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Currency option pricing with Wishart process

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ABSTRACT

It has been well-documented that foreign exchange rates exhibit both mean reversion and stochastic volatility. In addition to these, recent empirical evidence shows a stochastic skew of implied volatility surface from currency option data, which means that the slope of implied volatility curve of a given maturity is stochastically time varying. This paper develops a currency option pricing model which accommodates for this phenomena. The proposed model postulates that the log-currency value follows a mean reverting process with stochastic volatility driven by Wishart process under risk-neutral measure. Pricing formula for European currency option is derived in terms of Fourier Transform. Benchmarking against the Monte Carlo simulation, our numerical examples reveal that the pricing formula is accurate and remarkably efficient. The proposed model is also generalized to include jumps. The ability of the our model on capturing stochastic skew is illustrated through a numerical example.

1. Introduction

It is well-documented that the foreign exchange rates exhibit mean reversion (for example, [1,2]) and stochastic volatility (for example, [3–5]). It is important that option pricing models should be able to capture these features. Wong and Lo [6] assume that the log-currency value follows a mean reverting process and the volatility follows the Heston model. To further allow for multi-scale stochastic volatility, Wong and Zhao [7] assume that the volatility is driven by two Heston volatility factors.

Another important issue in financial modeling is "stochastic skew", which means that the slope of the implied volatility curve is stochastically time varying, documented in currency option markets in [8] and in index option markets in [4].

To incorporate the above three essential features, namely mean reversion, stochastic volatility and stochastic skew, into the dynamics of the foreign exchange rate, we generalize the two-factor Heston stochastic volatility model of Wong and Zhao [7] through the use of the Wishart process. The Wishart process, a matrix extension of Heston's [5] volatility model, is proposed in [9] and is then used in [10–12] to model stochastic volatility and covariance dynamics.

This paper is organized as follows. Section 2 provides the empirical evidence of stochastic skew in the EUR/USD currency option market. In Section 3, we formulate the currency option pricing model with the Wishart process and derive the pricing formula of currency options. We extend the model to include Poisson shocks, generalizing the framework of O'Hara and Pillay [13]. The implementation of the pricing formulas are demonstrated and contrasted with Monte Carlo simulation. In

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Section 4, we investigate the model property and show that the Wishart model provides additional freedom to better capture stochastic skew than the multi-factor Heston model. Section 5 concludes the paper.

2. Empirical evidence

To show the presence of stochastic skew in practice, we collect from Bloomberg daily mid quotes of the exchange rate of EUR/USD and the currency option written on EUR/USD from 1-Jan-2008 to 31-Oct-2010. The option quotes have 8 fixed time-to-maturities, namely, 1, 2, 3, 6, 9, 12, 18 and 24 months. We also collect quotes of 10-delta risk reversal of currency options[1] (R10) of all maturities, which is the difference between the implied volatilities of a 10-delta call option and a 10-delta put option. Hence, the risk reversal can be regarded as a measure of the slope of the implied volatility curve. Fig. D.1 shows that, for all maturities, the risk reversal is stochastically time varying, similar to the observation of Carr and Wu [8]. Hence, the empirical evidence strongly suggests that a currency option pricing model should be flexible enough to capture not only mean reversion and stochastic volatility but also stochastic skew.

3. The model

In this section, we generalize the two-factor Heston model of Wong and Zhao [7] through the use of the Wishart process. We derive the characteristic function of log-currency value and the futures-price calibrated characteristic function and provide the pricing formula for European currency call option. Finally, Monte Carlo simulation is used to demonstrate the quality of pricing formula.

We define the following notation: 1. $[A_{ij}]_{i,j=1,\ldots,n}$ as a constant square matrix ($[A_t^{ij}]_{i,j=1,\ldots,n}$ as a matrix-valued process) of order n with A_{ij} (A_t^{ij}) being its i,j-th element; 2. $[B]_{i,j}$ as the i,j-th element of the matrix B; 3. $\mathrm{Tr}(A)$ as the trace of the matrix A; and, 4. A^{T} as the transpose of the matrix A.

3.1. The dynamics of Wishart process and log-currency value

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q})$ be a filtered probability space where the square matrix Brownian motions $W_t = [W_t^{ij}]_{i,j=1,\ldots,n}$ and $Z_t = [Z_t^{ij}]_{i,j=1,\ldots,n}$ are defined for all $t \geq 0$ and adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and W_t^{ij} and Z_t^{ij}, for $i, j = 1, \ldots, n$, are independent scalar Brownian motions. Under a risk-neutral probability measure $\mathbb{Q}$[2], the Wishart process $\Sigma_t = [\Sigma_t^{ij}]_{i,j=1,\ldots,n}$ is an n-by-n symmetric positive definite matrix process which follows the dynamics

$$d\Sigma_t = \left(\Omega\Omega^{\mathrm{T}} + M\Sigma_t + \Sigma_t M^{\mathrm{T}}\right)dt + \sqrt{\Sigma_t}dW_tQ + Q^{\mathrm{T}}(dW_t)^{\mathrm{T}}\sqrt{\Sigma_t}, \tag{1}$$

where $M = [M_{ij}]_{i,j=1,\ldots,n}$, $Q = [Q_{ij}]_{i,j=1,\ldots,n}$ and $\Omega = [\Omega_{ij}]_{i,j=1,\ldots,n}$ are constant real square matrices, with Ω invertible, and $\sqrt{\Sigma_t}$ (the matrix square root of Σ_t) is defined as the unique symmetric positive definite matrix such that $\sqrt{\Sigma_t}\sqrt{\Sigma_t} = \Sigma_t$, that is, $\Sigma_t^{ij} = \sum_{l=1}^n \sigma_t^{il}\sigma_t^{lj} = \sum_{l=1}^n \sigma_t^{il}\sigma_t^{jl}$. Matrices M and Q are the matrix counterparts of mean reversion and vol-of-vol coefficients in the Heston model[3] (see [16]).[4] To guarantee the mean reverting feature and strict positivity of the Wishart process, M is required to be negative semi-definite and

$$\Omega\Omega^{\mathrm{T}} = \beta Q^{\mathrm{T}}Q,$$

with the real parameter $\beta > n - 1$ (see [9]).

Let S_t be a given currency in terms of another currency (that is, exchange rate) for which the risk-neutral process is postulated as

$$\begin{aligned} S_t &= \exp\left(X_t\right), \\ dX_t &= \left(\theta(t) - \kappa X_t - \frac{1}{2}\mathrm{Tr}\left(\Sigma_t\right)\right)dt + \mathrm{Tr}\left(\sqrt{\Sigma_t}dZ_t\right), \\ d\Sigma_t &= \left(\beta Q^{\mathrm{T}}Q + M\Sigma_t + \Sigma_t M^{\mathrm{T}}\right)dt + \sqrt{\Sigma_t}dW_tQ + Q^{\mathrm{T}}(dW_t)^{\mathrm{T}}\sqrt{\Sigma_t}, \end{aligned} \tag{2}$$

[1] For the details of R10 (and other related terminologies frequently used in currency option markets), please refer to Reiswich and Wystup [14].

[2] Appendix A presents a change of measure transformation between the real-world measure $\mathbb{P}$ and the risk-neutral measure $\mathbb{Q}$ using Girsanov's Theorem so that the affine nature of the dynamics is preserved in both measures.

[3] Unlike the multi-factor Heston model, we allow M and Q to be non-diagonal to enrich the dynamic interaction among components of Σ_t. An illustrative example using a 2 dimensional Wishart process can be found in [15].

[4] The existence and uniqueness of weak and strong solutions to matrix-valued affine processes are studied in [9,17,18].

where the constant κ is the mean reversion speed for the log-currency value, the deterministic function $\theta(t)$ represents the equilibrium mean level of the log-currency value against time and Z_t is correlated with W_t the following way (see [15]):

$$Z_t = W_t R^{\mathrm{T}} + B_t \sqrt{\mathbf{I}_n - RR^{\mathrm{T}}}, \tag{3}$$

where B_t is a matrix Brownian motion independent of W_t, $R = [R_{ij}]_{i,j=1,\ldots,n}$ and $\mathbf{I}_n$ is the identity matrix of order n.

Remarks. The proposed model embraces many existing models as its special cases. When $n = 1$, it is reduced to the model of Wong and Lo [6]; when $n = 2$ and the matrices M, Q and Σ_t in (1) and the matrix R in (3) are diagonal, it is reduced to the model of Wong and Zhao [7]; further, when $\theta(t) \equiv r$, the risk-free rate, and $\kappa = 0$, the proposed model is reduced to the two-factor Heston model of Christoffersen et al. [4].

