Some new results on \( k \)-free numbers

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Abstract

In this paper we obtain an improved asymptotic formula on the frequency of \( k \)-free numbers with a given difference. We also give a new upper bound of Barban–Davenport–Halberstam type for the \( k \)-free numbers in arithmetic progressions.

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1. Introduction

In this paper, we give some new results on the average behaviour of the remainder terms of the \( k \)-free numbers in arithmetic progressions. Such result is analogous to the Barban–Davenport–Halberstam theorem for the primes in arithmetic progressions.

Let \( \mu_k(n) \) be the characteristic function of the \( k \)-free numbers, then we have

\[
\mu_k(n) = \sum_{d^k | n} \mu(d),
\]

where \( \mu(n) \) is the Möbius function.

For fixed positive integer \( r \) and positive real number \( \varepsilon > 0 \), in [10] Mirsky obtained

\[
\sum_{n \leq x} \mu_k(n)\mu_k(n + r) = \prod_p \left( 1 - \frac{2}{p^k} \right) \prod_{p^k | r} \left( \frac{p^k - 1}{p^k - 2} \right) x + O_{r,k}(x^{2/k + \varepsilon}),
\]

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and in [11], Mirsky improved this to

\[
\sum_{n \leq x} \mu_k(n) \mu_k(n + r) = \prod_p \left(1 - \frac{2}{p^k}\right) \prod_{p^k \mid r} \left(\frac{p^k - 1}{p^k - 2}\right) x + O_{r,k}\left(x^{\frac{2}{k+1}}(\log x)^{\frac{k+2}{k+1}}\right).
\]

We improve this result as follows:

**Theorem 1.** For fixed positive integer \( r \geq 1, k \geq 2), we have

\[
\sum_{n \leq x} \mu_k(n) \mu_k(n + r) = \prod_p \left(1 - \frac{2}{p^k}\right) \prod_{p^k \mid r} \left(\frac{p^k - 1}{p^k - 2}\right) x + O_k\left(((x + r) \log \log 3r)^{\frac{2}{k+1}}\right).
\]

Especially, we have

\[
\sum_{n \leq x} \mu_k(n) \mu_k(n + r) = \prod_p \left(1 - \frac{2}{p^k}\right) \prod_{p^k \mid r} \left(\frac{p^k - 1}{p^k - 2}\right) x + O_{r,k}\left(x^{\frac{2}{k+1}}\right).
\]

For \( k = 2, r = 1\), Heath-Brown [6] obtained \( O(x^{7/11}(\log x)^7) \) in place of \( O(x^{2/3}(\log x)^{4/3}) \) in Mirsky’s theorem.

Write

\[
M(x; q, a) = \sum_{n \leq x, \, n \equiv a \pmod{q}} \mu_k(n), \quad \Phi_a(m) = \begin{cases} m, & \text{if } m \mid a, \\ 0, & \text{otherwise,} \end{cases}
\]

and

\[
F(x; q, a) = \frac{x}{q} \prod_p \left(1 - \frac{\Phi_a(p^k)}{p^k}\right) = xq^{-1} \sum_{d=1}^{\infty} \frac{\mu(d)(d^k, q)}{d^k},
\]

where the product is over prime numbers. This formula can be obtained as Hooley had done in [8,9].

When \( a \) and \( q \) are positive integers, define \( E(x; q, a) \) by the relation

\[
M(x; q, a) = F(x; q, a) + E(x; q, a).
\]

Also, define

\[
S(x, Q) = \sum_{q \leq Q} \sum_{1 \leq a \leq q} E^2(x; q, a).
\]
When $k = 2$, Warlimont [15] obtained the following bounds for $S(x, Q)$:

$$S(x, Q) \ll \begin{cases} x^{1+\varepsilon} Q, & \text{for } 1 \leq Q \leq x, \\ x Q, & \text{for } 1 \leq Q \leq x^{1/3}, \\ x Q, & \text{for } x^{1/3} \log^{10/3} x \leq Q \leq x. \end{cases}$$

He also proved that

$$S(x, x^\alpha) \ll x^{2(2\alpha+1)/3} + \varepsilon, \quad \text{for } \frac{1}{2} \leq \alpha \leq 1.$$  

In [16], Warlimont improved the above results for $x^{3/4} \leq Q \leq x$, namely

$$S(x, Q) \ll x^{1/2} Q^{3/2} + x^{5/3+\varepsilon}.$$  

Later, Croft [4] improved these results for $x^{5/8} \leq Q \leq x^{2/3}$, that is

$$S(x, Q) \ll x^{1/2} Q^{3/2} + x^{3/2} \log^{7/2} x.$$  

When $k > 2$, Brüdern and others [2,3] obtained

$$S(x, Q) \ll x^{-k/2} Q^{2-k/2}, \quad \text{for } 1 \leq Q \leq x.$$  

(1.5)

In this paper, we obtain the following:

**Theorem 2.** Suppose that $Q$ and $x$ are positive real numbers greater than 1, and that $k$ is an integer with $k > 2$. Then

$$S(x, Q) \ll x^{1/k} Q^{2-1/k} + x^{k+4/(k+2)} (\log x)^{3k+10/(k+2)}, \quad \text{for } 1 < Q \leq x,$$

(1.6)

where the $\ll$—constant will depend at most on $k$.

This theorem improves (1.5), for $x^{k^2+2k-4/(k+2)} < Q < x$.

Theorem 2 also improves Theorem 1.2 of Vaughan [13] in the case $a_n = \mu_k(n)$.

It should be noticed that we cannot use (1.6) to the case $k = 2$ directly.

When $k > 2$, we cannot obtain asymptotic formula of Montgomery–Hooley type as in the prime numbers in arithmetic progressions [5,7,14].

