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Conjugacy classes and growth conditions

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Abstract

In this paper we show using a purely combinatorial argument that a finitely generated infinite group such that $f_E(n) \leqslant an^s$, where a is a constant, admits for every ϵ a sequence $\{g_{i,\epsilon}\}$ of non-unit elements whose centralizer contains more than $i^{1/2-\epsilon}$ elements of length less than i. Of course, the interest of this result is in the fact that it excludes the possibility that the group is a pure torsion group, since otherwise the existence of the sequence $\{g_{i,\epsilon}\}$ is obvious. As an application of this result, we show that, in the case where $r < 3/2$, there exists an element whose centralizer has finite index in G.

1. Introduction

Let G be a finitely generated group and E a finite generating system. For $g \in G$ define the E-length of g as the minimal length of an expression of g as a product of elements in G and denote it by $l_E(g)$. Denote by $f_E(n)$ the function that gives the number of elements of G whose length is less or equal than n. We will say that G has *polynomial growth* if there exist $a, s \in \mathbb{R}_+$ such that $f_E(n) \leqslant an^s$. It can be shown that the polynomiality of growth does not depend on the choice of the generating system, so that having polynomial growth is an intrinsic property of a group. This notion was introduced by J. Milnor [4] in order to study the relationship between curvature and the growth of the volume of the spheres in a Riemannian manifold. A famous result of M. Gromov [1] states that a group of polynomial growth contains a subgroup of finite index that is nilpotent.

In this paper we study the relationships between growth and conjugacy. We show using a combinatorial argument that if G is a finitely generated infinite group of polynomial

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growth then, for every ϵ and for arbitrarily large n, we can find an element $g \in G$ whose centralizer grows locally more than $n^{1/2-\epsilon}$, that is $f_E^{C_G(g)}(n) \geqslant n^{1/2-\epsilon}$. If, moreover, $r < 3/2$, we will see that there exists an element whose centralizer has finite index.

2. First definitions and lemmas

Throughout this paper we will suppose that G is a finitely generated infinite group of polynomial growth, as defined in Section 1, and E a finite generating subset. We will suppose that we have chosen an order on E. By G_n we will denote the set of the elements of G whose length is less than n, by $G_{m,n}$ the set of the elements whose length is greater or equal than m and less than n, and by $V_{n,x}$ the set $G_{xn,(x+1)n}$. As usual, for $g \in G$, g^S will denote the set of the elements of G of the form $s^{-1}gs$, with $s \in S$. If $H < G$, we will denote by $f_E^H(n)$ the function that gives the number of elements of H whose E-length is less or equal than n. Notice that H needs not to be finitely generated.

For short, since no confusion can arise from the context, we will adopt the convention of writing $C(g)$ for $C_G(g)$ (the centralizer of g in G), $l(g)$ for $l_E(g)$, $f(n)$ for $f_E(n)$, and $f^H(n)$ for $f_E^H(n)$. Moreover, by $\log(n)$ we will mean $\log_2(n)$.

Definition 2.1. Set $S = \{t \in \mathbb{R}_+ \mid \exists n_0 \in \mathbb{N},\ f(n) \leqslant n^t,\ \forall n > n_0\}$. As, by hypothesis, $S \neq \emptyset$, we denote by r the infimum of S. We will call it the degree of growth of G relative to E.

An immediate property of r is the following lemma.

Lemma 2.2. *For every $\epsilon > 0$ we have $f(n) \leqslant n^{r+\epsilon}$ for sufficiently large n and $f(n) > n^{r-\epsilon}$ for infinitely many n.*

Here and in the following, ϵ will be positive.

Lemma 2.3. *Let $x, y \in G$ then $|l(xy) - l(x)| \leqslant l(y)$.*

Proof. Immediate. □

Lemma 2.4. *Let $A \subseteq G_n$ and $g \in G$. Then $|g^A| \geqslant |A|/(f^{C(g)}(2n))$.*

Proof. If $a, b \in A$ then $g^a = g^b$ only if $a = bx$, with $x \in C(g)$ and $l(x) \leqslant 2n$, since $x = b^{-1}a$. □

3. Intervals of rapid growth

The condition $f(n) \leqslant an^s$ does not say much on the cardinality of the sets $V_{n,i}$, whose variation can in principle be very erratic. Nevertheless, we can give an estimate on the set of the indices i such that $V_{n,i}$ does not increase or decrease too rapidly w.r.t. $V_{n,i-1}$.

