Convex polygons with few intervertex distances

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Abstract

Thirty years ago E. Altman proved that every convex \( n \)-gon (\( n \geq 3 \)) has at least \([n/2]\) different distances between vertices and that a convex \( n \)-gon for odd \( n \) has exactly \((n - 1)/2\) intervertex distances if and only if it is regular. We prove that a convex \( n \)-gon for even \( n \geq 8 \) has exactly \( n/2 \) intervertex distances if and only if it is a regular \( n \)-gon or a regular \((n + 1)\)-gon with a vertex removed. Three hexagons have exactly three intervertex distances, and four quadrilaterals have exactly four intervertex distances. Moreover, 15 pentagons have exactly three intervertex distances, and five heptagons have exactly four intervertex distances. The latter five are the regular octagon minus a vertex and the four dissimilar versions of a regular nonagon with two vertices removed.

1. Introduction

Many years ago Erdős [3] conjectured that every convex \( n \)-gon (\( n \) vertices, \( n \) sides, \( n \geq 3 \)) has at least \([n/2]\) different distances between vertices. Altman [1,2] proved this and noted for odd \( n \) that \([n/2]\) is attained only by regular polygons. He remarked also that for even \( n \) both a regular \( n \)-gon and a regular \((n + 1)\)-gon minus one vertex have exactly \( n/2 \) intervertex distances.

This paper completes Altman's remark by identifying all convex \( n \)-gons for even \( n \) that have exactly \( n/2 \) intervertex distances. We also consider for odd \( n \) the convex \( n \)-gons that have exactly \((n + 1)/2\) intervertex distances, one more than the minimum. We solve this only for \( n \in \{3,5,7\} \) but note that the \( n = 7 \) result suggests a plausible solution for odd \( n \geq 9 \).

The following proposition is a consequence of our results for even \( n \) in conjunction with Altman's theorem.
**Proposition 1.** For every \( n \geq 7 \) there is a largest nonnegative integer \( f(n) \) such that every convex \( n \)-gon with no more than \([n/2] + f(n)\) intervertex distances has all \( n \) vertices on a circle, and these vertices are among those of some regular polygon.

Our results and a few other observations show that \( f(7) = 1 \) and \( f(8) = 0 \). An upper bound on \( f(n) \) is noted at the end of the paper. We conjecture that \( f \) is unbounded.

To state our results let \( \mathcal{E}_n \) for \( n \geq 3 \) be the class of all convex \( n \)-gons in the plane. The Euclidean distance between points \( x \) and \( y \) in the plane is \( d(x, y) \). We write \( C \approx D \) if polygons \( C \) and \( D \) are similar, i.e., if \( D \) can be mapped into \( C \) by rotation about a point, reflection about a line, translation and uniform rescaling. We say that a subclass of \( \mathcal{E}_n \) contains \( N \) polygons if there are \( C_1, \ldots, C_N \in \mathcal{E}_n \) such that \( C_i \neq C_j \) whenever \( i \neq j \), and the subclass consists of all \( C \in \mathcal{E}_n \) such that \( C \approx C_i \) for some \( i \). Such \( C_i \) form a system of representatives for the subclass under the similarity equivalence relation.

A regular \( n \)-gon is denoted by \( R_n \). A regular \( n \)-gon with \( k \leq n - 3 \) vertices deleted, which is in \( \mathcal{E}_{n-k} \), is denoted by \( R_n - k \). When \( k \geq 2 \), dissimilar versions of \( R_n - k \) are obtained by removing different combinations of vertices from \( R_n \). If the vertices of \( R_n \) are labeled \( 1, 2, \ldots, n \) clockwise, the set of all \( R_n - 2 \) for fixed \( n \geq 5 \) contains \( N = \lfloor n/2 \rfloor \) polygons with \( C_i \) given by the removal of vertices \( i \) and \( i+1 \) for \( i = 1, 2, \ldots, \lfloor n/2 \rfloor \).

Suppose \( C \) in \( \mathcal{E}_n \) has vertex set \( \{1, \ldots, n\} \). We define \( m(C) \) as the number of different intervertex distances in \( C \):

\[
m(C) = | \{ d(i, j) : i \neq j \} |.
\]

Clearly \( m(R_n) = \lfloor n/2 \rfloor \), and if \( n > 2k \) then \( m(R_n - k) = \lfloor n/2 \rfloor \). In general, \( |n/2| \leq m(C) \leq \binom{n}{2} \). Let

\[
M_n(t) = \{ C \in \mathcal{E}_n : m(C) = t \}.
\]

**Theorem 1** (Altman). For every \( n \geq 3 \), \( m(C) \geq \lfloor n/2 \rfloor \) for all \( C \in \mathcal{E}_n \). If \( n \) is odd then \( M_n((n-1)/2) \) contains 1 polygon, \( R_n \).

**Theorem 2.** \( M_4(2) \) contains 4 polygons, and \( M_4(3) \) contains 3 polygons: see Fig. 1. For every even \( n \geq 8 \), \( M_n(n/2) \) contains 2 polygons, \( R_n \) and \( R_{n+1} - 1 \).

Our figures use \( \ldots, \ldots \), \( \ldots \), and so forth to denote different distances, and we shall partially identify specific polygons by their multiplicity vectors. The multiplicity vector for \( C \in \mathcal{E}_n \) with \( m(C) = t \) is \( r = (r_1, r_2, \ldots, r_t) \), in which \( r_1 \geq r_2 \geq \cdots \geq r_t \geq 1 \), \( \sum r_i = \binom{n}{2} \), and each \( r_i \) is the number of times a particular distance occurs between vertices.

We omit the straightforward proof for \( M_4(2) \) in Theorem 2. The \( M_4(3) \) proof is in the next section. Section 3 presents the proof for \( n \geq 8 \). Sections 4 and 5 prove our next result for \( n = 5 \) and \( n = 7 \) respectively.
Theorem 3. $M_5(2)$ is the class of all nonequilateral isosceles triangles. $M_5(3)$ contains the 15 pentagons shown in Fig. 2. $M_5(4)$ contains 5 polygons, namely $R_8 - 1$ and the four dissimilar versions of $R_9 - 2$.

The result for $n = 7$ suggests that for odd $n \geq 9$, $M_5((n + 1)/2)$ contains $(n + 3)/2$ $n$-gons, namely $R_{n+1} - 1$ and the $(n + 1)/2$ dissimilar versions of $R_{n+2} - 2$.

We conclude this introduction by listing special notations and lemmas that are used repeatedly in the proofs.

Let $x, y, \ldots, z$ be distinct points in the plane. Then

$$x = _d(y, \ldots, z) \text{ or } x = _d y \cdots z$$

means that each of $y, \ldots, z$ is the same distance from $x$. When $d(x, y) = d$, we often write $xy = d_j$ and say that $xy$ is $d$. By similar abbreviation, $xy > (\geq, =)zw$ means that $d(x, y) > (\geq, =)d(z, w)$. The perpendicular bisector of line segment $xy$ is $\perp_{xy}$. If the lines that contain segments $xy$ and $zw$ are parallel, then $xy \parallel zw$; otherwise, $xy \parallel zw$. The triangle with vertices $x, y$ and $z$ is $\Delta(x, y, z)$ or $\Delta xyz$, and its interior angle at $y$ is $\Delta xyz$. Angles are usually denoted by lower case Greek letters. A circle is denoted $\odot$, and the unique circle through $x, y$ and $z$ when they are not collinear is $\odot(x, y, z)$ or $\odot xyz$.

Suppose $n$-gon $C$ has vertices $1, 2, \ldots, n$ labeled consecutively clockwise or counterclockwise around $C$'s perimeter. Then $\langle i, i + 1, \ldots, j \rangle$ is the convex polygon whose vertices are the consecutive vertices of $C$ from $i$ to $j$ inclusive, going from $i$ in the
direction of $i + 1$. By this notation, $C = \langle 1, 2, \ldots, n \rangle = \langle n - 1, n - 2, \ldots, n \rangle = \langle 5, 6, \ldots, 4 \rangle$, and so forth. The number of vertices of subpolygon $\langle i, i + 1, \ldots, j \rangle$ is $|\langle i, i + 1, \ldots, j \rangle|$.

Unless stated otherwise, when $m(C) = t$ we label the $t$ intervertex distances in decreasing magnitude as

$$d_1 > d_2 > \cdots > d_t.$$ 

When $C$'s multiplicity vector is $(r_1, r_2, \ldots, r_t)$, each $r_i$ corresponds to a $d_j$, but not necessarily in the same order. For example, the shortest distance $d_i$ could have maximum multiplicity $r_1$.

There are a number of elementary facts of plane Euclidean geometry that we refer to collectively as Lemma 0. Specific pieces of Lemma 0 used later include:
(0.1) three noncollinear points lie on exactly one circle;
(0.2) two distinct circles intersect in at most two points;
(0.3) if x, y, z and w are vertices of a convex polygon listed in sequence clockwise (no more than one revolution), and if xy = yz = zw and xz = yw, then w ∈ ∩ xyz;
(0.4) ⊥ xy passes through the center of every circle that contains x and y;
(0.5) z = d xy ⇔ z ∈ ⊥ xy;
(0.6) xy < xz ⇔ z is on y’s side of ⊥ xy;
(0.7) the sum of the lengths of the diagonals of a quadrilateral exceeds the sum of the lengths of two opposite sides;
(0.8) if x, y, z and w are distinct, and z and w lie on the same side of the line containing xy, with xz = yw and xw = yz, then xy || zw;
(0.9) the relation || between line segments is transitive;
(0.10) if x, y, z and w are four vertices of a convex polygon, and z and w lie on the same side of xy, then z = d xy and w = d xy are jointly impossible.

For clarity, we sometimes refer to a piece of Lemma 0 by its (0.k) designation.

Lemmas from Altman [1] are central in some of our proofs. A side xy of a convex polygon with intervertex distances d₁ > d₂ > · · · is max if xy = d₁ and is uniquely max if xy = d₁ and no other side or diagonal has length d₁.

Lemma 1. If a side of convex n-gon C is max, then m(C) ≥ n - 2. If a side of C is uniquely max, then m(C) > n - 1.

In our other two Altman lemmas we denote the n vertices of convex n-gon C by 1, 2, . . . , n consecutively around the perimeter.

Lemma 2. If (1, n) is uniquely max and m(C) = n - 1 then (k, n - k + 1) = d₂k-1 for k = 1, 2, . . . , [n/2], and (k, n - k + 2) = (k - 1, n - k + 1) = d₂k-2 for k = 2, 3, . . . , [(n + 1)/2].

