^g y\) and \(x \perp^g z\), then \(\langle Jx, y \rangle = 0\) and \(\langle Jx, z \rangle = 0\), respectively. This yields the equality
\[
\langle Jx, y \rangle + \langle Jx, z \rangle = \langle Jx, y + z \rangle = 0,
\]
which means that \(x \perp^g (y + z)\). \(\square\)
\textbf{Theorem 2.7.} Let $f \in B^\ast$. Then $x \perp y$ for all $y \in \text{Ker}(f)$ if and only if $|f(x)| = \|f\|_\ast \|x\|$. 

\textbf{Proof.} For $f = Jx \in B^\ast$ with $f(y) = \langle Jx, y \rangle$, we have

$$\text{Ker}(f) = \{ y \in B : \langle Jx, y \rangle = 0 \}$$

and

$$|f(x)| = |\langle Jx, x \rangle| = \|Jx\|_\ast \|x\| \leq \|f\|_\ast \|x\| = \|x\|^2. \quad \square$$

If to compare these properties with the corresponding properties of the James orthogonality in [9], one can observe that they coincide. It is easy to show that the James orthogonality and generalized orthogonality are equivalent in smooth Banach spaces. Indeed, using the definition of gradient of the functional $u(x) = \|x\|^2$ and the definition of James orthogonality, we deduce from (2.1) that

$$2\langle Jx, y \rangle = \lim_{t \to 0} \frac{\|x + ty\|^2 - \|x\|^2}{t} \geq 0.$$ 

If now $y$ is replaced by $-y$, then

$$-2\langle Jx, y \rangle = \lim_{t \to 0} \frac{\|x - ty\|^2 - \|x\|^2}{t} \geq 0.$$ 

These two inequalities imply that $\langle Jx, y \rangle = 0$.

Conversely, let $\langle Jx, y \rangle = 0$, i.e., $x \perp y$. Since the functional $g(x)$ is convex in $B$, one gets

$$\|x + ty\|^2 - \|x\|^2 \geq 2t \langle Jx, y \rangle = 0, \quad \forall t \in \mathbb{R}.$$ 

This means that $x \perp y$.

\textbf{Remark 2.8.} Kato has shown in [12] that in an arbitrary Banach space $B$, (2.1) holds if and only if there is $j \in Jx$ such that Re$\langle j, y \rangle \geq 0$.

\textbf{Remark 2.9.} If an element $y \in B$ is $j$-orthogonal to $x \in B$, then

$$W(x, y) = \|x\|^2 + \|y\|^2.$$ 

In Hilbert space this equality gives

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2.$$

\textbf{Proposition 2.10.} If $x \in B$ is an arbitrary fixed point and $M$ is a closed subspace of $B$, then an element $\bar{x} = P_Mx$ minimizes $\|x - \xi\|$ relatively to $\xi \in M$ if and only if

$$\langle J(x - P_Mx), v \rangle = 0, \quad \forall v \in M.$$ 

The proof follows from (1.8) and (1.9). By analogy with (1.12), this assertion shows that the projection operator $P_M$ is $j$-orthogonal to $M$ in $B$.

The following proposition gives the upper and lower estimates of the dual product $\langle Jx, P_Mx \rangle$ in a uniformly convex and uniformly smooth Banach space $B$ (cf. (1.13)).
Proposition 2.11. Let $\theta_B \neq x \in B$. Then $\langle Jx, PMx \rangle \geq 0$ and
\[ 2LR^2 \rho_B (4R^{-1}\|PMx\|) \geq \|Jx - P_Mx\| \geq (2L)^{-1}R^2 \delta_B (2R^{-1}\|PMx\|). \]
where $R = \|x\|$.

