Multipulses of Nonlinearly Coupled Schrödinger Equations

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The capacity of coupled nonlinear Schrödinger (NLS) equations to support multibump solitary-waves is investigated. A detailed analysis is undertaken for a system of quadratically coupled equations that describe the phenomena of second harmonic generation and parametric wave interaction in non-centrosymmetric optical materials. Utilising the framework of homoclinic bifurcation theory, and employing a Lyapunov-Schmidt reduction method developed by Hale, Lin, and Sandstede, a novel mechanism for the generation of multipulses is identified, which arises from a resonant semi-simple eigenvalue configuration of the linearised steady-state equations. Conditions for the existence of multipulses, as well as a description of their geometry, are derived from the analysis.

1. INTRODUCTION

The nonlinear properties of optical materials have attracted a great deal of attention in recent years. It has been suggested that by exploiting the nonlinear response of matter, the bit-rate capacity of optical fibres can be increased dramatically (see [4, 5] and the references therein); in addition, ultra-high-speed signal processing, through the development of all-optical switches, couplers, and other devices that implement a “light guiding light” concept, may become realisable. This would allow for great improvement in the speed and economy of data transmission and manipulation; and even traditional linear transmission systems, which employ a “non-return-to-zero” (NRZ) data-encoding scheme, still need to account for the non-linearity inherent in optical fibres.

In nonlinear optics, models that take the form of two or more Schrödinger equations coupled together in a nonlinear fashion crop up in many different contexts. Examples include the systems that describe light propagation in birefringent Kerr ($\chi^{(3)}$) materials [6, 7], phase-matched third-harmonic generation (THG) in Kerr media [8], three-wave parametric
interaction in $\chi^{(2)}$ media \([9, 10]\), or $\chi^{(2)} + \chi^{(3)}$ competing nonlinearities \([11]\). Most of these models have polynomial nonlinearities, as they incorporate only the lowest order terms that represent some given type of material response to light; but there are also systems which model, for example, saturation effects (their nonlinearities could be rational functions, for instance), as well as systems which have coupling in their linear terms, e.g., fibre array equations. Only a few coupled nonlinear Schrödinger (CNLS) systems, such as the Manakov equations, are known to be completely integrable via the inverse scattering transform. Numerical simulation is frequently a valuable tool in helping to illuminate the behaviour manifested by such models; however, it provides few clues towards understanding the mechanisms that underlie the observed behaviour, and can occasionally give misleading results. In this article, and in the accompanying works \([12, 13]\), see also \([14–16]\), we identify, develop, and apply rigorous analytical methods appropriate for studying CNLS equations, and which reinforce or complement the numerical approaches. The example analysed in this article and in \([13]\) will be the equations that model “$\chi^{(2)}$ second-harmonic generation (SHG) of Type I.” Some background information on this system will be provided later on in this section.

1.1. The General Framework

The mathematical objects of particular relevance to optical communications in the nonlinear regime are “solitary-waves”: standing or travelling waves which are localised in time or in the transverse spatial direction(s)—depending on whether we are modelling light pulses in optical fibres (temporal case) or self-guided electromagnetic beams in waveguide materials (spatial case). The convention of optical scientists will be used throughout this article and \([13]\), so that the “evolution variable” of the partial differential equations (PDEs) is the coordinate, called $z$ or $\zeta$, which measures propagation distance, i.e., distance along the optical fibre or longitudinal distance along a waveguide. In the spatial case with one transverse dimension (a slab waveguide), or in the temporal case, we shall refer to the second independent variable—transverse distance or retarded time—as $t$. After transforming the PDEs to an appropriate moving frame, solitary-waves can then be characterised as “steady-state” solutions that are “homoclinic” as functions of $t$. Note that for the spatial case it makes sense to specify boundary conditions at $|t| = \infty$ because the width of a light beam is assumed to be extremely narrow compared to the width of the slab waveguide in which it propagates. Solitary-waves which are homoclinic to the zero state are called “bright,” whereas those homoclinic to some other continuous-wave equilibrium state are called “dark.” We will use the word “pulse” to refer to such solutions in general, regardless of whether the physical setting is temporal or spatial. An important issue for optical
applications is whether the device under consideration is capable of supporting “multipulses.” These are pulses which possess more than one hump, and resemble several concatenated copies of the basic single-humped pulse. They would represent the propagation of multiple pieces of information along a device, with potential applications in multiplexing; it has also been suggested that the number of humps of a multipulse could be used as the basis of a code—an alphabet, for instance. It is clearly of great value to know what kinds of models can support multihumped solitary-wave solutions, and to understand the mathematical mechanisms whereby such solutions are generated. Also of interest are waves which are heteroclinic connections from one continuous-wave equilibrium state to another, different, one; they are known in the optics literature as “kinks,” and often occur in pairs—forming so-called “domain walls” [17]—due to the symmetries CNLS equations usually possess. In the recent decade, significant attention has been paid to kinks and dark solitary-waves because of their potential role, as self-induced waveguides, in the design of all-optical signal processing devices; see, e.g., [18, 19]. Periodic waves and other, more complicated, types of behaviour and patterns have also been studied by some researchers.

The primary mathematical questions we seek to answer are: (1) whether a given system of equations admits solitary-wave solutions, and (2) whether such solutions are “stable,” i.e., whether they retain their shape upon propagation. We aim to show, by analytical means, that pulses exist in certain parameter regimes, and then investigate what happens to them as parameters are varied (bifurcation to multipulses can occur in certain cases), and as they are propagated along the optical fibre or waveguide.

When studying the sort of PDEs under consideration here, the interplay of two different, but complementary, mathematical viewpoints is fruitful. Solitary-waves are stationary with respect to the propagation variable ζ (after changing the frame of reference, if necessary), and for conservative CNLS systems, such ζ-stationary solutions can be cast as critical points of some energy functional. Taking a static, calculus of variations/critical point theory approach then seems very natural. In contrast, when trying to describe the shape of these solutions as functions of t, or when analysing their stability with respect to evolution in ζ, it would seem appropriate to use techniques of dynamical systems theory; for example, the generation of multipulses can often be studied as a global bifurcation problem organised around homoclinic or heteroclinic orbits. A mixture of these “static” and “dynamic” approaches makes for powerful machinery with which to tackle conservative PDE systems analytically.

Before introducing any specific models, let us outline the programme adopted here and in [12–14]:
(1) First of all, we determine the existence of solitary-waves, by means of a variational method. After identifying the appropriate functional(s) to work with, “critical points”—corresponding to solitary-waves—can be sought by applying various minimax or minimisation theorems and techniques.

(2) Next, we investigate the geometric properties of these pulse solutions, focusing especially on the behaviour of the system at and near “candidate bifurcation points.” These are parameter values at which something unusual occurs, for instance when the spectrum of the linearisation at an equilibrium point undergoes a qualitative change, or when a homoclinic orbit converges to the equilibrium in a non-generic way. Such candidate bifurcation points, and the “primary pulse” solutions that exist at these parameter values, frequently act as “organising centres” for interesting dynamics occurring nearby. In some cases, they can be identified as generating points of multipulses.

(3) To see whether or not multipulses are generated for a given system at a candidate bifurcation point, we analyse the appropriate homoclinic bifurcation problem, aiming, in particular, to construct multi-humped waves by patching together piecewise continuous solutions that lie close to the primary pulses.

(4) If the existence of pulses and multipulses can be demonstrated, we then proceed to analyse their stability. Much of the information needed for the determination of stability or instability (mostly concerned with the spectra of certain linearised operators) can be gleaned from the preceding existence and bifurcation analyses.

We will now focus on the two-equation system of “Type I” $\chi^{(2)}$ second harmonic generation (SHG). Some other CNLS systems and directions of generalisation will be discussed in Section 4.

1.2. The $\chi^{(2)}$ SHG Equations

In non-centrosymmetric materials, i.e., those which do not possess inversion symmetry at the molecular level, the lowest order nonlinear effects originate from the second-order susceptibility $\chi^{(2)}$; this means that the nonlinear response of the matter to the electric field is quadratic. Quadratic nonlinearities are long known to be responsible for phenomena such as “second-harmonic generation” (frequency doubling), whereby laser light with frequency $\omega$ can be partially converted to light of frequency $2\omega$ upon passing it through a crystal with $\chi^{(2)}$ response [20]. Since the early nineties there has been a resurgence of interest in parametric wave interactions in quadratic optical media, as it was observed experimentally that “cascaded $\chi^{(2)}$ effects” can produce a variety of phenomena that might find practical application in ultra-fast all-optical signal processing (e.g., switching, or
dragging and steering of light beams), as well as long-distance communications. For more background on the physics or engineering aspects of $\chi^{(2)}$ consult [17, 21–26] and the references therein.

In two dimensions—either space + time (pulse propagation in fibres) or two space dimensions (beams in slab waveguides)—the basic equations describing Type I $\chi^{(2)}$ SHG are

$$i \frac{\partial E_1}{\partial z} + i \delta_1 \frac{\partial E_1}{\partial x} + \gamma_1 \frac{\partial^2 E_1}{\partial x^2} + \chi_1 E_1 E_2^* e^{i(\beta_1)} e^{i z} = 0,$$

$$i \frac{\partial E_2}{\partial z} + i \delta_2 \frac{\partial E_2}{\partial x} + \gamma_2 \frac{\partial^2 E_2}{\partial x^2} + \chi_2 E_1^2 e^{-i(\beta_2)} e^{i z} = 0.$$  

After non-dimensionalisation, the use of moving-frame coordinates ($t = \tau - c \zeta$, where $\tau$, $\zeta$ are dimensionless variables), and the insertion of an ansatz of “bound state” form

$$w(\zeta, t) e^{i(\Omega t + \beta \zeta)}, \quad v(\zeta, t) e^{2i(\Omega t + \beta \zeta)},$$

the coupled nonlinear Schrödinger system

$$i \frac{\partial w}{\partial \zeta} + r \frac{\partial^2 w}{\partial \tau^2} - \theta w + w v = 0$$

$$i \sigma \frac{\partial v}{\partial \zeta} + s \frac{\partial^2 v}{\partial \tau^2} - \alpha v + \frac{1}{2} w^2 = 0$$

may be derived (i.e., finite values of $c$ and $\Omega$ determined) provided $\gamma_1 \neq 2 \gamma_2$, or $\gamma_1 = 2 \gamma_2$ and also $\delta_1 = \delta_2$. A detailed description of this normalisation procedure is contained in [14], and a brief physical interpretation of the conditions on $\gamma_i$ and $\delta_i$ (zero walk-off) can be found in [12, 14]. In (1), $w(\zeta, t)$ and $v(\zeta, t)$ are complex variables that represent the envelope amplitudes of the fundamental and second-harmonic waves, respectively; thus (1) describes the “interaction” of these harmonics. We have $r, s = \pm 1$, since they represent the signs of $\gamma_1$ and $\gamma_2$, the dispersion/diffraction coefficients (temporal/spatial cases, respectively), while the constant $\sigma$ measures the ratios of the dispersions/diffractions ($\sigma = 2$ for the spatial case). The $\theta$ and $\alpha$ are dimensionless real parameters, with $\alpha$ incorporating the wavevector mismatch between the two harmonics.

System (1) is an infinite-dimensional conservative system with an $\zeta$-invariant energy integral

$$\int_{\mathbb{R}} \left( r \left| \frac{\partial w}{\partial \tau} \right|^2 + s \left| \frac{\partial v}{\partial \tau} \right|^2 + \theta |w|^2 + \alpha |v|^2 + \text{Re}(w^2 v^*) \right) dt.$$
Solutions of (1) that are stationary with respect to the variable \( \zeta \) (these represent travelling waves for the original SHG equations) may be obtained upon setting \( w = w(t), \ v = v(t) \). This yields the ordinary differential equations

\[
\begin{align*}
    r \frac{d^2 w}{dt^2} - \theta w + w^* v &= 0 \\
    s \frac{d^2 v}{dt^2} - \varpi v + \frac{1}{2} w^2 &= 0
\end{align*}
\]  

and, further simplifying the problem, we first look for real solutions; we remark that real solutions also form an invariant set in the phase-space of (2). Taking \( w(t), \ v(t) \) to be real variables, the coupled ODEs make up a two degree-of-freedom Hamiltonian system

\[
\begin{align*}
    w' &= r p_w \\
    v' &= s p_v \\
    p_w' &= \theta w - w v \\
    p_v' &= \varpi v - \frac{1}{2} w^2
\end{align*}
\]  

(note that since \( r, s = \pm 1 \), we can identify \( \frac{1}{r} \) with \( r \), and \( \frac{1}{s} \) with \( s \)). Its Hamiltonian is

\[
H(w, v; p_w, p_v) = \frac{r}{2} p_w^2 + \frac{s}{2} p_v^2 + \frac{1}{2} (w^2 v - \theta w^2 - \varpi v^2),
\]

which belongs to the “Henon–Heiles” family, and is known to be nonintegrable (see, for example, [17; 24; 27, Sect. 1.3; 28]). It is not possible to find any other integral of motion for (3).

The equilibrium points of (3) are \((w, v, p_w, p_v) = (0, 0, 0, 0)\) and also \((\pm \sqrt{2\theta}, \theta, 0, 0)\) if \( \theta > 0 \). The spectrum of the linearisation of (3) at each of its equilibrium points exhibits many different scenarios in the \((r\theta, ss)\) parameter plane (depicted in [14, Sect. 1.4]).

A few explicit solutions are known for the ODEs. The simplest ones occur when we take \( r = s = +1 \) and \( \theta = \varpi \), in which case the two equations \( rw'' = \theta w - w v \) and \( sv'' = \varpi v - \frac{1}{2} w^2 \) reduce to the same one upon substituting \( w = \pm \sqrt{2v} \). This equation \( rw'' = \theta v - v^2 \) is a one degree-of-freedom Hamiltonian system that is easy to analyse and possesses \( \text{sech}^2 \)-type homoclinic solutions.