Lemma 3. If (1, n) is max and m(C) = n - 2, then (k, n - k + 1) = d₂k-2 for k = 2, 3, . . . , [n/2], and (k, n - k + 2) = (k - 1, n - k + 1) = d₂k-3 for k = 2, 3, . . . , [(n + 1)/2].

The lemmas are illustrated in Fig. 3.

2. Theorem 2 for n = 6

We apply the preceding lemmas to determine M₆(3).

Assume that n = 6 and m(C) = 3 = n - 3. We label C’s vertices as 1 through 6 clockwise. Lemma 1 implies that no side is max.
Suppose $d_1$ holds between two vertices adjacent to a third vertex. For definiteness take $13 = d_1$. Lemma 1 says that pentagon $\langle 34561 \rangle$ with side $13 = d_1$ has at least 3 intervertex distances. Hence it has exactly 3. By Lemma 3, $14 = 36 = 13 = d_1$, $46 = d_2$ and $45 = 56 = d_3$ (Fig. 4, top). By Lemma 0, $45 < 35$ and $56 < 15$, so $35$ and $15$ are in $\{d_1, d_2\}$.

With $13 = d_1$, suppose $35 = d_1$. By analogy to $13 = d_1$, $35 = d_1$ in $\langle 56123 \rangle$ gives $25 = 35 = 36 = d_1$, $26 = d_2$ and $12 = 16 = d_3$ (Fig. 4, left). If $15 = d_1$ also, then (by analogy) $24 = d_2$ and $23 = 34 = d_3$, and this yields $A_6$ of Fig. 1. On the other hand, if $15 = d_2$ then (Lemma 0) $2, 1, 6, 5$ and $4$ are on the circle with center at $3$ and radius $d_1$. This requires $23 = d_1$, which contradicts $23 < 24$ by Lemma 0 ($1 = \alpha 34$). So if either $35$ or $15$ is $d_1$, so is the other and we get $A_6$.

Continuing with $13 = d_1$, suppose $15 = 35 = d_2$ (Fig. 4, right). Then triangles $365$ and $145$ are congruent, so $13 \parallel 46$ and $16 = 34$. Since $34 < 35$, $16 = 34 = d_3$ as shown on the figure. This implies that 1, 6, 5, 4 and 3 are on a circle [see (0.3)] with center slightly below the intersection of 13 and $\perp_{12}$. Let $\alpha = 645$. The other angles labeled $\alpha$ follow from congruence and parallelism. The three angles labeled $\beta$ are the same by congruence and the fact that $\chi_{654} = \chi_{543}$ with two $\alpha$'s in each as indicated. By $\Delta 654$, $4\alpha + \beta = \pi$, and, by isosceles $\Delta 134$, $\beta = 3\alpha$. Hence $7\alpha = \pi$, so the interior angle of $C$ at each of 4, 5 and 6 is $5\pi/7$. This says that 1, 6, 5, 4 and 3 are consecutive vertices of $R_7$. 

Fig. 3. Altman's lemmas.
This leaves vertex 2 for the right picture of Fig. 4. If \(2 \in \perp_{13}\), \(23 < 24 < 25\) since the center of the \(\odot_1\) that contains the other five vertices is on \(\perp_{13}\) below 13. But then \((23, 24) = (d_3, d_2)\), which forces 2 to be on \(\odot_1\) above 3 to the right of \(\perp_{13}\). Therefore 2 \(\not\in \perp_{13}\). Assume for definiteness that 2 is left of \(\perp_{13}\). Then \(21 < 26 < 25\), so \((21, 26) = (d_3, d_2)\) and 2 is on \(\odot_1\) above 1. It follows that \(C = R_7 - 1\).

Suppose finally that \(d_1\) never holds between two vertices adjacent to a third vertex. Assume for definiteness that \(14 = d_1\). By hypothesis, 14 is uniquely max in each of \(\langle 1234 \rangle\) and \(\langle 4561 \rangle\). Lemmas 1 and 2 applied to each quadrilateral yield \(13 = 24 = 15 = 46 = d_2\) and \(23 = 56 = d_3\), so also \(23 \parallel 14 \parallel 56\).

Suppose \(xy = d_1\) for no pair other than 14. Remove vertex 4 to get \(m(\langle 56123 \rangle) = 2\). Then \(\langle 56123 \rangle = R_5\) by Theorem 1, and this contradicts \(23 \parallel 56\). Hence another opposites diagonal has length \(d_1\). Let it be 25 (Fig. 4, bottom). By analogy to our analysis for \(14 = d_1\), \(25 = d_1\) implies \(24 = 35 = 26 = 15 = d_2\) and \(34 = 16 = d_3\). It follows that \(12 \parallel 45\) and \(15 \parallel 24\), so 1245 is a rectangle with \(12 = 45\). If \(45 \in \{d_1, d_2\}\), 3 and 6 would be inside the rectangle, thus violating convexity. So \(12 = 45 = d_3\) and it follows that \(C = R_6\).

We have shown that \(M_6(3)\) contains precisely the three polygons noted in the middle of Fig. 1.
3. Larger even $n$

We are to show that $M_{2N}(N)$ contains precisely $R_{2N}$ and $R_{2N+1} - 1$ for $N = 4, 5, \ldots$. By removing a vertex from $C \in M_8(4)$ and then adding a vertex convexly to the resulting heptagon, it is easily seen from $M_3(3) = \{R_7\}$ and $M_4(4) = \{R_8 - 1, \text{four dissimilar versions of } R_9 - 2\}$ (proof in Section 5) that $C$ is either $R_8$ or $R_9 - 1$.

The following assumptions and conventions will be used in our general proof for $M_{2N}(N)$, $N \geq 4$.

Assumption 1. $C \in M_{2N}(N)$; the vertices of $C$ are labeled clockwise as $1, 2, \ldots, N, N + 1, \ldots, 2N$; the $N$ intervertex distances are $d_1 > d_2 > \cdots > d_N$; vertex 1 has the maximum number of $d_1$ instances to the other vertices.

Lemma 4. $d(1, N + 1) = d_1$. If $d(1, k) = d_1$ for $k \neq N + 1$, then $k \in \{N, N + 2\}$.

Proof. Suppose $d(1, k) = d_1$. If $k \notin \{N, N + 1, N + 2\}$, then Lemma 1, applied to the largest polygon in $C$ with side $1k$, gives the contradiction $m \geq N + 1$. If $1N = d_1$, Lemmas 1 and 3 imply $d(1, N + 1) = d(2, N + 2) = d_1$; similarly, if $d(1, N + 2) = d_1$, then $d(1, N + 1) = d(2, N + 2) = d_1$. $\Box$

We assume without loss of generality that $1N = d_1$ whenever $C$ has more than one $d_1$ segment from 1. Our general proof divides naturally into two parts:

Part I: $(1, N + 1)$ is the unique $d_1$ segment from vertex 1;

Part II: $d(1, N) = d(1, N + 1) = d_1$.

We obtain $C = R_{2N}$ from Part I and $C = R_{2N+1} - 1$ from Part II.

Part I

When $|k - j| = N$, we refer to vertices $j$ and $k$ as opposites and to segment $jk$ as an opposites segment. Two opposites segments are adjacent if their vertices are the endpoints of two sides of $C$.

We assume for Part I that $(1, N + 1)$ is the only $d_1$ segment for vertex 1. By Assumption 1, no other vertex has more than one $d_1$ instance.

Lemma 5. Every $d_1$ segment in $C$ is an opposites segment. Every opposites segment is a $d_1$ or $d_2$ segment.

Proof. The first assertion follows immediately from our assumptions and the Lemma 4 proof procedure. Consequently, if an opposites segment is $d_1$, it is uniquely max in each of the $(N + 1)$-gons in $C$ that use it as a side and, by Lemma 1, each of those $(N + 1)$-gons uses all $N d_1$. If an opposites segment is $d_2$, it is max in each of the $(N + 1)$-gons that have it as a side and, by Lemma 1, each of those $(N + 1)$-gons use all $d_i$ for $i \geq 2$. 
Suppose $j \leq N$ and $(j, N+j)$ is $d_1$ or $d_2$. Then, by Lemma 2 for the $d_1$ case, and Lemma 3 for the $d_2$ case, applied to each $(N+1)$-gon, we get

\[ d(j, N+j+1) = d(j-1, N+j) = d_2, \]

\[ d(j+1, N+j+1) = d(j-1, N+j) = d_2. \]

Therefore $N+j = d(j-1, j+1)$, so $d(j-1, N+j+1) > d(j+1, N+j-1) = d_3$ since $N+j$ is on the same side of $j, j+1, j+2$ as $j+1$. Hence $d(j-1, N+j-1) \geq d_2$; similarly, $d(j+1, N+j+1) \geq d_2$.

We conclude that each adjacent opposites segment of a $d_1$ or $d_2$ opposites segment is also $d_1$ or $d_2$, hence that all opposites segments are $d_1$ or $d_2$. \( \square \)

The results of Lemma 5 in conjunction with further applications of Lemmas 1–3 to every $(N+1)$-gon that uses an opposites segment as a side imply that every side of $C$ is a $d_{N-1}$ segment and every diagonal of $C$ whose vertices are adjacent to a third vertex is a $d_{N-1}$ segment. It follows that all vertices are on a circle $[0.3]$ of Lemma 01 and, since they are evenly spaced, that $C = R_{2N}$.

Part II

We now assume that $d(1, N) = d(1, N+1) = d_1$. Let

- $C_0$ be the $(N+2)$-gon $<N, N+1, \ldots, 2N, 1>$,
- $C_1$ be the $(N+1)$-gon $<N+1, N+2, \ldots, 1>$,
- $C_2$ be the $(N+1)$-gon $<N, N+1, \ldots, 2N>$.

By Lemma 1, $m(C_0) = N$, so by Lemma 3, $d(N, 2N) = d(N+1, 1) = d_1$, $d(N+1, 2N) = d_2$, $d(N+1, 2N-1) = d(N+2, 2N) = d_3$, $d(N+2, 2N-1) = d_4$, and so forth. The key for Part II is whether $(1, N+1)$ and $(2N, N)$ are uniquely max in $C_1$ and $C_2$ respectively.

**Lemma 6.** Suppose $(1, N+1)$ is uniquely max in $C_1$ and $(2N, N)$ is uniquely max in $C_2$. Then the vertices of $C_0$ are equally spaced at distance $d_N$ on a circle whose center is between $(1, N)$ and $(1, N+1) \cap (2N, N)$ and on $\perp_{1N}$.

**Lemma 7.** $(1, N+1)$ and $(2N, N)$ are uniquely max in $C_1$ and $C_2$ respectively.

Fig. 5A illustrates Lemma 6. We prove Lemmas 6 and 7 shortly. They are presumed in the following proof completion for Part II.