Proof. If $y = P_Mx$, then Proposition 2.10 implies
\[ \langle J(x - P_Mx), PMx \rangle = 0 \quad \text{and} \quad \langle Jx - J(x - P_Mx), PMx \rangle = \langle Jx, PMx \rangle. \]
The assertion results from the estimate (1.1), where $R = \|x\|$, since, by Definition 1.1, $\|x - P_Mx\| \leq \|x\|$. \(\square\)

Let $M$ be a subspace of $B$. Introduce the sets $C_M$ and $C'_M$ by the formulas:
\[ C_M = \{x \in B: PMx = \theta_B\} \quad \text{(2.3)} \]
and
\[ C'_M = \{x \in B: \Pi_Mx = \theta_B\}. \quad \text{(2.4)} \]
We have proved in [4] that $C_M$ and $C'_M$ are closed sets of $B$. It is easy to check that they coincide. Indeed, assume that $x \in C_M$ but $x \notin C'_M$. Then $PMx = \theta_B$ and, by [4, Corollary 1.16], $Jx \in M^\perp$. On the other side, $\Pi_Mx \neq \theta_B$ and, by the same corollary, $Jx \notin M^\perp$. This contradiction implies the claim.

Definition 2.12. We say that a Banach space $B$ is the James orthogonal sum of the closed manifolds $K_1 \subset B$ and $K_2 \subset B$ and denote $B = K_1 \cup K_2$, if:

(1) each element $x \in B$ has a unique decomposition $x = k_1 + k_2$, where $k_1 \in K_1$ and $k_2 \in K_2$;
(2) $K_2 \perp_{J} K_1$, and
(3) $K_1 \cap K_2 = \{\theta_B\}$.

Theorem 2.13. If $M \subset B$ is a closed subspace and $M^\perp \subset B^*$ is its annihilator, then
\[ B = M \cup J^* M^\perp \quad \text{(2.5)} \]
and
\[ B^* = M^\perp \cup JM. \quad \text{(2.6)} \]

Proof. From [4, Corollary 1.16] we know that the inclusion $Jx \in M^\perp$ implies $x \in C_M$. Consequently,
\[ JC_M = M^\perp. \quad \text{(2.7)} \]
The projection operator $P_M$ is conditionally linear. Therefore,$\]
\[ PM(x - PMx) = PMx - P_MP_Mx = PMx - PMx = \theta_B. \]
Using the decomposition (1.8), one gets
\[ PM(J^* \Pi_M \perp Jx) = PM(x - P_Mx) = \theta_B. \quad \text{(2.8)} \]
Since $C_M$ and $C'_M$ coincide, it results that

$$\Pi_M(x - P_Mx) = \Pi_M(J^*\Pi_M Jx) = \theta_B. \quad (2.9)$$

So, if $x \in C_M$, then $Jx \in M^\perp$, and this means that $\langle Jx, v \rangle = 0$ for all $v \in M$, i.e., $x \perp^I M$. Since $x$ is an arbitrary element of $C_M$, we have $C_M \perp^I M$. Then the equality $P_M(J^*\Pi_M Jx) = \theta_B$ implies the inclusion $J^*\Pi_M Jx \in C_M$. Consequently, the first element $P_Mx$ in the decomposition (1.8) belongs to the given subspace $M$ and the second element $J^*\Pi_M Jx$ belongs to $C_M$. Conversely, if arbitrary $y \in M$ and $z \in C_M$, then $y + z \in B$. It follows from this that $B = M \cup C_M$, and the conclusion (2.5) holds by (2.7). Emphasize that $C_M$ is a nonlinear set, in general, because $P_M(\lambda x + \mu y) \neq \theta_B$ for $x, y \in C_M$.