We shall concentrate on the parameter regime \( r = s = +1 \) and \( \theta, \varpi > 0 \), in which case it is not hard to show that a rescaling allows one to set \( \theta = 1 \)
without loss of generality. Currently, this is the most interesting regime from a physics and engineering viewpoint, as in experiments \( \chi^{(2)} \) phenomena have been observed successfully in the spatial, rather than temporal, situation—and the spatial model always has \( r = s = +1 \). We are primarily interested in solutions homoclinic to the origin, which represent bright pulses of the PDE system (1). At the distinguished value of \( \alpha = \theta = 1 \), (2) then possesses the explicit real pulse solutions

\[
\tilde{w}^\pm = \pm \frac{3}{\sqrt{2}} \operatorname{sech}^2 \left( \frac{t}{2} \right), \quad \tilde{v}^\pm = \frac{3}{2} \operatorname{sech}^2 \left( \frac{t}{2} \right)
\]

found by a number of authors (see, e.g., [22, 29]). In [12], the existence of bright solitary-waves was proved, for all values of \( \alpha > 0 \), by formulating the problem in terms of finding critical points of a certain energy functional. Since solitary-waves are \( \zeta \)-stationary solutions of (1), this functional closely resembles the Hamiltonian invariant of the PDEs. A Mountain Pass Theorem and “Concentration-Compactness” arguments were utilised to show the existence of non-trivial critical points. In [15], a related constrained minimisation problem is discussed, which allows us to define a “ground-state” solution in a natural way.

The variational approach taken in [12, 15], while proving the existence, for every positive value of \( \alpha \), of \( \zeta \)-stationary solutions that decay to 0 as \( t \to \pm \infty \), does not provide us with much information as to the shape of such solitary-wave solutions. Many authors ([12, 22, 24–26], etc.) have observed, from numerical simulations, the existence of solitary-waves of the \( \chi^{(2)} \) SHG system that possess a multiple number of “humps” (peaks or troughs). For all of the numerically computed multihumped pulses, with parameter values \( r = s = +1 \) and \( \alpha, \theta \) positive, the \( w \)-component appears to undergo a phase rotation of \( \pi \) from each hump to the next, while the same is not observed for the \( v \)-component; moreover, multihumped pulses seem to exist only for values of \( \alpha \) between 0 and \( \theta \); none have been found for \( \alpha > \theta \). In this article, we will make use of dynamical systems techniques (with \( t \) as the evolution variable) to elucidate the geometry of solitary-wave solutions in phase space. We begin with proving some geometric properties, such as non-degeneracy, of the explicit “1-pulse” pair at \( \alpha = \theta \), and offer some basic descriptions of the phase space nearby. Within the framework of homoclinic bifurcation theory, we then uncover a mechanism that gives rise to pulses which possess more than one hump. These “\( N \)-pulses” are found (for \( \alpha \) sufficiently near \( \theta \)) by patching together piecewise solutions that lie close to the pair of 1-pulses. This technique, developed by X.-B. Lin and B. Sandstede, is functional-analytic in essence, but with strong geometric underpinnings. The homoclinic bifurcation problem reduces to solving a set of algebraic equations, and from the
process of solving these equations, we are able to deduce that \( N \)-pulses exist only for \( \alpha < \theta \), and must be of "strictly alternating" type. This particular bifurcation occurs when the eigenvalues of the ODE linearised about zero are in resonant semi-simple configuration. We identify this scenario (which arises in many coupled nonlinear Schrödinger systems) as a mechanism of multipulse generation, the analysis of which has not appeared previously in the literature. In addition to demonstrating the presence of multihumped pulses, our mathematical analysis also helps to clarify the range of parameter values at which these \( N \)-pulses exist, proves their uniqueness up to \((w, v) \mapsto (-w, v)\) reflection, and provides detailed qualitative information about them, thus rigorously validating the observations previously obtained by various researchers from numerical simulation. Theorem 2.1 is the main result of this paper, and Section 3 contains the details of its proof. Section 4 includes discussion on the implications of results proved in earlier sections, especially with regard to systems of conservative coupled nonlinear Schrödinger equations. In the companion paper [13], the stability of the multipulses described in this article is analysed, and more potential directions of generalisation and other related problems are suggested.

2. BACKGROUND INFORMATION

Our purpose in this section and the next is to uncover a mechanism that generates multipulses. First, we analyse what happens near the "candidate bifurcation point," namely the parameter values at which the spectrum of the linearisation of (5) consists of a resonant double semi-simple pair of eigenvalues. We shall then be concerned with the question of whether multihumped pulses bifurcate from the pair of single-humped pulses which are known to exist there. This section contains preliminary lemmas and a statement of the main theorem, whose proof is the subject of Section 3.

The behaviour of real \( \zeta \)-stationary solutions of (1) is captured by the system of ODEs

\[
\begin{align*}
  rw'' &= \theta w' - wv \\
  sv'' &= \alpha v - \frac{1}{2} w^2,
\end{align*}
\]

where we use prime to denote differentiation with respect to \( t \). As noted in Section 1, when \( r = s \) and \( \theta = \pi \), explicit 1-pulse solutions exist, which have a localised "sech" shape. In particular, for \( r = s = 1 \) and, without loss of generality, \( \theta = 1 \), we have the first-order Hamiltonian system...
\[
\begin{cases}
  w' = p_w \\
  v' = p_v \\
  p_w' = w - wv \\
  p_v' = \sigma v - \frac{1}{2} w^2
\end{cases}
\]  \quad (5)

with a pair of 1-pulses existing at \( \alpha = 1 \), given by

\[
w = \pm \sqrt{2} \tilde{v}(t), \quad v = \tilde{v}(t) = \frac{3}{2} \operatorname{sech}^2 \left( \frac{t}{2} \right).
\]

This \( \tilde{v} \) is the unique homoclinic solution of the scalar equation \( v'' = v - v^2 \).

The conserved quantity (Hamiltonian) for (5) is

\[
H(w, v; p_w, p_v) = \frac{1}{2} \left( p_w^2 + p_v^2 - w^2 - \alpha v^2 + w v \right)
\]  \quad (6)

and the linearisation of the right hand side of (5) about the equilibrium point \((0, 0, 0, 0)\) is just

\[
J \nabla^2 H(0), \quad \text{where} \quad J = \left( \begin{array}{cc}
  0 & I_2 \\
  -I_2 & 0
\end{array} \right)
\]

\((I_2)\) represents the \(2 \times 2\) identity matrix. Let us define

\[
\mu = \sqrt{\alpha} - 1
\]

(the reason for this should become evident a little later), then

\[
\nabla^2 H(0) = \left( \begin{array}{cccc}
  -1 & 0 & 0 & 0 \\
  0 & -\alpha & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{array} \right)
\]

To emphasise the \(\mu\)-dependence, we will write \(\nabla^2 H(0, \mu)\) in the future. The following symmetries of (5) (and of the more general (3)) will be useful:

- reflection in \( w \), \((w, v, p_w, p_v) \mapsto (-w, v, -p_w, p_v)\); represented by the matrix

\[
S = \left( \begin{array}{cccc}
  -1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & -1 & 0 \\
  0 & 0 & 0 & 1
\end{array} \right).
\]
• reversibility, \( t \mapsto -t \) and \((w, v, p_w, p_v) \mapsto (w, v, -p_w, -p_v)\); the latter transformation (a linear involution) is represented by the matrix
\[
R = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix};
\]

• another reversibility: \( t \mapsto -t \) and \((w, v, p_w, p_v) \mapsto (-w, v, p_w, -p_v)\); the involution is represented by the matrix
\[
\tilde{R} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix};
\]

which is none other than the product of \( R \) and \( S \).

The pulse which is positive in both its \( w \) and \( v \) components will be denoted
\[
\tilde{q}(t) = (\tilde{w}(t), \tilde{v}(t), \tilde{w}'(t), \tilde{v}'(t))
\]

where
\[
\tilde{w}(t) = \frac{3}{\sqrt{2}} \, \text{sech}^2\left( \frac{t}{2} \right), \quad \tilde{v}(t) = \frac{3}{2} \, \text{sech}^2\left( \frac{t}{2} \right),
\]

and the other 1-pulse is then \( Sq(t) \); see Fig. 1. This pair of pulses \( \tilde{q}(t) \) and \( Sq(t) \) form a figure-eight in the 4-dimensional \((w, v, p_w, p_v)\) phase space; see Fig. 2. The fixed point space of \( R \) is the set
\[
\text{Fix}(R) = \{ (w, v, w', v') : w' = v' = 0 \}.
\]

We have \( \tilde{q}(0), Sq(0) \in \text{Fix}(R) \). Let \( \Sigma_f, \Sigma_S \subset \text{Fix}(R) \) be sections transverse to the vector field placed, respectively, at \( \tilde{q}(0), Sq(0) \).

The linearisation \( J^{\mathcal{H}} H(0, \mu) \) of (5) about \((0, 0, 0, 0)\) has \( \pm 1 \) and \( \pm \sqrt{\mu} \) (i.e., \( \pm (1 + \mu) \)) as eigenvalues. As \( \mu \) is increased from below to above 0, corresponding to \( \pi \) being increased from below to above 1, the two positive eigenvalues merge and then “pass through” each other, still remaining on
FIG. 1. \( w \) (solid curve) and \( v \) (dashed curve) depicted as functions of \( t \). (a) \( Sq \) has negative \( w \) component; (b) \( q \) has positive \( w \) component.

the real axis; similarly for the negative eigenvalues. See Fig. 3. The eigenvectors associated with \( \pm 1, \pm (1 + \mu) \) are, respectively,

\[
\begin{pmatrix} 1 \\ 0 \\ \pm 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ \pm (1 + \mu) \end{pmatrix}
\]

Note that they are always distinct and linearly independent, even when \( \mu = 0 \); so at \( \mu = 0 \) the resonant eigenvalues at \( \pm 1 \) not only have algebraic multiplicity 2, but also have geometric multiplicity equal to 2. This means that there is no Jordan block associated with these eigenvalues, and they are referred to as “semi-simple.” At first sight, the scenario depicted in Fig. 3 may seem non-generic, as an often encountered situation is the one in which two real eigenvalues merge and then split off the real axis (this occurs when the double eigenvalue created by the merge has geometric multiplicity 1). However, as we shall discuss in Section 4, the situation we

FIG. 2. The double-loop (figure-eight) configuration.
see here—the “resonant semi-simple eigenvalue scenario”—is typical for problems derived from coupled nonlinear Schrödinger equations.

### 2.1. Existence of Multihumped Pulses

We are interested in the existence of “N-pulses” for the system (5). These are homoclinic solutions lying in a small tubular neighbourhood of the figure-eight depicted in Fig. 2 and following the loops $\tilde{q}(t)$ and $S\tilde{q}(t)$ in an arbitrary order such that the total number of loop-like excursions is equal to $N$. More precisely, an $N$-pulse intersects $\Sigma_I \cup \Sigma_S$ exactly $N$ times, and its shape depends on the order in which it intersects the transverse sections $\Sigma_I$ and $\Sigma_S$, which encodes the order in which it follows $\tilde{q}(t)$ or $S\tilde{q}(t)$. Recall that apart from $R$, $R=RS$ also represents a reversibility of the system; we say a homoclinic solution $q(t)$ is symmetric if either $q(0) # \text{Fix}(R)$ or $q(0) # \text{Fix}(\tilde{R})$ holds after a suitable choice of $t=0$, i.e., if $q(t)$ is either $R$-symmetric or $\tilde{R}$-symmetric. The main result of this chapter is the following:

**Theorem 2.1.** Consider (5) and let $\alpha > 0$, then for any positive integer $N$, there exists an $\varepsilon_N > 0$ such that the following holds:

Whenever $\alpha < 1$ and $1 - \sqrt{\alpha} < \varepsilon_N$, there exists a multihumped solution $q_{N,\alpha}(t)$, which intersects the sections $\Sigma_I$ and $\Sigma_S$ $N$ times alternately. The separation in $t$ between consecutive humps tends to infinity as $\alpha \to 1$ with rate proportional to $(1 - \sqrt{\alpha})^{-1}$. All bifurcating $N$-pulses must be of such alternating type; in fact, for any $N$ and $\alpha$ satisfying the above conditions, $q_{N,\alpha}(t)$ is the unique $N$-pulse, up to $S$-reflection.

Multihumped solutions do not exist for $\alpha \geq 1$ (with $\alpha$ close to 1).

$N$-pulses with an odd number of humps are $R$-symmetric, while $N$-pulses with an even number of humps are $\tilde{R}$-symmetric. Theorem 2.1 is a bifurcation result, proved only for $\alpha \approx 1$. These bifurcating $N$-pulses can, however, be numerically continued to all values of $\alpha$ in $(0, 1)$ (an approach taken in [12]; also see [30, Sect. 5]) using, for instance, the tools for homoclinic and reversible-homoclinic continuation included in the software package AUTO97 [31]. Analytical continuation may also be possible, via topological degree arguments, for instance.

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**FIG. 3.** Movement of the eigenvalues when $\alpha \approx 1$. (a) $\alpha < 1$; (b) $\alpha = 1$; (c) $\alpha > 1$.
Since \( \theta \) has been scaled to 1 without loss of generality, we emphasise that all statements and results in this article and in [13] also hold when \( \theta \) assumes other positive values, with appropriate adjustments. For instance, the eigenvalues of the linearisation about \((0, 0, 0, 0)\) are \( \pm \sqrt{\theta} \) and \( \pm \sqrt{x} \) for general \( \theta > 0 \), and one would take \( \mu = \sqrt{\theta} - \sqrt{x} \) instead; then unique alternating multipulses exist when \( x < \theta \).

2.2. Preliminary Lemmas

In this section we examine in greater detail the properties of the 1-pulse pair at \( \alpha = 1 \) (\( \mu = 0 \)), as well as the phase space geometry for \( \alpha \) close to 1. Two lemmas will be presented which are crucial for both the bifurcation and stability analyses of multipulses. From now on when we refer to “the 1-pulse at \( \alpha = 1 \),” we shall mean the one with positive \( w \)-component, namely \( \hat{q}(\cdot) \). By reflection, of course, all statements will hold equally for the \( w < 0 \) pulse, \( \hat{S}q(\cdot) \).

