Edge coloring complete uniform hypergraphs with many components

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Abstract

Let H be a hypergraph. For a k-edge coloring $c: E(H) \to \{1, \ldots, k\}$ let $f(H, c)$ be the number of components in the subhypergraph induced by the color class with the least number of components. Let $f_k(H)$ be the maximum possible value of $f(H, c)$ ranging over all k-edge colorings of H. If H is the complete graph K_n then, trivially, $f_1(K_n) = f_2(K_n) = 1$. In this paper we prove that for $n \geqslant 6, f_3(K_n) = \lfloor n/6 \rfloor + 1$ and supply close upper and lower bounds for $f_k(K_n)$ in case $k \geqslant 4$. Several results concerning the value of $f_k(K_n^r)$, where K_n^r is the complete r-uniform hypergraph on n vertices, are also established.
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1. Introduction

All graphs and hypergraphs considered here are finite, unordered and simple. For standard terminology the reader is referred to [7]. Let H be a hypergraph. For a k-edge coloring $c: E(H) \to \{1, \ldots, k\}$ let $f(H, c)$ be the number of components in the subhypergraph induced by the color class with the least number of components. Isolated vertices are not considered as components in a subhypergraph induced by edges. To avoid trivialities we always assume $k \leqslant e(H)$. Let $f_k(H)$ be the maximum possible value of $f(H, c)$ ranging over all k-edge colorings of H. Trivially, if H has n vertices then $1 \leqslant f_k(H) \leqslant \lfloor n/r \rfloor$ where r is the minimum cardinality of an edge of H, and $f_1(H) = c(H)$ where $c(H)$ is the number of components of H that are not isolated vertices.

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In case H is the complete r-uniform hypergraph K_n^r it is not difficult to prove (see last part of Theorem 1.3) that $f_k(K_n^r) = 1$ for $k \leqslant r$. In fact, this is a generalization of the graph theoretic case, $r = 2$, where $f_2(K_n) = 1$ is merely the well-known fact that either a graph or its complement is connected. For fixed $k \geqslant r + 1$ it is not difficult to show that $f_k(K_n^r)$ is linear in n. However, determining the exact value is a non-trivial task.

This paper contains several results concerning the parameter $f_k(K_n^r)$. In the graph-theoretic case, we completely settle the case $k = 3$ and the cases $k = n - 1$ and n. For other fixed values of k we supply close upper and lower bounds that are also valid for the hypergraph case.

Our main results are summarized in the following theorems. For simplicity we use the notation $f(n, k, r)$ instead of $f_k(K_n^r)$, and $f(n, k)$ for the graph-theoretic case $r = 2$. The first theorem on $f(n, k)$ is an exact result dealing with the lower end of the scale of colors, namely $k = 3$.

Theorem 1.1. $f(4, 3) = f(5, 3) = 2$. *Otherwise* $f(n, 3) = \lfloor n/6 \rfloor + 1$.

The next theorem is an exact result dealing with values of k in the upper end of the scale.

Theorem 1.2.

1. $f(n, n - 1) = \lfloor n/2 \rfloor$.
2. $f(n, n) = \lfloor (n - 1)/2 \rfloor$.
3. *If* $k \geqslant n - 1$ *and* k *divides* $\binom{n}{2}$ *then* $f(n, k) = n(n - 1)/(2k)$.
4. *Suppose* $t \geqslant 1$ *and* rt *divides* n, *then for* $k = t(n - 1)!/((r - 1)!(n - r)!), f(n, k, r) = n/(rt)$.
5. *If* $k \geqslant \binom{n}{r} - \binom{n-r}{r}$ *then* $f(n, k, r) = \lfloor \binom{n}{r}/k \rfloor$.