We insert the definition of $E(x; q, a)$ in (1.4) and square out. Thus

$$S(x, Q) = \sum_{q \leq Q} \sum_{1 \leq a \leq q} (M(x; q, a) - F(x; q, a))^2 = J_1 - 2J_2 + J_3,$$

(1.7)

where

$$J_1 = \sum_{q \leq Q} \sum_{1 \leq a \leq q} M(x; q, a)^2,$$

$$J_2 = \sum_{q \leq Q} \sum_{1 \leq a \leq q} M(x; q, a) F(x; q, a),$$

$$J_3 = \sum_{q \leq Q} \sum_{1 \leq a \leq q} F(x; q, a)^2.$$  

(1.8)

We shall prove Theorem 1 in Section 2, and Theorem 2 in Section 5.
Notation. Let \( k > 1 \) denote a positive integer. Throughout, the implicit constants in Vinogradov’s notation \( \ll \), and in Landau’s \( O \)-notation, will depend at most on \( k \) unless it is pointed out depend upon the corresponding parameters. \( n \equiv a \pmod{q} \) may be written as \( n \equiv a(q) \). The greatest common divisor and the least common multiple of integers \( a, b \) are denoted by \( (a, b) \) and \([a, b]\), respectively; \( \mu(n) \) denotes the Möbius function and \( \tau(n) \) denotes the divisor function; \( \omega(n) \) denotes the number of distinct prime factors of \( n \); \( [e] \) denotes the integer part of \( e \); \( \psi(v) = [v] - v + \frac{1}{2} \). The letter \( p \) denotes a prime number, and write \( p^l \parallel n \) when \( p^l \mid n \) but \( p^{l+1} \nmid n \). Let \( x \) denote a sufficiently large real number and \( Q \) be a positive real number with \( Q \leq x \).

2. Lemmas for \( J_1 \) and the proof of Theorem 1

By (1.2), we have

\[
J_1 = \sum_{q \leq Q} \sum_{1 \leq a \leq q} \left( \sum_{n \leq x \atop n \equiv a(q)} \mu_k(n) \right)^2 = J_{11} + 2J_{12},
\]

where

\[
J_{11} = \sum_{q \leq Q} \sum_{1 \leq a \leq q} \mu_k(n)_n^{2}, \quad J_{12} = \sum_{q \leq Q} \sum_{1 \leq a \leq q} \sum_{m \leq n \leq x \atop n \equiv m \equiv a(q)} \mu_k(m)\mu_k(n).
\]

Now, \( J_{11} = \sum_{q \leq Q} \sum_{n \leq x} \mu_k(n) \). Using (2) in [12], we have

\[
\sum_{n \leq x} \mu_k(n) = \xi^{-1}(k)x + O(x^{\frac{1}{k}}),
\]

therefore

\[
J_{11} = \xi^{-1}(k)x[Q] + O(Qx^{\frac{1}{k}}).
\]

Write

\[
T_q = \sum_{1 \leq h \leq x \atop q \mid h} \sum_{h < n \leq x} \mu_k(n)\mu_k(n - h) = \sum_{l \leq x^{-1}q} \sum_{lq < n \leq x} \mu_k(n)\mu_k(n - lq),
\]

and for a fixed positive integer \( r \)

\[
W_r(x) = \sum_{r < n \leq x} \mu_k(n)\mu_k(n - r),
\]

then

\[
J_{12} = \sum_{q \leq Q} T_q.
\]
For fixed integers \( u \) and \( v \), we define the function \( N(u, v, x, r) \) by
\[
N(u, v, x, r) = \sum_{uc - vd = r} 1,
\]
where \( c \) and \( d \) are positive integers.

Lemma 2.1. For positive integers \( u \) and \( v \), we have
\[
N(u, v, x, r) = \begin{cases} \frac{x - r}{uv}(u, v) + O(1), & \text{if } (u, v) \mid r, \\ 0, & \text{otherwise.} \end{cases}
\]

Proof. If \( (u, v) \nmid r \), the lemma is obvious. We suppose that \( (u, v) \mid r \). Let \( c_0, d_0 \) be any fixed solution of the equation \( uc - vd = r \), then all the solutions of this equation may be represented as \( c = c_0 + \frac{v}{(u, v)} t, \ d = d_0 + \frac{u}{(u, v)} t \), where \( t \) is any integer, and \( uc_0 - vd_0 = r \). We need to count the number of \( t \) that satisfy the following conditions:
\[
uc_0 + \frac{uv}{(u, v)} t \leq x, \quad c_0 + \frac{v}{(u, v)} t \geq 1, \quad d_0 + \frac{u}{(u, v)} t \geq 1.
\]
Then, we have
\[
\max \left\{ \frac{u - uc_0}{uv}, \frac{v - vd_0}{uv} (u, v) \right\} \leq t \leq \frac{x - uc_0}{uv} (u, v),
\]
the number of such \( t \) is \( \frac{x - r}{uv}(u, v) + O(1) \), and the lemma follows. \( \square \)

Lemma 2.2. For positive integer \( r \) and positive real number \( y \), we have
\[
W_r(x) = (x - r) \sum_{(a, b)^k \mid r} \frac{\mu(a)\mu(b)(a, b)^k}{a^kb^k} - (x - r) \sum_{(a, b)^k \mid r \ ab > y} \frac{\mu(a)\mu(b)(a, b)^k}{a^kb^k}
\]
\[
+ \sum_{a^k c - b^k d = r \ a^k c \leq x, \ ab > y} \mu(a)\mu(b) + O \left( \sum_{(a, b)^k \mid r \ ab \leq y} \mu^2(a)\mu^2(b) \right).
\]

Proof. By (1.1), we have
\[
W_r(x) = \sum_{r < n \leq x} \sum_{a^k \mid n} \mu(a) \sum_{b^k \mid n - r} \mu(b) = \sum_{a^k c - b^k d = r \ a^k c \leq x} \mu(a)\mu(b) = \sum_1 + \sum_2,
\]
where
\[
\sum_1 = \sum_{a^k c - b^k d = r \ a^k c \leq x, \ ab \leq y} \mu(a)\mu(b), \quad \sum_2 = \sum_{a^k c - b^k d = r \ a^k c \leq x, \ ab > y} \mu(a)\mu(b).
\]
By Lemma 2.1, we have

\[
\sum_1 = \sum_{(a,b)^k \mid r \atop ab \leq y} \mu(a)\mu(b)N(a^k, b^k, x, r) = \sum_{(a,b)^k \mid r \atop ab \leq y} \mu(a)\mu(b)\left(\frac{(a,b)^k}{a^k b^k}(x-r) + O(1)\right)
\]

\[
= (x-r) \sum_{(a,b)^k \mid r \atop ab \leq y} \frac{\mu(a)\mu(b)(a,b)^k}{a^k b^k} - (x-r) \sum_{(a,b)^k \mid r \atop ab > y} \frac{\mu(a)\mu(b)(a,b)^k}{a^k b^k}
\]

\[
+ O\left( \sum_{(a,b)^k \mid r \atop ab \leq y} \mu^2(a)\mu^2(b) \right),
\]

and the lemma follows.  \(\Box\)