In the following two sections, we will suppose $r \geqslant 1$. In this section, we will suppose also that n is fixed.

Definition 3.1. Let $c > 1$ and $h > 0$. We will say that an interval $[h,k]$ is an increasing (respectively decreasing) c-interval if, for every i in $[h,k]$ we have $c < |V_{n,i+1}|/|V_{n,i}|$ (respectively $c < |V_{n,i-1}|/|V_{n,i}|$). We will say that it is a maximal increasing or decreasing interval, if it is maximal w.r.t. this property.

Definition 3.2. Set $C = \bigcup [h,k]$, where $[h,k]$ are the maximal increasing 2-intervals. Set $D = \bigcup [h,k]$, where $[h,k]$ are the maximal decreasing 2-intervals. Set $T = (C \cup D)$ and $I = \mathbb{N} \setminus T$.

Lemma 3.3. *Let $i \in I$. Then $|V_{n,i+1}|/|V_{n,i}| \leqslant 2$ and $|V_{n,i-1}|/|V_{n,i}| \leqslant 2$.*

Corollary 3.4. *Let $i \in I$. Then $|G_{(i-1)n,(i+2)n}|/|V_{n,i}| \leqslant 5$.*

Proof. Immediate, by Lemma 3.3, since $G_{(i-1)n,(i+2)n} = V_{n,i-1} \cup V_{n,i} \cup V_{n,i+1}$. □

Lemma 3.5. *Let $[h,k]$ be a maximal increasing (respectively decreasing) 2-interval. Then $k+1 \in I$ (respectively $h-1 \in I$).*

Proof. Let $[h,k]$ be increasing. Then $2 < |V_{n,k+1}|/|V_{n,k}|$ and $|V_{n,k}|/|V_{n,k+1}| < 1/2$, which implies that $k+1$ cannot belong to a decreasing 2-interval. On the other hand, by the maximality of $[h,k]$, $k+1$ cannot belong to an increasing 2-interval. Let $[h,k]$ be decreasing. Then $2 < |V_{n,h-1}|/|V_{n,h}|$, which implies that $h-1$ does not belong to an increasing interval. It cannot belong to a decreasing interval either, by the maximality of $[h,k]$. □

Lemma 3.6. *Let G be infinite. Let $\epsilon < 1/4$. Then, for any integer sequence $l_{n,\epsilon} \in \mathbb{N}$ such that $n^{1-4\epsilon} \leqslant l_{n,\epsilon} \leqslant n^{1-2\epsilon}$ one has $f(nl_{n,\epsilon}) \leqslant n^{2r-\epsilon}$ for sufficiently large n.*

Proof. Choose n sufficiently large and $l_{n,\epsilon}$ so that $n^{1-4\epsilon} \leqslant l_{n,\epsilon} \leqslant n^{1-2\epsilon}$. For every $\delta > 0$, by Lemma 2.2, we have

$$f(nl_{n,\epsilon}) \leqslant (nl_{n,\epsilon})^{r+\delta} \leqslant n^{r+\delta} n^{(1-2\epsilon)(r+\delta)} = n^{2r+2\delta-2\epsilon r-2\epsilon\delta}$$

for sufficiently large n. Since $r \geqslant 1$, we have

$$2\delta_0 - 2\epsilon r - 2\epsilon\delta_0 \leqslant -\tfrac{3}{2}\epsilon$$

for some sufficiently small δ_0. Then

$$f(nl_{n,\epsilon}) \leqslant n^{2r-3\epsilon/2} \leqslant n^{2r-\epsilon}$$

for sufficiently large n. □

Definition 3.7. Set $I_{n,\epsilon} = I \cap [1, l_{n,\epsilon}]$ and $T_{n,\epsilon} = T \cap [1, l_{n,\epsilon}]$.