3.2. The characteristic function

Denote the characteristic function of the log-currency value X_T as

$$\Psi(x, \Sigma, \tau; u) = \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(\mathrm{i}uX_T\Big)|X_t = x, \Sigma_t = \Sigma\Big], \tag{4}$$

where $\mathbb{E}^{\mathbb{Q}}[\cdot]$ denotes the expectation under $\mathbb{Q}$, $\tau = T - t \geq 0$ and $\mathrm{i} = \sqrt{-1}$. The following proposition holds.

Proposition 3.1. *If X_t follows the dynamics in (2), then the characteristic function for X_T in (4) is given by*

$$\Psi(x, \Sigma, \tau; u) = \exp\Big(\mathrm{Tr}\Big(A(\tau; u)\Sigma\Big) + b(\tau; u)x + c(\tau; u)\Big), \tag{5}$$

where

$$A(\tau; u) = H(\tau; u)^{-1} G(\tau; u), \tag{6}$$

$$b(\tau; u) = \mathrm{i}ue^{-\kappa\tau}, \tag{7}$$

$$c(\tau; u) = \mathrm{i}u \int_0^\tau \theta(T - s)e^{-\kappa s} ds - \frac{\beta}{2}\mathrm{Tr}\Big(\ln H(\tau; u) + M^{\mathrm{T}}\tau + \frac{2\mathrm{i}u}{\kappa}(1 - e^{-\kappa\tau})RQ\Big), \tag{8}$$

with

$$\frac{d}{d\tau}\Big(G(\tau; u) \quad H(\tau; u)\Big) = \Big(G(\tau; u) \quad H(\tau; u)\Big)\begin{pmatrix} M & -2Q^{\mathrm{T}}Q \\ \frac{1}{2}\mathrm{i}u(\mathrm{i}ue^{-2\kappa\tau} - e^{-\kappa\tau})\mathbf{I}_n & -\Big(M^{\mathrm{T}} + 2\mathrm{i}ue^{-\kappa\tau}RQ\Big) \end{pmatrix}, \tag{9}$$

where $H(0; u) = \mathbf{I}_n$ and $G(0; u) = \mathbf{0}_n$, $\mathbf{I}_n$ and $\mathbf{0}_n$ being the identity and zero matrices of order n, respectively.

Proof. Since the Wishart process is a matrix-valued affine process, following Da Fonseca et al. [15] and Duffie and Kan [19], we consider that the characteristic function of X_T is exponentially affine in the state variables, that is,

$$\Psi(x, \Sigma, \tau; u) = \exp\Big(\mathrm{Tr}\Big(A(\tau; u)\Sigma\Big) + b(\tau; u)x + c(\tau; u)\Big),$$

$$\Psi(x, \Sigma, 0; u) = \exp\Big(\mathrm{i}ux\Big),$$

so that $A(0; u) = \mathbf{0}_n$, $b(0; u) = \mathrm{i}u$ and $c(0; u) = 0$. Our strategy is to apply the Feynman–Kac argument to obtain a set of ordinary differential equations for each of the functions.

Denote $d\langle X, \Sigma^{ij}\rangle_t = \varrho_{ij}dt$, where $\langle X, \Sigma^{ij}\rangle_t$ is the covariation of X and Σ^{ij} at time t. The joint infinitesimal generator of (X, Σ) can be expressed in the form of

$$\mathcal{L}_{X,\Sigma} \triangleq \left(\theta(t) - \kappa x - \frac{1}{2}\mathrm{Tr}\Big(\Sigma\Big)\right)\frac{\partial}{\partial x} + \frac{1}{2}\mathrm{Tr}\Big(\Sigma\Big)\frac{\partial^2}{\partial x^2}$$
$$+ \mathrm{Tr}\Big(\Big(\beta Q^{\mathrm{T}}Q + M\Sigma + \Sigma M^{\mathrm{T}}\Big)D + 2\Sigma DQ^{\mathrm{T}}QD\Big) + \sum_{i,j=1}^{n} \varrho_{ij}\frac{\partial^2}{\partial \Sigma^{ij}\partial x}, \tag{10}$$

where $D = [\frac{\partial}{\partial \Sigma^{ij}}]_{i,j=1,\ldots,n}$ is a matrix differential operator. The first line and second line are the infinitesimal generators of the log-currency value process and the Wishart process (see [9]), respectively. ϱ_{ij} in the last line of Eq. (10) is derived as follows. Using the fact that

$$\begin{aligned}
\varrho_{ij}dt &= \mathbb{E}_t\left[\left(\sum_{l,k,h=1}^{n}\sigma_t^{lk}dW_t^{kh}R_{lh}\right)\left(\sum_{l,k=1}^{n}\sigma_t^{il}dW_t^{lk}Q_{kj}+\sum_{l,k=1}^{n}\sigma_t^{jl}dW_t^{lk}Q_{ki}\right)\right]\\
&= \sum_{l,k,h=1}^{n}\sigma_t^{hl}R_{hk}\Big(\sigma_t^{il}Q_{kj}+\sigma_t^{jl}Q_{ki}\Big)dt\\
&= \sum_{k,h=1}^{n}R_{hk}\left(\left(\sum_{l=1}^{n}\sigma_t^{il}\sigma_t^{hl}\right)Q_{kj}+\left(\sum_{l=1}^{n}\sigma_t^{jl}\sigma_t^{hl}\right)Q_{ki}\right)dt\\
&= \sum_{k,h=1}^{n}\Big(\Sigma_t^{ih}R_{hk}Q_{kj}+\Sigma_t^{jh}R_{hk}Q_{ki}\Big)dt\\
&= \Big([\Sigma_t RQ]_{i,j}+[Q^{\mathrm{T}}R^{\mathrm{T}}\Sigma_t]_{i,j}\Big)dt
\end{aligned}$$

with $\Sigma_t = \Sigma$, we have

$$\begin{aligned}
\sum_{i,j=1}^{n}\varrho_{ij}\frac{\partial^2}{\partial\Sigma^{ij}\partial x} &= \sum_{i,j=1}^{n}\left([\Sigma RQ]_{i,j}\frac{\partial}{\partial\Sigma^{ij}}+[Q^{\mathrm{T}}R^{\mathrm{T}}\Sigma]_{i,j}\frac{\partial}{\partial\Sigma^{ij}}\right)\frac{\partial}{\partial x}\\
&= \sum_{i,j=1}^{n}\left([\Sigma RQ]_{i,j}\frac{\partial}{\partial\Sigma^{ji}}+[\Sigma RQ]_{j,i}\frac{\partial}{\partial\Sigma^{ij}}\right)\frac{\partial}{\partial x}\\
&= 2\mathrm{Tr}\Big(\Sigma RQD\Big)\frac{\partial}{\partial x}.
\end{aligned}$$

Following Da Fonseca et al. [15], we impose the conditions that the characteristic function is twice-differentiable in x and Σ, and is differentiable in time.[5] Applying the Feynman–Kac argument on the characteristic function (5) by using the joint infinitesimal generator (10) gives the following partial differential equation for the characteristic function

$$\begin{aligned}
\frac{\partial\Psi}{\partial\tau} &= \left(\theta(T-\tau)-\kappa x-\frac{1}{2}\mathrm{Tr}\Big(\Sigma\Big)\right)\frac{\partial}{\partial x}\Psi+\frac{1}{2}\mathrm{Tr}\Big(\Sigma\Big)\frac{\partial^2}{\partial x^2}\Psi\\
&\quad+\mathrm{Tr}\Big(\Big(\beta Q^{\mathrm{T}}Q+M\Sigma+\Sigma M^{\mathrm{T}}\Big)D+2\Sigma DQ^{\mathrm{T}}QD\Big)\Psi+2\mathrm{Tr}\Big(\Sigma RQD\Big)\frac{\partial}{\partial x}\Psi\\
\Psi(x,\Sigma,0;u) &= \exp\Big(\mathrm{i}ux\Big).
\end{aligned}$$