First, we provide a portrayal of the behaviour of solutions that lie in the local stable manifold of the equilibrium point at 0; a similar description exists for solutions in the local unstable manifold. The following statement is a modification of [3, Lemmas 1.5 and 1.7] which is needed to treat our case of double semi-simple leading eigenvalues.

**Lemma 2.1** Consider the ordinary differential equation

\[ u' = A(\mu) u + F(u, \mu), \quad (u, \mu) \in \mathbb{R}^n \times V, \]

where \( V \subseteq \mathbb{R}^n \) is an open set containing 0, \( F \in C^k \) for some \( k \geq 2 \) and \( F(0, \mu) = D_2 F(0, \mu) = 0 \) for all \( \mu \in V \). Suppose that the spectrum of \( A(\mu) \) can be decomposed as

\[ \sigma(A(\mu)) = \{ -\hat{\lambda}_1(\mu), -\hat{\lambda}_2(\mu) \} \cup \sigma'(\mu) \]

with

\[ \text{Re} \sigma'(\mu) < -\alpha' < -\alpha'' < -\hat{\lambda}_1(\mu) < -\alpha' < 0 \quad \text{for} \quad i = 1, 2 \]

uniformly in \( \mu \) (within the set \( V \)); \( \alpha', \alpha'' \) are positive constants chosen to satisfy \( 2\alpha' > \alpha'' \). The \( -\hat{\lambda}_i(\mu) \) are assumed to be real eigenvalues, \( C^k \) in \( \mu \), and simple for \( \mu \neq 0 \). Also assume \( \hat{\lambda}_1(0) = \hat{\lambda}_2(0) \). Let \( Q_0^0, Q_0^c, P_0^c \) be the spectral projections associated with \( -\hat{\lambda}_1(\mu), -\hat{\lambda}_2(\mu), \sigma' \), respectively, with \( Q_0^0 + Q_0^c + P_0^c = I \) for all \( \mu \in V \), and assume these projections depend smoothly on \( \mu \), so that without loss of generality they may be taken independent of \( \mu \).
When the above hypotheses are satisfied, there exists a $\delta > 0$ with the following property: consider small initial data $u_0(\mu) \in C^k(V, U_0(0))$ with corresponding solutions $u(t, \mu)$, then the following estimates hold with $k = 0, 1$,

$$\|D^k_t(u(t, \mu) - e^{-\lambda_1(\mu)t}v_1(\mu) - e^{-\lambda_2(\mu)t}v_2(\mu))\| = O(e^{-2\delta t}) + O(e^{-\delta t}) P_0,$$

where $v_i(\cdot) \in C^k(V, \mathbb{R}Q_i^0)$, $i = 1, 2$.

(Notation: $U_0(0)$ refers to a $\delta$-neighbourhood of $0$; $\mathbb{R}Q_i^0$ refers to the range of $Q_i^0$, i.e., the eigenspace associated with eigenvalue $-\lambda_i$.) We remark that although the conclusion of this lemma closely resembles that of [3, Lemma 1.7(ii)], the same proof does not work here. For [3, Lemma 1.7], a spectral gap between the two weakest eigenvalues (both of them simple) was assumed; the local strong stable manifold can then be flattened out so that it lies in the strong stable eigenspace, and a crucial estimate in the proof depends on this condition. The proof of our lemma is based on the ideas in Lemma 1.5 of [3].

Proof. Let $P_0 = Q_1^0 + Q_2^0$ and $A_s(\mu) = A(\mu) P_0^s$, $A_r(\mu) = A(\mu) P_0^r$. Saying $\mathbb{R}Q_1^0 + \mathbb{R}Q_2^0$ spans the complement of $\mathbb{R}P_0^s$ for any $\mu \in V$ is equivalent to saying that the eigenvectors $e_1, e_2$ associated with $-\lambda_1(\mu), -\lambda_2(\mu)$ must be linearly independent for all $\mu \in V$, and hence $A_s(\mu)$ can be diagonalised. In fact, there is a matrix $M$, whose first two columns are $e_1, e_2$, such that taking $u = My$ transforms our ODE to

$$y' = M^{-1}AMy + g(y, \mu) \quad \text{with} \quad M^{-1}AM = \begin{pmatrix} B_s(\mu) & 0 \\ 0 & B_r(\mu) \end{pmatrix},$$

and moreover

$$B_s(\mu) = \begin{pmatrix} -\lambda_1(\mu) & 0 \\ 0 & -\lambda_2(\mu) \end{pmatrix}.$$

We shall write

$$A_s = \begin{pmatrix} B_s & 0 \\ 0 & 0 \end{pmatrix}, \quad A_r = \begin{pmatrix} 0 & 0 \\ 0 & B_r \end{pmatrix}.$$

Note that $\sigma(B_s) = \sigma(A_s) = \sigma^r$ and $g(y, \mu)$ is the nonlinearity in new coordinates which, like $F$, is at least quadratic.
Using the variation of constants formula and the spectral projections $P_s^*, P_r^*$, we can write the above (transformed) ODE as a set of two integral equations

$$
\begin{align*}
P_s^* y(t, \mu) &= e^{\tilde{A}_s(t-s)} P_s^* y(s, \mu) + \int_s^t e^{\tilde{A}_s(t-\tau)} P_s^* g(y(\tau, \mu), \mu) \, d\tau, \\
P_r^* y(t, \mu) &= e^{\tilde{A}_r(t-s)} P_r^* y_0(\mu) + \int_s^t e^{\tilde{A}_r(t-\tau)} P_r^* g(y(\tau, \mu), \mu) \, d\tau.
\end{align*}
$$

(8)

Multiplying the first equation through by $e^{-\tilde{A}_s t}$ gives

$$
e^{-\tilde{A}_s t} P_s^* y(t, \mu) - e^{-\tilde{A}_s t} P_s^* y(s, \mu) = \int_s^t e^{-\tilde{A}_s \tau} P_s^* g(y(\tau, \mu), \mu) \, d\tau.
$$

We estimate

$$
\left| \int_s^t e^{-\tilde{A}_s \tau} P_s^* g(y(\tau, \mu), \mu) \, d\tau \right| \leq C \int_s^t e^{c_0 \tau} - e^{-2\pi^2 \tau} d\tau
$$

(9)

($C$ is a generic positive constant). By assumption $\pi^m < 2\pi^s$, so the integral in (9) approaches zero uniformly in $\mu$ as $t, s \to \infty$. This means that $\{ e^{-\tilde{A}_s(\mu) \tau} P_s^* y(t, \mu) \}$ is a Cauchy sequence indexed by $t$, it then follows that the limit

$$
v(\mu) := \lim_{t \to \infty} e^{-\tilde{A}_s(\mu) \tau} P_s^* y(t, \mu)
$$

exists, and is $C^k$ in $\mu$. Now take $s = 0$ in the first equation of (8), and multiply through by $e^{-\tilde{A}_s t}$ to obtain

$$
e^{-\tilde{A}_s (t-s)} P_s^* y(t, \mu) = P_s^* y(0, \mu) + \int_0^t e^{-\tilde{A}_s (\tau-s)} P_s^* g(y(\tau, \mu), \mu) \, d\tau.
$$

Taking $t \to \infty$ then gives

$$
v(\mu) = P_s^* y(0, \mu) + \int_0^\infty e^{-\tilde{A}_s (\tau-s)} P_s^* g(y(\tau, \mu), \mu) \, d\tau \in \mathcal{A} P_s^*.
$$

Plugging this back into (8) with $s = 0$, we obtain

$$
P_s^* y(t, \mu) = e^{\tilde{A}_s (t-s)} v(\mu) + \int_s^t e^{\tilde{A}_s (t-\tau)} P_s^* g(y(\tau, \mu), \mu) \, d\tau.
$$
Now
\[ y(t, \mu) = P_0^s y(t, \mu) + P_0^r y(t, \mu) \]
\[ = e^{\tilde{\lambda}_1(\mu) t} y(\mu) + \int_0^t e^{\tilde{\lambda}_3(\mu)(\tau-t)} P_0^r g(\gamma(\tau, \mu), \mu) d\tau \]
\[ + e^{\tilde{\lambda}_1(\mu) t} P_0^r \gamma(\mu) + e^{\tilde{\lambda}_1(\mu) t} \int_0^t e^{\tilde{\lambda}_3(\mu)(\tau-t)} P_0^r g(\gamma(\tau, \mu), \mu) d\tau. \]

For the first integral term we use an estimate similar to that in (9) with \( s = \infty \) (so note that here \( \tau > t \)), namely
\[ \left| \int_0^t e^{\tilde{\lambda}_3(\mu)(\tau-t)} P_0^r g(\gamma(\tau, \mu), \mu) d\tau \right| \leq \int_0^t C e^{\lambda_0(\mu)(\tau-t)} e^{-2\tilde{\lambda}_3(\mu) t} d\tau \]
\[ \leq C e^{-\tilde{\lambda}_3(\mu) t} |e^{(\lambda_0(\mu) - 2\tilde{\lambda}_3(\mu)) \eta} | \]
\[ \leq C e^{-2\lambda_0^r \eta}, \]
where \( \lambda_0^r = \max \{ \tilde{\lambda}_1, \tilde{\lambda}_2 \} \), \( \tilde{\lambda}^t = \min \{ \tilde{\lambda}_1, \tilde{\lambda}_2 \} \). The terms involving \( P_0^r \) can be collected as \( C(e^{-s^t}) P_0^r \). Therefore we obtain
\[ y(t, \mu) = e^{\tilde{\lambda}_1(\mu) t} y(\mu) + C(e^{-s^t}) + C(e^{-s^t}) P_0^r \]

Now transform back into original coordinates: in the new \( (\gamma) \) coordinate system \( v(\mu) \) is of the form \( (c_1(\mu), c_2(\mu), 0, ..., 0)^T \), with \( c_1, c_2 \) being \( C^k \) in \( \mu \), so
\[
e^{\tilde{\lambda}_1(\mu) t} y(\mu) = \begin{pmatrix} e^{-\tilde{\lambda}_1(\mu) t} & e^{-\tilde{\lambda}_2(\mu) t} \\ 0 & \ddots \\ \vdots & \ddots & \ddots \\ 0 & \ddots & \ddots & c_1(\mu) e^{-\tilde{\lambda}_1(\mu) t} \\ \vdots & \ddots & \ddots & \ddots & c_2(\mu) e^{-\tilde{\lambda}_2(\mu) t} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \end{pmatrix} \begin{pmatrix} c_1(\mu) \\ c_2(\mu) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \]
and hence
\[
  u(t, \mu) = My(t, \mu) = c_1(\mu)e^{-\lambda_1(\mu)t}v_1 + c_2(\mu)e^{-\lambda_2(\mu)t}v_2 + e(\epsilon^{\alpha t})P_0',
\]
where \(i = 1, 2\) \(v_i(\mu) \in \mathbb{H}Q_0'\) are \(C^k\)-smooth in \(\mu\).

In a similar manner, the expansion for the derivative may also be verified.

Note that at \(\mu = 0\) we know \(\lambda_1(0) = \lambda_2(0)\), hence for \(|\mu|\) small, we also have \(|\lambda_1(\mu) - \lambda_2(\mu)|\) being small. This means that provided \(|\mu|\) is sufficiently small, \(\pi', \pi''\) can indeed be chosen close enough so that the assumption \(2\pi' > \pi''\) is satisfied.

Lemma 2.1 allows us to obtain asymptotic expansions for certain solutions contained in the local stable manifold of the origin:

**Corollary 2.1.** Suppose \(q^+(t, \mu) \in W_{loc}^s(0)\) is a perturbed version of the \(\tilde{q}(t)\) orbit, such that \(q^+(t, \mu)\) satisfies Eq. (5) for \(\mu \geq 0\) (i.e., \(\pi \approx 1\)) and \(q^+(t, 0) = \tilde{q}(t)\) for \(t \gg 1\); then for large positive \(t\), \(q^+(t, \mu)\) has the following asymptotic expansion
\[
  q^+(t, \mu) = b(\mu)\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}e^{-t} + d(\mu)\begin{pmatrix} 0 \\ 1 \\ 0 \\ -(1 + \mu) \end{pmatrix}e^{-(1 + \mu)t} + e(\epsilon^{2\alpha t}),
\]
where \(b(\mu), d(\mu)\) are smooth in \(\mu\), with \(b(0) = 6\sqrt{2}, d(0) = 6\).

**Proof.** Take \(k = 2\) and
\[
  \lambda_1(\mu) = 1, \quad \lambda_2(\mu) = \sqrt{\pi} = 1 + \mu, \quad \sigma' = \emptyset
\]
in Lemma 2.1. The eigenvectors corresponding to eigenvalues \(-\lambda_1, -\lambda_2\) are respectively
\[
  \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ -(1 + \mu) \end{pmatrix}.
\]
Note that they depend smoothly on \(\mu\) for all \(\mu\) in some open interval \(V\) containing 0; they are also always linearly independent, forming a basis for the stable eigenspace, even when \(\lambda_1 = \lambda_2\) at \(\mu = 0\). Thus \(Q_0^1 + Q_0^2 = I\), since

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here $P_0^\prime = 0$. The two vectors $v_1(\mu), v_2(\mu)$ of Lemma 2.1 are smooth in $\mu$
and lie in the eigenspaces of $-\lambda_1, -\lambda_2$ respectively. Since $v_1(\mu)$ is $C^2$ in $\mu$
and lies in $\mathcal{H}Q_0^1 = \text{Span}\{(1, 0, -1, 0)^T\}$, we can write

$$v_1(\mu) = b(\mu) \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

for some $C^2$ function $b(\mu)$.

Similarly,

$$v_2(\mu) = d(\mu) \begin{pmatrix} 0 \\ 1 \\ 0 \\ -(1 + \mu) \end{pmatrix}$$

for some $C^2$ function $d(\mu)$.