**Lemma 8.** Let $V = \{N, N+1, \ldots, 2N, 1\}$ and let $\cap_1$ denote the circle specified in Lemma 6 that contains $V$. Then $k \in \cap_1$ for each $k \in \{2, 3, \ldots, N-1\}$, and $C = R_{2N+1}$.
Proof. Given Lemmas 6 and 7 and the hypotheses of Lemma 8, consider \( k \in \{2, 3, \ldots, N - 1\} \). We have \( kv \in \{d_1, \ldots, d_N\} \) for each \( v \in V \). Vertex \( k \) cannot have the same \( d_i \) to three \( v \in V \), else it would be the center of \( C \) and violate convexity. Since \( |V| = N + 2 \), there are distinct \( d_i \) and \( d_j \) such that \( kv = d_i \) for two \( v \in V \) and \( kv = d_j \) for two other \( v \in V \). It follows from the structure of \( C \) that there are \( u_1, u_2 \in V \) with \( N + 1 < u_1 < u_2 < 2N \) and \( u_2 - u_1 < 2 \) such that \( d(k, u_1 - p) = d(k, u_2 + p) \) for \( p = 0, 1, \ldots \) so long as both \( u_1 - p \) and \( u_2 + p \) are in \( V \). Moreover, since convexity requires \( k \) outside of \( C \) (above \( 1N \) on Fig. 5A), the distance from \( k \) to \( v \) decreases as \( v \) moves counterclockwise from \( u_1 \) toward \( N \) or clockwise from \( u_2 \) toward \( 1 \). And if \( u_2 - u_1 = 2 \) then \( d(k, v_1 + 1) > d(k, v_1) \). See Fig. 5B.

If \( u \) and \( v \) are adjacent vertices in \( V \), \( x \) is a point above \( 1N \) and either \( xu = xv - d_i \) or \( xu = d_{i+1} \) for some \( i \geq 1 \), then \( x \in C \) by congruent triangles. Suppose \( k \notin C \) and, with no loss of generality, assume that \( |\{v_2, v_2 + 1, \ldots, 2N, 1\}| > |\{v_1, v_1 - 1, \ldots, N\}| \). If \( v_2 - v_1 = 1 \) then

\[
d(k, v_2) \leq d_2, \quad d(k, v_2 + 1) \leq d_4, \quad d(k, v_2 + 2) \leq d_6, \quad \ldots
\]
and, since \(|\{v_2, v_2 + 1, \ldots, 1\}| \geq \lceil (N + 3)/2 \rceil\), we obtain the contradiction that \(d(k, 1) < d_N\). If \(v_2 - v_1 = 2\) then

\[d(k, v_2 - 1) < d_1, \quad d(k, v_2) \leq d_3, \quad d(k, v_2 + 1) < d_5, \quad \ldots\]

and, since \(|\{v_2, v_2 + 1, \ldots, 1\}| \geq \lceil (N + 1)/2 \rceil\), we again get \(d(k, 1) < d_N\).

It follows that \(k \in \mathcal{O}_i\) for every \(k \in \{2, 3, \ldots, N - 1\}\). If \(k = d_i, i > 2\), then as we proceed counterclockwise from 1 through \(V\) the successive distances from \(k\) are \(d_{i-1}, d_{i-2}, \ldots, d_2, d_1, d_2, \ldots\), so each \(k\) has instances of \(d_i\) to adjacent vertices in \(V\). Since \(N = d(1, 2N)\) and \(1 = d(N, N + 1)\), this leaves \(N - 1\) pairs of adjacent vertices in \(V\) whose two members have \(d_i\) to a point above \(1N\) on \(\mathcal{O}_i\). Since \(|\{2, 3, \ldots, N - 1\}| = N - 2, N - 2\) of those \(N - 1\) points must be \(2, 3, \ldots, N - 1\), and we conclude that \(C = R_{2N+1} - 1\).

**Proof of Lemma 6.** We assume that \(1N = d_1\) and that \((1, N + 1)\) and \((2N, N)\) are the unique \(d_i\) segments in \(C_1\) and \(C_2\) respectively. Lemmas 1 and 3 for \(C_0\) and 1 and 2 for \(C_1\) and \(C_2\) give

\[d_2 = (1, N + 2) = (2N, N + 1) = (2N - 1, N),\]
\[d_3 = (2N, N + 2) = (2N - 1, N + 1),\]
\[d_4 = (2N, N + 3) = (2N - 1, N + 2) = (2N - 2, N + 1),\]
\[d_5 = (2N - 1, N + 3) = (2N - 2, N + 2)\]

and so forth: see Fig. 5C. The picture up from \(d_N = d(3N/2 + 2, 3N/2 + 1) = d(3N/2 + 1, 3N/2) = d(3N/2, 3N/2 - 1)\) for even \(N\) and from \(d_N = d((3N + 1)/2 + 1, (3N + 1)/2) = d((3N + 1)/2, (3N + 1)/2 - 1)\) for odd \(N\) appears at the bottom of the figure for \(N = 14\) and 13. By Lemma 0, applied bottom up, all horizontals in Fig. 5C are parallel and have the same perpendicular bisector. In general,

\[\perp_{1N} = \perp_{2N, N+1} = \perp_{2N, jN+1, j} \quad \text{for} \quad j = 1, \ldots, \lfloor N/2 - 1 \rfloor.\]

The following lemma in the spirit of Lemma 0 will help show that the vertices in Fig. 5C lie on a circle with distance \(d_N\) between adjacent vertices.

**Lemma 9.** Let \(x, y, v, z\) and \(w\) be counterclockwise successive points on a circle \(\mathcal{O}_i\) with \(v\) midway between \(y\) and \(z\). Suppose the chord of \(\mathcal{O}_i\) through \(v\) that is parallel to \(xw\) is no longer than \(xw\). Let \(p\) be the interior-direction point on \(\perp_{yz}\) that satisfies

\[d(y, p) = d(z, p) = d(x, w).\]

Then the perpendicular to the line through \(x\) and \(w\) that contains \(p\) intersects the line through \(x\) and \(w\) between \(x\) and \(w\).

**Proof of Lemma 9.** Assume for definiteness that \(d(x, v) \leq d(v, w)\). Figs. 6A and 6B illustrate cases in which (A) \(x\) through \(w\) lie within a semicircular arc of \(\mathcal{O}_i\) and (B) \(x\) through \(w\) cover more than a semicircle. For convenience take \(xw\) horizontal, let \(c\) denote the center of \(\mathcal{O}_i\), and let \(d_a = d(x, w)\). Let \(t\) be the interior direction point on
Fig. 6. Constructions for Lemmas 6 and 9.

\[ \perp_{yz} \text{ at distance } d_a \text{ from } x, \text{ and let } x' \text{ be the other point on } \odot_1 \text{ at distance } d_a \text{ from } t. \]

Clearly, \( t \) is left of \( w \). If \( vt \leq \text{radius}(\odot_1) \), the conclusion of Lemma 9 is obvious, so suppose that \( vt > \text{radius}(\odot_1) \). Then, between \( x \) and \( x' \), the circle centered at \( t \) with radius \( d_a \) is inside \( \odot_1 \), so \( vt > d_a \). It follows that \( p \), on \( \perp_{yz} \) at distance \( d_a \) from \( y \) and \( z \), is left of \( t \), so \( p \) (near \( t \) on the figures) is left of \( w \) and right of \( x \). \( \square \)

To complete the proof of Lemma 6, suppose first that \( N \) is even. Let \( K = 3N/2 \). Fig. 6C, where label \( N - k \) denotes \( d_{N-k} \), mimics the bottom left of Fig. 5. Let \( \odot_1 \) be the circle containing \( K + 2, K + 1, K \) and \( K - 1 \). Since

\[ d_K = (K, K - 1) < (K, K - 2) < (K + 1, K - 2) = d_{N-2}, \]

we have \( (K, K - 2) = d_{N-1} \) and then \( (K - 1, K - 2) = d_{N-2} \). Therefore triangles \( (K + 1, K, K - 1) \) and \( (K, K - 1, K - 2) \) are congruent, so \( K - 2 \in \odot_1 \). Symmetrically, \( K + 3 \in \odot_1 \).

Next, \( (K + 1, K - 3) > d_{N-3} \) since \( (K + 1, K - 3) > (K + 1, K - 2) = d_{N-2} \). If \( (K + 1, K - 3) = d_{N-4} \), Lemma 9 with \( (x, y, z, w, p) = (K + 3, K + 2, K + 1, K - 2, K - 3) \) gives the contradiction that \( K - 3 \) is left of \( K - 2 \). If \( (K + 1, K - 3) > d_{N-4} \), \( K - 3 \) would be even farther left. Hence \( (K + 1, K - 3) < d_{N-4} \), so \( (K + 1, K - 3) = d_{N-3} \). Then triangles \( (K + 3, K + 2, K - 2) \) and \( (K + 2, K + 1, K - 3) \) are congruent, so \( K - 3 \in \odot_1 \). Symmetrically, \( K + 4 \in \odot_1 \). Moreover, \( K - 3 \) is \( d_N \), \( d_{N-1} \) and \( d_{N-2} \) from \( K - 2, K - 1 \) and \( K \) respectively.
We continue up the figure in a similar manner. In the next step, \((K + 2, K - 4) > (K + 2, K - 3) = d_{N_5}\), so \((K + 2, K - 4) = d_{N_6}\). Congruence of \(\Delta(K + 4, K + 3, K - 3)\) with \(\Delta(K + 3, K + 2, K - 4)\) shows that \(K - 4 \in \mathcal{O}_1\). At the final step (top of Fig. 5C), all distances below \((2N, N + 1)\) are known. Then \((2N - 2, N) > (2N - 2, N + 1) = d_2\), so \((2N - 2, N) \geq d_2\) is impossible by Lemma 9 applied to \((x, y, z, w, p) = (2N, 2N - 1, 2N - 2, N + 1)\), and therefore \((2N - 2, N) = d_3\). Congruence of \(\Delta(2N, 2N - 1, N + 1)\) with \(\Delta(2N - 1, 2N - 2, N)\) gives \(N \in \mathcal{O}_1\) with \((N, N + 1) = d_{N_7}, (N, N + 2) = d_{N_8}\), and so forth. Because \(1N = (1, N + 1) = d_1\), the center of \(O_1\), which is obviously on \(\perp_{K+1}, K \in \perp_{N+1}\), is between \(1N\) and \((1, N + 1)\).

