A necessary and sufficient condition for oscillation of neutral type hyperbolic equations

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Abstract


In the present paper a necessary and sufficient condition is obtained for oscillation of the solutions of neutral type linear hyperbolic differential equations of the form

\[ u_{xx}(x, t) + \sum_{t} a_{t} u_{x}(x, t - \tau_{t}) - [\Delta u(x, t) + \sum_{j} b_{j}(t) \Delta u(x, \rho_{j}(t))] 
+ cu(x, t) + \sum_{k} c_{k} u(x, t - \sigma_{k}) = 0, \quad (x, t) \in \Omega \times (0, \infty) \equiv G, \]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^{n} \) with a piecewise smooth boundary and \( \Delta u(x, t) = \sum_{i=1}^{n} u_{x_{i}x_{i}}(x, t) \).

Keywords: Oscillation, neutral type hyperbolic equations.

1. Introduction

In recent years the fundamental theory of partial differential equations with a deviating argument developed intensively. The qualitative theory of these important classes for the applications of partial differential equations, however, is still in an initial stage of its development. Thus, for instance, to the oscillation theory for this class of equations only a small
number of papers are devoted, published in the period since 1984. Sufficient conditions for oscillation of the solutions of hyperbolic differential equations with delay were obtained in [2]. Conditions for oscillation of the solutions of neutral type hyperbolic differential equations were obtained in [3,4,7].

2. Statement of the problem

In the present paper a necessary and sufficient condition for oscillation of the solutions of neutral type linear hyperbolic equations of the form

\[ u_{tt}(x, t) + \sum_{i} a_i u_{t}(x, t - \tau_i) - \left[ \Delta u(x, t) + \sum_{j} b_j(t) \Delta u(x, \rho_j(t)) \right] + c u(x, t) + \sum_{k} c_k u(x, t - \sigma_k) = 0, \quad (x, t) \in \Omega \times (0, \infty) \equiv G, \]  

is obtained, where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with a piecewise smooth boundary, \( \Delta u(x, t) = \sum_{i=1}^{n} u_{xx_i}(x, t) \), \( I, J, K \) are finite sets of successive positive integers containing the number 1.

Consider boundary conditions of the form

\[ \frac{\partial u}{\partial n} = 0, \quad (x, t) \in \partial \Omega \times [0, \infty), \]  

and

\[ u = 0, \quad (x, t) \in \partial \Omega \times [0, \infty). \]  

We shall say that conditions (H) are met if the following conditions hold.

(H1) \( b_j(t) \in C([0, \infty); \mathbb{R}) \) for \( j \in J \).

(H2) \( \rho_j(t) \in C([0, \infty); \mathbb{R}) \) and \( \lim_{t \to \infty} \rho_j(t) = \infty \) for \( j \in J \).

(H3) \( c, a_i, c_k \in \mathbb{R} \) for \( i \in I, k \in K \).

(H4) \( \tau_i = \text{const.} > 0, \sigma_k = \text{const.} > 0 \) for \( i \in I, k \in K \).

Definition 1. The solution \( u(x, t) \in C^2(G) \cap C^1(\overline{G}) \) of problem (1), (2), (1), (3) is said to oscillate in the domain \( G \) if for any positive number \( \alpha \) there exists a point \( (x_0, t_0) \in \Omega \times (\alpha, \infty) \) such that the equality \( u(x_0, t_0) = 0 \) holds.

3. Main results

In the subsequent theorems a necessary and sufficient condition for oscillation of the solutions of problems (1), (2) and (1), (3) in the domain \( G \) is obtained.

With each solution \( u(x, t) \in C^2(G) \cap C^1(\overline{G}) \) of problem (1), (2) we associate the function

\[ v(t) = \int_{\Omega} u(x, t) \, dx, \quad t \geq 0. \]
Lemma 2. Let conditions (H) hold and let \( u(x, t) \in C^2(G) \cap C^1(\bar{G}) \) be a solution of problem (1), (2). Then the function \( v(t) \) defined by (4) satisfies the differential equation

\[
v''(t) + \sum_l a_l v''(t - \tau_l) + cv(t) + \sum_k c_k u(t - \sigma_k) = 0, \quad t \geq t_0,
\]

where \( t_0 \) is a sufficiently large positive number.