From the proof of Lemma 2.1, we can actually use $\lambda^* = \min\{\lambda_1, \lambda_2\}$ in
place of $\pi^*$, so we have proved the validity of the claimed expansion, which
also be differentiated once. By expanding sech$^2(t/2)$ as $(4e^{-t} - 8e^{-2t} + \cdots)$ for $t$ large and positive, we express the explicit solution (7) as

$$\begin{pmatrix} 6\sqrt{2} \\ 6 \\ -6\sqrt{2} \\ -6 \end{pmatrix} e^{-t} + O(e^{-2t}),$$

whence we obtain $b(0) = 6\sqrt{2}$ and $d(0) = 6$. $\blacksquare$

It is clear that for $\mu > 0$, $b(\mu)(1, 0, -1, 0)^T e^{-t}$ is the leading order term
in the expansion of $q^*$, so we would expect the trajectory to approach the
origin tangent to the $(1, 0, -1, 0)^T$ eigendirection; similarly, for $\mu < 0$, we
expect the trajectory to approach zero along the $(0, 1, 0, -(1 + \mu))^T$ eigendirection.
This certainly agrees with earlier numerical evidence; see, e.g., [24].

In the remainder of this section “non-degeneracy” of the explicit 1-pulse
will be proved. This property is intimately entwined with the bifurcation
analysis in this paper and the stability analysis in [13].

A solution $q(t)$ homoclinic to an equilibrium point $p$ of an ODE
$u' = f(u)$ (where $f: \mathbb{R}^n \to \mathbb{R}^n$ is an at least $C^1$-smooth vector field) is said to
be non-degenerate if $q(t)$ spans the space of all globally bounded solutions
of the equation of variations; so $\dim(T_{q(t)}W^s(p) \cap T_{q(t)}W^u(p)) = 1$ for any
$t \in \mathbb{R}$, where $T_{q(t)}W^s(p)$ and $T_{q(t)}W^u(p)$ are the tangent spaces of, respectively,
the stable and unstable manifolds of $p$ at the point $q(t)$, for some
fixed $t$. This means that along the homoclinic orbit, the stable and unstable manifolds of $p$ intersect as “cleanly” as possible. Non-degenerate homoclinic orbits appear stably in conservative systems; see [32]. If we denote the conserved quantity by $H$ (which may, for example, be a Hamiltonian), then non-degeneracy of a homoclinic solution $q(t)$ for which $VH(q(0)) \neq 0$ implies that $W^s(p)$ and $W^u(p)$ intersect transversely within the $\{H = H(p)\}$ level-set; it follows that such a homoclinic solution persists under conservative perturbations to the system. It is also true that certain homoclinic orbits appear stably in reversible systems: if $n = 2m$, and the system is $R$-reversible for some involution $R$, then a $R$-symmetric homoclinic solution (one which has $q(t) = Rq(-t)$ for all $t$) persists under $R$-reversible perturbations of the vector field, provided $W^s(p)$ and Fix($R$) are transverse at the unique intersection point $q(0)$ of $q(t)$ and Fix($R$); see [32]. The transversality notion here is weaker than non-degeneracy; thus if a $R$-symmetric homoclinic orbit is non-degenerate, then it is certainly persistent under $R$-reversible perturbations.

We claim that for system (5),

**LEMMA 2.2.** The 1-pulse $q(t)$ at $\alpha = 1$, given by (7), is non-degenerate.

Because our system is both conservative and reversible for any value of $\alpha$, the following then holds:

**COROLLARY 2.2.** There exists a $C^1$ branch of 1-pulses parametrised by $\alpha$, for $\alpha$ close to 1, which contains the explicit solution (7) at $\alpha = 1$.

**REMARK.** Smoothness of the branch is due to smoothness of the vector field of (5) with respect to the parameter $\alpha$. By appending (5) with the equation $\dot{\alpha} = 0$, and using the fact that $W^{cs}(0, \alpha)$ or $W^{cu}(0, \alpha)$ (centre-stable or centre-unstable manifolds) are as smooth as the (augmented) vector field, one can deduce smoothness with respect to the parameter $\alpha$.

To prove Lemma 2.2, observe that the equation of variations about $q(t)$ can be written as two coupled second-order linear ODEs,

$$X'' = X - \bar{v}(t) X - \bar{w}(t) Y$$

$$Y'' = Y - \bar{w}(t) X$$

or

$$\begin{pmatrix}
-d^2/dt^2 + 1 - \bar{v}(t) & -\bar{w}(t) \\
-\bar{w}(t) & -d^2/dt^2 + 1
\end{pmatrix}
\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

(10)

where $X(t)$, $Y(t)$ are the deviations from $\bar{w}$, $\bar{v}$, respectively.
The matrix operator on the left-hand side of (10) can be diagonalised. Remember $\tilde{\psi} = \sqrt{2} \psi$; multiplying (10) on the right by
\[
\begin{pmatrix}
\frac{2}{3} & -\frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{3} & 1
\end{pmatrix}
\]
equivalent to the change of variables $X = (2/3) Z_1 - (\sqrt{2}/2) Z_2$, $Y = (\sqrt{2}/3) Z_1 + Z_2$, and on the left by its inverse
\[
\begin{pmatrix}
1 & \frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{3} & \frac{2}{3}
\end{pmatrix}
\]
gives us
\[
\begin{pmatrix}
-\frac{d^2}{dt^2} + 1 - 2\tilde{\psi} & 0 \\
0 & -\frac{d^2}{dt^2} + 1 + \tilde{\psi}
\end{pmatrix}
\begin{pmatrix}
Z_1 \\
Z_2
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
The fact that the $2 \times 2$ matrix differential operator of (10) is diagonalisable is due to one component of our 1-pulse being a constant multiple of the other. Indeed, diagonalisability appears to be a trait shared by the variational equations derived from a number of different coupled nonlinear Schrödinger systems. We will come back to this point later, in Section 4.

Now consider the two decoupled equations
\[
L_u Z_1 = -Z_1^+ + Z_1 - 2\tilde{\psi}(t) Z_1 = 0 \quad \text{and} \quad L_{\tilde{\psi}} Z_2 = -Z_2^+ + Z_2 + \tilde{\psi}(t) Z_2 = 0.
\]
The first one has a globally bounded solution $\tilde{\psi}(t)$, because $L_u Z_1 = 0$ is precisely the equation of variations of $\psi = \psi - \psi^2$ about its homoclinic solution $\tilde{\psi}(\cdot)$. Another solution that is linearly independent of $\tilde{\psi}'$ can be obtained by the reduction-of-order technique, and is seen to be unbounded at both $\pm \infty$. There are thus no other globally bounded solutions to $L_u Z_1 = 0$, except for scalar multiples of $\tilde{\psi}'$.
As for $L_l Z_2 = 0$, we shall show slightly more generally that

**Lemma 2.3.** The equation

$$Z'' = (1 - \lambda) Z + C \sech^2 \left( \frac{t}{2} \right) Z$$

(11)

has no non-trivial solutions that are bounded over $(-\infty, +\infty)$, provided $C > 0$ and $\lambda < 1$.

Observe that $L_l Z_2 = 0$ is the special case with $\lambda = 0$ and $C = \frac{1}{2}$. This lemma will be used again in the analysis of linear stability [13].

**Proof.** Suppose (11) does possess a non-trivial globally bounded solution $Z(t)$, then $Z(t)$ and $Z'(t)$ decay exponentially as $t \to \pm \infty$ following, for example, the arguments in [33, Sect. 3]. Therefore $Z(t)$ and $Z'(t)$ are square-integrable over $\mathbb{R}$ (we will use this fact again in [13]). However, if one multiplies (11) through by $Z(t)$ and integrates, one obtains a negative quantity $-\int_{-\infty}^{+\infty} Z''^2 \, dt$ on the left hand side, and $\int_{-\infty}^{+\infty} \left( (1 - \lambda) + C \sech^2 \left( \frac{t}{2} \right) \right) Z^2 \, dt$ on the right hand side, which is strictly positive for $\lambda < 1$ and $C > 0$, thus reaching a contradiction.

An alternative, geometric, proof is presented in [14], which also illustrates how globally bounded solutions lie in certain stable and unstable manifolds, and hence must be exponentially decaying at $\pm \infty$.

Putting the results about $L_u$ and $L_l$ together, one sees that

$$\begin{pmatrix} L_u & 0 \\ 0 & L_l \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has the unique (up to scalar multiplication) non-zero globally bounded solution $(Z_1, Z_2) = (\tilde{v}'(t), 0)$. This, upon translation into the original variables $X$ and $Y$, is just the same as $(X, Y) = (\sqrt{2} \tilde{v}'(t), \tilde{v}'(t))$, i.e., $(w'(t), \tilde{v}'(t))$. We have therefore shown that the 1-pulse at $z = 1$ is indeed non-degenerate.

Later, when studying stability in [13], we shall again need to look at (10) and the operators $L_u, L_l$, as well as another, similar, system of second-order linear differential operators.

### 3. THE BIFURCATION OF MULTIPULSES

In this section, we give the detailed proof of Theorem 2.1. Homoclinic bifurcations are often analysed using Poincaré maps; we adopt a different
approach here, which is briefly described in the following subsection. This method, unlike some of the other techniques, provides considerable control over the return-times (times of flight between consecutive sections).

3.1. A Homoclinic Lyapunov–Schmidt Reduction

As an alternative to the “traditional” Poincaré map-based techniques for studying homoclinic or heteroclinic bifurcations, some authors, notably J. Hale (see, e.g., [1, 34]), have demonstrated the efficacy of adopting a functional-analytic approach to obtain bifurcation functions to the problem. In this section we will give an overview of one such method, which was first introduced by X.-B. Lin in [2], and further developed by B. Sandstede in [3] to treat a wide range of bifurcation problems. The bifurcation equations are obtained via Lyapunov–Schmidt reduction and the use of exponential dichotomies.

Consider the ODE

\[ u' = f(u, \mu) \]

where \((u, \mu) \in \mathbb{R}^n \times V\) \((0 \in V \subseteq \mathbb{R}^p)\) and \(f\) is a sufficiently smooth vector field. Assume that \(p_1, \ldots, p_m\) are hyperbolic equilibrium points for all \(\mu \in V\), and we assume that, at \(\mu = 0\), the ODE admits a non-degenerate heteroclinic cycle \(\{q_i(t), \ldots, q_m(t)\}\). This means that

\[
\begin{align*}
\lim_{t \to -\infty} q_i(t) &= p_i \quad i \mod m \\
\lim_{t \to +\infty} q_i(t) &= p_{i+1}
\end{align*}
\]

(see Fig. 4) and \(T_q(0) W^u(p_i) \cap T_q(0) W^s(p_{i+1}) = \text{span}\{q'_i(0)\}\) for all \(i\) (non-degeneracy). The \(q_i(\cdot)\) are referred to as “primary” orbits. Let \(\Sigma_i\) be codimension-1 transverse sections to the orbits \(q_i(t)\) at \(t = 0\), and let

\[
\bar{Z}_i = [T_q(0) W^u(p_i) + T_q(0) W^s(p_{i+1})]^{\perp}.
\]

Due to non-degeneracy, each \(\bar{Z}_i\) is one-dimensional, and we can write \(\bar{Z}_i = \text{span}\{\Psi_i^0\}\) for some vector \(\Psi_i^0\). Non-degeneracy also implies that for each \(i\), the adjoint variational equation

\[
x' = -[D_x f(q_i(t), 0)]^* x
\]

has a unique, up to constant multiples, bounded solution \(\Psi_i(t)\), which satisfies \(\Psi_i(0) = \Psi_i^0\), and moreover \(\Psi_i(t) \perp [T_q(0) W^u(p_i) + T_q(0) W^s(p_{i+1})]\) for all \(t\).

When \(\mu \neq 0\) and small, in general we would not expect the heteroclinic cycle to persist. It can be shown, however, that there exist unique functions \(\{q_i^\pm(t, \mu)\}_{i=1, \ldots, m}\), such that
FIG. 4. The heteroclinic cycle at $\mu = 0$.

FIG. 5. Piecewise continuous solutions in the stable and unstable manifolds of the equilibria, at $\mu \neq 0$ but close to 0. The trace of the $\mu = 0$ heteroclinic cycle is shown with dashed curves.
\[
\begin{align*}
q_+^x (\cdot, \mu) & : [0, +\infty) \to \mathbb{R}_+^n, \quad q_-^x (\cdot, \mu) : (-\infty, 0] \to \mathbb{R}^n \\
q_+^x (t, \mu) &= f(q_+^x, \mu) \\
q_-^x (t, \mu) &\in W^q(p_{i+1}), \quad q_-^x (t, \mu) \in W^q(p_i) \\
q_+^x &\text{ are smooth with respect to } \mu \\
q_+^x (\cdot, 0) &= \tilde{q}_i(\cdot) \\
q_+^x (0, \mu) &\in \Sigma_i \\
q_-^x (0, \mu) - q_+^x (0, \mu) &= \tilde{z}_i^x (\mu) \Psi_i^0 \in \tilde{Z}_i \\
\frac{d}{dt} \tilde{z}_i^x &|_{\mu=0} = \int_{-\infty}^{+\infty} \langle \Psi_j(t), D_\mu f(\tilde{q}_i(t), 0) \rangle \, dt.
\end{align*}
\]

In other words, there exist piecewise continuous solutions in the stable and unstable manifolds of the equilibria \( p_i \), and the discontinuities (jumps) are restricted to lying along a particular direction (namely \( \tilde{Z}_i = \text{span} \{ \Psi_i^0 \} \)) in each transverse section \( \Sigma_i \). It can also be shown that the adjoint equation \( x' = -[D_\mu f(q_+^x (t, \mu), \mu)]^* x \) possesses a unique bounded solution \( \Psi_i(t, \mu) \) \((t \in \mathbb{R})\), for each \( i \), such that \( \Psi_i(\cdot, 0) = \Psi_i(\cdot) \), and \( \Psi_i(0, \mu) \) spans a one-dimensional subspace \( Z_i(\mu) \) in \( \Sigma_i \). See Fig. 5.