Suppose \(N\) is odd. Let \(K = (3N + 1)/2\): see Fig. 6D. Let \(O_1\) be the circle containing \(K + 1, K\) and \(K - 1\). Since \((K - 1, K - 2) < (K, K - 2) = d_{N_9}\), we have \((K - 1, K - 2) = d_{N_9}\), so \(K - 2 \in \mathcal{O}_1\). Symmetrically, \(K + 2 \in \mathcal{O}_1\).

Next, \((K, K - 3) > (K, K - 2) = d_{N_9}, so (K, K - 3) \geq d_{N_9}. If (K, K - 3) = d_{N_9}, Lemma 9 with \((x, y, z, w, p) = (K + 2, K + 1, K, K - 2, K - 3)\) yields the contradiction that \(K - 3\) is left of \(K - 2\), and it follows that \((K, K - 3) = d_{N_9}. Congruence of \(\Delta(K + 2, K + 1, K - 2)\) with \(\Delta(K + 1, K, K - 3)\) shows that \(K - 3 \in \mathcal{O}_1\) with \(K - 3\) distance \(d_N\) from \(K - 2\) and \(d_{N-1}\) from \(K - 1\).

The rest of the proof for \(N\) odd is similar to that for \(N\) even. □

**Proof of Lemma 7.** Given \((1, N) = (1, N + 1) = d_1\), we show that \((1, N + 1)\) is the only \(d_1\) segment in \(C_1 = \langle N + 1, N + 2, \ldots, 1 \rangle\). The proof for \(C_2\) is similar.

Our initial configuration from the first paragraph of the proof for Part II is shown in Fig. 7A. To prove the claim of Lemma 7 for \(C_1\), we suppose that \(C_1\) has another \(d_1\) segment and obtain a contradiction. Suppose \(xy\) is an unlined segment on Fig. 7A for \(C_1\). If \(\{x, y\} \notin \{1, N + 2\}\) then \(C\) has at least \(N + 3\) vertices clockwise from \(x\) to \(y\) or counterclockwise from \(x\) to \(y\), so \(xy = d_1\) gives a contradiction by Lemma 1 to \(C \in M_{2N}(N)\). We suppose henceforth that

\[d(1, N + 2) = d_1.\]

Since \(|\langle 1, 2, \ldots, N + 2 \rangle| = N + 2\), Lemmas 1 and 3 augment Fig. 7A to produce Fig. 7B.

The rest of the proof of Lemma 7 is divided into two cases:

Case A: \((N, 2N - 1) = d_1;\)

Case B: \((N, 2N - 1) < d_1.\)

**Lemma 7, Case A**

Since \(|\langle 2N - 1, 2N, 1, \ldots, N \rangle| = N + 2\), Lemmas 1 and 3 with \((N, 2N - 1) = d_1\) give \((2N - 1, N - 1) = (2N, N) = d_1, (2N, N - 1) = d_2, (2N, N - 2) = (1, N - 1) = d_3, (1, N - 2) = d_4, and so forth. The final segments of this series are shown at the top of Fig. 7C by dashed lines. Their companions from \((1, N + 2) = d_1\) as the base are
shown as solid lines. Both $N$ even and odd have the same arrangement near $(1, N)$ at the bottom of the figure. Distance labels are omitted from the top but are easily supplied. For $N$ even, $(K - 1, K) = d_N$, $(K - 2, K) = (K - 1, K + 1) = d_{N-1}$, $(K - 2, K + 1) = d_{N-2}, \ldots$; for $N$ odd, $(K - 2, K - 1) = (K - 1, K) = d_N$, $(K - 2, K) = d_{N-1}$, $(K - 2, K + 1) = (K - 3, K) = d_{N-2}$, and so forth. For $N$ even, $\perp_{K,K+1} = \perp_{K-1,K+2}$ (hence $(K, K + 1) = d_N) = \cdots = \perp_{1N}$. For $N$ odd, $\perp_{K-1,K+1} = \perp_{K-2,K+2} = \cdots = \perp_{1N}$, and $(K - 1, K + 1) = d_{N-1}$ since $(K - 1, K + 1) > (K, K + 1) = d_N$ (the $K-1, K$ cuts $(K-1, K+3)$) and $(K - 1, K + 1) < (K - 2, K + 1) = d_{N-2}$. So the top four ($N$ even) or five ($N$ odd) vertices lie on the same circle, say $\odot_1$.

Suppose $N$ is even. Since $(K - 2, K - 1) < (K - 2, K) = d_{N-1}$, we have $(K - 2, K - 1) = d_N$, so $K - 2, K + 3 \in \odot_1$ with $(K - 1, K + 2) = (K - 2, K + 1) = d_{N-2}$ by similar chords. Next, $d_{N-1} = (K - 2, K) < (K - 3, K) < (K - 3, K + 1) = d_{N-3}$, so $(K - 3, K) = d_{N-2}$ and, by congruence of $\Delta(K - 3, K, K + 1)$ with $\Delta(K - 2, K +
1, \( K + 2 \)), we have \( K - 3, K + 4 \in \odot_1 \) with \((K - 3, K + 2) = (K + 3, K + 4) = \Delta_N \). At this point,

\[(K - 2, K + 3) = (K - 3, K + 2) = \Delta_N \text{ by similar chords,}
\]

but \((K - 3, K + 4)\) is undecided. Continuing downward in a similar manner, it follows that all vertices in \( \langle 1, 2, \ldots, N \rangle \) are on \( \odot_1 \) with \( \Delta_N \) between adjacent vertices. We have \((2, N - 1) = (1, N - 2) = \Delta_4 \), but the procedure does not prescribe \((1, N)\). A similar result holds when \( N \) is odd.

Segment \( 1N \) breaks the pattern since it is \( \Delta_1 \) instead of \( \Delta_2 \) and \( \Delta_2 \) never appears in \( \langle 1, 2, \ldots, N \rangle \). This leads to a contradiction for case A as follows.

Let \( p \) be the point near 1 that is \( \Delta_3 \) from \( N - 2 \) and \( \Delta_3 \) from \( N - 1 \). Then triangles \((p, N - 2, N - 1)\) and \((1, N - 1, N)\) are congruent, so \( p \) is on \( \odot_1 \) at distance \( \Delta_N \) below 1. By our case A hypothesis of \((2N - 1, N) = \Delta_1 \), we noted earlier that \( 2N \) is \( \Delta_3 \) from \( N - 2 \) and \( \Delta_2 \) (not \( \Delta_1 \)) from \( N - 1 \). Therefore \( 2N \) is below \( p \) and

\[(1, 2N) > \Delta_N .
\]

Hence \((2N, N - 3) \neq \Delta_4 \), else congruent triangles put \( 2N \) on \( \odot_1 \) at \( \Delta_N \) below 1. Also, since \( N = \Delta(1, 2N) \), \((2N, N - 3) > (1, N - 3) = \Delta_2 \), and therefore

\[(2N, N - 3) \geq \Delta_3 .
\]

Consequently, \( \perp_{N-2, N-3} \) goes through or above \( 2N \). In addition, \( N = \Delta(2N, 2N - 1) \), so

\[(2N - 1, N - 3) > (2N - 1, N - 2) > (2N, N - 2) = \Delta_3 ,
\]

and therefore \((2N - 1, N - 3) = \Delta_1 \). However, \(|\langle N - 3, N - 2, \ldots, 2N - 1 \rangle| = N + 3\), so Lemma 1 implies the contradiction that \( m(C) \geq N + 1 \).

**Lemma 7, Case B**

This case begins with Fig. 7B and \((N, 2N - 1) < \Delta_1 \). We also assume that \((N + 2, 3) < \Delta_1 \), else case A (relabel counterclockwise) gives a contradiction. Then the \((N + 1)\)-gon \( \langle 2, 3, \ldots, N + 2 \rangle \) has the unique \( \max(2, N + 2) = \Delta_1 \), so Lemmas 1 and 2 yield the additions to Fig. 7B shown in Figs. 8A and 8B.

Suppose \( N \) is even. Because \( N = \Delta(1, 2N) \), we have \((1, N - 1) < (2, N - 1)\), so \((1, N - 1) \leq \Delta_2 \). Moreover, \((1, N - 1)\) is the only possible \( \Delta_2 \) segment in \( \langle 1, 2, \ldots, N \rangle \). For example, if \((1, N - 2) = \Delta_2 \) then \((2N, N - 2) = \Delta_1 \), but this gives a contradiction by Lemma 1 since \(|\langle 2N, 2N - 1, \ldots, N - 2 \rangle| = N + 3\). And \((2, N - 1) < (1, N - 1)\) since \( N + 2 = \Delta(1, 2) \).

Suppose in fact that \((1, N - 1) < \Delta_2 \). Then \( \langle 1, 2, \ldots, N \rangle \) has no \( \Delta_2 \) segment and uses exactly the \( N - 1 \) different distances \( \Delta_1, \Delta_3, \ldots, \Delta_N \) with unique \( \max(1, N) = \Delta_1 \). We can then use Lemma 2 on \( \langle 1, 2, \ldots, N \rangle \) to force additional known distances, namely \((1, N - 1) = \Delta_1, (2, N - 1) = \Delta_1, (2, N - 2) = \Delta_3, \ldots, (K - 1, K + 2) = \Delta_{N-2} \) and \((K, K + 1) = \Delta_N \). With \( K \) shifted by 1, this gives a picture similar to Fig. 6C and implies by our proof of Lemma 6 that \( 2, 3, \ldots, N \) lie on a circle with distance \( \Delta_N \).
between adjacent vertices. This extends to $N + 1$ since triangles $(N + 1, 3, 4)$ and $(N, 2, 3)$ are congruent. However, triangles $(N + 1, N, 2)$ and $(N, N - 1, 1)$ are not congruent since $(N + 1, 2) < (N, 1)$, and it follows from these that $(1, 2) < d_N$, a contradiction. We conclude that $(1, N - 1) = d_1$.