Lemma 3.8. *Let $\epsilon < 1/8$ and let $l_{n,\epsilon}$ be as in Lemma 3.6. There exists a constant c such that if $[h,k] \subseteq [1, l_{n,\epsilon}]$ is a 2-interval (increasing or decreasing), then $k - h \leqslant c \log(l_{n,\epsilon})$.*

Proof. Let us suppose that $[h,k]$ is increasing. Since $|V_{n,h}| \geqslant 1$, we have

$$2^{k-h} \leqslant 2^{k-h}|V_{n,h}| \leqslant |V_{n,k}|.$$

But $|V_{n,k}| \leqslant f(n^2) \leqslant n^{2r+\delta}$, where δ is a constant, for sufficiently large n, by Lemma 2.2. Then $2^{k-h} \leqslant n^{2r+2\delta}$ and

$$k - h \leqslant (2r + 2\delta)\log(n) \leqslant \frac{c}{2}\log(n) = c\log\bigl(n^{1/2}\bigr) \leqslant c\log\bigl(n^{1-4\epsilon}\bigr) \leqslant c\log(l_{n,\epsilon})$$

for some constant c, for $\epsilon < 1/8$ and for sufficiently large n, by Lemma 3.6. The proof is analogous if $[h,k]$ is decreasing. □

Lemma 3.9. *For every ϵ and for sufficiently large n there exists a constant d such that $l_{n,\epsilon}^{1-\epsilon}/(d\log(l_{n,\epsilon})) \leqslant |I_{n,\epsilon}|$.*

Proof. Consider $T_{n,\epsilon}$. Suppose that, for sufficiently large n, one has $|T_{n,\epsilon}| \leqslant l_{n,\epsilon}^{1-\epsilon}$. Then, since, by definition, $[1, l_{n,\epsilon}] \setminus T_{n,\epsilon} \subseteq I_{n,\epsilon}$, we have $l_{n,\epsilon} - l_{n,\epsilon}^{1-\epsilon} \leqslant |I_{n,\epsilon}|$, and then we have the claim, since $l_{n,\epsilon}^{1-\epsilon}/(c\log(l_{n,\epsilon})) \leqslant l_{n,\epsilon} - l_{n,\epsilon}^{1-\epsilon}$, for sufficiently large n. Otherwise $l_{n,\epsilon}^{1-\epsilon} < |T_{n,\epsilon}|$. In this case consider that, by Lemma 3.8, $T_{n,\epsilon}$ contains at least $l_{n,\epsilon}^{1-\epsilon}/(c\log(l_{n,\epsilon}))$ increasing or decreasing 2-intervals. By Lemma 3.5, this implies that $I_{n,\epsilon}$ contains at least $l_{n,\epsilon}^{1-\epsilon}/(2c\log(l_{n,\epsilon}))$ elements. □

4. The growth of the conjugacy classes

Throughout this section, we will suppose that G is infinite.

Lemma 4.1. *Let $i \in I_{n,\epsilon}$. Then, there exists a constant c such that, for sufficiently large n, for every $s \in V_{n,i}$ there are at least $c\lfloor |G_{n/4}|^2/|V_{i,n}|\rfloor$ elements $p \in G_n$ such that $l(s^{-1}ps) \leqslant n$.*

Proof. Consider the function $f_s : G_{n/4} \times G_{n/4} \to G_{(i-1)n,(i+2)n}$ defined by $f_s(x,y) = xsy$. (By Lemma 2.3, if $s \in V_{n,i}$ and $x, y \in G_{n/4}$ then $xsy \in G_{(i-1)n,\,(i+2)n}$.) There is at least one element $m \in f(G_{n/4} \times G_{n/4})$ such that

$$\bigl|f_s^{-1}(m)\bigr| \geqslant \left\lfloor \frac{|G_{n/4}|^2}{|f(G_{n/4} \times G_{n/4})|} \right\rfloor \geqslant \left\lfloor \frac{|G_{n/4}|^2}{|G_{(i-1)n,\,(i+1)n}|} \right\rfloor \geqslant \left\lfloor \frac{1}{5}\,\frac{|G_{n/4}|^2}{|V_{n,i}|} \right\rfloor \geqslant c\left\lfloor \frac{|G_{n/4}|^2}{|V_{n,i}|} \right\rfloor$$

where c is a constant. Define the set L_m as

$$L_m = \{ g \in G_{n/4} \mid f_s(g, h_g) = m, \text{ for some } h_g \in G_{n/4} \}.$$