Next, we look for a sequence of functions \( \{Q_j^x (t, \mu)\}_{j \in \mathbb{Z}} \) which satisfy

\[
\begin{align*}
Q_j^+ (\cdot, \mu) & : [0, T_j] \to \mathbb{R}^n; \quad Q_j^- (\cdot, \mu) : [-T_{j-1}, 0] \to \mathbb{R}^n \\
Q_j^x &= f(Q_j^x, \mu) \\
|Q_j^x (\cdot, \mu) - q_+^x (\cdot, \mu)| &\text{ small in an appropriate function space} \\
Q_j^+ (T_j, \mu) &= Q_{j-1}^- (-T_{j-1}, \mu) \\
Q_j^+ (0, \mu) &\in \Sigma_j \\
Q_j^- (0, \mu) - Q_j^+ (0, \mu) &\in \text{span} \{ \Psi_j(0, \mu) \} \equiv Z_j(\mu)
\end{align*}
\]

where each \( q_+^x \) is just one of the \( \{q_i^x : i = 1, \ldots, m\} \) and \( \Sigma_j \) is the corresponding section from the collection \( \{\Sigma_i : i = 1, \ldots, m\} \). Let \( T = \{T_j\}_{j \in \mathbb{Z}} \) and \( Q_j^- (0, \mu) - Q_j^+ (0, \mu) = \tilde{z}_j \Psi_j(0, \mu) \), so \( \tilde{z}_j = \tilde{z}_j(T, \mu) \) are the jump sizes along the directions \( Z_j(\mu) \) in the transverse sections. In [3] it is proved that with \(|\mu|\) sufficiently small, choosing any sequence \( \{T_j\}_{j \in \mathbb{Z}} \) such that the \( T_j \) are all sufficiently large, there exists a unique sequence of functions \( \{Q_j^x\}_{j \in \mathbb{Z}} \) satisfying the above conditions, for some sequence of jump sizes \( \{\tilde{z}_j(T, \mu)\}_{j \in \mathbb{Z}} \). See Fig. 6. Moreover, the jump sizes can be expressed as

\[
\tilde{z}_j(T, \mu) = \tilde{z}_j^\infty (\mu) + \langle \Psi_j(T, \mu) - T_{j-1}, \mu), q_{j-1}^-(T_{j-1}, \mu) \rangle \\
- \langle \Psi_j(T, \mu), q_{j+1}^+(T_{j}, \mu) \rangle + B_j(T, \mu) .
\]
FIG. 6. Piecewise continuous orbits at $\mu \approx 0$. Now the solutions $q^\pm_i$ are shown with dashed curves, and the trace of the heteroclinic cycle with dotted curves.

Estimates for the remainder term $R_j(T, \mu)$ and its derivatives with respect to the parameter $\mu$ and return-times $T_j$ are also given in [3]. In examples, one needs to check to make sure $R_j(T, \mu)$ is really of higher order compared to the inner product terms. Observe that the $\{Q_j^\pm(\cdot, \mu)\}_{j \in \mathbb{Z}}$ together make up a sequence of piecewise trajectories, whose times of flight between consecutive sections are specified by the sequence $\{2T_j\}_{j \in \mathbb{Z}}$. Jump discontinuities are only allowed at the sections, and in addition are forced to lie along the directions $Z_j(\mu)$. Time is parametrised so that the sections are always hit at $t = 0$; but although jump sizes are calculated at $t = 0$, observe that information in the inner product terms all comes from the vicinity of the equilibrium points ($t = \pm T_k$), i.e., from the “tails” of the primary solutions. Note that if $p_1 = \cdots = p_m$, then the “heteroclinic cycle” becomes a homoclinic orbit or a collection of homoclinic loops.

To see whether interesting orbits exist for $\mu$ close to 0, one would attempt to solve the system of algebraic equations

$$z_j(T, \mu) = 0 \quad \text{for all } J_j,$$

with some special type of sequence $\{T_j\}_{j \in \mathbb{Z}}$. For example, to investigate the existence of $N$-periodic solutions, we try to find out whether the system can be satisfied by some $\mu$ and some $N$-periodic sequence $\{T_j\}$; to look for $N$-looped homoclinic orbits ($N$-pulses), we concentrate on $\{T_j\}_{0 \leq j \leq N}$ such
that $T_1, \ldots, T_{N-1}$ are finite and $T_0 = T_N = \infty$. In both these cases, we would have $N$ equations to solve simultaneously. The idea is to apply the Implicit Function Theorem about $\mu = 0$ and the primary heteroclinic/homoclinic solution which exists there, represented by the “return-time” $\infty$.

3.2. Deriving the Bifurcation Equations

We seek solutions of (5) that follow the primary 1-pulses $\bar{q}(t)$ and $\bar{S}(t)$ in a given order $\{\bar{y}_i\}_{i=1, \ldots, N}$, where $\bar{y}_i \in \{I, S\}$ (no other restrictions are placed on the sequence $\{\bar{y}_i\}_{i=1, \ldots, N}$ at this point). The equilibria $p_1, \ldots, p_m$ of the method description in Subsection 3.1 are all equal to the origin in this case, and the heteroclinic cycle is the figure-eight configuration of two homoclinic loops shown in Fig. 2. A possible 2-pulse solution is drawn in Fig. 7.

Consider the jumps at sections $\Sigma_i$,

$$\xi_i(T, \mu) = \xi_i^0(\mu) + \langle \Psi(T_{i-1}, \mu), q_{i-1}^+(T_{i-1}, \mu) \rangle$$

$$- \langle \Psi(T_i, \mu), q_{i+1}^-(T_i, \mu) \rangle + \beta_i(T, \mu)$$

for $i = 1, \ldots, N$.

In this case the primary 1-pulse homoclinics persist when $\mu$ is nonzero but small, from Corollary 2.2, and this implies that $\xi_i^0(\mu) = 0$ and

![Diagram of a 2-pulse solution](image.png)
\( q^+ (t, \mu) = q_i (t, \mu) \) for all such \( \mu \), where the \( q_i (\cdot, \mu) \) are now just single-looped orbits with no discontinuities. In fact, 
\[ q_i (t, \mu) = \gamma_i q_i (t, \mu), \quad \gamma_i \in \{ I, S \}, \]
where \( q (\cdot, \mu) \) is the positive (in both \( w \) and \( v \) ) 1-pulse existing at parameter value \( \mu \), with \( q (\cdot, 0) = \bar{q} (\cdot) \). Next, let us consider the inner product terms. Now \( q_i (t, \mu) \) is perpendicular to the tangent spaces of the stable and unstable manifolds of 0, so for conservative systems with a first integral \( H \), a natural candidate for \( \Psi_i (t, \mu) \) is \( \nabla H (q_i (t, \mu), \mu) \). By virtue of non-degeneracy, this must be the unique candidate (up to scalar multiples). Therefore, after expanding in Taylor series, we have
\[ \Psi_i (-T_i, \mu) = \nabla^2 H (0, \mu) q_i (-T_i, \mu) + \mathcal{O} (|q_i (-T_i, \mu)|^2) \]
\[ \Psi_i (T_i, \mu) = \nabla^2 H (0, \mu) q_i (T_i, \mu) + \mathcal{O} (|q_i (T_i, \mu)|^2). \]
Reversibility allows one to write
\[ q_i (-T_i, \mu) = R q_i (T_i, \mu) \]
for each \( i \). Putting the above information together, we find that the inner product terms become
\[ \langle \nabla^2 H (0, \mu) R \gamma_i q_i (T_i, \mu), \gamma_i^{-1} q (T_{i-1}, \mu) \rangle \]
\[ - \langle \nabla^2 H (0, \mu) \gamma_i q_i (T_i, \mu), R \gamma_{i+1} q (T_{i-1}, \mu) \rangle \]
\[ + \mathcal{O} (|q_i (T_{i-1}, \mu)|^3 + |q_i (T_i, \mu)|^3). \]
Furthermore, we only need to consider \( i \) running from 1 to \( (N-1) \), because, due to the Hamiltonian structure, \( \zeta_i (T, \mu) = 0 \) for \( i = 1, ..., (N-1) \) implies \( \zeta_N (T, \mu) = 0 \) as well (see Lemma 3.2 of [30]). Note that if one is interested only in symmetric orbits, which are believed to be generic in a reversible system, then one can just consider \( i = 1, ..., [N/2] \), because \( \zeta_i = -\zeta_{N-i} \) and \( T_i = T_{N-i} \) (see, for example, [30, Lemma 3.1]). However, we shall keep things more general here.

The matrices \( R, \gamma_i \) are easily seen to commute with each other and with \( \nabla^2 H (0, \mu) \); also \( R, \gamma_i \) are self-adjoint, so \( \zeta_i (T, \mu) \) may be further simplified to
\[ \zeta_i (T, \mu) = \langle M_{i-1} (\mu) q (T_{i-1}, \mu), q (T_{i-1}, \mu) \rangle - \langle M_i (\mu) q (T_i, \mu), q (T_i, \mu) \rangle \]
\[ + \mathcal{O} (|q_i (T_{i-1}, \mu)|^3 + |q_i (T_i, \mu)|^3) + \mathcal{O} \]
\[ + \mathcal{O} (|q_i (T_{i-1}, \mu)|^3 + |q_i (T_i, \mu)|^3) + \mathcal{O} (|q_i (T_{i-1}, \mu)|^3 + |q_i (T_i, \mu)|^3), \quad (12) \]
where
\[ M_j(\mu) = \nabla^2 H(0, \mu) R_{j' + 1} \]  \hspace{1cm} (13)

Let us treat these terms one by one. First, \( \mathcal{A}(T, \mu) \) can be estimated using the following expressions from [30, Theorem 3]; they hold if one has a reversible system with a \( \mathbb{Z}_2 \) symmetry and a non-degenerate homoclinic orbit,

\[
|\mathcal{A}(T, \mu)| \leq C \left( e^{-2\epsilon T_{j-1}} (|q(-T_{j-1}, \mu)| + |q(T_{j-1}, \mu)|)^2 \\
+ e^{-2\epsilon T} (|q(-T, \mu)| + |q(T, \mu)|)^2 \right) \\
+ (e^{-2\epsilon T_{j-1} - \epsilon^2 T_j} + e^{-2\epsilon T_j}) \tilde{\mathcal{A}},
\]

where

\[
|\tilde{\mathcal{A}}| \leq C \sup_{1 \leq k \leq N} \left\{ e^{-2\epsilon T_k} (|Q^\ast q(-T_k, \mu)| + |Q^\ast q(T_k, \mu)|) \right. \\
+ e^{-2\epsilon T_k} (|q(-T_k, \mu)|^2 + |q(T_k, \mu)|^2) \\
+ e^{-2\epsilon T_k} T_k (|q(T_k, \mu)| + |q(T_k, \mu)|) \right. \]

\[ \] and \( \pm \tilde{\epsilon}^* \) are the leading eigenvalues. Estimates for the derivatives of \( \mathcal{A}_j \) with respect to \( T_k \) and \( \mu \) are also given in [30, Theorem 3]. We use the asymptotic expansion for \( q(t, \mu) \) given in Corollary 2.1 and its analogue for solutions in the local unstable manifold of 0,

\[
q(t, \mu) = \begin{bmatrix} b(\mu) e^{-\epsilon|t|} \\
\epsilon(1 + \mu) e^{-\epsilon|t|} \\
b(\mu) e^{-\epsilon|t|} \\
(1 + \mu) e^{-\epsilon|t|} \end{bmatrix} + O(e^{-2\epsilon|t|})
\]

for \( |t| \) large, to deduce that

\[ |\mathcal{A}| \leq C \sup_{1 \leq k \leq N} \left\{ e^{-2\epsilon T_k} \right\} = C e^{-2\epsilon T}, \]

where \( T := \min_{1 \leq k \leq N} T_k \). Thus

\[ |\mathcal{A}(T, \mu)| = C (e^{-2\epsilon T} (e^{-2\epsilon T_{j-1}} + e^{-2\epsilon T_i}) + e^{-2\epsilon T_{j-1}} + e^{-2\epsilon T_i}), \]  \hspace{1cm} (14)

and also

\[ |q_j(T_{j-1}, \mu)|^3 + |q_j(T, \mu)|^3 = C (e^{-3\epsilon T_{j-1}} + e^{-3\epsilon T_i}), \]  \hspace{1cm} (15)
We remark that one can also work directly from the estimates in [3] (rather than use [30, Theorem 3]) to obtain (14). We then calculate

\[
M_i(\mu) = \begin{pmatrix}
-\delta_i & 0 & 0 & 0 \\
0 & -(1+\mu)^2 & 0 & 0 \\
0 & 0 & -\delta_i & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\]

where

\[
\delta_i = \begin{cases} 
+1, & \gamma_i = \gamma_{i+1} \\
-1, & \gamma_i \neq \gamma_{i+1}
\end{cases}
\]

with \( \gamma_j \in \{I, S\} \), the group of symmetries. Observe that \( \delta_i = -1 \) means that the \( i \)-th and the \( (i+1) \)-th excursions follow different loops, whereas \( \delta_i = +1 \) means that they follow the same loop. Plugging the above expression for \( M_i(\mu) \) into (12) gives

\[
\langle M_i(\mu) q(T_i, \mu), q(T_i, \mu) \rangle = -\delta_i b(\mu) e^{-T_i} - (1+\mu)^2 (d(\mu) e^{-T_i})^2 - \delta_i (b(\mu) e^{-T_i})^2 - (1+\mu) d(\mu) e^{-T_i} + \mathcal{O}(e^{-3\mu T_i})
\]

\[
= -2\delta_i b(\mu)^2 e^{-2T_i} - 2(1+\mu)^2 d(\mu)^2 e^{-2(1+\mu) T_i} + \mathcal{O}(e^{-3\mu T_i})
\]

for \( i = 1, \ldots, N-1 \). To look for \( N \)-pulse solutions we set \( T_0 = T_N = \infty \), and hence

\[
\xi_1(T, \mu) = 2[\delta_1 b(\mu)^2 e^{-2T_1} + (1+\mu)^2 d(\mu)^2 e^{-2(1+\mu) T_1}]
\]

\[
+ \mathcal{O}(e^{-3\mu T_1} e^{-2\mu T_1})
\]

\[
(17)
\]
and
\[
\dot{\zeta}_j(T, \mu) = -2 \delta_{j-1} b(\mu)^2 e^{-2T_{j-1}} + (1 + \mu)^2 d(\mu)^2 e^{-2(1 + \mu)T_{j-1}} \\
+ 2 \delta_{j} b(\mu)^2 e^{-2T_j} + (1 + \mu)^2 d(\mu)^2 e^{-2(1 + \mu)T_j} \\
+ \mathcal{O}(e^{-2T_j} (e^{-2T_{j-1}} + e^{-2T_j}))
\] (18)
for \( j = 2, \ldots, N-1 \).