Although the two preceding paragraphs focus on matters clockwise from 1, a similar situation holds when we go counterclockwise from 1 around the bottom of Fig. 8A. Under this orientation, the conclusion of the preceding paragraph translates into $(1, N + 3) = d_2$. We therefore assume henceforth for $N$ even that

$$(1, N - 1) = (1, N + 3) = d_2.$$ 

Since $N + 2 = q(1, 2)$, we have $(1, N + 3) < (2, N + 3)$. Therefore $(2, N + 3) = d_1$. Since $|\langle 2, 3, \ldots, N + 3 \rangle| = N + 2$, Lemmas 1 and 3 augment Fig. 8A with additional
specified distances which yield the conclusion that 3, 4, ..., \(N + 1\) lie on a circle, say \(\bigcirc_1\), with \(d_N\) between adjacent vertices. Since
\[d_2 = (N - 1, 3) < (N - 1, 2) < (N, 2) - d_3,\]
we have \((2, N - 1) = d_4\) and therefore get \(2 \in \bigcirc_1\) by congruence of \(\Delta(2, N, N - 1)\) with \(\Delta(3, N + 1, N)\). Then \(\Delta(2, 3, N + 1)\) has sides \(d_N, d_3\) and \(d_2\). Since Lemmas 1 \(\text{and} 3\) applied to \((2, N + 3) = d_1\) give \((4, N + 2) = d_3\) as well as \((3, N + 2) = d_2\), \(\Delta(3, 4, N + 2)\) has sides \(d_N, d_3\) and \(d_2\), and therefore \(N + 2 \in \bigcirc_1\). Then \(N, N + 1\) and \(N + 2\) are all on \(\bigcirc_1\). However, since vertex 1 is equidistant from all three, this implies that vertex 1 is the center of \(\bigcirc_1\), which is absurd.

We have thus arrived at a contradiction when \(N\) is even. When \(N\) is odd, so that Fig. 8B applies, we obtain the same contradiction under straightforward modifications that account for the slightly different arrangement at the top of the figure.

This concludes our proof of Lemma 7. \(\square\)

Because \(M_{2N}(N)\) contains a third polygon in addition to \(R_6\) and \(R_7 - 1\) when \(N = 3\) (see Fig. 1), the preceding proofs of this section do not fully apply to this case. However, they apply for all \(N \geq 4\). Moreover, lest there be any question about \(N = 4\), we have noted at the outset of the section that separate verification for \(N = 4\) follows almost immediately from the result for \(M_4(4)\) proved in Section 5.

4. Pentagons in \(M_5(3)\)

We begin our proof that \(M_5(3)\) contains 15 pentagons by adding vertex 5 to the four vertices \((1, 2, 3, 4)\) of a member of \(M_4(2)\). To get \(m = 3\), vertex 5 must be equidistant from two others.

Convexity with \(A_4\) requires \(5 = d_{12}\), where 12 is a side of \(A_4\). Convexity also requires \(15 \leq 12\), but then either 35 or 45 gives \(m \geq 4\).

Label \(B_4\)'s vertices as in Fig. 9A. Either \(5 = d_{12}\) or \(5 = d_{13}\). The former gives (5.3) of Fig. 2 with \(5 = x\); no other \(5 \in \perp_{12}\) has \(m = 3\). The latter gives (5.1) of Fig. 2 at \(5 = y\); no other \(5 \in \perp_{13}\) above 12 has \(m = 3\).

For \(R_4\), take \(5\) on \(\perp_{12}\) away from side 12: see Fig. 9B. We get (5.8) at \(5 = x\), (5.2) at \(5 = y\), and (5.9) at \(5 = z\).

Six positions on \(\perp_{12}\) in \(R_5 - 1\) provide candidates for \(M_5(3)\): see Fig. 9C. Three are unsuitable: \(R_5\) has \(m = 2\), \(u\) lies inside \(R_5 - 1\), and 2 lies on \(t4\). We get (5.6) at \(5 = x\), (5.5) at \(5 = y\), and (5.7) at \(5 = z\). It is easily seen that \(5 \in \perp_{13} \cup \perp_{23}\) forces \(m \geq 4\).

We conclude that precisely eight members of \(M_5(3)\) include a member of \(M_4(2)\):
(5.1)-(5.3) and (5.5)-(5.9). Two of these, (5.3) and (5.6), have \(d_1\) as a side. We examine this possibility further.

With \(d_1 > d_2 > d_3\), suppose pentagon \(C\) has \(d_1\) as a side and \(m(C) = 3\). By Lemmas 1 \(\text{and} 3\), it has the configuration of Fig. 9D, but perhaps with 12 \parallel 34. Suppose 12 \parallel 34 as shown in D. If \(13 = d_3\) then \(C = (5.3)\) if \(15 = d_1\), \(C - (5.13)\) if \(15 = d_2\), and \(15 = d_3\) is
impossible. If \(13 = d_2\) then \(15 > d_2\) and \(C = (5.6)\) if \(15 = d_1\). Alternatively, suppose that \(12 \parallel 34\) and assume that \(24 < 13\) so that \(13 \in \{d_1, d_2\}\). If \(13 = d_1\) then \(\Delta 123\) is equilateral and \(C = (5.4)\). If \(13 = d_2\), we get Fig. 9E which requires \(25 \in \{d_1, d_2\}\). If \(25 = d_2\) then 2 through 5 are on a circle \([0.3], Lemma 01\), but this is impossible since \(15 \parallel d_1\) and we conclude that 1 is both the center and not the center of the circle. If \(25 = d_1\) then, since \(13 < 15\), \(15 = d_1\) and equilateral \(\Delta 125\) forces a contradiction of \(13 = d_2\).

The preceding paragraph adds (5.4) and (5.13) to our list for \(M_5(3)\). Assume henceforth that no side of \(C\) has length \(d_1\). Then some 4-gon in \(C\), say \(C^*\), has side \(d_1\). Let \(12\) be a \(d_1\) side of \(C^*\) and let 3 and 4 be its other vertices. Vertex 5 of \(C\) is on the opposite side of 12 from 3 and 4. Since \(m(C^*) = 2\) was covered by the first part of this proof, assume \(m(C^*) = 3\) henceforth. Assume also that either \(C^*\) has another \(d_1\) segment or \(d_1 \in \{53, 54\}\), since otherwise \(C\) includes a 4-gon with \(m = 2\).

Suppose 12 is the only \(d_1\) segment in \(C^*\). Then Lemmas 1 and 2 imply the arrangement of Fig. 9F. We consider \(12 \parallel 34\) and \(12 \parallel 34\) in turn.
Suppose $12 \parallel 34$. Then $23 = 14 = d_3$ as shown in Fig. 9G, and we can presume that $5$ lies on one of the noted $\perp$ with $d_1 \in \{35, 45\}$ and $\max(15, 25) \leq d_2$. We examine the three $\perp$ possibilities.

Case 1: $5 \in \perp_{12}$ in Fig. 9G. Then $35 = 45 = d_1$. If $15 = d_2$, then $C = (5.14)$. If $15 = d_3$ then, by Lemma 0, 4152 are on one circle, as are 1523, and it must be the same circle so we get the contradiction to $m(C) = 3$ that $C = R_5$.

Case 2: $5 \in \perp_{13} \setminus \perp_{12}$ in Fig. 9G. If $5$ is left of $\perp_{12}$ then $35 > 45$, so $35 = d_1$, hence $15 = d_1$, a contradiction. If $5$ is right of $\perp_{12}$ then $35 < 45$ and $25 < 15$, so $45 = d_1$, $25 = d_3$ and $15 = 35 = d_2$. It follows that $C = (5.10)$.

Case 3: $5 \in \perp_{14} \setminus \perp_{12}$. If $5$ is right of $\perp_{12}$ then $35 < 45$ so $45 = d_1$ and $15 = d_1$, a contradiction. If $5$ is left of $\perp_{12}$ then $35 = d_1$, $25 = d_2$ and $15 = 45 = d_3$. But then $1, 5, 2$ and $3$ form a rectangle (parallelogram with equal diagonals), which is impossible because $1513 < 1514 \neq 60^\circ$.

Suppose for Fig. 9F that $12 \parallel 34$ with $23 < 14$. Then $23 = d_3$ and $14 = d_2$: see Fig. 9H. By convexity, 5 is right of $\perp_{34}$, so $35 < 45$ and $45 = d_1$. In addition, $35 = d_2$, $15 = d_3$ and $25 = d_2$. But then triangles $135, 253$ and $314$ are congruent, $23 \parallel 1134$, and therefore 3 lies on 24, a contradiction of convexity.

Our supposition that 12 is the only $d_1$ segment in $C^*$ yields (5.10) and (5.14) for $M_5(3)$.

Assume finally that every 4-gon in $C$ that has a $d_1$ side also has another $d_1$ segment which, by (0.7) of Lemma 0, shares a vertex with the $d_1$ side. We continue to assume that $C$ has no $d_1$ side and every 4-gon in $C$ has $m = 3$. For definiteness let $C^*$ on $\{1, 2, 3, 4\}$ have $d_1$ side 12 and diagonal $13 = d_1$. Then either $24 = d_1$ or $35 = d_1$, else one of the stated assumptions fails. With respect to $C$, $24 = d_1$ and $35 = d_1$ form similar patterns in the $d_1$ structure, so we assume without loss of generality that $24 = d_1$: see Fig. 9I. Because 5 lies above the perpendicular bisectors of 14 and 23, $15 < 45$ and $25 < 35$. If any more segments of $C$ are $d_1$, they could only be 35 or 45. We consider the possibilities.

Case 1: $35 = 45 = d_1$. Since all diagonals are $d_1$, three sides are $d_2$ or $d_1$ and the other two are $d_3$ or $d_2$. The three sides with the same $d_1$ cannot be consecutive, else some 4-gon has $m = 2$. We can therefore presume that either

(a) $15 = 23 = 34 = d_3$ and $14 = 25 = d_2$, or
(b) $15 - 23 - 34 - d_2$ and $14 - 25 - d_3$.

Both are impossible because when all diagonals are $d_1$ and $15 = 23 = 34$, we must have $R_5$. The hypotheses of the preceding sentence imply that $14 \parallel 35$ and $\perp_{35} = \perp_{14}$. Since $\perp_{14}$ goes through 2, it follows that $25 = 23$. Hence all five sides are equal.

Case 2: $35 \neq d_1 \neq 45$. Then $15 = 25 = d_3$ and $45 = 35 = d_2$. It follows that $12 \parallel 34$, so $14 = 23$. If $23 = d_2$ then isosceles triangle 235 implies that 3 is left of the perpendicular to 12 that is $3/4$ of the way from 1 to 2; symmetrically, 4 is right of the perpendicular to 12 that is $1/4$ of the way from 1 to 2. Therefore $34 < (1/2)d_1 < d_3$, a contradiction. Hence $14 = 23 = d_3$. Then, because no 4-gon has $m = 2$, we must have $34 = d_2$ for $\{1, 2, 3, 4\}$. The result is (5.12).

Case 3: $35 < 45 = d_1$. Then $25 = d_3$ and $35 = d_2$. Because $4 \in \perp_{25}, 23 < 35$ and
therefore $23 = d_3$ and $14 \parallel 35$. If $14 = d_4$ then our assumption of $m = 3$ for 4-gons implies that $15 = 34 = d_2$. The result is (5.15). If $14 = d_3$ then the 4-gon requirement for \{1, 3, 4, 5\} yields $15 = 34 = d_3$, which gives (5.11) of Fig. 2.