Proof. Let \( u(x, t) \in C^2(G) \cap C^1(\bar{G}) \) be a solution of problem (1), (2). From condition (H2) it follows that there exists a number \( \mu > 0 \) such that \( \rho_j(t) > 0 \) for \( t \geq \mu, \ j \in I \). Introduce the notation

\[
t_0 = \max\{\mu, \tau_i, \sigma_k: i \in I, k \in K\}.
\]

Integrating both sides of (1) with respect to \( x \) over the domain \( \Omega \), for \( t \geq t_0 \) we obtain

\[
\frac{d^2}{dt^2} \left[ \int_\Omega u(x, t) \, dx + \sum_l a_l \int_\Omega u(x, t - \tau_l) \, dx \right]
- \left[ \int_\Omega \Delta u(x, t) \, dx + \sum_j b_j(t) \int_\Omega \Delta u(x, \rho_j(t)) \, dx \right]
+ c \int_\Omega u(x, t) \, dx + \sum_k c_k \int_\Omega u(x, t - \sigma_k) \, dx = 0.
\]

From Green's formula it follows that

\[
\int_\Omega \Delta u(x, t) \, dx = \int_{\partial \Omega} \frac{\partial u}{\partial n} \, dS = 0,
\]

\[
\int_\Omega \Delta u(x, \rho_j(t)) \, dx = \int_{\partial \Omega} \frac{\partial u}{\partial n}(x, \rho_j(t)) \, dS = 0, \quad j \in J.
\]

Using (7) and (8), from (6) we obtain that

\[
v''(t) + \sum_l a_l v''(t - \tau_l) + cv(t) + \sum_k c_k u(t - \sigma_k) = 0,
\]

which was to be proved. \( \square \)

In the domain \( \Omega \) consider the Dirichlet problem

\[
\Delta U(x) + \alpha U(x) = 0, \quad x \in \Omega,
\]

\[
U(x) = 0, \quad x \in \partial \Omega,
\]

where \( \alpha = \text{const.} \). It is well known [5] that the smallest eigenvalue \( \alpha_0 \) of problem (9), (10) is positive and the corresponding eigenfunction \( \phi(x) \) can be chosen so that \( \phi(x) > 0 \) for \( x \in \Omega \).

Assume that the following two additional conditions are fulfilled.

(H5) \( b_j(t) = b_j \in \mathbb{R} \) for \( j \in J \).

(H6) \( \rho_j(t) = t - \mu_j \) for \( j \in J \), where \( \mu_j = \text{const.} > 0 \).

With each solution \( u(x, t) \in C^2(G) \cap C^1(\bar{G}) \) of problem (1), (3) we associate the function

\[
w(t) = \int_\Omega u(x, t) \phi(x) \, dx, \quad t \geq 0.
\]

We note that such an averaging was first used in [6].
Lemma 3. Let conditions (H3)–(H6) hold and let $u(x, t) \in C^2(G) \cap C^1(\overline{G})$ be a solution of problem (1), (3). Then the function $w(t)$ defined by (11) satisfies the differential equation

$$w''(t) + \sum_i a_i w''(t - \tau_i) + \alpha_0 \left[w(t) + \sum_j b_j w(t - \mu_j)\right] + c w(t) + \sum_{k} c_k w(t - \sigma_k) = 0, \quad t \geq t_0,$$

where $t_0$ is a sufficiently large positive number.

Proof. Let $u(x, t) \in C^2(G) \cap C^1(\overline{G})$ be a solution of problem (1), (3). Introduce the notation $t_0 = \max\{\tau_i, \mu_j, \sigma_k : i \in I, j \in J, k \in K\}$.

Multiply both sides of (1) by the eigenfunction $\phi(x)$ and integrate with respect to $x$ over the domain $\Omega$. For $t \geq t_0$ we obtain

$$\frac{d^2}{dt^2} \left[\int_\Omega u(x, t) \phi(x) \, dx + \sum_i a_i \int_\Omega u(x, t - \tau_i) \phi(x) \, dx\right]$$

$$- \left[\int_\Omega \Delta u(x, t) \phi(x) \, dx + \sum_j b_j \int_\Omega \Delta u(x, t - \mu_j) \phi(x) \, dx\right]$$

$$+ c \int_\Omega u(x, t) \phi(x) \, dx + \sum_{k} c_k \int_\Omega u(x, t - \sigma_k) \phi(x) \, dx \, dx = 0. \quad (13)$$

From Green's formula it follows that

$$\int_\Omega \Delta u(x, t) \phi(x) \, dx = \int_\Omega u(x, t) \Delta \phi(x) \, dx$$

$$= -\alpha_0 \int_\Omega u(x, t) \phi(x) \, dx = -\alpha_0 w(t), \quad (14)$$

$$\int_\Omega \Delta u(x, t - \mu_j) \phi(x) \, dx = \int_\Omega u(x, t - \mu_j) \Delta \phi(x) \, dx$$

$$= -\alpha_0 \int_\Omega u(x, t - \mu_j) \phi(x) \, dx = -\alpha_0 w(t - \mu_j). \quad (15)$$