Write

\[
K_k(n) = \prod_{p^m \parallel n \atop m \geq k} p^m.
\]

**Lemma 2.3.** For \(y \geq 2\), we have

\[
\sum_{(a,b)^k \mid r \atop ab > y} \frac{\mu(a)\mu(b)(a,b)^k}{a^k b^k} \ll y^{-\frac{k}{2}}2^{\omega(K_k(r))}\log y.
\]

If \(k \geq 3\) then

\[
\sum_{(a,b)^k \mid r \atop ab > y} \frac{\mu(a)\mu(b)(a,b)^k}{a^k b^k} \ll y^{-\frac{k}{2}}2^{\omega(K_k(r))}.
\]

**Proof.** Write

\[
I = \sum_{(a,b)^k \mid r \atop ab > y} \frac{\mu(a)\mu(b)(a,b)^k}{a^k b^k},
\]

then

\[
I = \sum_{t^k \mid r} \sum_{(a,b)=t \atop ab > y} \frac{\mu(a)\mu(b)(a,b)^k}{a^k b^k} \ll \sum_{t^k \mid r} \mu^2(t)t^{-k} \sum_{a \geq 1} a^{-k} \sum_{b > yr^{-2}a^{-1}} b^{-k}.
\]
We need to estimate the sums of $a$ and $b$ more carefully. We have

\[
\sum_{a \geq 1} a^{-k} \sum_{b > y^{1-2}a-1} b^{-k} \ll \sum_{a > y^{1-2}} a^{-k} + \sum_{a \leq y^{1-2}} a^{-k} (y a^{-1} t^{-2})^{1-k},
\]

hence

\[
I \ll \sum_{t^k | r} \mu^2(t) t^{-k} \left( \sum_{a > y^{1-2}} a^{-k} + y^{1-k} t^{2k-2} \sum_{a \leq y^{1-2}} a^{-1} \right) \ll I_1 + I_2 + I_3,
\]

where

\[
I_1 = \sum_{t^k | r \atop t^2 \leq y} \mu^2(t) t^{-k} \sum_{a > y^{1-2}} a^{-k}, \quad I_2 = \sum_{t^k | r \atop t^2 > y} \mu^2(t) t^{-k} \sum_{a \geq 1} a^{-k},
\]

\[
I_3 = y^{1-k} \sum_{t^k | r \atop t^2 \leq y} \mu^2(t) t^{k-2} \sum_{a \leq y^{1-2}} a^{-1}.
\]

We estimate $I_1$, $I_2$ and $I_3$ separately. We have

\[
I_1 \ll \sum_{t^k | r \atop t^2 \leq y} \mu^2(t) t^{-k} (y t^{-2})^{1-k} \ll y^{1-k} \sum_{t^k | r \atop t^2 \leq y} \mu^2(t) t^{k-2} \ll y^{1-k} y^{\frac{k-2}{2}} \sum_{t^k | r} \mu^2(t),
\]

therefore

\[
I_1 \ll y^{-\frac{k}{2}} 2^{o(K_k(r))}. \tag{2.5}
\]

Also

\[
I_2 \ll \sum_{t^k | r \atop t^2 > y} \mu^2(t) t^{-k} \ll y^{-\frac{k}{2}} \sum_{t^k | r} \mu^2(t) \ll y^{-\frac{k}{2}} 2^{o(K_k(r))}. \tag{2.6}
\]

Similarly, we have

\[
I_3 \ll y^{1-k} \log y \sum_{t^k | r \atop t^2 \leq y} \mu^2(t) t^{k-2} \ll y^{-\frac{k}{2}} \log y \sum_{t^k | r} \mu^2(t) \ll y^{-\frac{k}{2}} \log y 2^{o(K_k(r))}. \tag{2.7}
\]

By (2.5)–(2.7), we obtain the first part of the lemma.
If \( k \geq 3 \), then

\[
I_3 = y^{1-k} \left( \sum_{t^k | r \atop 9t^2 \leq y} \mu^2(t)t^{k-2} \sum_{a \leq yt^{-2}} a^{-1} + \sum_{t^k | r \atop y < 9t^2 \leq 9y} \mu^2(t)t^{k-2} \sum_{a \leq yt^{-2}} a^{-1} \right)
\]

\[
\ll y^{1-k} \left( \sum_{t^k | r \atop 9t^2 \leq y} \mu^2(t)t^{k-2} \log \frac{y}{t^2} + \sum_{t^k | r \atop y < 9t^2 \leq 9y} \mu^2(t)t^{k-2} \right) \ll y^{1-k} 2^\omega(K_k(r)),
\]

and the lemma follows. \( \square \)

**Lemma 2.4.** For \( y \geq 2 \), we have

\[
\sum_{(a,b)^k | r \atop ab \leq y} \mu^2(a)\mu^2(b) \ll y \log y.
\]

**Proof.** This is a trivial result, we give a proof for completeness. We have

\[
\sum_{(a,b)^k | r \atop ab \leq y} \mu^2(a)\mu^2(b) \ll \sum_{a \leq y} \sum_{b \leq y^{-1}} 1 \ll y \sum_{a \leq y} a^{-1} \ll y \log y,
\]

and the lemma follows. \( \square \)

Write

\[
f(r) = \sum_{(a,b)^k | r \atop ab \leq y} \frac{\mu(a)\mu(b)(a,b)^k}{a^k b^k}.
\]

(2.8)

**Lemma 2.5.** For \( r \leq x \) and \( y \geq 2 \), we have

\[
W_r(x) = (x-r)f(r) + \sum_{a^k c - b^k d = r \atop a^k c \leq x, ab > y} \mu(a)\mu(b) + O\left( (x-r)y^{-\frac{k}{2}} 2^\omega(K_k(r)) \log y + y \log y \right).
\]

If \( k \geq 3 \), then

\[
W_r(x) = (x-r)f(r) + \sum_{a^k c - b^k d = r \atop a^k c \leq x, ab > y} \mu(a)\mu(b) + O\left( (x-r)y^{-\frac{k}{2}} 2^\omega(K_k(r)) + y \log y \right).
\]
Proof. By Lemmas 2.2–2.4, we have

\[ W_r(x) = (x - r)f(r) + O((x - r)y^{-\frac{k}{2}}2^{\mu(K_r)}\log y) + \sum_{\substack{a^k c = b^k d = r \\
 a^k c \leq x, ab > y}} \mu(a)\mu(b) \]

\[ + O(y\log y), \]

and the first part of the lemma follows. We may obtain the second part similarly. □