By rearranging the terms,

$$\begin{aligned}
&\mathrm{Tr}\left(\frac{d}{d\tau}A(\tau;u)\Sigma\right)+\frac{d}{d\tau}b(\tau;u)x+\frac{d}{d\tau}c(\tau;u)\\
&\quad=\left(\theta(T-\tau)-\lambda m-\kappa x-\frac{1}{2}\mathrm{Tr}\Big(\Sigma\Big)\right)b(\tau;u)+\frac{1}{2}\mathrm{Tr}\Big(\Sigma\Big)b(\tau;u)^2\\
&\qquad+\mathrm{Tr}\left(\Big(\beta Q^{\mathrm{T}}Q+M\Sigma+\Sigma M^{\mathrm{T}}\Big)A(\tau;u)+2\Sigma A(\tau;u)Q^{\mathrm{T}}QA(\tau;u)\right)+2\mathrm{Tr}\Big(\Sigma RQA(\tau;u)\Big)b(\tau;u)\\
&\quad=\mathrm{Tr}\left(\left(\frac{1}{2}b(\tau;u)(b(\tau;u)-1)\mathbf{I}_n+A(\tau;u)M+\Big(M^{\mathrm{T}}+2b(\tau;u)RQ\Big)A(\tau;u)\right.\right.\\
&\qquad\left.\left.+2A(\tau;u)Q^{\mathrm{T}}QA(\tau;u)\right)\Sigma\right)-\kappa b(\tau;u)x+\theta(T-\tau)b(\tau;u)+\beta\mathrm{Tr}\Big(Q^{\mathrm{T}}QA(\tau;u)\Big),
\end{aligned}$$

[5] The imposed conditions, which are used by almost all affine models, allow us to solve the characteristic function in closed-form. We thank an anonymous referee for pointing out the required conditions.

and identifying the coefficients of X, Σ and constant terms respectively on both sides, we obtain the following system of ODEs:

$$\frac{d}{d\tau}A(\tau;u) = \frac{1}{2}b(\tau;u)(b(\tau;u)-1)\mathbf{I}_n + A(\tau;u)M + \left(M^{\mathrm{T}} + 2b(\tau;u)RQ\right)A(\tau;u) + 2A(\tau;u)Q^{\mathrm{T}}QA(\tau;u), \tag{11}$$

$$\frac{d}{d\tau}b(\tau;u) = -\kappa b(\tau;u), \tag{12}$$

$$\frac{d}{d\tau}c(\tau;u) = \theta(T-\tau)b(\tau;u) + \beta\mathrm{Tr}\left(Q^{\mathrm{T}}QA(\tau;u)\right), \tag{13}$$

with initial conditions

$$A(0;u) = \mathbf{0}_n, \tag{14}$$

$$b(0;u) = \mathrm{i}u, \tag{15}$$

$$c(0;u) = 0. \tag{16}$$

The solution to Eq. (12) with initial condition (15) is

$$b(\tau;u) = \mathrm{i}ue^{-\kappa\tau}.$$

By Radon's lemma (see [20]), Eq. (11) can be linearized with the following procedures. Let

$$G(\tau;u) = H(\tau;u)A(\tau;u),$$

with $H(\tau;u)$ invertible. With Eq. (11), differentiating both sides with respect to τ yields

$$\begin{aligned}\frac{d}{d\tau}G(\tau;u) &= \left(\frac{d}{d\tau}H(\tau;u)\right)A(\tau;u) + H(\tau;u)\frac{d}{d\tau}A(\tau;u)\\ &= \left(\frac{d}{d\tau}H(\tau;u)\right)A(\tau;u) + H(\tau;u)\left[\frac{1}{2}\mathrm{i}u(\mathrm{i}ue^{-2\kappa\tau} - e^{-\kappa\tau})\mathbf{I}_n \right.\\ &\quad \left. + A(\tau;u)M + \left(M^{\mathrm{T}} + 2\mathrm{i}ue^{-\kappa\tau}RQ\right)A(\tau;u) + 2A(\tau;u)Q^{\mathrm{T}}QA(\tau;u)\right]\\ &= G(\tau;u)M + \frac{1}{2}\mathrm{i}u(\mathrm{i}ue^{-2\kappa\tau} - e^{-\kappa\tau})H(\tau;u)\\ &\quad + \left(\frac{d}{d\tau}H(\tau;u) + 2G(\tau;u)Q^{\mathrm{T}}Q + H(\tau;u)\left(M^{\mathrm{T}} + 2\mathrm{i}ue^{-\kappa\tau}RQ\right)\right)A(\tau;u),\end{aligned}$$

and, then, matching both sides yields

$$\begin{cases}\dfrac{d}{d\tau}G(\tau;u) = G(\tau;u)M + \dfrac{1}{2}\mathrm{i}u(\mathrm{i}ue^{-2\kappa\tau} - e^{-\kappa\tau})H(\tau;u)\\ \dfrac{d}{d\tau}H(\tau;u) = -2G(\tau;u)Q^{\mathrm{T}}Q - H(\tau;u)\left(M^{\mathrm{T}} + 2\mathrm{i}ue^{-\kappa\tau}RQ\right).\end{cases}$$

The above system of ODEs can be re-written as follows:

$$\begin{aligned}&\frac{d}{d\tau}\begin{pmatrix}G(\tau;u) & H(\tau;u)\end{pmatrix}\\ &= \begin{pmatrix}G(\tau;u) & H(\tau;u)\end{pmatrix}\begin{pmatrix}M & -2Q^{\mathrm{T}}Q\\ \frac{1}{2}\mathrm{i}u(\mathrm{i}ue^{-2\kappa\tau} - e^{-\kappa\tau})\mathbf{I}_n & -\left(M^{\mathrm{T}} + 2\mathrm{i}ue^{-\kappa\tau}RQ\right)\end{pmatrix},\end{aligned}$$

where $H(0;u) = \mathbf{I}_n$ and $G(0;u) = \mathbf{0}_n$. Therefore, $A(\tau;u)$ is solved and given by (6) and (9). Finally, consider Eq. (13),

$$\begin{aligned}\frac{d}{d\tau}c(\tau) &= \theta(T-\tau)\mathrm{i}ue^{-\kappa\tau} + \beta\mathrm{Tr}\left(Q^{\mathrm{T}}QA(\tau;u)\right)\\ &= \theta(T-\tau)\mathrm{i}ue^{-\kappa\tau} + \beta\mathrm{Tr}\left(Q^{\mathrm{T}}QH(\tau;u)^{-1}G(\tau;u)\right)\end{aligned}$$

$$= \theta(T-\tau)\mathrm{i}ue^{-\kappa\tau} - \frac{\beta}{2}\mathrm{Tr}\left(Q^{\mathrm{T}}QH(\tau;u)^{-1}\left(\frac{d}{d\tau}H(\tau;u) + H(\tau;u)\left(M^{\mathrm{T}} + 2\mathrm{i}ue^{-\kappa\tau}RQ\right)\right)(Q^{\mathrm{T}}Q)^{-1}\right)$$

$$= \theta(T-\tau)\mathrm{i}ue^{-\kappa\tau} - \frac{\beta}{2}\mathrm{Tr}\left(H(\tau;u)^{-1}\frac{d}{d\tau}H(\tau;u) + M^{\mathrm{T}} + 2\mathrm{i}ue^{-\kappa\tau}RQ\right).$$

The solution of $c(\tau)$ in (8) can be obtained by directly integrating from 0 to τ with initial condition (16). □

As the matrix-valued differential equation (9) involves τ-dependent parameters and we use the classical fourth-order Runge–Kutta method to solve (9) (see Appendix B for the details of the Runge–Kutta method), we choose to apply the numerical method on the linear differential equation (9) but not on the matrix-valued differential Riccati equation (11) because the numerical stability of the former is much easier to control than the latter.

Now, we extend the dynamic of X_t in (2) by including the stochastic jump component as follows,

$$dX_t^{(J)} = dX_t - m\lambda dt + d\left(\sum_{i=1}^{N_t} J_i\right), \tag{17}$$

where $\sum_{i=1}^{N_t} J_i$ is a compound Poisson process in which N_t is a Poisson process with constant intensity λ and J_i, $i = 1, \ldots,$ are i.i.d. random variables with probability density function $f(x)$; the compound Poisson process is assumed to be independent of W_t and Z_t under the measure $\mathbb{Q}$, and $m = \int_{\mathbb{R}}(e^x - 1)f(dx)$.