Using (14) and (15), from (13) we obtain that

$$w''(t) + \sum_i a_i w''(t - \tau_i) + \alpha_0 \left[w(t) + \sum_j b_j w(t - \mu_j)\right]$$

$$+ c w(t) + \sum_{k} c_k w(t - \sigma_k) = 0,$$

which was to be proved. \square

From the lemmas above it follows that the finding of conditions for oscillation of the solutions of (1) in the domain $G$ is reduced to the investigation of the oscillatory properties of
neutral type ordinary differential equations of the form

\[ \frac{d^2}{dt^2} \left[ x(t) + \sum_{i} p_i x(t - \tau_i) \right] + \sum_{k} q_k x(t - \sigma_k) = 0, \quad t \geq t_0. \] (16)

Assume that the following conditions are fulfilled.

(H7) \( p_i \in \mathbb{R}, \ tau_i = \text{const.} \geq 0 \) for \( i \in I \).

(H8) \( q_k \in \mathbb{R}, \ sigma_k = \text{const.} \geq 0 \) for \( k \in K \).

**Definition 4.** The solution \( x(t) \) of the differential equation (16) is said to oscillate if the function \( x(t) \) has a sequence of zeros tending to \( +\infty \). Otherwise the solution is said to be nonoscillating.

In the proof of the subsequent theorems we shall use the following result of [1].

**Theorem 5** (Arino and Győri [1]). Let conditions (H7)-(H8) hold. A necessary and sufficient condition for (16) to have a nonoscillating solution is that the corresponding characteristic equation

\[ \lambda^2 \left[ 1 + \sum_{i} p_i e^{-\lambda \tau_i} \right] + \sum_{k} q_k e^{-\lambda \sigma_k} = 0 \] (17)

should have a real root.

A corollary of Lemma 2 and Theorem 5 is the following necessary and sufficient condition for oscillation of the solutions of problem (1), (2).

**Theorem 6.** Let conditions (H) hold. A necessary and sufficient condition for all solutions of problem (1), (2) to oscillate in the domain \( G \) is that the equation

\[ f(\lambda) = \lambda^2 \left[ 1 + \sum_{i} a_i e^{-\lambda \tau_i} \right] + c + \sum_{k} c_k e^{-\lambda \sigma_k} = 0 \] (18)

should have no real roots.

**Proof.** (Necessity) If \( \lambda_0 \in \mathbb{R} \) is a root of (18), then the function \( u(x, t) = e^{\lambda_0 t} \) is a nonoscillating positive solution of problem (1), (2) in the domain \( \Omega \times [t_0, \infty) \).

(Sufficiency) Suppose that the assertion is not true and let \( u(x, t) \) be a nonoscillating solution of problem (1), (2). Let \( u(x, t) > 0 \) for \( (x, t) \in G \). (The case when \( u(x, t) < 0 \) for \( (x, t) \in G \) is considered analogously.) From Lemma 2 it follows that the function \( v(t) \) defined by (4) is a positive solution of the differential equation (5). Then from Theorem 5 applied to (5) it follows that (18) has a real root, which contradicts the condition of the theorem. \( \Box \)

A corollary of Lemma 3 and Theorem 5 is the following necessary and sufficient condition for oscillation of the solutions of problem (1), (3).
Theorem 7. Let conditions (H3)-(H6) hold. Then a necessary and sufficient condition for all solutions of problem (1), (3) to oscillate in the domain $G$ is that the equation

$$g(\lambda) \equiv \lambda^2 \left[ 1 + \sum_i a_i e^{-\lambda \tau_i} \right] + \alpha_0 \left[ 1 + \sum_j b_j e^{-\lambda \mu_j} \right] + c + \sum_k c_k e^{-\lambda \sigma_k} = 0$$

(19)

should have no real roots.

Proof. (Necessity) If $\lambda_0 \in \mathbb{R}$ is a root of (19), then the function $u(x, t) = e^{\lambda_0 \phi(x)}$ is a nonoscillating positive solution of problem (1), (3) in the domain $\Omega \times [t_0, \infty)$.

(Sufficiency) Suppose that the assertion is not true and let $u(x, t)$ be a nonoscillating solution of problem (1), (3). Let $u(x, t) > 0$ for $(x, t) \in G$. (The case when $u(x, t) < 0$ for $(x, t) \in G$ is considered analogously.) From Lemma 3 it follows that the function $w(t)$ defined by (11) is a positive solution of the differential equation (12). Then from Theorem 5 applied to (12) it follows that (19) has a real root, which contradicts the condition of the theorem. \( \square \)

References