Write

\[ g(t) = \mu^2(t)t^{-k} \sum_{n=1}^{\infty} \mu(n)\tau(n)n^{-k}. \quad (2.9) \]

Lemma 2.6. For positive integer \( r \), we have

\[ f(r) = \sum_{t^k|r} g(t) = \prod_p \left(1 - \frac{2}{p^k}\right) \prod_{p^k|r} \left(\frac{p^k - 1}{p^k - 2}\right). \]

Proof. By the definition of \( f(r) \), we have

\[ f(r) = \sum_{t^k|r} \sum_{(a,b) = t} \frac{\mu(a)\mu(b)(a,b)^k}{a^k b^k} = \sum_{t^k|r} \mu^2(t)t^{-k} \sum_{(ab,t) = 1} \frac{\mu(ab)}{a^k b^k} \]

\[ = \sum_{t^k|r} \mu^2(t)t^{-k} \sum_{(n,t) = 1} \frac{\mu(n)\tau(n)}{n^k} = \sum_{t^k|r} g(t), \]

this is the first part of the formula. We continue in this fashion obtaining

\[ f(r) = \sum_{t^k|r} \mu^2(t)t^{-k} \prod_{p \nmid t} (1 - 2p^{-k}) = \prod_p \left(1 - 2p^{-k}\right) \sum_{t^k|r} \mu^2(t)t^{-k} \prod_{p \nmid t} (1 - 2p^{-k})^{-1} \]

\[ = \prod_p \left(1 - \frac{2}{p^k}\right) \prod_{p^k|r} \left(\frac{p^k - 1}{p^k - 2}\right), \]

and the lemma follows. □

Combining Lemma 2.5 and the method of Atkinson and Cherwell [1] gives

\[ \sum_{n \leq x} \mu_k(n)\mu_k(n + r) = \prod_p \left(1 - \frac{2}{p^k}\right) \prod_{p^k|r} \left(\frac{p^k - 1}{p^k - 2}\right)x + O_{r,k}(x^{\frac{2}{k+1}}\log x). \]
Write
\[
D_1(n) = \sum_{v^k|n, v \leq z} \mu(v) \sum_{u^k|n+r, u \leq z} \mu(u), \quad D_2(n) = \mu_k(n) \sum_{u^k|n+r, u \leq z} \mu(u),
\]
\[
D_3(n) = \sum_{v^k|n, v > z} \mu(v) \mu_k(n+r), \quad D_4(n) = -\sum_{u^k|n, u > z} \mu(v) \sum_{u^k|n+r, u \leq z} \mu(u).
\]

**Lemma 2.7.** For fixed positive integer \(r\) and positive real number \(z\), we have
\[
\mu_k(n) \mu_k(n+r) = D_1(n) + D_2(n) + D_3(n) + D_4(n).
\]

**Proof.** By (1.1),
\[
\mu_k(n) \mu_k(n+r) = \mu_k(n) \left( \sum_{u^k|n+r, u > z} \mu(u) + \sum_{u^k|n+r, u \leq z} \mu(u) \right) = D_2(n) + \mu_k(n) \sum_{u^k|n+r, u \leq z} \mu(u),
\]
and
\[
\mu_k(n) \sum_{u^k|n+r, u \leq z} \mu(u) = D_1(n) + \sum_{u^k|n+r, u \leq z} \mu(u).
\]

We have
\[
\sum_{v^k|n, v > z} \mu(v) \sum_{u^k|n, u \leq z} \mu(u) = \sum_{v^k|n} \mu(v) \mu_k(n+r) - \sum_{u^k|n} \mu(v) \sum_{u^k|n+r, u > z} \mu(u) = D_3(n) + D_4(n),
\]
and the lemma follows. \(\Box\)

Now, we have
\[
\sum_{n \leq x} \mu_k(n) \mu_k(n+r) = \sum_{j=1}^{4} \sum_{n \leq x} D_j(n) = \sum_{j=1}^{4} E_j, \quad \text{say},
\]
\[
E_1 = \sum_{u,v \leq z} \mu(u) \mu(v) \sum_{n \leq x} \frac{1}{(n+r \equiv 0(u^k), n \equiv 0(u^k))} \left( \frac{x}{[u,v]^k} + O(1) \right)
\]
\[
= x E_{11} + O(z^2), \quad \text{say},
\]
\[
E_{11} = \sum_{t^k|r} \sum_{m,s \leq z, t^{-1}} \mu(mt) \mu(st) (ms)^{-k} = f(r) + O\left( \sum_{t^k|r} \sum_{s > zt^{-1}} s^{-k} \right)
\]
\[
= f(r) + O(z^{1-k} \log \log 3r).
\]
Hence

\[ E_1 = f(r)x + O(xz^{1-k} \log \log 3r + z^2). \]

For \( j = 2, 3, \)

\[ E_j \ll \sum_{u \leq x+r \atop u > z} \sum_{n \leq x+r \atop u^k | n} 1 \ll (x + r)z^{1-k}. \]

For \( j = 4, \)

\[ E_4 \ll \sum_{n \leq x+r} \left( \sum_{u^k | n \atop u > z} 1 \right)^2 \ll \sum_{u > z \atop u, v > z} \sum_{[u^k, v^k] | n \atop n \leq x+r} 1 \ll (x + r) \sum_{u, v > z} (u, v)^k / u^k v^k \]

\[ \ll (x + r) \sum_{d=1}^{\infty} d^k \sum_{u, v > z} (uv)^{-k} \ll (x + r) \sum_{d=1}^{\infty} d^{-k} \left( \sum_{t > zd^{-1}} t^{-k} \right)^2 \]

\[ \ll (x + r) \left( \sum_{d \leq z} d^{-k} \left( \sum_{t > zd^{-1}} t^{-k} \right)^2 + \sum_{d > z} d^{-k} \right). \]

Hence

\[ E_4 \ll (x + r) \left( \sum_{d \leq z} d^{-k} \left( zd^{d-1} (1-k)^2 + z^{1-k} \right) \right) \ll (x + r) \left( z^{2-2k} \sum_{d \leq z} d^{k-2} + z^{1-k} \right) \]

\[ \ll (x + r)z^{1-k}. \]

Thus

\[ \sum_{n \leq x} \mu_k(n) \mu_k(n + r) = f(r)x + O((x + r)z^{1-k} \log \log 3r + z^2), \quad (2.10) \]

so that by choosing \( z = ((x + r) \log \log 3r)^{1/k} \) in (2.10), this completes the proof of Theorem 1.