Let $(x, h_x) \in f_s^{-1}(m)$ and consider the set $L_m^{-1}x$. Since, for every fixed x_0, the function $f_s(x_0, y)$ is injective, we have $|L_m| = |f_s^{-1}(m)|$, and then $|L_m^{-1}x| = |f_s^{-1}(m)|$. Let us show that, for $s \in V_{n,i}$ and $p \in L_m^{-1}x$, we have $l(s^{-1}ps) \leqslant n$. Since $p \in L_m^{-1}x$, then there exist $g_p, h_{g_p} \in G_{n/4}$ such that $p = g_p^{-1}x$ and $g_p s h_{g_p} = x s h_x$. By this relation, we have $l(s^{-1}ps) = l(s^{-1}g_p^{-1}xs) = l(h_{g_p}h_x^{-1}) \leqslant n$ since $h_x, h_g \in G_{n/4}$. Finally, note that $L_m^{-1}x \subseteq G_n$ since $x, g_p \in G_{n/4}$. This proves the claim. □

Corollary 4.2. *Denote by $P_{n,i}$ the set of the pairs $(x, y) \in G_n \times V_{n,i}$ such that $x^y \in G_n$. Then $|P_{n,i}| \geqslant c|G_{n/4}|^2 - c|V_{n,i}|$ for sufficiently large n.*

Proof. By Lemma 4.1 we have $|P_{n,i}| \geqslant c\lfloor |G_{n/4}|^2/|V_{n,i}| \rfloor |V_{n,i}|$, and, since $\lfloor x/y \rfloor \geqslant x/y - 1$, we have $|P_{n,i}| \geqslant c|G_{n/4}|^2 - c|V_{n,i}|$. □

Lemma 4.3. *Let $\epsilon < 1$. Set $P_n^\epsilon = \bigcup_{i \in I_{n,\epsilon/16}} P_{n,i}$. Then there exists a sequence $\{n_{i,\epsilon}\}$ such that $|P_{n_{i,\epsilon}}| \geqslant n_{i,\epsilon}^{2r+1-\epsilon}$.*

Proof. For short, we set $\epsilon' = \epsilon/16$. For any fixed n, the sets $P_{n,i}$ are disjoint, then $|P_n^\epsilon| = \sum_{i \in I_{n,\epsilon'}} |P_{n,i}|$. The sets $V_{n,i}$ are disjoint too, then

$$\left| \bigcup_{i \in I_{n,\epsilon'}} V_{n,i} \right| = \sum_{i \in I_{n,\epsilon'}} |V_{n,i}| \leqslant f(nl_{n,\epsilon'})$$

since $\bigcup_{i \in I_{n,\epsilon'}} V_{n,i} \subseteq G_{nl_{n,\epsilon'}}$. Then, by Lemma 3.6 and Corollary 4.2:

$$\begin{aligned} |P_n^\epsilon| &= \sum_{i \in I_{n,\epsilon'}} |P_{n,i}| \geqslant c \sum_{i \in I_{n,\epsilon'}} \left(|G_{n/4}|^2 - |V_{n,i}| \right) = c|I_{i,\epsilon'}||G_{n/4}|^2 - c \sum_{i \in I_{n,\epsilon'}} |V_{n,i}| \\ &\geqslant c|I_{n,\epsilon'}||G_{n/4}|^2 - cf(nl_{n,\epsilon'}) \geqslant c|I_{n,\epsilon'}||G_{n/4}|^2 - cn^{2r-\epsilon'} \\ &= c|I_{n,\epsilon'}||G_{n/4}|^2 - cn^{2r-\epsilon/16}. \end{aligned}$$

Since $\epsilon < 1$ and then $\epsilon' \leqslant 1/2$, by Lemmas 3.9 and 3.6 we have:

$$\begin{aligned} c|I_{n,\epsilon'}||G_{n/4}|^2 &\geqslant \frac{cl_{n,\epsilon'}}{d\log(l_{n,\epsilon'})}|G_{n/4}|^2 \geqslant \frac{cn^{1-4\epsilon'}}{d\log(n^{1-2\epsilon'})}|G_{n/4}|^2 \\ &= \frac{1}{1-2\epsilon'}\frac{cn^{1-4\epsilon'}}{d\log(n)}|G_{n/4}|^2 \geqslant \frac{cn^{1-4\epsilon'}}{d\log(n)}|G_{n/4}|^2. \end{aligned}$$