Our next theorem supplies close upper and lower bounds for all fixed values of k. Before we state the theorem we need a few definitions. Given an edge-coloring c of K_n^r, let $z(n, r, c, s)$ denote the fraction of the vertices incident with at least one edge whose color is s. Let $z(n, r, c)$ denote the maximum value of $z(n, r, c, s)$ taken over all colors appearing in c. For $k \geqslant 1$, let $z_k(n, r)$ denote the minimum possible value of $z(n, r, c)$ taken over all colorings that use at most k colors. Finally, let $z_{k,r}$ denote the infimum of $z_k(n, r)$ taken over all $n \geqslant r$. For $r = 2$ denote $z_k = z_{k,2}$. The following theorem relates $z_{k,r}$ with the function $f(n, k, r)$.

Theorem 1.3. *Let* $k \geqslant r + 1$. *Then,*

$$n\left(\frac{1}{r} - \frac{1}{rk^{1/r}}\right)(1 + o_n(1)) \geqslant f(n, k, r) \geqslant n\left(\frac{1}{r} - \frac{z_{k,r}}{r}\right)(1 + o_n(1)).$$

If $k \leqslant r$ *then* $z_{k,r} = 1$ *and* $f(n, k, r) = 1$.

We conjecture that the lower bound in the last theorem is the correct one:

Conjecture 1.4. *Let $k \geqslant r + 1$. Then,*

$$f(n, k, r) = n\left(\frac{1}{r} - \frac{z_{k,r}}{r}\right)(1 + o_n(1)).$$

In Section 2 we analyze the parameters $z_{k,r}$ and z_k. Infinitely many values of $z_{k,r}$ are known, and the values of infinitely many others are open problems. In particular, we determine (with varying difficulty of proofs depending on k) the following specific values: $z_{r+1,r} = r/(r+1), z_4 = \frac{3}{5}, z_5 = \frac{5}{9}, z_6 = \frac{1}{2}, z_7 = \frac{3}{7}, z_{12} = \frac{1}{3}, z_{6,3} = \frac{2}{3}, z_{14,3} = \frac{1}{2}$. More generally, $z_{p^2+p+1} = (p+1)/(p^2+p+1)$ whenever p is a prime power, and $z_{p^2+p} = 1/p$ whenever p is a prime power. We note that the values of z_k for $k = 8, 9, 10, 11$ can be determined precisely using the extensive case analysis appearing in the result of Mills [8]. It is also not difficult to bound z_k from below. In fact, we show $z_k \geqslant 1/\lceil \sqrt{k+1/4} - 1/2 \rceil$. Since, by definition, z_k is monotone decreasing and since for every integer s there is a prime (moreover a prime power) between s and $s + O(s^{2/3})$ [6] we have that $z_k = 1/\sqrt{k} + O(1/k)$. Together with Theorem 1.3 we have, in the graph-theoretic case:

Corollary 1.5. *Let $k \geqslant 3$ and let p be the largest prime power satisfying $p^2 + p + 1 \leqslant k$ then*

$$n\left(\frac{1}{2} - \frac{1}{2\sqrt{k}}\right)(1 + o_n(1)) \geqslant f(n, k) \geqslant n\left(\frac{1}{2} - \frac{p+1}{2(p^2+p+1)}\right)(1 + o_n(1)).$$

The last corollary, together with the argument concerning density of primes show that the upper and lower bounds in the last corollary are very close. Table 1 summarizes the best upper and lower bounds that we currently have for $\limsup f(n, k)/n$ and $\liminf f(n, k)/n$, respectively for some small values of k. For each specific k (except $k = 3$), the upper bound follows from Theorem 1.3 and in all cases, the lower bound follows from the best known upper bound for z_k (all these upper bounds are consequences of constructions that appear in Section 2). We note that it may be possible to prove Conjecture 1.4 without determining the precise value of z_k or $z_{k,r}$ for all k. On the other hand, although we know that, say, $z_4 = 3/5$, and $z_{6,3} = 2/3$, proving that $\lim f(n, 4)/n = 1/5$ or $\lim f(n, 6, 3)/n = 1/9$ is still an open problem. Currently, Conjecture 1.4 is open, in the graph theoretic case, for all $k \geqslant 4$, and for $r \geqslant 3$ it is open for all $k \geqslant r + 1$.