3. The formula for \( T_q \)

From now on we make the assumption: \( 2 \leq y \leq x^{2/k}, k > 2. \)

By Lemma 2.5 and (2.4), we have
$$T_q = \sum_{l \leq xq^{-1}} W_{lq}(x)$$

$$= \sum_{l \leq xq^{-1}} \left( (x - lq) f(lq) + \sum_{a^k c - b^k d = lq} \sum_{a^k c \leq x, ab > y} \mu(a)\mu(b) + O \left( (x - lq)y^{-\frac{k}{2}} 2^{\omega(K_k(lq))} + y \log y \right) \right)$$

$$= U_q + V_q + O(\mathcal{Z}_q^*),$$

where

$$U_q = \sum_{l \leq xq^{-1}} (x - lq) f(lq), \quad V_q = \sum_{l \leq xq^{-1}} \sum_{a^k c - b^k d = lq} \sum_{a^k c \leq x, ab > y} \mu(a)\mu(b),$$

$$\mathcal{Z}_q^* = \sum_{l \leq xq^{-1}} \left( (xq^{-1} - l)qy^{-\frac{k}{2}} 2^{\omega(K_k(lq))} + y \log y \right).$$

**Lemma 3.1.** We have

$$\mathcal{Z}_q^* \ll x^2 q^{-1} y^{-\frac{k}{2}} 2^{\omega(K_k(q))} \log x + xq^{-1} y \log x.$$

**Proof.** By noting that $\omega(K_k(lq)) \leq \omega(K_k(l)) + \omega(K_k(q))$, we have

$$\mathcal{Z}_q^* \ll 2^{\omega(K_k(q))} qy^{-\frac{k}{2}} \sum_{l \leq xq^{-1}} (xq^{-1} - l) \tau(l) + xq^{-1} y \log x.$$

By Dirichlet’s formula for divisor function, we have

$$\sum_{l \leq xq^{-1}} (xq^{-1} - l) \tau(l) \ll \int_1^{xq^{-1}} \tau(l) \, dl \ll (xq^{-1})^2 \log x,$$

and the lemma follows. \(\square\)

Write

$$Z_q = x^2 q^{-1} y^{-\frac{k}{2}} 2^{\omega(K_k(q))} \log x + xq^{-1} y \log x,$$

then

$$T_q = U_q + V_q + O(\mathcal{Z}_q),$$

and

$$J_{12} = \sum_{q \leq Q} T_q = \sum_{q \leq Q} (U_q + V_q + O(\mathcal{Z}_q)) = A + B + O(C), \quad (3.1)$$
where
\[ A = \sum_{q \leq Q} U_q, \quad B = \sum_{q \leq Q} V_q, \quad C = \sum_{q \leq Q} Z_q. \]

**Lemma 3.2.** We have
\[ C \ll x^2 y^{-\frac{k}{2}} \log^3 x + xy \log^2 x. \]

**Proof.** By the definition of \( C \), we have
\[ C \ll x^2 y^{-\frac{k}{2}} \log x \sum_{q \leq Q} q^{-1} \tau(q) + xy \log x \sum_{q \leq Q} q^{-1}, \]
again, by Dirichlet’s formula for the divisor function and noting that \( \sum_{q \leq Q} q^{-1} \ll \log Q \), we have
\[ \sum_{q \leq Q} q^{-1} \tau(q) \ll \log^2 x, \quad C \ll x^2 y^{-\frac{k}{2}} \log^2 x \log x + xy \log x \log x, \]
and the lemma follows. \( \square \)

**4. The formula for \( J_1 \)**

In this section we will give the formula for \( J_1 \).

**Lemma 4.1.** Suppose that \( f(t) \) and \( g(t) \) are defined by (2.8) and (2.9), respectively, that \( u > 0 \) and that \( v = u(t^k, q)t^{-k} \). We have
\[ \sum_{m \leq u} f(mq) = \sum_{t=1}^{\infty} g(t) \left( v - \frac{1}{2} \right) + \sum_{t=1}^{\infty} g(t) \psi(v) + O((uq)^{1/2}). \]

**Proof.** By Lemma 2.6, we have
\[ \sum_{m \leq u} f(mq) = \sum_{m \leq u} \sum_{t \mid mq \atop t \leq (uq)^{1/2}} g(t) = \sum_{m \leq u} g(t) \sum_{t \leq (uq)^{1/2}} 1 = \sum_{t \leq (uq)^{1/2}} g(t) \left[ u(t^k, q)t^{-k} \right] \]
\[ = \sum_{t \leq (uq)^{1/2}} g(t) \left( u(t^k, q)t^{-k} - \frac{1}{2} + \psi(u(t^k, q)t^{-k}) \right) \]
\[ = \sum_{t \leq (uq)^{1/2}} g(t) \left( v - \frac{1}{2} \right) + \sum_{t \leq (uq)^{1/2}} g(t) \psi(v). \]
By (2.9), we have $g(t) \ll t^{-k}$, and

$$
\sum_{t>(uq)^{\frac{1}{k}}} g(t)(v + 1) \ll \sum_{t>(uq)^{\frac{1}{k}}} t^{-k}(1 + ut^{-k}(q,t^k)) \ll (uq)^{\frac{1}{k}-1},
$$

and the lemma follows. \(\square\)

Write

$$
H(x,q) = 2q \sum_{t=1}^{\infty} g(t)t^k(q,t^k) x(q,t^k)q^{-1} t^{-k} - \int_{0}^{\infty} \psi(v) dv.
$$

**Lemma 4.2.** We have

$$
A = -\frac{1}{2} \zeta^{-1}(k) x[Q] + \frac{1}{2} \sum_{q \leq Q} H(x,q) + \frac{1}{2} x^2 \sum_{q \leq Q} q^{-1} \sum_{t=1}^{\infty} g(t)t^{-k}(q,t^k) + O(Qx^{\frac{1}{k}}).
$$