Corollary 3.1. *The characteristic function of $X_T^{(J)}$ in (17), conditional on $X_t^{(J)} = x$ and $\Sigma_t = \Sigma$, is given by*

$$\Psi^{(J)}(x, \Sigma, \tau; u) = \exp\left(\mathrm{Tr}\left(A(\tau;u)\Sigma\right) + b(\tau;u)x + \widetilde{c}(\tau;u)\right),$$

where $A(\tau;u)$ and $b(\tau;u)$ are defined in Proposition 3.1, and,

$$\widetilde{c}(\tau;u) = \mathrm{i}u\int_0^\tau \theta(T-s)e^{-\kappa s}ds - \mathrm{i}u\lambda m(1-e^{-\kappa\tau}) - \frac{\beta}{2}\mathrm{Tr}\left(\ln H(\tau;u)\right) + \Lambda(u)\tau$$
$$\Lambda(u) = \lambda\int_{\mathbb{R}}(e^{\mathrm{i}ux} - 1)f(dx).$$

Proof. Rewrite $dX_t^{(J)}$ as

$$dX_t^{(J)} = d\widetilde{X}_t + d\left(\sum_{i=1}^{N_t} J_i\right),$$

where $d\widetilde{X}_t = dX_t - m\lambda dt$. Hence,

$$\begin{aligned}
\Psi^{(J)}(x, \Sigma, \tau; u) &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\mathrm{i}uX_T^{(J)}\right)\middle| X_t^{(J)} = x, \Sigma_t = \Sigma\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\mathrm{i}u\left(\widetilde{X}_T + \sum_{i=1}^{N_\tau} J_i\right)\right)\middle| X_t^{(J)} = x, \Sigma_t = \Sigma\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\mathrm{i}u\widetilde{X}_T\right)\middle| X_t^{(J)} = x, \Sigma_t = \Sigma\right]\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\mathrm{i}u\left(\sum_{i=1}^{N_\tau} J_i\right)\right)\right].
\end{aligned}$$

The last line is obtained by using the independence of the compound Poisson process and the process $\widetilde{X}_T$. The first expectation in the last row can be derived from Proposition 3.1 by replacing $\theta(t)$ with $\theta(t) - m\lambda$ and the second one is the characteristic function of the compound Poisson process, which is given by

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\mathrm{i}u\left(\sum_{i=1}^{N_\tau} J_i\right)\right)\right] = \exp\left(\lambda\tau\int_{\mathbb{R}}(e^{\mathrm{i}ux} - 1)f(dx)\right). \quad \square$$

With Corollary 3.1, the dynamic of $X_t^{(J)}$ in (17) can be considered as a generalization of the model proposed in [13] in which the jump size is i.i.d. normally distributed with mean μ and variance γ^2.

3.3. Super-calibration to FX futures prices

Since FX futures are actively traded, it is important to ensure that the FX option prices derived are consistent with the FX futures prices. Under a risk-neutral measure $\mathbb{Q}$, the relationship between spot price and current futures price with maturity T is given by

$$\begin{aligned} F_T(t) &= \mathbb{E}^{\mathbb{Q}}\left[S_T | X_t = x, \Sigma_t = \Sigma \right] \\ &= \Psi(x, \Sigma, \tau; -\mathrm{i}) \\ &= \exp\left(\mathrm{Tr}\left(A(\tau; -\mathrm{i})\Sigma \right) + b(\tau; -\mathrm{i})x + c(\tau; -\mathrm{i}) \right). \end{aligned}$$

It is interesting to note that the first term of the function $c(\tau; -\mathrm{i})$ in (8), which is an integral with the time-dependent mean reversion level $\theta(t)$, can be absorbed into the term structure of futures prices so that the characteristic function can be re-written without the knowledge of the functional form of $\theta(t)$.

Proposition 3.2. *If X_t follows the dynamics in (2), then the characteristic function for X_T calibrated to the current futures price $F_T(t)$ is given by*

$$\Psi(x, \Sigma, \tau; u, F_T(t)) = \exp\left(\mathrm{i}u \ln F_T(t) + \mathrm{Tr}\left(\Delta A(\tau; u)\Sigma \right) + \Delta c(\tau; u) \right),$$

where

$$\begin{aligned} \Delta A(\tau; u) &= A(\tau; u) - \mathrm{i}uA(\tau; -\mathrm{i}), \\ \Delta c(\tau; u) &= -\frac{1}{2}\beta \mathrm{Tr}\Big(\ln H(\tau; u) - \mathrm{i}u \ln H(\tau; -\mathrm{i}) + M^{\mathrm{T}}\tau(1 - \mathrm{i}u) \Big) \end{aligned}$$

with $A(\tau; u)$ and $H(\tau; u)$ solved in Proposition 3.1.

Proof. Please refer to Appendix C. □

3.4. Pricing formula of FX options

An FX call option with strike price K and maturity T has the payoff

$$\max\{S_T - K, 0\}.$$

To price FX call option in the Wishart model, we adopt the fast Fourier Transform approach of Carr and Madan [21] so that rapid computation of option prices and Greeks can be achieved, which are essential for trading and hedging. Let $C(K, T)$ be the value of FX call option with strike price K and maturity T, where the underlying is $S_t = \exp(X_t)$ and the continuously compounded domestic risk-free rate is r. Then,

$$C(K, T) = \frac{e^{-\alpha \ln K}}{\pi} \int_0^\infty \frac{e^{-rT - \mathrm{i}\xi \ln K}\Psi(x, \Sigma, T; \xi - (\alpha + 1)\mathrm{i})}{\alpha^2 + \alpha - \xi^2 + (2\alpha + 1)\mathrm{i}\xi} d\xi, \tag{18}$$

for some constant $\alpha > 0$. The above integral is approximated by Simpson's rule:

$$C(K, T) \approx \frac{e^{-\alpha \ln K}}{\pi} \sum_{j=0}^{N-1} \frac{e^{-rT - \mathrm{i}\xi_j \ln K}\Psi(x, \Sigma, T; \xi_j - (\alpha + 1)\mathrm{i})}{\alpha^2 + \alpha - \xi_j^2 + (2\alpha + 1)\mathrm{i}\xi_j} w_j, \tag{19}$$

where N is the number of grid points, $\xi_j = j\eta$ and

$$w_j = \begin{cases} \frac{1}{3}\eta & \text{for } j = 0, N - 1 \\ \frac{1}{3}(3 + (-1)^{j+1})\eta & \text{for } j = 1, \ldots, N - 2. \end{cases}$$

The implementation of the fast Fourier Transform can be found in [21] so it is omitted.

3.5. Simulation study

To investigate the accuracy and efficiency of our model in which a numerical method is used, we compare the option prices computed by pricing formula (18) and by Monte Carlo simulation,[6] whose procedure is given in Appendix D. For the

Table 1
The call option prices and CPU times produced by the pricing formula and Monte Carlo simulation.

Strike price	PF	MC	% Error	PF	MC	% Error
		$T = 0.5$			$T = 1$	
0.85	0.3360	0.3364	−0.11	0.4025	0.4022	0.07
0.90	0.3119	0.3123	−0.11	0.3809	0.3806	0.07
0.95	0.2896	0.2899	−0.12	0.3606	0.3604	0.07
1.00	0.2689	0.2692	−0.12	0.3416	0.3414	0.06
1.05	0.2497	0.2500	−0.13	0.3238	0.3237	0.05
1.10	0.2320	0.2323	−0.13	0.3071	0.3070	0.04
1.15	0.2156	0.2159	−0.14	0.2915	0.2914	0.03
Total CPU time (s)	0.06	409		0.11	817	
Strike price	PF	MC	% Error	PF	MC	% Error
		$T = 1.5$			$T = 2$	
0.85	0.4310	0.4316	−0.14	0.4410	0.4409	0.03
0.90	0.4107	0.4113	−0.14	0.4215	0.4214	0.02
0.95	0.3915	0.3921	−0.14	0.4031	0.4030	0.02
1.00	0.3735	0.3741	−0.14	0.3857	0.3856	0.01
1.05	0.3566	0.3571	−0.15	0.3693	0.3693	0.01
1.10	0.3406	0.3411	−0.15	0.3539	0.3538	0.01
1.15	0.3255	0.3260	−0.15	0.3393	0.3392	0.00
Total CPU Time (s)	0.17	1223		0.22	1629	

numerical integration for the pricing formula, the number of grid point is $N = 64$, the grid size is $\eta = 0.25$ and the damping coefficient is $\alpha = 3$. Let $\theta = 0.1$, $\kappa = 0.25$, $S_0 = 1$ be the parameters of the log-currency value dynamics. The parameters of the Wishart process are chosen in similar scale with those of Wong and Zhao [7], namely, $\beta = 5$,

$$M = \begin{pmatrix} -0.50 & 0.00 \\ 0.00 & -0.05 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0.10 & 0.05 \\ 0.05 & 0.05, \end{pmatrix},$$
$$R = \begin{pmatrix} -0.04 & -0.02 \\ -0.02 & -0.04 \end{pmatrix}, \qquad \Sigma_0 = \begin{pmatrix} 0.50 & 0.20 \\ 0.20 & 0.50 \end{pmatrix}.$$

In Table 1, the 0.5-year, 1-year, 1.5-year and 2-year call option prices are computed and compared with MC simulation with 100,000 sample paths and time step of 1/100, where the prices and CPU times are reported ("PF" refers to pricing formula and "MC" to Monte Carlo simulation). As shown in the table, the model prices are very close to simulated prices and the CPU times for pricing formula are much less than those for MC simulation. Thus, the simulation study demonstrates that the analytical formula is correct and efficient.