It is not difficult to show that if $x \in M$, then $Jx$ belongs to the set $C_M^\perp$ defined as

$$C_M^\perp = \{\phi \in B^*: P_M\phi = \theta_B\}, \quad (2.10)$$

which is $j$-orthogonal to the annihilator (closed subspace) $M^\perp$ in the dual space $B^*$. Indeed, we proved in [4] the inclusion $Jx \in C_M^\perp$, where

$$C_M^\perp = \{\phi \in B^*: \Pi_M\phi = \theta_B\}. \quad (2.11)$$

In this case, $\langle x, \psi \rangle = 0$, i.e., $\langle J^* Jx, \psi \rangle = 0$. The latter means that $Jx \perp^I M^\perp$. Since $Jx$ is an arbitrary element of $C_M^\perp$, one gets that $C_M^\perp \perp^I M^\perp$. In the decomposition (1.10) the element $P_M^\perp \psi \in M^\perp$ and $J\Pi_M J^*\psi \in C_M^\perp$. Obviously it follows that $B^* = M^\perp \cup C_M^\perp$ thus, due to the fact that $(M^\perp)\perp = M$, we have (2.6).

It is easy to see by a contradiction that $C_M \cap M = \{\theta_B\}$. Indeed, if $x \neq \theta_B$ and $x \in C_M \cap M$, then on the one hand, $P_Mx = x \neq \theta_B$ because of $x \in M$. On the other hand, $P_Mx = \theta_B$ because of $x \in C_M$. Since $M$ is a subspace in a reflexive Banach space, we conclude that $P_Mx$ is unique that implies the claim. Similarly, $C_M^\perp \cap M^\perp = \{\theta_B^*\}$. The proof is accomplished. □

**Remark 2.14.** The decompositions (2.5) and (2.6) involve the relations $J^* M^\perp \perp^I M$ and $JM \perp^I M^\perp$, respectively. Thus, $B$ is the James orthogonal sum of $M$ and $J^* M^\perp$ and $B^*$ is the James orthogonal sum of $M^\perp$ and $J M$.

In a Hilbert space $H$, the decompositions (2.5) and (2.6) are well known (see, for instance, [16]). Each of them is the direct sum of a subspace $M$ and its orthogonal complement $M^\perp$, that is,

$$H = M \oplus M^\perp. \quad (2.12)$$

Indeed, duality mappings $J$ and $J^*$ are the identity operators in $H = H^*$ and then James orthogonality $J^* M^\perp \perp^I M$ and $JM \perp^I M^\perp$ mean the mutual orthogonality of $M$ and $M^\perp$ in the sense of the inner product in $H$.

More generally than (2.12), we call a subspace $M \subset B$ complemented if there exists another subspace $N \subset B$ such that each $x \in B$ is uniquely expressible in the form $x = x_M + x_N$, where $x_M \in M$ and $x_N \in N$. We also say that $B$ is decomposed into a direct sum of the subspaces $M$ and $N$ and write $B = M \oplus N$.

Lindenstrauss and Tzafriri have proved in [13] the following significant fact:
**Theorem 2.15.** An infinite-dimensional Banach space $B$ is isomorphic to a Hilbert space if and only if each closed subspace of $B$ is complemented.

In reality, we have shown in Theorem 2.13 that a uniformly convex and uniformly smooth Banach space $B$ is decomposed into a direct sum of the subspace $M$ and, in general, the nonlinear smooth manifold $J^*M^\perp$. The property of smoothness follows from the facts that $J^*$ is a strictly monotone and smooth mapping and $M^\perp$ is a smooth and strictly convex set in $B$ [1,11,19]. In addition, similarly to orthogonal complement $M^\perp$ in a Hilbert space $H$, nonlinear manifold $J^*M^\perp$ has the following remarkable property in $B$ which arises from (1.14) and (2.3): if $x \in J^*M^\perp$, then $\lambda x \in J^*M^\perp$ for all $-\infty < \lambda < +\infty$. The same property characterizes nonlinear manifold $JM$. Nevertheless, we abstain from using the symbol $\oplus$ of the direct summation because, by tradition, it deals with (closed) subspaces.