In Section 3 we prove the theorems and also consider the bipartite analog, namely, the parameter $f_k(K_{n,n})$.

2. Localized edge-coloring of complete hypergraphs

2.1. Lower bounds for $z_{k,r}$

We consider first the case $k \leqslant r$. We show that in this case $z_{k,r} = 1$. Namely, there must be a color that is incident with all vertices. In fact, we will show something

Table 1
Asymptotic upper and lower bounds for small values of k

k	$\operatorname{Lim} \sup f(n,k)/k \leqslant$	$\operatorname{Lim} \inf f(n,k)/k \geqslant$	$z_k \leqslant$	$z_k \geqslant$
3	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
4	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{3}{5}$
5	0.277	0.222	$\frac{5}{9}$	$\frac{5}{9}$
6	0.296	0.25	$\frac{1}{2}$	$\frac{1}{2}$
7	0.311	0.285	$\frac{3}{7}$	$\frac{3}{7}$
8	0.324	0.285	$\frac{3}{7}$	$\frac{3}{8}$
9	$\frac{1}{3}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{1}{3}$
10	0.342	0.3	$\frac{2}{5}$	$\frac{1}{3}$
11	0.35	0.318	$\frac{4}{11}$	$\frac{1}{3}$
12	0.356	0.333	$\frac{1}{3}$	$\frac{1}{3}$
13	0.362	0.346	$\frac{4}{13}$	$\frac{4}{13}$

somewhat stronger. There is always a color such that the subgraph of $K_{n,r}$ induced by the edges with this color is connected and spanning. In other words:

Lemma 2.1. *If $k \leqslant r$ then $f(n,k,r) = 1$.*

Proof. Clearly it suffices to prove $f(n,r,r) = 1$. We shall prove this by induction on r. For $r = 2$ we trivially have $f(n,2) = f(n,2,2) = 1$ since either a graph or its complement is connected. Assume the theorem holds for $r - 1$, and we prove it for r. The proof for r also proceeds by induction on n. If $n = r + 1$ then exactly one color appears in two (intersecting) edges and hence this color must be incident with $r + 1 = n$ vertices and we trivially have $f(r + 1, r, r) = 1$. Assuming the theorem holds for $n - 1$, we prove it for n. Fix a vertex v and consider the subhypergraph K_{n-1}^r obtained by removing v. By the induction hypothesis there is a color c such that the subhypergraph of K_{n-1}^r induced by this color is spanning and connected. If c also appears in some edge of K_n^r that contains v we are done. Otherwise, consider the hypergraph K_{n-1}^{r-1} obtained by removing v and coloring each $(r - 1)$-set f with the color of the edge $f \cup v$ in K_n^r. This coloring does not use the color c and hence is an $r - 1$ coloring of K_{n-1}^{r-1}. By the induction hypothesis it has a color c' such that the subhypergraph induced by this color is connected and spanning. By the definition of the coloring of K_{n-1}^{r-1}, the subhypergraph of K_n^r induced by color c' must also be connected and spanning. □

The following lemma supplies a lower bound for $z_{k,r}$ in case $k \geqslant r + 1$.

Lemma 2.2. *Let $k \geqslant r + 1$, and let $d \geqslant 2$ be a positive integer. Then,*

$$z_{k,r} \geqslant \max \left\{ \min \left\{ \left(\frac{d}{k} \right)^{\frac{1}{r-1}}, \frac{1}{d-1} \right\}, \quad \min \left\{ \frac{d}{k}, z_{d-1,r-1} \right\} \right\}.$$

Proof. Let $d \geq 2$ be a positive integer. Consider a coloring of K_n^r with at most k colors. For $i = 1, \ldots, k$, let s_i denote the number of $(r-1)$-sets that are contained in an edge whose color is i. For an $(r-1)$-set U, let b_U denote the number of distinct colors that appear in edges that contain U. Clearly, $\sum_{U \in \binom{[n]}{r-1}} b_U = s_1 + \cdots + s_k$. If