Now, let $\{n_{i,\epsilon}\}$ be a sequence such that $f(n_{i,\epsilon}/4) \geqslant n_{i,\epsilon}^{r-\epsilon/8}$, which exists by Lemma 2.2. We have:

$$\frac{cn_{i,\epsilon}^{1-4\epsilon'}}{d\log(n_{i,\epsilon})}|G_{n_{i,\epsilon}/4}|^2 \geqslant \frac{cn_{i,\epsilon}{}^{1-4\epsilon'}}{d\log(n_{i,\epsilon})}n_{i,\epsilon}^{2r-\epsilon/4} = \frac{cn_{i,\epsilon}^{2r+1-\epsilon/2}}{d\log(n_{i,\epsilon})} \geqslant n_{i,\epsilon}^{2r+1-3\epsilon/4}$$

for sufficiently large $n_{i,\epsilon}$. Then

$$|P_n^\epsilon| \geqslant n_{i,\epsilon}^{2r+1-3\epsilon/4} - cn^{2r-\epsilon/16} \geqslant n_{i,\epsilon}^{2r+1-\epsilon},$$

again, for sufficiently large $n_{i,\epsilon}$. Then, up to eliminating a finite number of elements of $\{n_{i,\epsilon}\}$ and renumbering it, we have the claim. □

Lemma 4.4. *For every ϵ, there exist two sequences $\{g_{i,\epsilon}\}$ and $\{m_{i,\epsilon}\}$ such that $f^{C(g_{i,\epsilon})}(2m_{i,\epsilon}^2) > m_{i,\epsilon}^{1-\epsilon}$.*

Proof. Again for short we set $\epsilon' = \epsilon/2$. Let $P_{n_{i,\epsilon'}}^{\epsilon'}$ and $n_{i,\epsilon'}$ be as in Lemma 4.3. Now, $P_{n_{i,\epsilon'}}^{\epsilon'}$ can be regarded as a subset of $G_{n_{i,\epsilon'}} \times \bigcup_{i\in I_{n,\epsilon'/16}} V_{n,i}$, so that we can define an equivalence relation on $P_{n_{i,\epsilon'}}^{\epsilon'}$ as $(x, y) \equiv (x', y') \Leftrightarrow x = x'$. There are obviously at most $|G_{n_{i,\epsilon'}}|$ classes on $P_{n_{i,\epsilon'}}^{\epsilon'}$; then there is an equivalence class with more than $|P_{n_{i,\epsilon'}}^{\epsilon'}|/|G_{n_{i,\epsilon'}}|$ elements. Let $(g_{i,\epsilon}, h_{i,\epsilon})$ be a representative of this class. Therefore the set $B = \{x \in G_{n_{i,\epsilon'}l_{n_{i,\epsilon'},\epsilon'}} \mid g_{i,\epsilon}^x \in G_{n_{i,\epsilon'}}\}$ has more than $|P_{n_{i,\epsilon'}}^{\epsilon'}|/|G_{n_{i,\epsilon'}}|$ elements. Now

$$|B| \geqslant \frac{|P_{n_{i,\epsilon'}}^{\epsilon'}|}{|G_{n_{i,\epsilon'}}|} \geqslant n_{i,\epsilon'}^{r+1-3\epsilon/4}$$

since, by applying Lemma 4.3, we have $|P_{n_{i,\epsilon'}}^{\epsilon'}| \geqslant n_{i,\epsilon'}^{2r+1-\epsilon/2}$ and $|G_{n_{i,\epsilon'}}| \leqslant n_{i,\epsilon'}^{r+\epsilon/4}$, by Lemma 2.2, for sufficiently large $n_{i,\epsilon'}$. Now, since $g_{n_{i,\epsilon'}}^B \subseteq G_{i,\epsilon'}$, we have $|g_{n_{i,\epsilon'}}^B| \leqslant |G_{n_{i,\epsilon'}}| \leqslant n_{i,\epsilon'}^{r+\epsilon/4}$, for sufficiently large $n_{i,\epsilon'}$. Then, by Lemma 2.4, we have $f^{C(g)}(2n_{i,\epsilon'}^2) \geqslant |B|/|g_{n_{i,\epsilon'}}^B| \geqslant n_{i,\epsilon'}^{1-\epsilon}$, for sufficiently large $n_{i,\epsilon'}$. Then we have the claim by setting $m_{i,\epsilon} = n_{i,\epsilon'}$. □

Now we can give the result on the centralizers.