Recall the definition introduced in [4]:

**Definition 2.16.**

1. An element $\psi \in B^*$ is called the $J$-co-ordinate sum of elements $y \in B$ and $\phi \in B^*$ if $\psi = Jy + \phi$.
2. An element $x \in B$ is called the $J^*$-co-ordinate sum of elements $y \in B$ and $\phi \in B^*$ if $x = y + J^*\phi$.

The representation (1.8) asserts that an element $x \in B$ is the $J^*$-co-ordinate sum of two mutually $d$-orthogonal projections $P_Mx$ and $\Pi_{M^\perp}Jx$. From this point of view, (2.5) and (2.6) show that $B$ is the $J^*$-co-ordinate sum of two mutually $d$-orthogonal subspace $M$ and $M^\perp$ and $B^*$ is the $J$-co-ordinate sum of the same subspaces. It is natural to call $M^\perp$ by the $J^*$-co-ordinate complement of $M$ in $B$ and $M$ by the $J$-co-ordinate complement of $M^\perp$ in $B^*$. In other words, each closed subspace of a uniformly convex and uniformly smooth Banach space $B$ is $J^*$-complemented.

**Example 2.17.** Let $\psi \in B^*$ be a fixed vector. Define $M_\psi = \{x \in B: \langle \psi, x \rangle = 0\}$. Then $M_\psi^\perp = \lambda \psi$, where $-\infty < \lambda < +\infty$. By Theorem 2.13, one gets $B = M_\psi \oplus \lambda J^*\psi$. This means that $\lambda \psi$ is $J^*$-co-ordinate complement of $M_\psi$ in $B$. Moreover, $M_\psi$ is complemented in $B$ because $\lambda J^*\psi$ forms one-dimensional subspace in $B$.

Next we present some corollaries from Theorem 2.13.

**Corollary 2.18.** Let the manifold $C_M$ be convex. If the metric projection operators $P_{CM}$ is conditionally linear and it commutes with $P_M$, i.e., $P_MP_{CM} = P_{CM}P_M$, then

$$x = P_Mx + P_{CM}x.$$  (2.13)
Proof. Since $C_M$ is convex and $B$ is a reflexive Banach space, $P_{C_M}$ is well defined. Since $P_{C_M}$ is conditionally linear and $J^*\Pi_M J x \in C_M$, we have

$$P_{C_M}^* x = P_{C_M} P_M x + J^*\Pi_M J x.$$  \hfill (2.14)

A commutativity of $P_M$ and $P_{C_M}$ gives

$$P_{C_M} P_M x = P_M P_{C_M} x = 0.$$

Then in view of (2.14), $P_{C_M} x = J^*\Pi_M J x$, and (2.13) follows from (1.8). \hfill $\blacksquare$

Corollary 2.19. If the generalized projection operator $\Pi_M$ is conditionally linear, then it is the metric projection operator $P_M$.

Proof. Let $x \in B$. Since $P_M x \in M$ and $\Pi_M \xi = \xi$ for all $\xi \in M$, we have by (1.8) and by (2.9)

$$\Pi_M x = \Pi_M P_M x - \Pi_M (J^*\Pi_M J x) = \Pi_M P_M x = P_M x.$$

The result follows. \hfill $\blacksquare$

Recall the following

Definition 2.20. A set $\Omega \subseteq B$ is called complete if its closed linear hull (span $\Omega$) coincides with $B$.

Corollary 2.21. A set $\Omega \subseteq B$ is complete if and only if there are no vectors in $B$, except $\theta_B$, which are $j$-orthogonal to this set.

Proof. Let $\Omega$ be complete and $\langle J y, x \rangle = 0$ for all $x \in \Omega$. By definition, span $\Omega = B$. Since the duality mapping $J$ is continuous in a smooth Banach space $B$, then the dual product $\langle J y, x \rangle$ is continuous with respect to both independent variables $x$ and $y$, and $\langle J y, x \rangle = 0$ for all $x \in B$. Therefore, taking $x = y$ we get $\langle J y, y \rangle = \|y\|^2 = 0$, that is, $y = \theta_B$.