4. Model property

In this section, we demonstrate the flexibility of the Wishart model on capturing stochastic skew over the multi-factor Heston model.

4.1. Stochastic instantaneous correlation

Under the stochastic volatility framework, the implied volatility skew is determined by the instantaneous correlation between the log-currency noise and volatility noise, which, for the Wishart model, is given by (see [15])

$$\rho_t^{\mathrm{Wis}} = \frac{\mathrm{Tr}\left(RQ\,\Sigma_t\right)}{\sqrt{\mathrm{Tr}\left(\Sigma_t\right)}\sqrt{\mathrm{Tr}\left(Q^{\mathrm{T}}Q\,\Sigma_t\right)}}. \tag{20}$$

With Eq. (20), the stochastic instantaneous correlation under the multi-factor Heston model, ρ_t^{Hes}, can be derived by taking the matrix RQ and Σ_t to be diagonal and is shown to be dependent solely on the variance factors Σ_t^{11} and Σ_t^{22}. In the case of

6 Although finite difference methods are widely used in option pricing under stochastic volatility models, it is difficult to implement here because of the dimensionality of the model. For instance, the mean reverting dynamics X_t with a 2-D Wishart process Σ_t has already involved 5 variables, namely, $t, X_t, \Sigma_t^{11}, \Sigma_t^{12}$ and Σ_t^{22}.

Σ_t being a 2×2 symmetric positive definite matrix and R being upper triangular, the stochastic instantaneous correlations under the Wishart model, ρ_t^{Wis}, is given by

$$\rho_t^{\text{Wis}} = \rho_t^{\text{Hes}} + \frac{R_{12} Q_{22}}{\sqrt{\Sigma_t^{11} + \Sigma_t^{22}} \sqrt{Q_{11}^2 \Sigma_t^{11} + Q_{22}^2 \Sigma_t^{22}}} \Sigma_t^{12}. \tag{21}$$

The above relation shows that, when compared with the multi-factor Heston model, the Wishart model contains extra parameter R_{12} and process Σ_t^{12}, which act independently from the variance factors, that offer extra flexibility to model stochastic skew. This flexibility is crucial when those volatility factors have been fitted to short-term and long-term volatility levels.

4.2. Numerical example

On 31-Dec-2009, the spot price of the 1 EUR is 1.4405 USD and the futures prices are shown below.

Maturity	Futures price (USD/EUR)
1-month	1.440437
2-month	1.440354
3-month	1.440260
6-month	1.439749
9-month	1.438766
12-month	1.437908
18-month	1.437360
24-month	1.438791

With Corollary 3.1, the form of $\theta(t)$ is not important when the term structure of futures prices are available. For the other model parameters, they are given by $\kappa = 0.25$, $\beta = 5$,

$$M = \begin{pmatrix} -10.00 & 0.00 \\ 0.00 & -0.05 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0.20 & 0.00 \\ 0.00 & 0.20 \end{pmatrix},$$
$$R = \begin{pmatrix} +0.10 & R_{12} \\ 0.00 & +0.10 \end{pmatrix}, \qquad \Sigma_0 = \begin{pmatrix} 0.05 & 0.04 \\ 0.04 & 0.05 \end{pmatrix}.$$

In our numerical experiment, we consider the two correlation structures, the first is $R_{12} = -0.4$ and the second is $R_{12} = +0.4$ in the matrix R in order to show that the parameter R_{12} and state variable Σ_t^{12} offer the Wishart model the freedom to control the implied volatility skew with respect to the two-factor Heston model.

Fig. D.2 displays the implied volatility curves generated by the two models, where the lines with *crosses* and lines with *stars* represent the curves generated by the Wishart model with $R_{12} = -0.4$ and $R_{12} = +0.4$ respectively, and the lines with *dots* represent the curves generated by two-factor Heston model. In Fig. D.2, we observe that the implied volatilities for the short maturity options, from one-month to six-month, are twisted from negative skew to positive skew pivoted around the spot price $S_0 = 1.4405$ when R_{12} changes from -0.4 to $+0.4$. The term structure of the implied volatility skew[7] in Fig. D.3 further confirms the conclusions which are made from Fig. D.2. In other words, the Wishart model offers extra flexibility to control the implied volatility skew without affecting the volatility level. The significance of the above experiment is that supposing the two-factor Heston model has a good fit to the short-term volatility by the process Σ_t^{11} and long-term volatility by the process Σ_t^{22} with suitable degree of correlation to the log-currency process, the Wishart model provides another parameter R_{12} to fit the variation of implied volatility skew that is not captured by the short-term and long-term volatilities.

5. Conclusion

We propose a model to simultaneously capture the three essential features observed in the currency market: mean reversion, stochastic volatility and stochastic skew. Using the non-diagonal elements of matrices in the Wishart process, our model offers extra control on the implied volatility skew as compared with the multi-factor Heston model. Analytical solutions are derived for the characteristic functions and European options. This enables our framework to be implemented accurately and efficiently for practical use as shown in our simulation study.

[7] Here, the skew of the implied volatility curve for a particular maturity is defined as the difference between the Black's implied volatilities at the right-end point and the left-end point of that volatility curve.

Acknowledgment

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Appendix A. The change of probability measure

As we attempt to develop a model which is calibrated to derivatives prices, it is necessary for us to start by the risk-neutral log-currency dynamics in (2) so that the expectation of currency value S_T under risk-neutral measure $\mathbb{Q}$ matches the term structure of futures price $F_T(t)$, that is,

$$F_T(t) = \mathbb{E}^{\mathbb{Q}}[S_T|\mathcal{F}_t].$$

The matching process determines the function $\theta(t)$ in the dynamics of X_t. However, when derivatives prices are not available, one has to estimate parameters from historical values of the underlying currency. The estimation leads to the dynamics under the real-world probability measure $\mathbb{P}$. To value derivatives, the real-world process has to transform into risk-neutral process. We propose a change of measure to bridge the real-world and risk-neutral process below.

Suppose that $W_t^{\mathbb{P}}$ and $B_t^{\mathbb{P}}$ are independent square matrix Brownian motions under $\mathbb{P}$, and $W_t^{\mathbb{Q}}$ and $B_t^{\mathbb{Q}}$ are independent square matrix Brownian motions under $\mathbb{Q}$. Consider the following relationships.