**Proof.** By Lemma 4.1, we have

$$
U_q = q \int_{0}^{xq^{-1}} \left( \sum_{m \leq u} f(mq) \right) du = q \int_{0}^{xq^{-1}} \left( \sum_{t=1}^{\infty} g(t) \left(v - \frac{1}{2}\right) + \sum_{t=1}^{\infty} g(t) \psi(v) + O((uq)^{\frac{1}{k}-1}) \right) dv 
$$

$$
= q \int_{0}^{xq^{-1}} \left( \sum_{t=1}^{\infty} g(t) \left(v - \frac{1}{2}\right) + \sum_{t=1}^{\infty} g(t) \psi(v) \right) dv + O \left( q \int_{0}^{(uq)^{\frac{1}{k}-1}} du \right).
$$

The error term is

$$
\ll q^{\frac{1}{2}}(xq^{-1})^{\frac{1}{2}} \ll x^{\frac{1}{k}}.
$$

Therefore,

$$
U_q = q \sum_{t=1}^{\infty} g(t) \int_{0}^{xq^{-1}} \left(v - \frac{1}{2}\right) dv + q \sum_{t=1}^{\infty} g(t) \int_{0}^{xq^{-1}} \psi(v) dv + O \left( x^{\frac{1}{k}} \right).
$$

Also

$$
\int_{0}^{xq^{-1}} \left(v - \frac{1}{2}\right) dv = \frac{1}{2} x^2 t^{-k}(q,t^k)q^{-2} - \frac{1}{2} xq^{-1},
$$
\[
\int_0^{xq^{-1}} \psi(v) \, dv = t^k(q, t^k)^{-1} \int_0^{xq^{-1}t^{-k}(q, t^k)} \psi(v) \, dv,
\]

\[
\sum_{t=1}^{\infty} g(t) = \zeta^{-1}(k).
\]

Thus
\[
U_q = \frac{1}{2} x^2 q^{-1} \sum_{t=1}^{\infty} g(t) t^{-k}(q, t^k) - \frac{1}{2} x \sum_{t=1}^{\infty} g(t) + \frac{1}{2} H(x, q) + O(x^{1/k}),
\] (4.1)
and the lemma follows. \(\square\)

**Lemma 4.3.** Suppose that \(k > 2\) and that \(2 \leq y \leq x^{\frac{2}{k}}\). Then
\[
B \ll x^2 y^{1-k} \log^{5/2} x + x^{3/2} y^{1-k} \log^{7/2} x + x^2 y^{1/2 - \frac{3k}{4}} \log^{3/2} x.
\]

**Proof.** We refine the argumentation in [4]. We have \(B \ll J(x)\), where
\[
J(x) = \sum_{q \leq Q} \sum_{l \leq xq^{-1}} \sum_{a^k c - b^k d = lq} 1 \ll \sum_{m \leq x} \tau(m) \sum_{a^k c - b^k d = m} \sum_{c \leq xa^{-k}} 1.
\]

We write
\[
K = K(x; a, b) = \sum_{m \leq x} \tau(m) \sum_{a^k c - b^k d = m} \sum_{c \leq xa^{-k}} 1,
\]
then
\[
J(x) \leq \sum_{a^k \leq x, b^k \leq x} K(x; a, b). \tag{4.2}
\]

By the Cauchy inequality and Dirichlet’s formula for the divisor function
\[
K \ll \left( \sum_{m \leq x} \tau^2(m) \right)^{1/2} L^{1/2} \ll x^{1/2} \log^{3/2} x L^{1/2},
\]
where
\[
L = L(x; a, b) = \sum_{a^k c - b^k d = a^k \gamma - b^k \delta} 1.
\]
Writing \( c - \gamma = u \) and \( d - \delta = v \), we have

\[
L \ll \frac{x^2}{ab^k} \sum_{a^k b^k \leq x} 1 \ll \frac{x^2}{ab^k} \left( \frac{(a, b)^k}{ab^k} + 1 \right) \ll \frac{x^3}{a^{2k}b^{2k}} (a, b)^k + \frac{x^2}{a^k b^k}.
\]

Hence

\[
L^{1/2} \ll \frac{x^{3/2}}{a^k b^k} (a, b) + \frac{x}{a^k b^k},
\]

and

\[
K \ll x^2 \log^{3/2} x \frac{(a, b)^{\frac{k}{2}}}{a^k b^k} + x^{3/2} \log^{3/2} x \frac{1}{a^k b^k}.
\]  \hspace{1cm} (4.3)

By (4.2) and (4.3),

\[
J(x) \ll x^2 \log^{3/2} x \sum_{a, b \leq x^{\frac{1}{k}}} (a, b)^{\frac{k}{2}} + x^2 \log^{3/2} x \sum_{a, b \leq x^{\frac{1}{k}}} \frac{1}{a^k b^k}.
\]

Thus

\[
J(x) \ll x^2 \log^{3/2} x L_1 + x^3/2 \log^{3/2} x L_2,
\]  \hspace{1cm} (4.4)

where

\[
L_1 = \sum_{a, b \leq x^{\frac{1}{k}}} \frac{(a, b)^{\frac{k}{2}}}{a^k b^k}, \hspace{1cm} L_2 = \sum_{a, b \leq x^{\frac{1}{k}}} \frac{1}{a^k b^k}.
\]

We have

\[
L_1 \ll \sum_{d \leq x^{\frac{1}{k}}} \frac{d^{\frac{k}{2} - 2k}}{\left( \frac{x}{d^2} \right)^{1/2}} \sum_{y \leq d^2 \leq n \leq x^{\frac{2}{k}}} \frac{\tau(n)}{n^k} \ll \sum_{1 \leq n \leq x^{\frac{2}{k}}} \frac{\tau(n)}{n^k} \sum_{\sqrt{\frac{x}{n}} \leq d \leq x^{\frac{1}{2}}n^{-1/2}} \frac{d^{-3k}}{\sqrt{\frac{x}{n}}}.
\]

Again, by Dirichlet's formula for the divisor function, we have

\[
L_1 \ll y^{1 - \frac{k}{2}} + y^{1-k} \log x.
\]  \hspace{1cm} (4.5)
We see at once that
\[
L_2 \ll \sum_{a,b \leq x^{\frac{1}{2}} \atop ab > y} \frac{1}{y^{\frac{1}{2}-1}ab} \ll y^{1-\frac{1}{k}} \log^2 x, \tag{4.6}
\]
and the lemma then follows from (4.4)–(4.6).