Conversely, assume that the assertion $y \perp_j \Omega$ implies $y = \theta_B$. Then by (2.5),

$$\bar{p} = p + \theta_B, \quad \forall \bar{p} \in B,$$

where $p \in \text{span} \, \Omega$. Thus, $B = \text{span} \, \Omega$. \hfill $\blacksquare$

The deficiency of a subspace $M \subset B$ (def $M$) in a Banach space $B$ is the dimension of the factor space $B \setminus M$. The subspace $M$ which has def $M = 1$ is said to be a hypersubspace [16, p. 47]. Let $M$ be a hypersubspace and

$$M(u_0) = \{u + u_0 : \forall u \in M\}$$

be a hyperplain. Suppose that $e$ is a vector in $C_M$ such that $e \perp_j M$ and $\langle Je, e \rangle = \|e\|^2 = 1$. Observe that due to Stiles [20], there exists at most one linearly independent vector $j$-orthogonal to $M$. It is easy to see that $x \in M(u_0)$ if and only if $\langle Je, x \rangle = \langle Je, u_0 \rangle$. Indeed, the last is equivalent to the equality $\langle Je, x - u_0 \rangle = 0$ which means that $e \perp_j (x - u_0)$, and then $x \in M(u_0)$. The equation of the hypersubspace $M$ for which $e \perp_j M$ is $\langle Je, x \rangle = 0$. 

Remark 2.22. The James orthogonal decompositions of Banach spaces can be also constructed on the basis of cones using the representations (2.7)–(2.10) in [4] (see also [7,17], where it was done for a Hilbert space).

Remark 2.23. The analysis of the proof of Theorem 2.13 shows that the conditions of uniform convexity and uniform smoothness of the space $B$ can be weakened up to level: $B$ is strictly convex and smooth together with its dual space $B^*$.

3. Relative decompositions in a Banach space

Let $x$ be a fixed vector of $B$ and $\psi_1 \in B^*$ with $\|\psi_1\|_* = 1$. Define the annihilator

$$M^1 = M^1_1 = \{ y \in B: \langle \psi_1, y \rangle = 0 \}.$$  \hspace{1cm} (3.1)

Then, by Theorem 2.13, we deduce that $B = M^1 \cup \lambda J^* \psi_1$ and $B^* = \lambda \psi_1 \cup J M^1$, where $-\infty < \lambda < +\infty$. Using Proposition 1.5, according to the decomposition formula (1.8), one gets

$$x = PM^1 x + \langle \psi_1, x \rangle J^* \psi_1.$$  \hspace{1cm} (3.2)

Take $\psi_2 \in J M^1 \subset B^*$ with $\|\psi_2\|_* = 1$. It defines the annihilator

$$M^2 = M^2_2 = \{ y \in M^1: \langle \psi_2, y \rangle = 0 \}.$$  \hspace{1cm} Then $M^1 = M^2 \cup \lambda J^* \psi_2$ and

$$PM^1 x = PM^2 P M^1 x + \langle \psi_2, P M^1 x \rangle J^* \psi_2.$$  \hspace{1cm} Substituting this expression for (3.2), one gets

$$x = \langle \psi_1, x \rangle J^* \psi_1 + \langle \psi_2, P M^1 x \rangle J^* \psi_2 + P M^2 P M^1 x.$$  \hspace{1cm} (3.3)

It is clear that $\langle \psi_1, J^* \psi_2 \rangle = 0$ because of (3.1). Therefore, $\psi_2 \perp^j \psi_1$.