We will then try to solve simultaneously the system of bifurcation equations
\[
\dot{\zeta}_i(T, \mu) = 0 \quad \text{for} \quad i = 1, \ldots, N-1.
\]

3.3. Solving the Bifurcation Equations

Define, for \( k = 1, \ldots, N-1 \)
\[
\eta_k = \frac{1}{2}(\zeta_1 + \cdots + \zeta_k)
\]
\[
= \delta_k b(\mu)^2 e^{-2T_k} + (1 + \mu)^2 d(\mu)^2 e^{-2(1 + \mu)T_k} + \mathcal{O}(e^{-3\lambda T_k}).
\]

Note how all the remainder terms have been grouped together to give the single error expression. Thus solving \( \zeta_i = 0 \) for \( i = 0, \ldots, N-1 \) is equivalent to solving \( \eta_k = 0 \) for \( k = 0, \ldots, N-1 \).

The idea is to use the Implicit Function Theorem (IFT) about the “point” \( T = 0: \mu = 0 \) corresponds to \( \varphi = 1 \), and \( T = \infty \) corresponds to all the \( T_i \) being \( \infty \), which simply represents the primary 1-pulse orbit, and this is explicitly known at \( \mu = 0 (\varphi = 1) \). However, applying IFT about infinity is a very singular problem, so we introduce the following “polar variables” to get around this difficulty. Let
\[
r = e^{-2(1 + \mu)T} \in [0, 1)
\]
\[
a_k = e^{-2(1 + \mu)T_k} \in [0, 1]
\]
\[
a_k r = e^{-2(1 + \mu)T_k} \in [0, 1)
\]
so \( r = 0 \) corresponds to \( T_i = \infty \) for all \( i \). In these new variables we have, denoting \( \{a_k\}_{1 \leq k \leq N} \) by \( a \),
\[
\eta_k(a, r, \mu) = \delta_k b(\mu)^2 (a_k r)^{1/(1 + \mu)} + (1 + \mu)^2 d(\mu)^2 (a_k r) + \mathcal{O}(r^{1+\omega})
\]
\[
= (a_k r)[\delta_k b(\mu)^2 (a_k r)^{1/(1 + \mu) - 1} + (1 + \mu)^2 d(\mu)^2] + \mathcal{O}(r^{1+\omega})
\]
for \( k = 1, \ldots, N-1 \), where \( \omega \) is some fixed positive number. (Note: \( e^{-3\lambda T_i} = e^{2/(1 + \mu)} \), so if \( x \leq 1 \), then \( \lambda' = 1 + \mu \) and hence \( \omega = 1/2 \). If \( x \gg 1 \), then \( \lambda' = 1 \); suppose \( |\mu| < \delta < 1/2 \), i.e., \( \varphi \) is sufficiently close to 1, then
Differentiability of the error term in the new variables may be checked using the estimates given in [30] for derivatives of $R_i$ with respect to $T_k, \mu$, as well as the relations between $a_k, r$ and $T_k$. To put the equations in a more amenable form, we further define

$$v = \frac{1}{1 + \mu} - 1.$$  

Observe that $\mu = 0$ if and only if $v = 0$, and otherwise they are of opposite signs. With an abuse of notation, we now write $b$ and $d$ as functions of $v$, i.e., $b(v)$ and $d(v)$; note that $b(0) = 6\sqrt{2}$ and $d(0) = 0$, and, since $\mu, v$ are smoothly related, $b$ and $d$ are also $C^1$ in $v$. Let

$$c(v) = \frac{b(v)^2(1 + v)^2}{d(v)^2} > 0;$$

then $c(v)$ is $C^1$ in $v$, and $c(0) = 2$. We thus obtain:

$$\frac{(1 + v)^2}{d(v)^2} \eta_k(a, r, v) = (a_k r c(v)(a_k r)^*) + c(r^1 + \omega),$$

and wish to solve

$$(a_k r c(v)(a_k r)^*) + c(r^1 + \omega) = 0$$

for $k = 1, ..., N - 1$. Observe that $r = 0$ is a solution for any $v$; this corresponds to the 1-pulse, present at each value of $\alpha$ close to 1. To seek other solutions, we factor out an $r$ and obtain

$$a_k [\delta_k c(v)(a_k r)^* + 1] + c(r^\alpha) = 0, \quad k = 1, ..., N - 1. \quad (20)$$

Let $n$ be the integer such that $T_n = T = \min \{ T_j \}$, then we have $a_n = 1$, and at $k = n$ we obtain the equation

$$\delta_n c(v) r^\alpha + 1 + c(r^\alpha) = 0.$$  

It is clear that if $\delta_n = +1$, then $\delta_n c(v) r^\alpha + 1 \geq 1$, therefore a solution is not possible unless $\delta_n = -1$. Taking $\delta_n = -1$, the resulting transcendental equation is treated in the following lemma (see, for example, [35]):

**Lemma 3.1.** Let $c = c(v)$ be a $C^1$ function such that $c(0) > 1$. Consider the equation

$$1 = c(v) r^\alpha + c(r^\alpha), \quad v > 0$$

(21)
with \( \alpha > 0 \). The error term is understood to be smoothly differentiable. Then, for \( \nu \leq 0 \), Eq. (21) cannot have solutions with small \( r \geq 0 \). For \( \nu > 0 \), Eq. (21) can be solved for \( \rho \equiv r^\nu \) as a \( C^1 \) function of \( \nu \), locally near \( r = 0 \), \( \rho_0 = \frac{1}{c(0)} \), by the Implicit Function Theorem.

Here we indeed have \( c(\nu) \) being \( C^1 \) and \( c(0) = 2 > 1 \). Therefore, a necessary condition for the \( k = n \) equation to be solvable is

\[
\nu > 0.
\]

This is then also a necessary condition for the whole set of bifurcation equations to be solvable.

Before proceeding further, we give a brief explanation of Lemma 3.1. The equation \(- c(\nu) r^\nu + 1 + \ell(r^\nu) = 0\) is transformed to

\[
-c(\nu) \rho^\nu + 1 + \ell(\rho^\nu) = 0
\]

upon setting \( \rho = r^\nu \). When \( \nu = 0 \), taking \( \rho = \frac{1}{c(0)} \) eliminates the leading order terms. For the remainder term to vanish, we require \( \rho^\nu \to 0 \) as \( \nu \downarrow 0 \) (with \( \alpha > 0 \) fixed), and this is easily seen to be true if \( \rho(0) < 1 \), i.e., \( c(0) > 1 \). To check that IFT is applicable, differentiate the equation’s left hand side with respect to \( \rho \); evaluating at \( r = 0 \) gives \(-c(0)\), which is clearly nontrivial.

Turning our attention to the other bifurcation equations of (20), we can see that, for each \( k \neq n \), its equation may be solved by IFT about one of the following points:

1. \( \nu = 0, a_k = 1, \rho(0) = \frac{1}{c(0)} \) (where \( \rho \equiv r^\nu \)).
2. \( r = 0, a_k = 0 \).

In particular, for Case 1, the assumption is that \( a_k [\delta_k c(\nu)(a_k r)^\nu + 1] \) is truly of leading order, so then:

- \( \delta_k = -1 \) is necessary, just as in the \( k = n \) case.
- IFT cannot be applied with respect to the variable \( a_k \).
- The solution can be expressed in the form

\[
(a_k r)^\nu = \frac{1}{c(0)} \left( 1 - \frac{D c(0)}{c(0)} \nu + o(\nu) \right).
\]

For clarification of the second and third points above, note that the derivative of

\[
a_k [\delta_k c(\nu)(a_k r)^\nu + 1] + \ell(\rho^\nu)
\]
with respect to $a_k$ is
\[(v + 1) \delta_k c(v) a_k^* r^* + 1 + \mathcal{O}(r^m).\]
Recall that $r^* \equiv \rho(v)$ and $\rho(0) = c(0)^{-1}$, so the above can be rewritten
\[(v + 1) \delta_k c(v) a_k^* \rho(v) + 1 + \mathcal{O}(r^m),\]
which, when evaluated at $v = 0$, $a_k = 1$, $\rho(0) = \frac{1}{c(0)}$, $\delta_k = -1$ gives zero. In order to avoid this difficulty, we define
\[\rho_k(v) = (a_k r)^v,\]
so that the bifurcation equations (before factoring out an $r$, see (19)) become
\[\rho_k^1 \left[ \delta_k c(v) \rho_k + 1 \right] + \mathcal{O}(r^{1+\omega^v}) = 0.\]
Dividing through by $\rho_k^1$ gives
\[\delta_k c(v) \rho_k + 1 + \mathcal{O}(r^m) = 0.\]
We then argue as for the case $k = n$: $\delta_k$ must be $-1$ for solvability; IFT can be applied about $v = 0$, $\rho_k(0) = \frac{1}{c(0)} < 1$, and the solution may be expressed as
\[\rho_k = \frac{1}{c(0)} \left( 1 - \frac{D_k c(0)}{c(0)} v + o(v) \right).\]
Now, for Case 2, assume $a_k \left[ \delta_k c(v)(a_k r)^v + 1 \right]$ is not really of leading order, thus
- $\delta_k$ may be either $+1$ or $-1$
- IFT can be applied with respect to $a_k$.
- $a_k = \mathcal{O}(r^m)$.
To clarify the second item, observe that, in contrast to Case 1, the derivative with respect to $a_k$, in this case evaluated at $r = 0$ and $a_k = 0$, gives 1. It is clear from (20) that $a_k$ needs to be of $\mathcal{O}(r^m)$.
In summary, we apply IFT with respect to the variables
\[\rho_m \quad \text{and} \quad a_l,\]
where
\[1 \leq l, m \leq N - 1 \quad \text{and} \quad l \neq m.\]
about

\[ v = 0, \quad \rho_m(0) = \frac{1}{c(0)}, \quad \text{and} \quad a_t = 0. \]

Note that \( n \) must be among the indices \( m \) above, and \( \rho_n = \rho = r^* \).

3.4. Interpretation of the Solutions

Here \(|v|\) is assumed to be small, i.e., \( \alpha \) is assumed to be close to 1.

1. For \( v \leq 0 \), which is equivalent to \( \mu \geq 0 \), or \( \alpha \geq 1 \), there does not exist a solution to the set of bifurcation equations. This means that no multihumped \( N \)-pulses exist for \( \alpha \geq 1 \).

2. For \( v > 0 \), which is equivalent to \( \mu < 0 \), or \( \alpha < 1 \).

- Take \( N \geq 2 \) (arbitrary), then for any \( v \) smaller than some \( \varepsilon_N \), there exists a solution to the set of bifurcation equations, with \( \delta_k = -1 \) for every \( k \), obtained by applying Case 1 for all \( k \). Recall that \( \delta_k = -1 \) means that \( \gamma_k \neq \gamma_{k+1} \), so such a solution is an \( N \)-humped wave which, in phase space, hits the sections \( \Sigma_I \) and \( \Sigma_S \) alternately. In other words, such an \( N \)-pulse is constructed by concatenating, alternately, the 1-pulse with positive \( w \)-component and the 1-pulse with negative \( w \)-component. The separation between consecutive humps is \( 2T_k \) and can be estimated as

\[
2T_k = -\frac{1}{1 + \mu} \ln(a_k r) \\
= -\frac{1}{1 + \mu} \left[ \frac{1}{v} \left( \ln \frac{1}{c(0)} + \ln \left( 1 - \frac{D_r c(0)}{c(0)} + o(v) \right) \right) \right] \\
= \frac{1}{1 + \mu} \left[ \ln c(0) + \frac{D_r c(0)}{c(0)} + o(v) \right].
\]

Use \( c(0) = 2 \) and the relations between \( v, \mu \) and \( \pi \) to deduce that

\[
2T_k \sim \frac{\ln 2}{1 - \sqrt{\pi}} \quad \text{as} \quad \pi \to 1.
\]

- If \( N = 2 \), such alternating solutions are the only ones that exist. In other words, all 2-pulses are of alternating type. This is because \( N - 1 = 2 - 1 = 1 \), i.e., we have only one equation to solve, which is therefore by necessity the \( n \)th equation. Trying to solve it via Case 2 would just give us two infinitely separated 1-pulse solutions.

- If \( N \geq 3 \), any solution that is not strictly alternating must have \( a_k = \ell(r^*) \) for some integer \( k \). Translated into the original variables, this
means $T_k \sim (1 + \omega) T$ for this particular $k$; so consecutive excursions which hit the same section in phase space must have a corresponding larger hump separation than those which alternate.

3.5. Uniqueness

The existence of non-alternating $N$-pulses can in fact be excluded, for every $N$, using an inductive uniqueness argument similar to the one in [30, Lemma 3.6]. We prepare the ground by refining the expression for the remainder terms. Instead of defining $\eta_k$ and grouping all remainder terms together as simply $O(e^{-3T_k})$ where $T = \min\{T_j\}$, let us recall the formulae (17), (18) for the jump-sizes, which we reproduce here,

$$\zeta_1(T, \mu) = 2[\delta_1 b(\mu)^2 e^{-2T_i} + (1 + \mu)^2 d(\mu)^2 e^{-2(1+\mu)T_i}]$$

$$+ O(e^{-2T_i} e^{-2dT_i}),$$

and

$$\zeta_j(T, \mu) = -2[\delta_j b(\mu)^2 e^{-2T_{j-1}} + (1 + \mu)^2 d(\mu)^2 e^{-2(1+\mu)T_{j-1}}]$$

$$+ 2[\delta_j b(\mu)^2 e^{-2T_j} + (1 + \mu)^2 d(\mu)^2 e^{-2(1+\mu)T_j}]$$

$$+ O(e^{-2T_j} (e^{-2T_{j-1}} + e^{-2dT_j}))$$

for $j = 2, \ldots, N - 1$. Changing the variables to $a_i$, $r$ and $v$, we get bifurcation equations of the form

$$(a_i r)[\delta_i c(v)(a_i r)^v + 1] + O(r^\infty(a_i r)) = 0,$$

and for $j = 2, \ldots, N - 1$,

$$-(a_{j-1} r)[\delta_{j-1} c(v)(a_{j-1} r)^v + 1] + (a_j r)[\delta_j c(v)(a_j r)^v + 1]$$

$$+ O(r^\infty(a_{j-1} r + a_j r)) = 0. \quad (22)$$

In the following, we will use the shorthand $a_i^0$ for the “initial value” of $a_i$, namely the value of $a_i$ about which we apply IFT; thus $a_i$ is either 0 or 1.