This exhausts the possibilities. Our final cases add (5.11), (5.12) and (5.14) to $M_5(3)$, thus completing the 15 pentagons of Fig. 2 that comprise $M_5(3)$.

5. Heptagons for $M_5(4)$

Fig. 10 pictures the five heptagons of $M_5(4)$ identified in Theorem 3. To prove that there are no others, label the vertices of $C \in \mathcal{C}_7$ clockwise from 1 through 7 and assume that $m(C) = 4$. No side of $C$ is a $d_i$ segment since this implies $m \geq 5$ by Lemma 1. We partition the other possibilities into three main parts:

Part I. $d(1, 3) = d_1$.

Part II. $d(1, 4) = d_1$, 14 is the unique $d_i$ segment in $\{4, 5, 6, 7, 1\}$, and $\max\{13, 24, \ldots, 61, 72\} \leq d_2$.

Part III. $d(1, 4) = d(1, 5) = d_1$, $\max\{13, 24, \ldots, 72\} \leq d_2$, and no 5-gon of consecutive vertices has a unique $d_i$ segment that is a side of the 5-gon.

We consider Parts I, II and III in sequence. As in Fig. 10, we use a solid line for $d_1$, a long-dashed line for $d_2$, a short-dashed line for $d_3$, and a dotted line for $d_4$.

Part I

With $13 = d_1$, Lemmas 1 and 3 give the distances configuration of Fig. 11A. We partition Part I into three cases:

Case A: 14 is the only $d_i$ in $\{4, 5, \ldots, 1\}$,
37 is the only $d_i$ in $\{3, 4, \ldots, 7\}$;

Case B: 14 is the only $d_i$ in $\{4, 5, \ldots, 1\}$,
37 is not the only $d_i$ in $\{3, 4, \ldots, 7\}$;

Case C: 14 is not the only $d_i$ in $\{4, 5, \ldots, 1\}$,
37 is not the only $d_i$ in $\{3, 4, \ldots, 7\}$.

Case A. Lemmas 1 and 2 along with $m = 4$ for the whole heptagon imply that $15 = 36 = d_2$ and $67 = 45 = d_4$. Then $13 \parallel 47 \parallel 56$, and it follows that $16 = 35 = d_3$ and $17 = 34 = d_4$, so 1, 7, \ldots, 3 are equally spaced on a circle whose center lies between 13 and 14. Suppose for definiteness that vertex 2 is on or left of $\perp_{13}$. If 2 is left of $\perp_{13}$

Fig. 10. $M_5(4)$.
then $21 < 27 < 26 < 25$, so $m = 4$ requires $21 = d_4$, $27 = d_3$, $26 = d_2$ and $25 = d_1$, with 2 on the circle of the others. By analogy with 1 versus 3 and 4, $24 = d_1$; by analogy with \( \Delta 367 \), $23 = d_2$. It follows that the heptagon is $R_9 - 2_1$ of Fig. 10. If vertex 2 $\in$ $\perp 13$ then $21 = 23 < 27 = 24 < 26 = 25$. To satisfy $m = 4$, it is easily checked that $d_4$ cannot be used for $21$, so $21 = d_3$, $27 = d_2$ and $26 = d_1$. The result is $R_9 - 2_2$ of Fig. 10.

**Case B.** The hypothesis for $\langle 3, 4, \ldots, 7 \rangle$ implies $36 = d_1$: if $35 = d_1$, Lemmas 1 and 3 also give $36 = 25 = d_1$. Since 14 is the only $d_1$ in $\langle 4, 5, \ldots, 1 \rangle$, $15 = d_2$ and $67 = d_4$. Also, $67 < 16 < d_1$. If $16 = d_2$ then $45 < 46$ ($1 =_d 56$) so $45 = d_4$; but then 4, 5, 6 and 7 are on a circle, which contradicts $14 = d_1$ in conjunction with $15 = 16 = d_2$. We conclude that $16 = d_3$; see Fig. 11B. Then $14 \parallel 75$ and $17 = d_4 = 45$. So now 1, 7, 6, 5 and 4 are on a circle whose center is at vertex 3 in view of $31 = 37 = 36 = d_1$. But this implies that side $34 = d_1$, a contradiction to Lemma 1 applied to the whole polygon. Hence this case is impossible.

**Case C.** The reasoning in the first sentence of the preceding paragraph gives $36 = 15 = d_1$. Since $1 =_d 45$, $45 < 46$ and therefore $45 = d_4$. Similarly, $67 = d_4$. Then $74 \parallel 65 \parallel 13$ so $17 = 34 < 16 = 35 < d_1$. If $16 = d_3$, we get the contradiction that 1, 3, $\ldots$, 7 lie on a circle with centers at 1 and 3. Hence $16 = d_2$: see Fig. 11C. Suppose $17 = d_4$. 

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**Fig. 11. Constructions for $M_r(4)$.**
Then the five \( d_1 \) segments and the five \( d_4 \) sides determine the hexagon up to similarity transformations, and in this case it turns out that \( 16 < 74 \), a contradiction. Hence \( 17 = 34 = d_3 \). Ignoring alleged \( d_2 \) segments, the others determine the hexagon up to similarity, and in this case \( 74 < 16 \). So case C is impossible.

**Part II**

We now assume that no two vertices adjacent to a third vertex are \( d_1 \) apart, and that 14 is the unique \( d_1 \) segment in \( \langle 4, 5, 6, 7, 1 \rangle \); see Fig. 11D, as required by Lemmas 1 and 2. We divide Part II into three cases according to potential lengths for 45.

Case A: \( 45 = d_2 \). If \( 16 = d_2 \) then \( 46 = d_1 \), a contradiction. Therefore 15 is the unique \( d_2 \) in \( \langle 5, 6, 7, 1 \rangle \), and it follows from Lemmas 1 and 2 that \( 16 = d_3 \). If \( 17 = d_3 \) also then \( \Delta 175 \) is isosceles with a suitably large \( \varepsilon \) to accommodate \( 16 = d_3 \) convexly. However 47, with \( 45 = d_2 \) and \( 41 = d_1 \), will then exceed \( d_2 \), a contradiction. (Alternatively, with \( 47 = d_2 \) also, \( \Delta 475 \) forces 4 left of \( \perp 15 \), while \( \Delta 145 \) forces 4 right of \( \perp 15 \).) Therefore \( 17 = d_4 \), so 1, 7, 6 and 5 lie on a circle with center on \( \perp 15 \) and 4 to the right of \( \perp 15 \); see Fig. 11E. Because 4 is right of \( \perp 15 \) and \( 76 \| 15, 46 < 47 \) and the only possibility is \( 46 = d_3 \). Let \( \theta \) denote the interior angle of a triangle with sides \( d_1, d_2 \) and \( d_3 \) where \( d_2 \) meets \( d_3 \). Then \( \zeta 465 \theta \) and \( \zeta 165 \theta \), a contradiction. We conclude that \( 45 < d_2 \) and symmetrically (see Fig. 11D) that \( 17 < d_2 \).

Case B: \( 45 = d_3 \). This gives Fig. 11F with \( 17 \in \{d_3, d_4 \} \). We consider the possibilities for 17.

Case B1: \( 17 = d_3 \). Then \( 14 \| 57 \) so \( 16 = 46 = d_2 \); see Fig. 11G. At the top, we assume without loss of generality that 2 is left of \( \perp 14 \). Then since \( \perp 75 = \perp 14, 27 < 25 \); since \( 1 = d \) 56, \( 25 < 26 \). Hence \( 27 < 25 < 26 \). We cannot have \( 25 = d_2 \) and \( 27 = d_3 \) (else 5 and 7 would both be on \( \perp 12 \)), and therefore \( 27 = d_4 \), which is shorter than \( 17 = d_3 \). Consequently, we cannot have \( 26 > d_2 \) since the circles centered at 7 with radius \( d_4 \) and at 6 with radius \( d_2 \) or \( d_1 \) do not intersect above 14. Hence \( 26 < d_3 \). But then \( d_4 < 25 < d_3 \), a contradiction to \( m = 4 \). So case B1 is impossible.

Case B2: \( 17 = d_4 \). Suppose \( 16 = d_2 \), approximately like Fig. 11F. Then \( 46 > 45 = d_3 \) since \( 1 = d \) 56, so \( 46 = d_2 \) and \( \Delta 164 \) is isosceles. Because \( 15 = 47 \) and \( 65 = 67 \), \( 14 \| 57 \) with \( \perp 14 = \perp 57 \). But then \( 17 = 45 \), a contradiction. Suppose \( 16 = d_3 \); see Fig. 11H. Then \( 15 \| 67, \perp 67 = \perp 15 \) and this perpendicular bisector goes left of 4, so \( 46 < 47 = d_2 \). Hence \( 46 < d_3 \). Comparison between quadrilaterals 1765 and 7654 shows that \( 46 > d_3 \), so \( 16 \neq d_3 \). Suppose finally for case B2 that \( 16 = d_4 \). Then \( \Delta 167 \) is equilateral and forces \( 74 > d_1 \). Hence case B2 is impossible.

We conclude that \( 45 < d_3 \). Symmetrically, \( 17 < d_3 \).

Case C: \( 45 = 17 = d_4 \). Fig. 11D gives an approximate picture, with \( 16 \in \{d_2, d_3 \} \). Suppose \( 16 = d_2 \). Assume for definiteness that 2 is left of \( \perp 14 = \perp 75 \). Then \( 27 < 25 < 26 \). Since circles of radius \( d_4 \) centered at 7 and radius \( d_2 \) or \( d_1 \) centered at 6 do not cross above 14, \( 27 \neq d_4 \). Therefore \( 27 = d_3 \), \( 25 = d_2 \) and \( 26 = d_1 \); see Fig. 11I. If \( 12 = d_4 \), then, considering 7 also, \( 16 \| 25 \) and \( \perp 16 = \perp 25 \), which contradicts \( 26 \neq 15 \). Since \( 12 < 24 < 14 \), we conclude that \( 12 = d_3 \) and \( 24 = d_2 \). But then the perpendicular
bisectors of 17 and 45 cross to the left of \( \perp_{14} \), a contradiction to the symmetry of 176 and 456 around \( \perp_{14} \). Hence 16 \( \neq d_1 \), so 16 = \( d_3 \).

We continue case C with 45 = 17 = \( d_4 \) and 16 = 46 = \( d_5 \); see Fig. 12A, with 1, 7, 6, 5 and 4 on a circle, say \( \odot_1 \). This will induce \( R_8 = 1 \) and \( R_9 - 2 \) but nothing else in \( M_7(4) \).