The next step is described as follows. Take $\psi_3 \in J M^2 \subset B^*$ such that $\|\psi_3\|_* = 1$ and construct

$$M^3 = M^3_3 = \{ y \in M^2: \langle \psi_3, y \rangle = 0 \}.$$  \hspace{1cm} This implies the representation $M^2 = M^3 \cup \lambda J^* \psi_3$. Note that we also have the following equality:

$$PM^3 P M^1 x = PM^3 P M^2 P M^1 x + \langle \psi_3, P M^2 P M^1 x \rangle J^* \psi_3.$$  \hspace{1cm} Then it results from (3.3) that

$$x = \langle \psi_1, x \rangle J^* \psi_1 + \langle \psi_2, P M^1 x \rangle J^* \psi_2 + \langle \psi_3, P M^2 P M^1 x \rangle J^* \psi_3$$  $$+ PM^3 P M^2 P M^1 x.$$  \hspace{1cm} (3.4)

It can be verified that $\langle \psi_1, J^* \psi_3 \rangle = 0$ and $\langle \psi_2, J^* \psi_3 \rangle = 0$, i.e., $\psi_3 \perp^j \psi_1$ and $\psi_3 \perp^j \psi_2$. By induction, we obtain the following decomposition:
\[ x = \langle \psi_1, x \rangle J^* \psi_1 + \langle \psi_2, P_{M^1} x \rangle J^* \psi_2 + \langle \psi_3, P_{M^2} P_{M^1} x \rangle J^* \psi_3 + \cdots \\
+ \langle \psi_n, P_{M^{n-1}} \cdots P_{M^2} P_{M^1} x \rangle J^* \psi_n + P_{M^n} \cdots P_{M^2} P_{M^1} x, \]

where

\[ P_{M^0} x = P_B x = I x = x, \]
\[ P_{M^1} x = x - \langle \psi_1, x \rangle J^* \psi_1, \]
\[ P_{M^2} P_{M^1} x = P_{M^1} x - \langle \psi_2, P_{M^1} x \rangle J^* \psi_2, \]
\[ P_{M^3} P_{M^2} P_{M^1} x = P_{M^2} P_{M^1} x - \langle \psi_3, P_{M^2} P_{M^1} x \rangle J^* \psi_3, \]
\[ \vdots \]
\[ P_{M^n} \cdots P_{M^2} P_{M^1} x = P_{M^{n-1}} \cdots P_{M^2} P_{M^1} x - \langle \psi_n, P_{M^{n-1}} \cdots P_{M^2} P_{M^1} x \rangle J^* \psi_n, \]
or, for short,

\[ x = \sum_{i=1}^{n} \left( \psi_i, \prod_{j=1}^{i-1} P_{M^j} x \right) J^* \psi_i + \prod_{j=1}^{n} P_{M^j} x, \tag{3.5} \]

where

\[ \prod_{j=1}^{n} P_{M^j} x = \prod_{j=1}^{n-1} P_{M^j} x - \left( \psi_n, \prod_{j=1}^{n-1} P_{M^j} x \right) J^* \psi_n, \quad \prod_{j=1}^{n} P_{M^j} = P_{M^n} \cdots P_{M^2} P_{M^1}, \]

and

\[ \langle \psi_m, J^* \psi_k \rangle = 0, \quad k \neq m, \quad 1 \leq m \leq k - 1, \tag{3.6} \]
\[ \langle \psi_k, J^* \psi_k \rangle = \| \psi_k \|_*^2 = 1, \tag{3.7} \]
\[ k = 1, 2, \ldots, n, \tag{3.8} \]

that is, the system \( \{ \psi_1, \psi_2, \ldots, \psi_n, \ldots \} \) is \( j \)-orthogonal and normed in \( B^* \). However, the question whether the series in (3.5) is convergent as \( n \to \infty \) is still open. Note that in a Hilbert space \( H \), (3.6)–(3.8) form the orthonormed system of elements.

The results of this paper were presented at the Haifa Technion (Israel) in January 2002, ICTP (Trieste, Italy) in October 2002 and University of Notre Dame (USA) in May 2003.

Acknowledgment

The author is extremely grateful to the referee for his careful reading and good suggestions for this manuscript.

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