**Lemma 4.4.** For \( k > 2 \), we have

\[
J_1 = \sum_{q \leq Q} H(x, q) + x^2 \sum_{q \leq Q} q^{-1} \sum_{t=1}^{\infty} g(t)t^{-k}(q, t^k)
\]

\[
+ O\left( Qx^{\frac{1}{2}} + x^2 y^{1-k} \log^{5/2} x + x^3 y^{1-\frac{k}{2}} \log^{7/2} x + x^2 y^{\frac{1}{2} - \frac{3k}{4}} \log^{3/2} x
\]

\[
+ x^2 y^{-\frac{k}{2}} \log^3 x + xy \log^2 x \right).
\]

**Proof.** By (2.1)–(2.3), we have

\[
J_1 = \zeta^{-1}(k)x[Q] + O\left( Qx^{\frac{1}{2}} \right) + 2J_{12}.
\]

By (3.1), (4.1), Lemmas 3.2, 4.2 and 4.3, we have

\[
J_{12} = -\frac{1}{2} \zeta^{-1}(k)x[Q] + \frac{1}{2} \sum_{q \leq Q} H(x, q) + \frac{1}{2} x^2 \sum_{q \leq Q} q^{-1} \sum_{t=1}^{\infty} g(t)t^{-k}(q, t^k)
\]

\[
+ O\left( x^2 y^{1-k} \log^{5/2} x + x^3 y^{1-\frac{k}{2}} \log^{7/2} x
\]

\[
+ x^2 y^{\frac{1}{2} - \frac{3k}{4}} \log^{3/2} x + x^2 y^{-\frac{k}{2}} \log^3 x + xy \log^2 x \right),
\]

and the lemma follows.

**5. The formulas for \( J_2, J_3 \) and the proof of Theorem 2**

In this section we give the complete proof of Theorem 2.

For \( k > 2 \), there are not simple properties for the function \( M(x; q, a) \) as in the case \( k = 2 \) [4,16], our method is suitable for all \( k \geq 2 \).

**Lemma 5.1.** For \( k \geq 2 \), we have

\[
J_2 = x^2 \sum_{q \leq Q} q^{-1} \sum_{t=1}^{\infty} g(t)t^{-k}(q, t^k) + O\left( x^{1+\frac{1}{k}} \log x \right).
\]

**Proof.** By (1.1)–(1.3) and (1.8),
\[ J_2 = \sum_{q \leq Q} \sum_{1 \leq a \leq q} \sum_{n \equiv a(q)}^{n \leq x} \sum_{t^k | n} \mu(t)xq^{-1} \sum_{d=1}^{\infty} \frac{\mu(d)(d^k, q)}{d^k} \sum_{a=1}^{q} \sum_{\frac{\mu(t)}{a}}^{t^k \leq x} \sum_{n \equiv a(q)}^{n \leq x} 1. \]

Write

\[ G = \sum_{a=1}^{q} \sum_{\frac{\mu(t)}{a}}^{t^k \leq x} \sum_{n \equiv a(q)}^{n \leq x} 1. \]

Writing \( a = (q, d^k) j \) and \( n = t^k m \), we have

\[ G = \sum_{j=1}^{q(q, d^k)^{-1}} \sum_{\frac{\mu(t)}{a}}^{t^k \leq x} \sum_{m \equiv a(q)}^{m \leq xt^{-k}} \sum_{t^k m \equiv (q, d^k) j(q)}^{1} = \left[ \frac{x(q, d^k, t^k)}{(q, d^k)t^k} \right] = \frac{x(q, d^k, t^k)}{(q, d^k)t^k} + O(1). \]

Hence

\[ J_2 = J_{21} + O(J_{22}), \quad (5.1) \]

where

\[ J_{21} = x^2 \sum_{q \leq Q} q^{-1} \sum_{d=1}^{\infty} \frac{\mu(d)}{d^k} \sum_{t \equiv x t^{-k}}^{t \leq x} \sum_{(q, d^k, t^k)}^{\mu(t)} \frac{t^{-k}}{t^{-k}}, \]

\[ J_{22} = x \sum_{q \leq Q} q^{-1} \sum_{d=1}^{\infty} \frac{\mu^2(d)(d, q)}{d^k} \sum_{t \equiv x t^{-k}}^{\mu^2(t)} \frac{t^{-k}}{t^{-k}}. \quad (5.2) \]

By (2.9), proceeding as in the proof of Lemma 2.6, we get

\[ J_{21} = x^2 \sum_{q \leq Q} q^{-1} \sum_{t=1}^{\infty} g(t)t^{-k}(q, t^k) - x^2 \sum_{q \leq Q} q^{-1} \Delta_{21}, \quad (5.3) \]

where,

\[ \Delta_{21} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^k} \sum_{\frac{\mu(t)}{a}}^{t \leq x} \sum_{n \equiv a(q)}^{n \leq x} \frac{t^{-k}}{t^{-k}}. \]

We have
\[ \Delta_{21} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^k} \sum_{m \mid d} (q, m^k) \sum_{t > x^{\frac{1}{k}} \text{ and } (t, d) = 1} \mu(t) t^{-k} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^k} \sum_{m \mid d} \mu(m) (q, m^k) m^{-k} \sum_{t > x^{\frac{1}{k}} m^{-1} \text{ and } (t, d) = 1} \mu(t) t^{-k} \]

\[ = \sum_{m=1}^{\infty} \mu(m) (q, m^k) m^{-k} \sum_{d=1}^{\infty} \frac{\mu(d)}{d^k} \sum_{m \mid d} \mu(m) \left( q, \frac{m}{k} \left( q, \frac{m}{k} \right) \right) m^{-k} \sum_{t > x^{\frac{1}{k}} m^{-1} \text{ and } (t, d) = 1} \mu(t) t^{-k} = \Delta_{211} + \Delta_{212}, \]

where

\[ \Delta_{211} = \sum_{m \leq x^{\frac{1}{k}}} \mu(m) (q, m^k) m^{-k} \sum_{d=1}^{\infty} \frac{\mu(d)}{d^k} \sum_{t > x^{\frac{1}{k}} \text{ and } (t, d) = 1} \mu(t) t^{-k}, \]

\[ \Delta_{212} = \sum_{m > x^{\frac{1}{k}}} \mu(m) (q, m^k) m^{-k} \sum_{d=1}^{\infty} \frac{\mu(d)}{d^k} \sum_{t > x^{\frac{1}{k}} m^{-1} \text{ and } (t, d) = 1} \mu(t) t^{-k}. \]