Initiating induction. Any 2-pulse must be of alternating type; it has $a_i^0 = 1$ for all $i$.

Inductive step. Assume that, for any $M < N$, the only $M$-pulses which exist are the ones which are strictly alternating, i.e., those which have $\delta_i = -1$ for all $i = 1, \ldots, M$. Now consider $N$-pulses:

- We know that if an $N$-pulse is not strictly alternating, then there exists at least one integer $k$ such that $a_k^0 = 0$. 
Observe from the bifurcation equations (22) that \( a_k^0 = 0 \) implies that the first \((k - 1)\) equations decouple from the remaining \((N - k)\) equations at the point about which IFT is applied.

Suppose first that within these smaller blocks, all \( a_0^i = 1 \). But then, we already know a solution that satisfies

\[
\begin{align*}
  a_0^i &= 1 & \text{for } i = 1, \ldots, (k - 1) \\
  a_k^0 &= 0 \\
  a_0^i &= 1 & \text{for } i = (k + 1), \ldots, (N - 1)
\end{align*}
\]

namely two distinct homoclinic orbits: a \((k - 1)\)-pulse and an \((N - k)\)-pulse, with “infinite separation” between them. By uniqueness of solutions obtained via IFT, this must then be the only solution, so actually \( a_k \equiv 0 \) for all \( r \) near 0 (representing the infinite separation). Therefore non-alternating pulses, which have \( \delta_k = +1 \) and \( a_k = \mathcal{O}(r^\alpha) \) (but not identically zero) for some \( k \), are non-existent.

Now suppose that within the smaller blocks (of size \( k - 1 \) and \( N - k \)), we also have \( a_0^i = 0 \) for some \( i \), but then we can simply split them into even smaller blocks, “separated” by the \( i \) for which \( a_0^i = 0 \), and repeat the argument above.

**Induction process.** From the above two steps one can therefore deduce that for arbitrary positive integers \( N \), \( N \)-pulses are always strictly alternating.

4. DISCUSSION

4.1. Some Comments Pertaining to Theorem 2.1 and Its Proof

In [14, Chap. 2; 15] it was pointed out that all homoclinic solutions of (5) should have \( v > 0 \), with the ground-state possessing the additional special property that \( w \) is either strictly positive or strictly negative. These statements are supported by the existence results of the current chapter and the numerical evidence presented in [12], from which we see that the \( w \)-component of an \( N \)-pulse (\( N \geq 2 \)) changes sign, but \( v \) always remains positive. In [12], the numerically calculated separation between consecutive humps of an \( N \)-pulse was found to scale like \( (1 - \sqrt{x})^{-1} \) as \( x \) approaches 1 from below (plotting separation against \( (1 - \sqrt{x})^{-1} \) produced an almost perfect straight line), exactly as predicted by Theorem 2.1.

Non-alternating multihumped pulses have never been found numerically, by physicists or mathematicians, for the \( X^{(2)} \) SHG system with \( r = s = +1 \), \( \theta, \pi > 0 \) (of comparable order). The theory in Section 3 gives the explanation
from an analytical and algebraic point of view, but it would also be worthwhile understanding what geometric/topological obstruction there is that prevents non-alternating solutions from existing. This piece of information would perhaps contribute to our understanding of the instability of N-pulses as well; see [13].

We remark that all the information used to derive and manipulate the bifurcation equations comes from the linear terms of the ODEs. The nonlinear terms merely serve to determine the specific form of the primary pulse solution (which includes information on whether the $c(0)$ of Lemma 3.1 is greater than or less than 1), and of the symmetries under which the ODE system is invariant. The same is true for the other coupled nonlinear Schrödinger systems mentioned in Section 1 and Subsection 4.2.

Comparing Subsection 3.5 with the results in [30, Theorem 1 and Lemma 3.6], we see that although similar arguments are used, N-pulses of the phase-sensitive amplifier (PSA) equation studied in [30] which have $T_k \sim (1 + \omega) T$ for some $k$ (i.e., $a_k = \epsilon(\tilde{r}^{m})$) cannot be excluded even through the uniqueness proof. In particular, solutions of that kind are shown to be impossible only among the N-pulses which are already known to be either strictly alternating or strictly non-alternating. The main difference between these two situations lies in the fact that the $\chi^{(2)}$ SHG steady state ODEs have an additional conservative structure, which is not present in the PSA case. First, note that the number of bifurcation equations needs to be reduced before attempting to apply IFT, otherwise we would have $N$ equations and $N - 1$ variables (aside from the bifurcation parameter $\mu$), namely $T_1, ..., T_{N-1}$ or $a_1, ..., a_{N-1}, r$, with one variable in the latter set being fixed in some way (e.g., the $a_n$ of this section). Now, for the $\chi^{(2)}$ SHG case, we used the Hamiltonian structure to reduce the number of equations by one, and we were then able to deduce that all pulses are symmetric (with respect to either $R$ or $\tilde{R}$). For the PSA equation, only symmetric solutions were sought in the N-pulse existence analysis, thus reducing the number of equations to $\lfloor N/2 \rfloor$; but whatever one is able to deduce from the $\lfloor N/2 \rfloor$ equations, one can never get any information from them about non-symmetric solutions. It is this lack of knowledge about non-symmetric pulses that limits the power of the uniqueness argument in [30]. We also remark that Subsection 3.5 concerns uniqueness in the local sense only; specifically, it is proved that a unique family of N-pulses bifurcates from the primary pulse as $\pi$ passes through 1 from above. However, global uniqueness of multihumped solutions has not been proved for $\pi \in (0, 1)$, and the global uniqueness or non-uniqueness of pulses in general has not been determined for $\pi > 1$. Variational methods may help to shed some light upon these issues.

Although we have not done so in this work, one could attempt to analyse an infinite sequence of bifurcation equations $\xi_j(T, \mu) = 0$ for $j \in \mathbb{Z}$
(where \( T = \{ T_j \} \subseteq \mathbb{Z} \) is an infinite sequence of numbers). Solving these equations simultaneously would give us a bounded solution of (5) that possesses infinitely many relative maxima. Even though a wave consisting of an infinite train of humps might not be as physically relevant as multi-pulses with a finite number of humps, its existence as a mathematical object would imply that the \( \varepsilon_\ast \) of Theorem 2.1 can be taken independent of \( N \); in other words, all the \( N \)-pulses would exist on some uniform interval.

4.2. Generalisations to a Class of Coupled Nonlinear Schrödinger Equations

The analysis in Sections 2 and 3 provides a first step towards an understanding of conservative coupled nonlinear Schrödinger (CNLS) equations in general. Here we will extract the key ingredients from those previous sections and discuss how these ideas might be applied to a wider range of situations.

Consider the conservative CNLS system

\[
\begin{align*}
    iu_j + s_j \partial_t u_j - \theta_j u_j + f_j(u) &= 0, & j = 1, \ldots, m, \\
\end{align*}
\]

where \( s_j = \pm 1 \) and \( u = (u_1, \ldots, u_m) \) with the \( u_j(z, x) \in \mathbb{C} \). The dot denotes differentiation with respect to \( z \), where \( z \in \mathbb{R} \) and \( \partial \) denotes the Laplacian operator in \( x \) where \( x \in \mathbb{R}^n \) (typically \( n = 1, 2 \) in optics problems). Assume

- the \( f_j \) are \( C^1 \) functions from \( \mathbb{C}^m \) to \( \mathbb{C} \)
- \( f_j(0) = 0 \) and \( D_u f_j(0) = 0 \) for all \( j \) (i.e., the \( f_j(u) \) are nonlinear terms)
- there exists a real-valued “potential energy” function \( V = V(u, u^*) \) such that \( V(0) = 0 \) and \( f_j = \partial V / \partial u^* \) for each \( j \)

Remark. Equations (23) can arise from inserting an ansatz of the form

\[
U_j = u_j(z, x) e^{i\omega_j t}
\]

(the \( \omega_j \) are real constants; certain constraints may need to be imposed upon them) into the equations

\[
\begin{align*}
    i\dot{U}_j + s_j \partial_t U_j + F_j(U) &= 0, & j = 1, \ldots, m
\end{align*}
\]

and then cancelling factors of \( e^{i\omega_j t} \). The functions \( F_j \) are allowed to contain linear terms; for instance, in the \( \chi^{(2)} \) SHG example, a linear term which comes from the wave-vector mismatch is present. We then regroup all linear terms into the \( \theta_j u_j \) terms of (23), and all nonlinear terms into the \( f_j(u) \). In some cases there are also extra coefficients multiplying the \( u_j \), and their effect would be to somewhat modify our skew operator \( J \); since the mathematical analysis would stay much the same, we will omit such coefficients for simplicity.
We can separate (23) into real and imaginary parts, with $u_j = u_R + i u_I$, $f_j = f_R + i f_I$, then we obtain the system

$$
\begin{align*}
\dot{u}_R &= -s_j A u_B + \theta_j u_B - f_B(u_R, u_I) \\
\dot{u}_I &= s_j A u_B - \theta_j u_B + f_B(u_R, u_I)
\end{align*}
$$

(24)

Equation (24) can be viewed as an infinite-dimensional Hamiltonian system

$$
\frac{\partial}{\partial \xi} \begin{pmatrix} u_R \\ u_I \end{pmatrix} = J \begin{pmatrix} \frac{\delta \mathcal{H}}{\delta u_R} \\ \frac{\delta \mathcal{H}}{\delta u_I} \end{pmatrix}, \quad \text{where} \quad J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}
$$

and

$$
\mathcal{H} = \frac{1}{2} \int \sum_{j=1}^{m} \left( s_j |u_{Rj}|^2 + s_j |u_{Ij}|^2 + \theta_j u_{Rj}^2 + \theta_j u_{Ij}^2 \right) - V(u_R, u_I) d\xi.
$$

(25)

The following is a list of salient features of CNLS systems that contribute to the generation of $N$-pulses with widely spaced humps via bifurcation from a resonant semi-simple eigenvalue scenario. We focus on the situation where $x \in \mathbb{R}$, and call this one-dimensional variable $t$, for consistency with previous sections.

- The ODEs that describe real stationary solutions of the PDE system are Hamiltonian (with $m$ degrees of freedom). For real stationary solutions $\Phi = (\Phi_R, 0)$ of (23), $\Phi_R = (\Phi_{R1}, \ldots, \Phi_{Rm})$ solves the equations

$$
\begin{align*}
s_j A \Phi_B - \theta_j \Phi_B + f_B(\Phi_R, \Phi_I) &= 0 \\
s_j A \Phi_R - \theta_j \Phi_R + f_R(\Phi_R, \Phi_I) &= 0
\end{align*}
$$

for $j = 1, \ldots, m$.

The following is a list of salient features of CNLS systems that contribute to the generation of $N$-pulses with widely spaced humps via bifurcation from a resonant semi-simple eigenvalue scenario. We focus on the situation where $x \in \mathbb{R}$, and call this one-dimensional variable $t$, for consistency with previous sections.
where the $u_j$ are now taken to be real variables, and prime denotes differentiation with respect to the scalar variable $t$. This is equivalent to the first order system

$$
\begin{align*}
    u_j' &= s_j p_j \\
    p_j' &= \theta_j u_j - f_{\mu_j}(u, 0)
\end{align*}
$$

which is conservative with Hamiltonian function given by

$$
H = \frac{1}{2} \sum_{1 \leq j \leq m} (s_j p_j^2 - \theta_j u_j^2) + V|_{u_1=0, u_2=0}.
$$

Notice that the ODE Hamiltonian consists of the very terms that appear in the integrand of the PDE Hamiltonian, but with sign changes. Indeed, the fact that (23) is Hamiltonian and does not contain any first derivative terms in $t$ guarantees that the steady state ODEs derived from it will be Hamiltonian too. This fact also implies that the ODEs are reversible in $t$.

- 0 is a hyperbolic equilibrium point for (25). Observe that, since

$$
\nabla^2 H(0) = \begin{pmatrix}
-\theta_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & -\theta_m
\end{pmatrix},
$$

and we have the usual

$$
J = \begin{pmatrix}
0 & I_m \\
-I_m & 0
\end{pmatrix},
$$

the eigenvalues of the linearisation at 0 are $\pm \sqrt{s_j \theta_j}$, $j = 1, ..., m$, so they are either pure real or pure imaginary. Hyperbolicity requires that $s_j \theta_j > 0$ for all $j$.

- Occurrence of resonant semi-simple eigenvalues: the above eigenvalues $\pm \sqrt{s_j \theta_j}$ have corresponding eigenvectors

$$
e^{\pm}_j = (0, ..., 0, 1, 0, ..., 0, \pm \sqrt{s_j \theta_j}, 0, ..., 0)^T
$$

where 1 appears as the $j$th entry and $\pm \sqrt{s_j \theta_j}$ as the $(m + j)$th entry; their projections onto $(u_1, ..., u_m)$ space (“position” space) are simply the standard basis vectors for $\mathbb{R}^m$. When all the $s_j \theta_j$ are equal to some value $\theta > 0$, we have resonant (i.e., symmetric with respect to the imaginary axis,
a property due to \( t \)-reversibility) \( m \)th order eigenvalues at \( \pm \sqrt{\theta} \). Not only is the algebraic multiplicity for each of \( \pm \sqrt{\theta} \) equal to \( m \), so is the geometric multiplicity. This is because the \( s_j^+ \) are distinct, linearly independent eigenvectors, spanning respectively the unstable and stable eigenspaces, for all values of \( s_j \theta_j > 0 \), even when the eigenvalues overlap. Such a situation is referred to as “semi-simple”; there are no Jordan blocks. Of course, for \( m > 2 \), we could have some of the \( s_j \theta_j \) being equal, but different from all the others; these resonant \( k \)th order eigenvalues (with \( 2 \leq k < m \)) would also be semi-simple.