$b_U \geq d$ for all $U \in \binom{[n]}{r-1}$ then $s_1 + \cdots + s_k \geq \binom{n}{r-1} d$ and hence at least some s_i satisfies $s_i \geq \binom{n}{r-1} d/k$. Thus if x is the number of vertices appearing in an edge colored i, we must have $\binom{x}{r-1} \geq s_i \geq \binom{n}{r-1} d/k$. This implies that

$$\left(\frac{x}{n}\right)^{r-1} \geq \frac{x(x-1)\cdots(x-r+2)}{n(n-1)\cdots(n-r+2)} \geq \frac{d}{k}$$

and hence $x \geq n(d/k)^{1/(r-1)}$. Thus, the fraction of vertices incident with color i is at least $(d/k)^{1/(r-1)}$. Otherwise, there exists an $(r-1)$-subset U such that $b_U \leq d-1$. Assume, without loss of generality, that only the colors $1, \ldots, d-1$ appear in edges that contain U. Then each vertex of K_n^r appears in an edge whose color is one of the colors $1, \ldots, d-1$. Thus, for some color i the fraction of vertices appearing in an edge colored i is at least $1/(d-1)$. We have shown that $z_{k,r} \geq \min\{(d/k)^{1/(r-1)}, 1/(d-1)\}$.

Next, we show that $z_{k,r} \geq \min\{d/k, z_{d-1,r-1}\}$. For completeness, define $z_{t,1} = 1/t$ (in a t-coloring of n singletons at least $\lceil n/t \rceil$ elements obtain the same color, and this bound is realized). If there is a vertex v of K_n^r that is incident only with $t \leq d-1$ colors then consider a coloring of K_{n-1}^{r-1} obtained by removing vertex v and coloring each $(r-1)$-set U with the color of the edge $U \cup \{v\}$ in K_n^r. This defines a t-coloring of K_{n-1}^{r-1} and hence, by definition of $z_{t,r-1}$, some color is incident with at least $(n-1)z_{t,r-1}$ vertices. Since this color is also incident with v and since $z_{d-1,r-1} \leq z_{t,r-1}$ we have found a color incident with a fraction of at least $z_{d-1,r-1}$ vertices of K_n^r. Otherwise, each vertex of K_n^r is incident with at least d colors, and hence there is a color that is incident with at least nd/k vertices. In any case we have shown $z_{k,r} \geq \min\{d/k, z_{d-1,r-1}\}$. $\square$

In the case $r = 2$, by choosing $d = \lceil \sqrt{k+1/4} + 1/2 \rceil$ and using the fact that for this choice of d, $d/k \geq 1/(d-1)$, we get

Corollary 2.3. *For all $k \geq 3$, $z_k \geq 1/\lceil \sqrt{k+1/4} - 1/2 \rceil$.*

Notice that for $r \geq 3$ it is best to take $d \approx k^{1/r}$ in Lemma 2.2, except for small values of k where the lower bound of $\min\{d/k, z_{d-1,r-1}\}$ does better.

2.2. Upper bounds and precise values of $z_{k,r}$

Upper bounds for z_k and $z_{k,r}$ are demonstrated by construction. Consider a trivial coloring of K_n^r with $\binom{n}{r}$ colors, each edge colored with a unique color. The

fraction of vertices incident with each color is trivially r/n. This shows, in particular, that

Corollary 2.4. $z_{\binom{n}{r},r} \leqslant r/n$, $z_{n(n-1)/2} \leqslant 2/n$, $z_{r+1,r} \leqslant r/(r+1)$.

Corollary 2.4, together with Lemma 2.2 yield the following:

Corollary 2.5. $z_3 = 2/3$, $z_{r+1,r} = r/(r+1)$, $z_6 = 1/2$.