Lemma 4.5. *Let G be a finitely generated group of polynomial growth. Suppose that there exists a finite generating subset E such that the degree of growth of G relative to E is greater or equal than 1. Then, for every ϵ, there exist a sequence $\{g_{i,\epsilon}\}$, $g_{i,\epsilon} \in G$, and an increasing sequence $\{n_{i,\epsilon}\}$, $n_{i,\epsilon} \in \mathbb{N}$, such that $f^{C(g_{i,\epsilon})}(n_{i,\epsilon}) \geqslant n_{i,\epsilon}^{1/2-\epsilon}$.*

Proof. Consider $g_{i,\epsilon}$ and $n_{i,\epsilon} = 2m_{i,\epsilon}^2$, where $m_{i,\epsilon}$ and $g_{i,\epsilon}$ are as in Lemma 4.4. We have $f^{C(g_{i,\epsilon})}(n_{i,\epsilon}) \geqslant n_{i,\epsilon}^{1-\epsilon} = (2n_{i,\epsilon}^2)^{(1-\epsilon)/2} \geqslant n_{i,\epsilon}^{1/2-\epsilon}$, for sufficiently large $n_{i,\epsilon}^{1/2-\epsilon}$. □

This, together with the characterization of the case where $r < 1$ (see [3,5] or [2]), gives a combinatorial proof of the following result.

Theorem 4.6. *Let G be a finitely generated group, E a finite generating subset and a, s two positive constants such that $f_E(n) \leqslant an^s$. Then, for every ϵ, there exist a sequence $\{g_{i,\epsilon}\}$, $g_{i,\epsilon} \in G$, and an increasing sequence $\{n_{i,\epsilon}\}$, $n_{i,\epsilon} \in \mathbb{N}$, such that $f^{C(g_{i,\epsilon})}(n_{i,\epsilon}) \geqslant n_{i,\epsilon}^{1/2-\epsilon}$.*

5. The growth of the subgroups

In this section, we apply Theorem 4.6 to the case of a group having a degree of growth relative to a finite generating subset E which is less than $3/2$. In order to do it, we give a result based on a local condition on the growth of the subgroups of a group of polynomial growth.

Definition 5.1. As usual, denote by E^* the set of the finite sequences of symbols of the set E and denote by $|w|$ the length of $w \in E^*$. Let H be a subgroup of G and G/H be the set of its right cosets. Let denote $\phi_H : G/H \to E^*$ the map which associates to a coset x the element $w \in E^*$ affording its minimal representative in G w.r.t. the length and the lexicographic order $<_E$ induced by the order on E. We will say that w is the minimal representative of x. For short, we will also say that w is minimal and that $|w|$ is the length of x.

Lemma 5.2. *The set $\phi(G/H)$ is closed under taking prefixes.*

Proof. Let $x \in G/H$ and let $\Phi_H(x) = e_{n_1} \cdots e_{n_a}$, with $e_{n_1}, \ldots, e_{n_a} \in E$ be its minimal representative. The claim is equivalent to saying that, if $a > 1$, then $e_{n_1} \cdots e_{n_{a-1}}$ is minimal. In order to show it, let $e_{m_1} \cdots e_{m_{a'}}$ be the minimal representative of $He_{n_1} \cdots e_{n_{a-1}}$. We have that $a - 1 \leqslant a'$ since otherwise $e_{m_1} \cdots e_{m_{a'}} e_{n_a}$ would be a representative of x of length at most $a - 1$. On the other hand, $a' \leqslant a - 1$ since $e_{n_1} \cdots e_{n_{a-1}}$ is a representative of $He_{n_1} \cdots e_{n_{a-1}}$ of length $a-1$ and $e_{m_1} \cdots e_{m_{a'}}$ is minimal. Then $a' = a-1$. Now, $e_{n_1} \cdots e_{n_a}$ and $e_{m_1} \cdots e_{m_{a'}} e_{n_a}$ belong to the same right coset by the definition of $e_{m_1} \cdots e_{m_{a'}}$ and by the above they have both length a. Then, since $e_{n_1} \cdots e_{n_a}$ is minimal, $e_{n_1} \cdots e_{n_a} \leqslant_E e_{m_1} \cdots e_{m_{a'}} e_{n_a}$ and then $e_{n_1} \cdots e_{n_{a-1}} \leqslant_E e_{m_1} \cdots e_{m_{a'}}$. Again, by the definition of $e_{m_1} \cdots e_{m_{a'}}$, we have that $e_{n_1} \cdots e_{n_{a-1}}$ and $e_{m_1} \cdots e_{m_{a'}}$ belong to the same right coset. Then, since $e_{n_1} \cdots e_{n_{a-1}}$ has length a' and it is less or equal to $e_{m_1} \cdots e_{m_{a'}}$ w.r.t. the lexicographic order, and since $e_{m_1} \cdots e_{m_{a'}}$ is minimal, we have $e_{m_1} \cdots e_{m_{a'}} = e_{n_1} \cdots e_{n_{a-1}}$. $\square$