$$dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} + \frac{b-\theta(t)}{n}(\sqrt{\Sigma_t})^{-1}R^{\mathrm{T}}dt,$$
$$dB_t^{\mathbb{Q}} = dB_t^{\mathbb{P}} + \frac{b-\theta(t)}{n}(\sqrt{\Sigma_t})^{-1}\sqrt{\mathbf{I}_n - RR^{\mathrm{T}}}dt,$$
$$dZ_t^{\mathbb{Q}} = dW_t^{\mathbb{Q}}R + dB_t^{\mathbb{Q}}\sqrt{\mathbf{I}_n - RR^{\mathrm{T}}},$$

where b is some constant. Then, $Z_t^{\mathbb{Q}}$ is also a square matrix Brownian motion under $\mathbb{Q}$. Now, we perform the change of probability measure from risk-neutral measure $\mathbb{Q}$ to physical measure $\mathbb{P}$.[8]

$$\begin{aligned}
dX_t &= \left(\theta(t) - \kappa X_t - \frac{1}{2}\mathrm{Tr}\left(\Sigma_t\right)\right)dt + \mathrm{Tr}\left(\sqrt{\Sigma_t}dZ_t^{\mathbb{Q}}\right)\\
&= \left(\theta(t) - \kappa X_t - \frac{1}{2}\mathrm{Tr}\left(\Sigma_t\right)\right)dt + \mathrm{Tr}\left(\sqrt{\Sigma_t}dW_t^{\mathbb{Q}}R + \sqrt{\Sigma_t}dB_t^{\mathbb{Q}}\sqrt{\mathbf{I}_n - RR^{\mathrm{T}}}\right)\\
&= \left(\theta(t) - \kappa X_t - \frac{1}{2}\mathrm{Tr}\left(\Sigma_t\right)\right)dt\\
&\quad + \mathrm{Tr}\left(\sqrt{\Sigma_t}\left(dW_t^{\mathbb{P}} + \frac{b-\theta(t)}{n}(\sqrt{\Sigma_t})^{-1}R^{\mathrm{T}}dt\right)R\right)\\
&\quad + \mathrm{Tr}\left(\sqrt{\Sigma_t}\left(dB_t^{\mathbb{P}} + \frac{b-\theta(t)}{n}(\sqrt{\Sigma_t})^{-1}\sqrt{\mathbf{I}_n - RR^{\mathrm{T}}}dt\right)\sqrt{\mathbf{I}_n - RR^{\mathrm{T}}}\right)\\
&= \left(\theta(t) - \kappa X_t - \frac{1}{2}\mathrm{Tr}\left(\Sigma_t\right)\right)dt + \mathrm{Tr}\left(\sqrt{\Sigma_t}dW_t^{\mathbb{P}}R + \sqrt{\Sigma_t}dB_t^{\mathbb{P}}\sqrt{\mathbf{I}_n - RR^{\mathrm{T}}}\right)\\
&\quad + \mathrm{Tr}\left(\frac{b-\theta(t)}{n}\mathbf{I}_nR^{\mathrm{T}}R\right)dt + \mathrm{Tr}\left(\frac{b-\theta(t)}{n}\left(\mathbf{I}_n - RR^{\mathrm{T}}\right)\right)dt\\
&= \left(\theta(t) - \kappa X_t - \frac{1}{2}\mathrm{Tr}\left(\Sigma_t\right)\right)dt + \mathrm{Tr}\left(\sqrt{\Sigma_t}dW_t^{\mathbb{P}}R + \sqrt{\Sigma_t}dB_t^{\mathbb{P}}\sqrt{\mathbf{I}_n - RR^{\mathrm{T}}}\right)\\
&\quad + \frac{b-\theta(t)}{n}\mathrm{Tr}\left(R^{\mathrm{T}}R\right)dt + \frac{b-\theta(t)}{n}\mathrm{Tr}\left(\mathbf{I}_n - RR^{\mathrm{T}}\right)dt\\
&= \left(b - \kappa X_t - \frac{1}{2}\mathrm{Tr}\left(\Sigma_t\right)\right)dt + \mathrm{Tr}\left(\sqrt{\Sigma_t}\left(dW_t^{\mathbb{P}}R + dB_t^{\mathbb{P}}\sqrt{\mathbf{I}_n - RR^{\mathrm{T}}}\right)\right).
\end{aligned}$$

---

8 As we propose the dynamics of X_t and Σ_t under a risk-neutral measure $\mathbb{Q}$, we should deduce their dynamics under the physical measure $\mathbb{P}$ with a suitable transformation so that the affine property is preserved. Reversing the transformation allows one to deduce the dynamics under $\mathbb{Q}$ from $\mathbb{P}$.

Define

$$dZ_t^{\mathbb{P}} = dW_t^{\mathbb{P}} R + dB_t^{\mathbb{P}} \sqrt{\mathbf{I}_n - RR^{\mathrm{T}}}.$$

Then, $Z_t^{\mathbb{P}}$ is a square matrix Brownian motion under $\mathbb{P}$. The mean reversion level of X_t under physical measure $\mathbb{P}$ is thus assumed to be a constant b.

For the Wishart process,

$$\begin{aligned}
d\Sigma_t &= \left(\beta Q^{\mathrm{T}} Q + M\Sigma_t + \Sigma_t M^{\mathrm{T}}\right)dt + \sqrt{\Sigma_t}\, dW_t^{\mathbb{Q}} Q + Q^{\mathrm{T}} (dW_t^{\mathbb{Q}})^{\mathrm{T}} \sqrt{\Sigma_t} \\
&= \left(\beta Q^{\mathrm{T}} Q + M\Sigma_t + \Sigma_t M^{\mathrm{T}}\right)dt + \sqrt{\Sigma_t}\left(dW_t^{\mathbb{P}} + \frac{b-\theta(t)}{n}(\sqrt{\Sigma_t})^{-1} R^{\mathrm{T}} dt\right) Q \\
&\quad + Q^{\mathrm{T}}\left(dW_t^{\mathbb{P}} + \frac{b-\theta(t)}{n}(\sqrt{\Sigma_t})^{-1} R^{\mathrm{T}} dt\right)^{\mathrm{T}} \sqrt{\Sigma_t} \\
&= \left(\beta Q^{\mathrm{T}} Q + M\Sigma_t + \Sigma_t M^{\mathrm{T}}\right)dt + \sqrt{\Sigma_t}\, dW_t^{\mathbb{P}} Q + Q^{\mathrm{T}} (dW_t^{\mathbb{P}})^{\mathrm{T}} \sqrt{\Sigma_t} \\
&\quad + \frac{b-\theta(t)}{n} R^{\mathrm{T}} Q dt + \frac{b-\theta(t)}{n} Q^{\mathrm{T}} R dt \\
&= \left(\left(\beta Q^{\mathrm{T}} Q + \frac{b-\theta(t)}{n} R^{\mathrm{T}} Q + \frac{b-\theta(t)}{n} Q^{\mathrm{T}} R\right) + M\Sigma_t + \Sigma_t M^{\mathrm{T}}\right)dt \\
&\quad + \sqrt{\Sigma_t}\, dW_t^{\mathbb{P}} Q + Q^{\mathrm{T}} (dW_t^{\mathbb{P}})^{\mathrm{T}} \sqrt{\Sigma_t}.
\end{aligned}$$

Thus, it can be seen that the matrix of mean reversion of Wishart process is time-dependent but non-stochastic.

Appendix B. The Runge–Kutta method

In the numerical solution of differential equations, the Taylor-series method is simple to implement but has the drawback of requiring higher order derivatives and some error analysis prior to implementation. To circumvent these problems, the family of Runge–Kutta method imitates the Taylor-series method by means of clever combinations of the values of the first derivative through the repeating use of chain rule of differentiation. Now, we give below the classical fourth-order Runge–Kutta method used in this paper. Given the time to maturity T, we partition the interval $[0, T]$ into N equal subintervals with partition points $0 = \tau_0 < \tau_1 < \cdots < \tau_n < \cdots < \tau_N = T$, where $\tau_n = \tau_0 + nh$ and $h = \frac{T}{N}$ for $n = 0, \ldots, N-1$. Denote

$$X(\tau) = \left(G(\tau; u) \quad H(\tau; u)\right),$$

$$f(\tau) = \begin{pmatrix} M & -2Q^{\mathrm{T}} Q \\ \frac{1}{2} \mathrm{i}u(\mathrm{i}ue^{-2\kappa\tau} - e^{-\kappa\tau})\mathbf{I}_n & -\left(M^{\mathrm{T}} + 2\mathrm{i}ue^{-\kappa\tau} RQ\right) \end{pmatrix}.$$

Let X_n be the numerical approximation of $X(\tau)$ at τ_n, for $n = 1, 1, \ldots, N$, with $X_0 = X(0)$. In our case, the method is given by, for $n = 0, 1, \ldots, N-1$,

$$\begin{aligned}
Y_1 &= X_n, \\
Y_2 &= X_n + \frac{h}{2} Y_1 f(\tau_n), \\
Y_3 &= X_n + \frac{h}{2} Y_2 f\left(\tau_n + \frac{h}{2}\right), \\
Y_4 &= X_n + h Y_3 f\left(\tau_n + \frac{h}{2}\right), \\
X_{n+1} &= X_n + \frac{h}{6}\left[Y_1 f(\tau_n) + 2Y_2 f\left(\tau_n + \frac{h}{2}\right) + 2Y_3 f\left(\tau_n + \frac{h}{2}\right) + Y_4 f(\tau_n + h)\right].
\end{aligned}$$