Assume for definiteness henceforth in Part II that 2 is left of \( \perp_{14} \) with 12 < 24 \(< d_2 \) since 24 \( \neq d_2 \) by hypothesis. We consider 2 when 12 = \( d_2 \) and then when 12 = \( d_3 \).

Case C1: 12 = \( d_2 \). If 27 = \( d_3 \) or 26 = \( d_2 \) or 25 = \( d_1 \) then 2 is on \( \odot_1 \) with 27 = \( d_3 \), 26 = \( d_2 \), and 25 = \( d_1 \). Suppose 2 \( \notin \odot_1 \). Then 25 \( \leq d_2 \) and 27 \( \neq d_3 \) and, since 27 < 25 in any case, 27 = \( d_4 \). Since circles with radii \( d_4 \) centered at 7 and \( d_3 \) centered at 6 do not intersect above 14, 27 = \( d_4 \) forces 26 = \( d_4 \). This implies the contradiction that 6 lies above 75. Hence 2 \( \in \odot_1 \). If 24 = \( d_3 \) then, since 17 \( \parallel 26 \) and 46 = \( d_3 \), 4 would lie on \( \perp_{14} \), contrary to 14 \( \neq 74 \). Therefore 24 = \( d_2 \). In summary:

\[
12 = d_4 \implies 27 = d_3 , \quad 26 = 24 = d_2 , \quad 25 = d_1 .
\]

Because 14 lies on \( \perp_{27} \), the center of \( \odot_1 \) is at the intersection of 14 and \( \perp_{14} \). We also have \( \angle 275 = \pi/2 \) and \( \angle 164 = \pi/2 \), and it follows that \( \langle 2, 1, 7, 6, 5, 4 \rangle \) is the instance of \( R_8 - 2 \) in which the two vertices removed from \( R_8 \) are adjacent. By analogy, placement of 3 on \( \perp_{14} \) with 32 = \( d_2 \), or to the right of \( \perp_{14} \) with 34 = \( d_4 \), yields \( R_8 - 1 \). The only other possibility for vertex 3 is on \( \perp_{24} \) with 32 = 34 = \( d_3 \), 31 = 35 = \( d_2 \), and 37 = 36 = \( d_1 \). However, these equations are jointly infeasible given \( R_8 - 2 \), so 12 = \( d_4 \) implies that heptagon C is \( R_8 - 1 \).

Case C2: 12 = \( d_3 \). Then 24 = \( d_2 \) since 12 < 24 < \( d_1 \), and 27 \( \leq d_2 \) by hypothesis; see Fig. 12B. Suppose 2 \( \notin \odot_1 \). Then 27 \( \neq d_2 \), 26 < \( d_1 \) and 25 < \( d_1 \) (else \( \triangle 245 \) is congruent

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Fig. 12. More constructions for \( M_7(4) \).
to \( \Delta 471 \). Since \( 27 \neq d_4 \) and \( 27 < 25 \), it follows that \( 27 = d_3 \) and \( 25 = d_2 \). We cannot have \( 26 \leq d_3 \) since appropriate radii circles centered at 6 and 5 would not cross above 14. Therefore \( 26 = d_2 \), and \( 24 = 25 = 26 = d_2 \) gives 2 as the center of \( \odot 1 \). But this contradicts the fact that the center is on \( \perp_{14} \). We conclude that \( 2 \in \odot 1 \) with \( 27 = d_2 \) and \( 26 = 25 = d_1 \).

Finally, consider vertex 3 for case C2. If 3 is left of \( \perp_{14} \), preceding analyses contradict membership in \( \mathcal{M}_7 \). If \( 3 \in \perp_{14} \), we have \( 23 < 31 = 34 < 37 = 35 < 36 \) (the last by \( 2 = d_5 \)). So \( 23 = d_4 \), \( 31 = 34 = d_3 \), and so forth. Because \( 12 = 13 = d_3 \), 3 lies below 2. On the other hand, analogy between \( \Delta 461 \) and \( \Delta 613 \) implies that \( 3 \in \odot 1 \), a contradiction. Hence 3 is right of \( \perp_{14} \). If \( 34 = d_4 \), then the analysis for case C1 gives a contradiction, so \( 34 = d_4 \). By analogy to the preceding paragraph, 3 is on \( \odot 1 \) with \( 31 = d_2 \), \( 35 = d_2 \), and \( 36 = 37 = d_1 \). Because 2, 3 \( \in \odot 1 \), comparison of \( \Delta 123 \) and \( \Delta 571 \) shows that \( 23 = d_4 \) and hence that heptagon \( C \) is \( R_9 - 23 \).

**Part III partition**

The preceding proofs show that the only members of \( \mathcal{M}_7 \) that satisfy the hypotheses of Part I or Part II are \( R_8 - 1 \), \( R_9 - 2 \), \( R_9 - 2_1 \) and \( R_9 - 2_2 \). For Part III we assume that nothing in \( \{13, 24, \ldots, 72\} \) is \( d_1 \) and no 5-gon of consecutive vertices has a \( d_1 \) side that is the only \( d_1 \) segment in the 5-gon. With no loss of generality assume \( 14 = d_1 \) and, for the 5-gon requirement, that \( 15 = d_1 \). There may be other \( d_1 \) segments, so we consider exhaustive cases as follows:

- **Case A**: 14 and 15 are the only \( d_1 \) segments;
- **Case B**: \( 14 = 15 = 36 = 37 = d_1 \), and there are no other \( d_1 \) segments: see Fig. 12C;
- **Case C**: \( 14 = 15 = 47 = d_1 \) and there may be other \( d_1 \) segments, but no four have the pattern of Fig. 12E: see Fig. 12D.
- **Case D**: \( 14 = 15 = 47 = 25 = d_1 \) (see Fig. 12E) and there may be other \( d_1 \) segments.

We shall argue that cases A, B and C do not contribute to \( \mathcal{M}_7 \) and that case D adds only \( R_9 - 2_4 \) to the four others in \( \mathcal{M}_7 \).

**Case III-A**

Suppose with \( m(C) = 4 \) that only 14 and 15 are \( d_1 \) segments. Remove vertex 1 to get a hexagon in \( \mathcal{M}_6 \), which could only be \( A_6 \), \( R_6 \) or \( R_7 - 1 \). It is easily seen to be impossible to add a seventh vertex to anything in \( \mathcal{M}_6 \) that is equidistant from a pair of adjacent vertices and adds only one new (and longest) distance to the other three. So the hypotheses of this case cannot occur.

**Case III-B**

Suppose \( m(C) = 4 \) and the \( d_1 \) segments are 14, 15, 36 and 37. Remove 1 and 3 to get a pentagon in \( \mathcal{M}_5 \), which by the proof in the preceding section must be one of (5.1) through (5.15) on Fig. 2. Examination of each of these shows that it is impossible to add
two vertices in the manner of 1 and 3 of Fig. 12C. More specifically, if a line through 
(5, k) cuts off one vertex from the other four, and if two points on this line are 
equidistant with the same new longest distance from pairs of those four in the manner of 
Fig. 12C, then either the resulting figure violates convexity or a second new distance 
arises. Hence III-B is impossible.

Case III-C

We assume that 14, 15 and 57 are \(d_1\) segments. Since the pattern in Fig. 12E is 
forbidden, the only possibilities for other \(d_1\) segments are 26 and 36.

With reference to Fig. 12D we have 23 < 13 (since \(25 \leq d_2\), \(1 \perp 12\) intersects \(15\) 
between 1 and 5), 23 < 24, 13 < 37 and 24 < 25, so 
\[23 - d_4, \quad 13 - 24 - d_3 \quad \text{and} \quad 37 - 25 - d_2.\]

Since 24 < 34 implies that 2 is below 3 (14 assumed horizontal), and 13 < 12 implies 
that 3 is below 2, at least one of 34 < 24 and 12 < 13 holds. Assume 34 < 24 for 
definiteness, so 34 = \(d_4\). We partition the possibilities for 12 versus 13.

Case III-C1: 12 < 13. Then 12 = \(d_4\), 23 \(\parallel\) 14 \(\parallel\) 75, and 35 = 27 = \(d_3\) because 34 < 35 < 25. Vertices 1, 2, 3 and 4 are on a circle. If 45 = 17 = \(d_4\) then 7 and 5 are on 
the same circle, which contradicts \(14 \neq 25\) (similar chords). Hence 45 = 17 = \(d_3\) since 
46 > 45, and we also have 46 = 16 = \(d_2\) with 6 \(\in\) \(\perp\) 14. Then since 5 \(\in\) \(\perp\) 34 and 6 is left 
of 5, we require 36 = 26 < 46 = \(d_2\). But this forces 6 above 57, contradicting convexity: 
see Fig. 12F.

Case III-C2: 13 < 12. As above, 23 = 34 = \(d_4\), 13 = 24 = \(d_3\) and 37 = 25 = \(d_2\). Now 
\(d_3 = 13 < 12 < 27 < d_2\), so 12 = \(d_3\) and 27 = \(d_2\). But then both 1 and 6 are on \(\perp 23\), a 
contradiction.

Hence case III-C is impossible.

Case III-D

Suppose \(m(C) = 4\) and the \(d_1\) segments include 14, 15, 25 and 47 as in Fig. 12E. 
Inequalities among other segments include 
\[23 < 13 < 37, \quad 67 < 16 < 26, \quad 34 < 35 < d_2 \quad \text{and} \quad 23 < 24 < d_3,\]

but none of these force specific \(d_i\) on segments. Other possible \(d_i\) segments are 26, 37 
and 36. By our 5-gon restriction of Part III, \(36 = d_1 \Rightarrow (26 = d_1 \text{ or } 37 = d_1)\). Taking 
account of relabeling, there are four distinct possibilities for other \(d_1\) segments:

III-D1: None of 26, 37 and 36 is \(d_1\);
III-D2: \(26 = d_1 \text{ and } \max(37, 36) < d_2\);
III-D3: \(26 = 37 = d_1 \text{ and } 36 < d_2\);
III-D4: \(26 = 37 = 36 = d_1\).

Case III-D1. The inequalities given above yield 23 = 67 = \(d_4\), 13 = 16 = \(d_3\) and 
37 = 26 = \(d_2\). Consider pentagon \(\langle 6, 7, 1, 2, 3\rangle\), which has no \(d_1\) segment, versus the 
members of \(M_5(3)\) on Fig. 2. We see there that 36 = \(d_4\) is impossible, 36 = \(d_3\) implies
Fig. 13. Penultimate constructions for $M_7(4)$.