Lemma 5.3. *If H has infinite index in G, then for every n there exists an element of $\phi_H(G/H)$ of length n.*

Proof. Since $\phi_H(G/H) \subseteq E^*$ is injective, then, since $[G : H]$ is infinite, we can find an increasing sequence $\{n_i\}$, $i, n \in \mathbb{N}$, such that for every $i \in \mathbb{N}$ there exists $x_i \in G/H$ of length n_i. Now, define $P \subseteq E^*$ as the set of the prefixes of the elements $\phi_H(x_i)$. By

Lemma 5.2, we have $P \subseteq \phi_H(G/H)$. Moreover, for every $i \in \mathbb{N}$, and for every $k \in [1, i]$, $\phi_H(x_i)$ has a prefix of length k. Then the set $\phi_H^{-1}(P)$ yields the claim. □

The Splitting Lemma in [1, p. 59] can be stated as follows.

Lemma 5.4. *Let $a, r, s \in \mathbb{R}_+$ be such that $f_E(n) \leqslant an^r$, $s > r - 1$, and let H be a finitely generated subgroup of G. Then, if $f^H(n) \geqslant n^s$, $\forall n \in \mathbb{N}$, H has finite index in G.*

The following is a local version of the Splitting Lemma, in which H is not necessarily finitely generated.

Lemma 5.5. *Let $a, r, s \in \mathbb{R}_+$ be such that $f_E(n) \leqslant an^r$, $s > r - 1$. Fix n_0 such that $n_0^{s-r+1} > a2^r$. Then, if $H < G$ is such that $f^H(n) \geqslant n^s$, $\forall n \leqslant n_0$, H has finite index in G.*

Proof. We argue by contradiction and assume that H has infinite index. Then by Lemma 5.3, G_{n_0} contains n_0 representatives of distinct right cosets of H. Let denote them $g_1, \ldots, g_{n_0}$. Now,

$$\left| \bigcup_{i=1}^{n_0} H_{n_0} g_i \right| = n_0 |H_{n_0}| \geqslant n_0^{s+1}.$$

But

$$\bigcup_{i=1}^{n_0} H_{n_0} g_i \subseteq G_{2n_0}.$$

Then we have $n_0^{s+1} \leqslant |G_{2n_0}| \leqslant a2^r n_0^r$, contradicting the assumption on n_0. □

Now we can show the main result of this section.

Theorem 5.6. *Let G be a finitely generated group and E a finite generating subset such that $f_E(n) \leqslant an^r$, with $r < 3/2$. Then there exists $g \in G$ such that $[G : C(g)] < \infty$.*

Proof. Let ϵ' be such that $f_E(n) \leqslant an^{3/2-\epsilon'}$. Let $\epsilon < \epsilon'$ and let n_0 be as in Lemma 5.5, where $r = 3/2 - \epsilon'$ and $s = 1/2 - \epsilon$. Let the sequences $\{g_{i,\epsilon}\}$, $g_{i,\epsilon} \in G$, and $\{n_{i,\epsilon}\}$, $n_{i,\epsilon} \in \mathbb{N}$, be as in Theorem 4.6 and let i_0 be such that $n_{i_0,\epsilon} > n_0$. Now, by Theorem 4.6, we have $f^{C(g_{i_0,\epsilon})}(n_{i_0,\epsilon}) \geqslant n_{i_0,\epsilon}^{1/2-\epsilon}$. Then, since $n_{i_0,\epsilon} > n_0$, by Lemma 5.5 we have that $C(g_{i_0,\epsilon})$ has finite index in G. □

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