The fourth-order Runge–Kutta method is called an *explicit one-step method* and involves *local truncation error* of order 4 so, as a one-step method, it is *convergent*. Moreover, it is *strongly stable*. For the efficiency of implementation, we fix the stepsize;

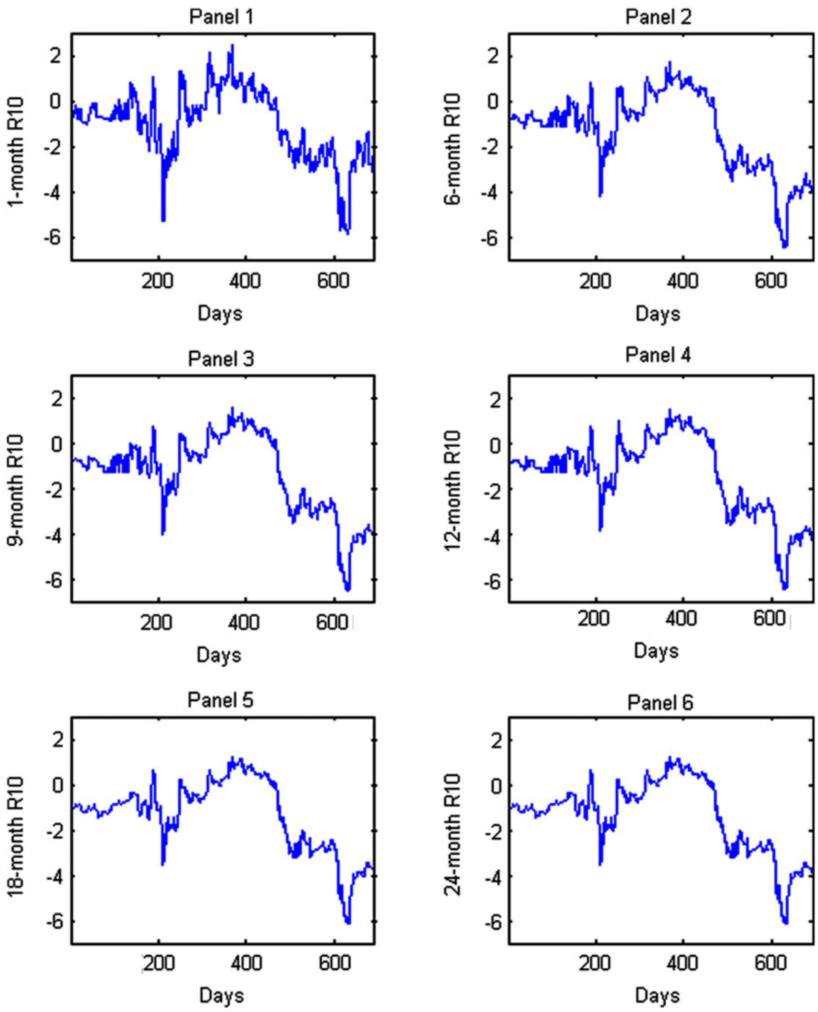

Fig. D.1. The time series of 1, 6, 9, 12, 18, 24-month 10-delta risk reversals from 1-Jan-2008 to 31-Oct-2010.

however, to impose error control, *adaptive* algorithms, for example, the Runge–Kutta–Fehlberg Method, which vary stepsize to accommodate local peculiarities in the solution, should be implemented. These standard results can be found in Chapter 5 of Burden and Faires [22] or Chapter 7 of Allen and Isaacson [23].

Appendix C. Proof of Proposition 3.2

The current futures price with maturity T is given by

$$
\begin{aligned}
F_T(t) &= \mathbb{E}\left[S_T | X_t = x, \Sigma_t = \Sigma \right] \\
&= \Psi(x, \Sigma, \tau; -\mathrm{i}) \\
&= \exp\left(\mathrm{Tr}\left(A(\tau; -\mathrm{i})\Sigma \right) + b(\tau; -\mathrm{i})x + c(\tau; -\mathrm{i}) \right)
\end{aligned}
$$

where $A(\tau; -\mathrm{i})$ is solved in Proposition 3.1 and

$$
\begin{aligned}
b(\tau; -\mathrm{i}) &= e^{-\kappa\tau}, \\
c(\tau; -\mathrm{i}) &= \int_0^\tau \theta(T-s)e^{-\kappa s}ds - \frac{1}{2}\beta \mathrm{Tr}\left(\ln H(\tau; -\mathrm{i}) + M^{\mathrm{T}}\tau + \frac{2}{\kappa}(1 - e^{-\kappa\tau})RQ \right).
\end{aligned}
$$

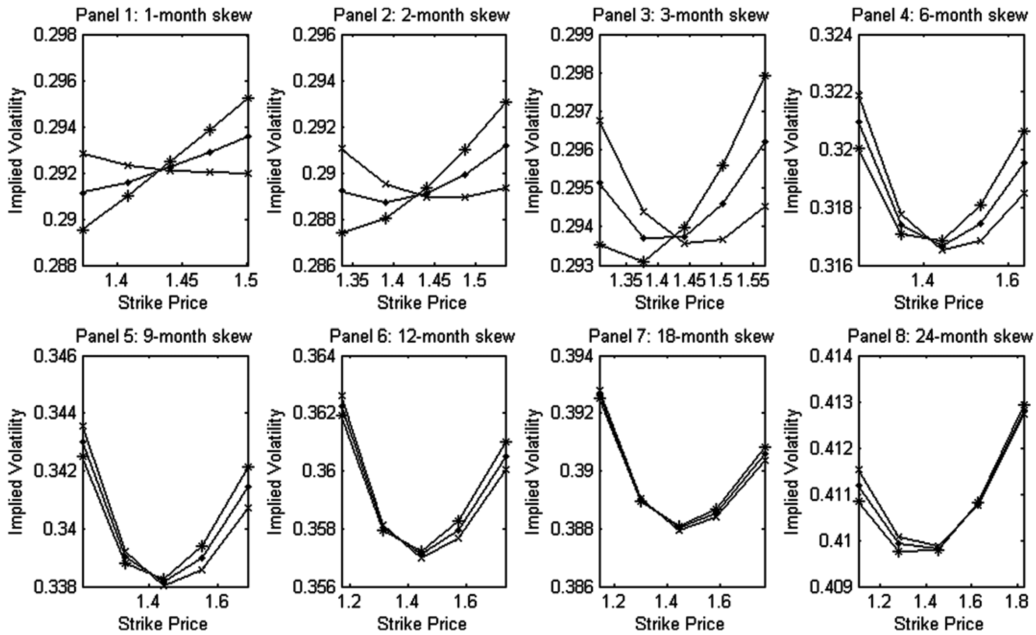

Fig. D.2. The implied volatility curves generated by the Wishart model (lines with *crosses* represent the case of $R_{12} = -0.4$, lines with *stars* represent the case of $R_{12} = +0.4$) and two-factor Heston model (lines with *dots*).

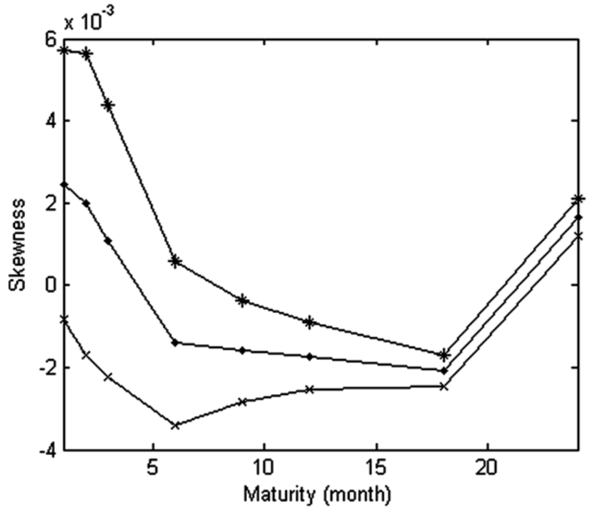

Fig. D.3. The term structure of implied volatility skew generated by the Wishart model (lines with *crosses* represent the case of $R_{12} = -0.4$, lines with *stars* represent the case of $R_{12} = +0.4$) and the two-factor Heston model (lines with *dots*).