We have

\[ \Delta_{211} \ll \sum_{m \leq x^{\frac{1}{k}}} (q, m^k) m^{-2k} \sum_{t > x^{\frac{1}{k}} m^{-1}} t^{-k} \ll \sum_{m \leq x^{\frac{1}{k}}} (q, m^k) m^{-2k} (x^{\frac{1}{k}} m^{-1})^{1-k} \]

\[ \ll x^{\frac{1}{k}-1} \sum_{d \leq x^{\frac{1}{k}}} (q, d^k) d^{-k-1}, \]

hence

\[ x^2 \sum_{q \leq Q} q^{-1} \Delta_{211} \ll x^2 \sum_{q \leq Q} q^{-1} x^{\frac{1}{k}-1} \sum_{d \leq x^{\frac{1}{k}}} (q, d^k) d^{-k-1} \ll x^{1+\frac{1}{k}} \sum_{d \leq x^{\frac{1}{k}}} d^{-k-1} \sum_{q \leq Q} q^{-1} (q, d^k) \]

\[ \ll x^{1+\frac{1}{k}} \sum_{d \leq x^{\frac{1}{k}}} d^{-k-1} \sum_{m \mid d^k} \sum_{m \mid d^k} q^{-1} \ll x^{1+\frac{1}{k}} \log x \sum_{d \leq x^{\frac{1}{k}}} \tau(d^k) d^{-k-1}. \]

Thus

\[ x^2 \sum_{q \leq Q} q^{-1} \Delta_{211} \ll x^{1+\frac{1}{k}} \log x. \quad (5.4) \]

Since \( \Delta_{212} \ll \sum_{m > x^{\frac{1}{k}}} (q, m^k) m^{-2k}, \) we have

\[ x^2 \sum_{q \leq Q} q^{-1} \Delta_{212} \ll x^2 \sum_{q \leq Q} q^{-1} \sum_{m > x^{\frac{1}{k}}} (q, m^k) m^{-2k} \ll x^2 \sum_{q \leq Q} q^{-1} \sum_{m > x^{\frac{1}{k}}} m^{-k} \]

\[ \ll x^2 \log x \left( x^{\frac{1}{k}} \right)^{1-k}, \]
therefore,

\[ x^2 \sum_{q \leq Q} q^{-1} \Delta_{212} \ll x^{1 + \frac{1}{k}} \log x. \]  

(5.5)

By (5.2), we have

\[ J_{22} \ll x^{1 + \frac{1}{k}} \sum_{q \leq Q} q^{-1} \sum_{d=1}^{\infty} \frac{\mu^2(d)(q, d^k)}{d^k} \ll x^{1 + \frac{1}{k}} \sum_{d=1}^{\infty} \sum_{q \leq Q} q^{-1}(q, d^k), \]

as in the estimation of (5.4), we have

\[ J_{22} \ll x^{1 + \frac{1}{k}} \log x, \]  

(5.6)

and the lemma follows from (5.1), (5.3)–(5.6). □

**Lemma 5.2.** For \( k \geq 2 \), we have

\[ J_3 = x^2 \sum_{q \leq Q} q^{-1} \sum_{t=1}^{\infty} g(t)t^{-k}(q, t^k). \]

**Proof.** By the definition of \( J_3 \), we have

\[
J_3 = \sum_{q \leq Q} \sum_{1 \leq a \leq q} \left( xq^{-1} \sum_{d=1}^{\infty} \frac{\mu(d)(d^k, q)}{d^k} \right)^2 = x^2 \sum_{q \leq Q} q^{-1} \sum_{d_1, d_2=1}^{\infty} \frac{\mu(d_1)\mu(d_2)}{d_1^{k}d_2^{k}}(q, d_1^k, d_2^k)
\]

\[ = x^2 \sum_{q \leq Q} q^{-1} \sum_{t=1}^{\infty} g(t)t^{-k}(q, t^k), \]

and the lemma follows. □

By (1.7), Lemmas 4.4, 5.1 and 5.2, for \( k > 2 \),

\[
S(x, Q) = \sum_{q \leq Q} H(x, q) + O \left( Qx^{\frac{1}{k}} + x^2y^{1-k}\log^{5/2}x + x^{3/2}y^{1-k}\log^{7/2}x + x^2y^{1-k}\frac{1}{\log x} + x^{1 + \frac{1}{k}} \log x \right)
\]

\[ + O \left( x^2y^{-\frac{k}{2}} \log^3x + xy \log^2x + x^{1 + \frac{1}{k}} \log x \right). \]

A good choice for \( y \) is

\[ y = (x \log x)^{\frac{2}{k+2}}, \]

then

\[ S(x, Q) = \sum_{q < Q} H(x, q) + O \left( x^{\frac{k+4}{k+2}} \log x^{\frac{2k+6}{k+2}} \right). \]  

(5.7)
Lemma 5.3. For $k \geq 2$, we have

$$\sum_{q \leq Q} H(x, q) \ll x^{\frac{1}{k}} Q^{2 - \frac{1}{k}}.$$ 

Proof. We have

$$g(t) \ll t^{-k}, \quad \left| \int_0^t \psi(v) dv \right| \leq 1, \quad |\psi(v)| \leq 1.$$ 

Hence

$$H(x, q) \ll q \left( \sum_{t < (xq^{-1})^{\frac{1}{k}}} (q, t^k)^{-1} + \sum_{t \geq (xq^{-1})^{\frac{1}{k}}} xq^{-1}t^{-k} \right) \ll q \left( xq^{-1} \right)^{\frac{1}{k}} + qxq^{-1}(xq^{-1})^{\frac{1-k}{k}} \ll x^{\frac{1}{k}} q^{1 - \frac{1}{k}},$$

and

$$\sum_{q \leq Q} H(x, q) \ll x^{\frac{1}{k}} \sum_{q \leq Q} q^{1 - \frac{1}{k}} \ll x^{\frac{1}{k}} Q^{2 - \frac{1}{k}},$$

and the lemma follows. \(\Box\)

Finally, by Lemma 5.3 and (5.7), we have

$$S(x, Q) \ll x^{\frac{1}{k}} Q^{2 - \frac{1}{k}} + x^{\frac{k+4}{k+2}} (\log x)^{\frac{2k+6}{k+2}},$$

and Theorem 2 follows.

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References