- Aside from reversibility in \( t \), which is represented by the \( 2m \times 2m \) diagonal matrix \( R \) whose first \( m \) entries are \(+1\) and whose remaining \( m \) entries are \(-1\), the system (53) is invariant under a discrete group \( G \) of symmetries (with representations as \( 2m \times 2m \) matrices), such as the \( S \) in Section 2. We further assume that each symmetry commutes with \( R \). We allow the possibility that \( G \) is just \( \{I\} \), the trivial group.

- At a parameter value \( \mu = \mu_1 \) (note, \( \mu \) can be a vector, and \( \mu_1 \) is usually taken to be 0), where we have leading resonant \( k \)th order semi-simple eigenvalues (for some \( 2 \leq k \leq m \)), there exists a non-degenerate primary homoclinic solution \( \tilde{q}(t) \). By virtue of the symmetry, \( \{ S \tilde{q}(t) : S \in G \} \) forms a set of non-degenerate homoclinic solutions at \( \mu = \mu_1 \). Non-degeneracy, in conjunction with the conservative or reversible properties of (26), implies persistence of these primary pulses for \( \mu \) close to \( \mu_1 \).

We use the phase-space coordinates \((u_1, \ldots, u_m, p_1, \ldots, p_m)\). Without loss of generality suppose the leading eigenvalues at \( \mu = \mu_1 \) are \( \pm \sqrt{s_1 \theta_1} = \cdots = \pm \sqrt{s_k \theta_k} = \pm 1 \) (i.e., set \( \theta = 1 \)), and let \( \mu = (\mu^{(1)}, \ldots, \mu^{(k)}) \) where \( \mu^{(j)} = \sqrt{s_j \theta_j} - 1 \). Lemma 2.1 in a slightly more general form provides the expansion

\[
q^\pm(t, \mu) = \sum_{j=1}^{k} (b_j(\mu) e_j^\pm e^{-(1+\mu^{(j)}|t|)}) + O(e^{-2\lambda^*(|t|)}) + O(e^{-\lambda^*(|t|)}) P^0
\]

for solutions in \( W^u_{loc}(0) \) or \( W^s_{loc}(0) \) (with \( |t| \gg 1 \)), where \( \lambda^* = \min \{1 + \mu^{(1)}, \ldots, 1 + \mu^{(k)}\} \), and \( \lambda^* = \min \{1 + \mu^{(k+1)}, \ldots, 1 + \mu^{(m)}\} \). Following the steps in Subsection 3.2 we arrive at the following expression for \( \xi_t(T, \mu) \),

\[
\left\langle M_{t-1}(\mu) \sum_{j=1}^{k} (b_j(\mu) e_j^\pm e^{-(1+\mu^{(j)} T_{r-1})}) \right\rangle \sum_{j=1}^{k} (b_j(\mu) e_j^\pm e^{-(1+\mu^{(j)} T_{r-1})}) \right\rangle

- \left\langle M_{t}(\mu) \sum_{j=1}^{k} (b_j(\mu) e_j^\pm e^{-(1+\mu^{(j)} T_{r})}) \right\rangle \sum_{j=1}^{k} (b_j(\mu) e_j^\pm e^{-(1+\mu^{(j)} T_{r})}) \right\rangle

+ O(e^{-2\lambda^* T_r (e^{-2\lambda^* T_r} + e^{-2\lambda^* T_r})} + O(e^{-2\lambda^* T_r} + e^{\min\{3\lambda', 2\lambda^* T_r\} T_r} + e^{\min\{3\lambda', 2\lambda^* T_r\} T_r} ),
\]

\[
\left\langle M_{t}(\mu) \sum_{j=1}^{k} (b_j(\mu) e_j^\pm e^{-(1+\mu^{(j)} T_{r})}) \right\rangle \sum_{j=1}^{k} (b_j(\mu) e_j^\pm e^{-(1+\mu^{(j)} T_{r})}) \right\rangle

- \left\langle M_{t}(\mu) \sum_{j=1}^{k} (b_j(\mu) e_j^\pm e^{-(1+\mu^{(j)} T_{r})}) \right\rangle \sum_{j=1}^{k} (b_j(\mu) e_j^\pm e^{-(1+\mu^{(j)} T_{r})}) \right\rangle

+ O(e^{-2\lambda^* T_r (e^{-2\lambda^* T_r} + e^{-2\lambda^* T_r})} + O(e^{-2\lambda^* T_r} + e^{\min\{3\lambda', 2\lambda^* T_r\} T_r} + e^{\min\{3\lambda', 2\lambda^* T_r\} T_r} ),
\]

\[
\left\langle M_{t}(\mu) \sum_{j=1}^{k} (b_j(\mu) e_j^\pm e^{-(1+\mu^{(j)} T_{r})}) \right\rangle \sum_{j=1}^{k} (b_j(\mu) e_j^\pm e^{-(1+\mu^{(j)} T_{r})}) \right\rangle

- \left\langle M_{t}(\mu) \sum_{j=1}^{k} (b_j(\mu) e_j^\pm e^{-(1+\mu^{(j)} T_{r})}) \right\rangle \sum_{j=1}^{k} (b_j(\mu) e_j^\pm e^{-(1+\mu^{(j)} T_{r})}) \right\rangle

+ O(e^{-2\lambda^* T_r (e^{-2\lambda^* T_r} + e^{-2\lambda^* T_r})} + O(e^{-2\lambda^* T_r} + e^{\min\{3\lambda', 2\lambda^* T_r\} T_r} + e^{\min\{3\lambda', 2\lambda^* T_r\} T_r}).
\]
where
\[ M_i(\mu) = V^2 R(0, \mu) R_i, \quad R_i, R_i+1 \in G. \]

To seek \( N \)-pulses, we look for solutions of the set of algebraic equations
\[ \xi_i(T, \mu) = 0, \quad i = 1, \ldots, N-1. \]

The symmetries which commonly exist for CNLS systems are ones that can be expressed as matrices of the form
\[ \gamma_i = \begin{pmatrix} d_{i1} & 0 \\ 0 & d_{i2} \end{pmatrix}, \]

where the \( \gamma_i \) are \( m \times m \) matrices, typically diagonal with \( \pm 1 \) as entries (\( \mathbb{Z}_2 \) symmetries); e.g., for the \( \chi^{(2)} \) SHG equations, \( d_i = \text{diag}( \pm 1, 1 ) \). Let us therefore concentrate on such situations, whereby
\[ \gamma_i(\gamma_i+1) = \text{diag}( \delta_i^{(1)}, \ldots, \delta_i^{(m)}, \delta_i^{(1)}, \ldots, \delta_i^{(m)}) \]

with \( \delta_i^{(j)} = \pm 1 \) for \( i = 1, \ldots, N-1 \) and \( j = 1, \ldots, m \). The bifurcation equations then become
\[
-2 \sum_{j=1}^{k} (s_i \delta_i^{(j)} (1 + \mu^{(j)})^2 b_j(\mu)^2 e^{-2(1+\mu^{(j)}) T_{i-1}} \\
+ 2 \sum_{j=1}^{k} (s_i \delta_i^{(j)} (1 + \mu^{(j)}) b_j(\mu)^2 e^{-2(1+\mu^{(j)}) T_i} \\
+ \mathcal{E}(e^{-2\Gamma T_{i-1}} + e^{-2\Gamma T_i} + e^{-2\Gamma T_{i-1}} + e^{-2\Gamma T_i} = 0
\]

for \( i = 1, \ldots, N-1 \) (where \( T_0 = \infty \)). Polar variables \( a_i, r \) of the kind presented in Subsection 3.3 may be used to transform the equations into a form suitable for an application of the Implicit Function Theorem. In particular, setting \( r = e^{-2(1+\mu^{(j)}) T} \) for some \( 1 \leq j \leq k \) makes the error term expressible as \( \mathcal{E}(r^{1+\omega}) \), where \( \omega = \min\{ \chi^{(1+\omega)}(1+\mu^{(j)}) \} \) is positive provided \( |\mu| \) is small enough.

For the sake of clarity, let us restrict our attention to the case \( k = 2 \). As in Subsection 3.3, for any \( N \geq 2 \), the problem reduces to solving a basic algebraic equation of the form
\[
a_1 \left[ 1 \delta_i^{(1)} + s_2 \delta_i^{(2)} \right] \left( \frac{1+\mu^{(2)}}{1+\mu^{(1)}} \right)^2 b_2(\mu)^2 \left( \frac{b_1(\mu)}{b_1(\mu)} \right)^2 (a_1 r)^{-\mu^{(2)}(1+\mu^{(1)})} + \mathcal{E}(r^{\omega}) = 0
\]
if $r$ is taken to be $e^{-2(1 + \mu^{(1)})\tau}$, or

$$a \left[ s_2 \delta_1^{(2)}(t) + s_1 \delta_2^{(1)}(t) \right] \frac{1}{1 + \mu^{(1)}} \left( \frac{b_1(\mu)}{b_2(\mu)} \right)^2 (\alpha_\rho)(\mu^{(1)}(1 + \mu^{(2)})) + C(\sigma^{\prime\prime}) = 0$$

if $r$ is taken to be $e^{-2(1 + \mu^{(2)})\tau}$. Using the arguments contained in Subsections 3.3 and 3.5 we make the following observations:

- For $N$ pulses to exist, $s_1 \delta_1^{(1)}$ and $s_2 \delta_2^{(2)}$ cannot be of the same sign. If $s_1$, $s_2$ have the same sign, this means that one of $\delta_1^{(1)}$, $\delta_1^{(2)}$ has to be $-1$, i.e., an $N$-pulse must be of alternating type, as following the same primary loop during consecutive excursions would necessitate $\delta_i^{(j)} = +1$ for all $i$ (we are considering only $\mathbb{Z}_2$ symmetries here); at the same time, an $N$-pulse cannot alternate in all of its components, because also one of $\delta_1^{(1)}$, $\delta_1^{(2)}$ needs to be $+1$. Note that for $s_1$, $s_2$ of the same sign, $N$-pulses do not exist if $G$ is trivial, but if $s_1 s_2 = -1$, there is a possibility of $N$-pulses existing even when the system has no non-trivial symmetries.

- The quantity $c(0) := b_1(0)/b_2(0)$ determines the direction of bifurcation. If $|c(0)| \neq 1$, the bifurcation is “one-sided,” i.e., $N$-pulses may exist for either $\mu^{(1)} < \mu^{(2)}$ or $\mu^{(1)} > \mu^{(2)}$, but not both, and not at $\mu^{(1)} = \mu^{(2)}$. In particular (looking at the two basic equations above and Lemma 3.1),

$$|c(0)| > 1 \Rightarrow N\text{-pulses exist for } \mu^{(1)} > \mu^{(2)}$$

$$|c(0)| < 1 \Rightarrow N\text{-pulses exist for } \mu^{(1)} < \mu^{(2)}.$$ 

If $|c(0)|$ happens to be exactly equal to 1, it is possible for $N$-pulses to exist at $\mu^{(1)} = \mu^{(2)}$, and on both sides of it. A model for which this situation occurs is the $\chi^{(3)}$ birefringence CNLS system. Essentially, $c(0)$ measures the slope of the path along which the primary homoclinic $q(t)$ approaches 0. Often, as for the $\chi^{(2)}$ SHG system, the components of the primary pulse are linearly related with one another. For example, $m = k = 2$, and $\Phi_R = (K\psi, \psi)^T$ at $\mu = \mu_1$; we then have $c(0) = K$.

There are certainly situations that are more difficult than the ones we have focused on in the latter part of this section. For instance, $k$ may be greater than 2 (e.g., in the $\chi^{(3)}$ three-wave interaction/Type II SHG system), although the same ideas are basically valid. We could also encounter more complicated symmetries, or primary pulses with either $c(0)$ or $c(0)^{-1}$ equal to zero, e.g., the “circularly polarised” waves of $\chi^{(3)}$ birefringence, which have one of their $u_j$ components identically equal to zero; more sophisticated techniques would be called for to investigate the existence of $N$-pulses whose loops might trace, say, a primary pulse with $|c(0)| = 1$ and another primary pulse with $c(0) = 0$. 
We remark that for a number of CNLS systems which arise as physical models, we have:

- at \( s_1 \theta_1 = \cdots = s_m \theta_m = \theta (>0) \), there exists a real 1-pulse solution \( \Phi \) with \( \Phi_{ij} = 0 \) for all \( j \) and \( \Phi_{Rj} = K_j \phi \), where \( \phi(x) \to 0 \) exponentially as \( |x| \to \infty \), and the \( K_j \) are real constants;
- is a polynomial which is purely of \( p \)th power in \((u_R, u_I)\), for some \( p \geq 3 \).

These conditions imply that \( L_R \) can be diagonalised, where \( L_R \) is the matrix differential operator representing the equation of variations of (25) about the real 1-pulse (see [14] for more explanation). Diagonalisation of \( L_R \) is usually an important step towards proving non-degeneracy of the 1-pulse.

The observations in this section have been of some help in understanding the bifurcation (or non-existence) of widely spaced \( N \)-pulses for the CNLS systems that describe \( \chi^{(3)} \) birefringence effects [6, 7], \( \chi^{(3)} \) third-harmonic generation (THG) [8], \( \chi^{(2)} + \chi^{(3)} \) competing nonlinearities [11], and \( \chi^{(2)} \) three-wave interaction or SHG of Type II [9, 10]. See [14] for a brief discussion of these examples.

Finally, a worthwhile extension would be the investigation of bifurcation(s) occurring at a resonant semi-simple eigenvalue scenario in the more general context of reversible ODEs. For reversible ODEs, the spectrum of the linearisation at a reversible equilibrium (one that lies in the fixed-point space of the involution) is symmetric with respect to the imaginary axis, so we always get resonant pairs of eigenvalues. The analysis in this paper depends heavily on the additional Hamiltonian structure our equations possess. Studying possible bifurcations without recourse to a conservative structure would require alternative ways of representing \( \mathcal{P}(t) \).

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