Rearrangement of terms yields

$$\int_0^\tau \theta(T-s)e^{-\kappa s}ds = \ln F_T(t) - \mathrm{Tr}\left(A(\tau;-\mathrm{i})\Sigma\right) - xe^{-\kappa\tau} + \frac{1}{2}\beta\mathrm{Tr}\left(\ln H(\tau;-\mathrm{i}) + M^{\mathrm{T}}\tau + \frac{2}{\kappa}(1-e^{-\kappa\tau})RQ\right).$$

Now, substituting the above expression into the characteristic function (5) yields

$$\Psi(x,\Sigma,\tau;u,F_T(t)) = \exp\left(\mathrm{Tr}\left(A(\tau;u)\Sigma\right) + \mathrm{i}uxe^{-\kappa\tau} + \mathrm{i}u\left[\ln F_T(t) - \mathrm{Tr}\left(A(\tau;-\mathrm{i})\Sigma\right) - xe^{-\kappa\tau}\right.\right.$$
$$\left.\left. + \frac{1}{2}\beta\mathrm{Tr}\left(\ln H(\tau;-\mathrm{i}) + M^{\mathrm{T}}\tau + \frac{2}{\kappa}(1-e^{-\kappa\tau})RQ\right)\right]\right.$$

$$- \frac{1}{2}\beta \mathrm{Tr}\left(\ln H(\tau;u) + M^{\mathrm{T}}\tau + \frac{2\mathrm{i}u}{\kappa}(1-e^{-\kappa\tau})RQ\right)\Bigg)$$

$$= \exp\Bigg(\mathrm{i}u \ln F_T(t) + \mathrm{Tr}\left(\left(A(\tau;u) - \mathrm{i}uA(\tau;-\mathrm{i})\right)\Sigma\right)$$

$$- \frac{1}{2}\beta \mathrm{Tr}\left(\ln H(\tau;u) - \mathrm{i}u \ln H(\tau;-\mathrm{i}) + M^{\mathrm{T}}\tau(1-\mathrm{i}u)\right)\Bigg).$$

Appendix D. The simulation of log-currency value

To simulate the log-currency value X_t in the Wishart model, we apply the OU-discretization scheme proposed in [24] with β being positive integers.

D.1. The relationship between OU process and Wishart process

Let $\beta \in \mathbb{N}$ and let $\{U_{m,t} : t \geq 0\}_{1\leq m\leq \beta}$ be independent OU processes in $\mathbb{R}^n$ which follow the dynamics

$$dU_{m,t} = MU_{m,t}dt + Q^{\mathrm{T}}dW_{m,t},$$

where $\{W_{m,t} : t \geq 0\}_{1\leq m\leq\beta}$ are independent vector Brownian motions in $\mathbb{R}^n$, M and Q are real-valued square matrices of order n. Then, the matrix process defined by

$$\Sigma_t = \sum_{m=1}^{\beta} U_{m,t}U_{m,t}^{\mathrm{T}}$$

follows the dynamics

$$d\Sigma_t = (\beta Q^{\mathrm{T}}Q + M\Sigma_t + \Sigma_t M^{\mathrm{T}})dt + \sqrt{\Sigma_t}dW_tQ + Q^{\mathrm{T}}(dW_t)^{\mathrm{T}}\sqrt{\Sigma_t},$$

where W is a square matrix Brownian motion. Note that

$$\sqrt{\Sigma_t}dW_t \sim \sum_{m=1}^{\beta} U_{m,t}dW_{m,t}^{\mathrm{T}}.$$

D.2. The simulation procedure

Under risk-neutral measure, the log-currency value follows the dynamics

$$X_{t+\Delta t} = X_t + \int_t^{t+\Delta t}\left(\theta(s) - \kappa X_s - \frac{1}{2}\mathrm{Tr}\left(\Sigma_s\right)\right)ds + \int_t^{t+\Delta t}\mathrm{Tr}\left(\sqrt{\Sigma_t}dW_sR^{\mathrm{T}}\right)ds$$
$$+ \int_t^{t+\Delta t}\mathrm{Tr}\left(\sqrt{\Sigma_t}dB_s\sqrt{\mathbf{I}_n - RR^{\mathrm{T}}}\right)ds,$$

where B and W are independent matrix Brownian motions. The stochastic integrals in the above expression can be approximated as follows:

$$\int_t^{t+\Delta t}\mathrm{Tr}\left(\Sigma_s\right)ds \approx \frac{\Delta t}{2}\mathrm{Tr}\left(\Sigma_{t+\Delta t} + \Sigma_t\right),$$

$$\int_t^{t+\Delta t}\mathrm{Tr}\left(\sqrt{\Sigma_s}dW_sR^{\mathrm{T}}\right)ds = \mathrm{Tr}\left(\sum_{m=1}^{\beta}\int_t^{t+\Delta t}U_{m,s}dW_{m,s}^{\mathrm{T}}R^{\mathrm{T}}\right)$$
$$\approx \sqrt{\Delta t}\mathrm{Tr}\left(\sum_{m=1}^{\beta}U_{m,t}\epsilon_{m,t+\Delta t}^{\mathrm{T}}R^{\mathrm{T}}\right),$$

$$\int_t^{t+\Delta t}\mathrm{Tr}\left(\sqrt{\Sigma_s}dB_s\sqrt{\mathbf{I}_n - RR^{\mathrm{T}}}\right)ds \sim N\left(0, \int_t^{t+\Delta t}\mathrm{Tr}\left(\Sigma_s(\mathbf{I}_n - RR^{\mathrm{T}})\right)ds\right)$$
$$\approx \sqrt{\frac{\Delta t}{2}\mathrm{Tr}\left((\Sigma_t + \Sigma_{t+\Delta t})(\mathbf{I}_n - RR^{\mathrm{T}})\right)}Z,$$

where $\{\epsilon_{m,t+\Delta t}\}_{1\le m\le\beta}$ are vectors in $\mathbb{R}^n$ of independent standard normal random variables for the time interval $[t, t+\Delta t]$ and Z is a standard normal random variable.

Here is the simulation procedure:

Step 1: Partition the interval $[0, T]$ into N equal subintervals $\{t_i, t_{i+1}\}_{i=0}^{N-1}$ such that $t_0 = 0$, $t_{i+1} = t_i + \Delta t$ and $\Delta t = T/N$.

Step 2: To initialize the simulation, an eigenvalue decomposition on the symmetric positive definite matrix Σ_0 yields

$$\Sigma_{t_0} = \sum_{m=1}^{n} \lambda_m \phi_m \phi_m^{\mathrm{T}},$$

where λ_m are the eigenvalues and ϕ_m are the corresponding eigenvectors of Σ_{t_0}. Thus, the initial state of the vector OU process is, for $m = 1, \ldots, \beta$,

$$U_{m,t_0} = 1_{\{1\le m\le n\}}\sqrt{\lambda_m}\phi_m.$$

Step 3: To generate the OU process, for $k = 1, \ldots, \beta$,

$$U_{m,t_{i+1}} = U_{m,t_i} + \Delta t M U_{m,t_i} + \sqrt{\Delta t} Q^{\mathrm{T}} \epsilon_{m,t_{i+1}},$$

where $\{\epsilon_{m,t_{i+1}}\}_{1\le m\le\beta}$ are vectors in $\mathbb{R}^n$ of independent standard normal random variables for the time interval $[t_i, t_{i+1}]$.

Step 4: The Wishart process is generated by

$$\Sigma_{t_{i+1}} = \sum_{m=1}^{\beta} U_{m,t_{i+1}} U_{m,t_{i+1}}^{\mathrm{T}}.$$

Step 5: To generate log-currency value under the Wishart model,

$$X_{t_{i+1}} = X_{t_i} + \Delta t\left(\theta(t_i) - \kappa X_{t_i} - \frac{1}{4}\mathrm{Tr}\left(\Sigma_{t_i} + \Sigma_{t_{i+1}}\right)\right) + \sqrt{\Delta t}\,\mathrm{Tr}\left(\sum_{m=1}^{\beta} U_{m,t_i}\epsilon_{m,t_{i+1}}^{\mathrm{T}} R^{\mathrm{T}}\right) + \sqrt{\frac{\Delta t}{2}\mathrm{Tr}\left((\Sigma_{t_i} + \Sigma_{t_{i+1}})(\mathbf{I}_n - RR^{\mathrm{T}})\right)}\,Z.$$

Step 6: Repeat Steps 3 to 5 for $i = 1, \ldots, N-1$.

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