(5.10) or (5.12), and, in view of Lemmas 1 and 3, $36 = d_2$ implies (5.13), with vertex 1 at the top in each possible (5.5). In each of these three (5.k), it is impossible to add two vertices at the bottom which have the presumed new longest $d_1$ distance to top pairs and use only original distances for other new segments. For example, in (5.13), which is an $R_7 - 2$, the perpendicular bisector of the top point and a neighbor goes through a missing vertex of $R_7$, and the new vertex on that segment will lie outside the circle of the pentagon and force a new distance besides $d_1$. Hence case III-D1 is impossible.

Case III-D2. Given $26 = d_1$, $37 \leq d_2$ and $36 \leq d_2$, we have $23 = d_1$, $13 = d_3$, and $37 = d_2$ since $23 < 13 < 37$. Because $2 = d_5$, $35 < 36$ and therefore $(34, 35, 36) = (d_4, d_3, d_2)$. It follows that $24 \parallel 15 \parallel 67$, hence that $12 < 13$, so $12 = 45 = d_4$: see Fig. 13A. Suppose $24 = d_3$. Then 1, 2, 3, 4 and 5 are on a circle whose center lies on $\perp_{15}$ between 14 and 15, and $\perp_{23}$ goes through the center and to the left of vertex 6. Since we would have $56 = d_4$ if 36 were $d_1$ instead of $d_2$, it follows that $56 < d_4$, a contradiction. Hence $24 = d_2$. If $46 = 27 = d_2$ so that $6 \in \perp_{34}$ and $7 \in \perp_{23}$, we conclude that $67 < d_4$. Hence $46 = 27 = d_3$ and, since $46 > 56$, $56 = 17 = d_4$: see Fig. 13B. Despite appearances, $45 \parallel 36$. If $57 = d_2$, then $45 \parallel 27$ (because 2 is $d_2$ from 4 and $d_1$ from 5, and 7 is $d_2$ from 5 and $d_1$ from 4) and $\perp_{45} = \perp_{36} = \perp_{27}$, which contradict the unequal diagonals of quadrilateral 2367. Hence $57 = 16 = d_3$ and $67 = d_4$. It follows that 3, 4, 5, 6, 7, 1 and 2 are on the same circle, equally spaced, and this contradicts $24 = d_2 \neq 35 = d_3$. Hence case III-D2 is impossible.

Case III-D3. Given $26 = 37 = d_1$ and $36 \leq d_2$, the $d_1$ segments imply $45 < 46 < 36$, $45 < 35 < 36$, $56 < 46$ and $34 < 35$, so

\[ 45 = 56 = 34 = d_4, \quad 35 = 46 = d_3, \quad \text{and} \quad 36 = d_2. \]
Fig. 14. Final constructions for $M_7(4)$.

with $\perp_{36} = \perp_{45} = \perp_{27}$ and 3, 4, 5 and 6 on a circle, say $\odot_1$: see Fig. 13C. Since $23 < 13$, 23 is $d_3$ or $d_4$. Suppose $23 = d_3$. Then $67 = d_3$ and $24 = 57 = d_2$, so by congruent triangles ($\triangle 234$, $\triangle 356$), 2 and 7 are on $\odot_1$. In addition, $23 = d_3$ implies $13 = 16 = d_2$. Then 1 is on $\odot_1$ by congruence of $\triangle 154$ and $\triangle 265$. It follows that $17 = 12 = d_4$ and that the resulting heptagon is $R_9 - 2_4$.

Suppose $23 = d_4$ for case III-D3. Then vertices 2 and 7 are not on $\odot_1$ because $63 \neq 52$ (cf. $\triangle 653$ and $\triangle 542$). Therefore $24 \neq d_3$, so $24 = 57 = d_2$. We require $27 \in \{d_2, d_3\}$: see Fig. 13D for $27 = d_2$. In either case $\angle 763 < \pi/2$ and $\angle 457 > \pi/2$. However, by side lengths, $\triangle 763$ and $\triangle 457$ are supposedly congruent. So case III-D3 with $23 = d_4$ is impossible.

Case III-D4. Our final supposition is that all seven diagonals whose end points are not adjacent to a third vertex are $d_1$ segments. Each side is $d_3$ or $d_4$, and each non-$d_1$ diagonal is $d_2$ or $d_3$. The latter must be $d_2$ if either side in its outer triangle is $d_3$. We divide the proof of III-D4 according to the maximum number $s$ of consecutive $d_4$ sides.

Case D4a: $s \leq 1$ or $s \geq 6$. If $s \leq 1$ then every non-$d_1$ diagonal is $d_2$ and $C = R_7$. If $s = 7$, $C = R_7$; if $s = 6$, it is easily seen that exactly one $d_3$ side gives a contradiction.

Case D4b: $s = 5$. Then the other two sides are adjacent $d_3$ sides: see Fig. 14A. We have $13 \parallel 74 \parallel 65$ with $75 = 46$ and $16 = 35$. Suppose $46 = d_2$, so $75 = d_2$ also. Let $\odot_1$ be the circle on which 7, 6, 5 and 4 lie. Because triangles 643 and 754 are congruent, $3 \in \odot_1$, so 35 and 16 are $d_2$ segments. Then all seven non-$d_1$ diagonals are $d_2$, which gives the contradiction that $C = R_7$. Suppose $46 = d_3$. By analogy to the preceding subcase, 1, 7, . . . , 4 and 3 lie on a circle with $16 = 75 = 64 = 53 = d_3$. Then, because triangles 763 and 652 are congruent, 2 is on the same circle and, considering $\{1, 7, 6, 4, 3\}$ versus $\{7, 6, 5, 3, 2\}$, forces the contradiction that $23 = d_4$. 

Case D4c: $s = 4$. This gives Fig. 14B in which 23 might be either $d_4$ or $d_3$, and 23 $\parallel$ 14 $\parallel$ 75. Suppose 23 $\parallel$ $d_4$. Then 12 $\parallel$ 37 and 36 $\parallel$ 27. If 16 $= d_2$ then 46 $\parallel$ 12, so 46 $\parallel$ 37, a contradiction. Hence 16 $= 46 = d_3$. However, this puts 4 equidistant from 3 and 6 which, in view of $\perp_{27} = \perp_{35}$, contradicts 24 $\neq 74$. Suppose 23 $= d_3$. Then 1, 2, 3 and 4 lie on a circle. Comparison of $\Delta 124$ and $\Delta 235$ shows that 5 is also on the circle, which forces the contradiction that 45 $= d_3$.

Case D4d: $s = 3$. This gives Fig. 14C with $\perp_{24} = \perp_{15} = \perp_{76}$, so 23 $= 34$. Suppose 23 $= d_3$. Then 24 $= d_2$ and 1, 2, 3, 4 and 5 are on a circle. It follows from congruent triangles (cf. $\Delta 145$ and $\Delta 734$) that 7 and 6 are also on the circle. This forces 17 $= d_3$, a contradiction. Suppose 23 $= d_4$. Then 13 $\parallel$ 47. Since 12 $> 23$, this gives 72 $> 24$, hence 24 $= d_3$. Then, since $\perp_{12} = \perp_{37} = \perp_{46}$, 16 $= 57 = d_3$: see Fig. 14D. We have 26 $\parallel$ 35 with $\perp_{26} = \perp_{35}$. Since 1 $= d_2$, 6 $\in \perp_{26}$, and this contradicts 13 $\neq 15$.

Case D4e: $s = 2$. This gives Fig. 14E with $\perp_{23} = \perp_{14} = \perp_{57}$, 6 $\in \perp_{14}$ and not all of 12, 23 and 34 $d_4$'s since otherwise we have the preceding case D4d. If 12 $= d_4$ then $\perp_{17} = \perp_{26} = \perp_{35}$, which forces 23 $= d_4$. But then 12 $= 23 = 34 = d_4$. Therefore 12 $= d_3$, so 13 $= 24 = d_2$. This gives 3 $\in \perp_{15} = \perp_{24}$, so 23 $= 34 = d_3$. It follows that 7, 1, 2, 3, 4 and 5 are equally spaced on a circle. Congruent triangles (236, 174) imply that 6 is on the same circle, which could only be true (compare $\Delta 345$ to $\Delta 456$) if 56 $= d_3$ and $C$ were $R_7$.

This completes our analysis of case III-D, hence of Part III, and proves that $M_7(4)$ contains exactly the five heptagons of Fig. 10.

6. Discussion

We have completely specified the set $M_n(t)$ of all convex n-gons with exactly t intervertex distances for $t = n/2$ when n is even and for $(n, t) \in ((5, 3), (7, 4))$. Our results imply that every $n \geq 7$ has a largest nonnegative integer $f(n)$ such that every convex n-gon with no more than $\lfloor n/2 \rfloor + f(n)$ intervertex distances is a regular $(n + k)$-gon with $k \geq 0$ vertices removed. The theorems and a few examples show that $f(7) = 1$ and $f(8) = f(10) = 0$.

A main open problem is to determine $f(n)$ for all $n \geq 9$. For even $n$, an upper bound on $f(n)$ is suggested by generalizing $A_6$ in Fig. 1. By interweaving the vertices of two copies of $R_N$ with the same center but different diameters we obtain a 2N-gon $A_{2N}$ for which

$$m(A_{2N}) = 3N/2 - 1 \quad \text{if } N \text{ is even},$$
$$m(A_{2N}) = 3(N - 1)/2 \quad \text{if } N \text{ is odd}.$$

Hence

$$f(2N) \leq N/2 - 2 \quad \text{for even } N \geq 4,$$
$$f(2N) \leq (N - 5)/2 \quad \text{for odd } N \geq 5.$$
Thus $f(8) = 0$, $f(10) = 0$, $f(12) \leq 1$, $f(14) \leq 1$, and so forth. Removal of a vertex from $A_{2N}$ gives
\[
\begin{align*}
f(2N - 1) &\leq \frac{N}{2} - 1 \quad \text{for even } N \geq 4, \\
f(2N - 1) &\leq \frac{N - 3}{2} \quad \text{for odd } N \geq 5,
\end{align*}
\]
so $f(7) \leq 1$, $f(9) \leq 1$, $f(11) \leq 2$, $f(13) \leq 2$, and so forth. Our result for $M_7(4)$ shows that $f(7) = 1$. Similarly, if $M_8(5)$ consists of $R_{10} - 1$ and versions of $R_{11} - 2$, then $f(9) = 1$. [Added in proof: This is verified for $M_8(5)$ in P. Erdős and P. Fishburn, Convex nonagons with five intervertex distances, *Geometriae Dedicate* (in press).]

A variety of related distance problems for convex polygons are discussed in [4–6].

**References**