Proof. Upper bounds follow from Corollary 2.4. $z_{r+1,r} \geqslant r/(r+1)$ follows from Lemma 2.2 by taking $k = r+1$ and $d = r$ (and recalling that $z_{r-1,r-1} = 1$ by Lemma 2.1). z_3 is a special case of $z_{r+1,r}$. $z_6 \geqslant 1/2$ by taking $r = 2, k = 6$ and $d = 3$ in Lemma 2.2. $\square$

In many cases we can find non-trivial constructions that match the lower bound that follows from Lemma 2.2. For example, the smallest nontrivial Steiner Triple System shows that K_7 can be decomposed into 7 triangles. In other words, there is a coloring of K_7 with 7 colors such that each color induces a triangle. We therefore get $z_7 \leqslant 3/7$. On the other hand, applying Lemma 2.2 with $r = 2, k = 7$ and $d = 3$ gives $z_7 \geqslant 3/7$. Thus, $z_7 = 3/7$. More generally we can prove the following:

Proposition 2.6. *Let p be a prime power. Then, $z_{p^2+p+1} = (p+1)/(p^2+p+1)$ and $z_{p^2+p} = 1/p$.*

Proof. Whenever p is a prime power there exists a projective plane $PG(2,p)$. This projective plane corresponds to the existence of a $2 - (p^2+p+1, p+1, 1)$ design (see, e.g., [2]), which, in turn, corresponds to the fact that K_{p^2+p+1} decomposes into p^2+p+1 copies of K_{p+1}. Hence, we have that $z_{p^2+p+1} \leqslant (p+1)/(p^2+p+1)$ and using Lemma 2.2 with $r = 2$ and $d = p+1$ we get $z_{p^2+p+1} = (p+1)/(p^2+p+1)$. Similarly, when p is a prime power there exists an affine plane $AG(2,p)$. This affine plane corresponds to the existence of a $2 - (p^2, p, 1)$ design [2], which, in turn, corresponds to the fact that K_{p^2} decomposes into p^2+p copies of K_p and using Lemma 2.2 with $r = 2, d = p+1$ we get $z_{p^2+p} = 1/p$. $\square$

Notice that Proposition 2.6 gives, in particular, $z_{13} = 4/13$ and $z_{12} = 1/3$.

Hanani has shown the existence of $3 - (n, 4, 1)$ designs for every even n not divisible by 6 (cf. [2]). In other words, K_4^3 decomposes K_n^3 for $n = 2, 4 \bmod 6$. Hence if $k = n(n-1)(n-2)/24$ and $n = 2, 4 \bmod 6$ then $z(k, 3) \leqslant 4/n$. In particular $z(14, 3) \leqslant 1/2$. This upper bound has a matching lower bound that follows from Lemma 2.2 by taking $r = 3, k = 14$ and $d = 7$ and recalling that $z_{6,2} = z_6 = 1/2$. Thus, we obtain the sporadic value $z_{14,3} = 1/2$.

In all previous constructions, all color classes induced the same clique (or hyperclique) size. Constructions using non-isomorphic color classes (and even color

classes that are not cliques) are also very useful. In fact, sometimes using non-isomorphic color classes is provably an optimal strategy. Consider the case $k = 4$. Color K_5 with four colors as follows: For $i = 1, 2, 3$, color i appears in the edges $(i, 4)$ and $(i, 5)$ and color 4 appears in the edges $(1, 2), (1, 3), (2, 3)$. The edge $(4, 5)$ is colored arbitrarily by one of the colors $1, 2$ or 3. Notice that each color is incident with precisely three vertices. Thus, $z_4 \leqslant 3/5$. Notice that we cannot match this upper bound with Lemma 2.2 so in order to prove that this is an optimal strategy we need to explicitly prove:

Proposition 2.7. $z_4 \geqslant 3/5$.

Proof. We need to show that in any coloring of a complete graph K_n with at most four colors, at least one color is incident with at least $3n/5$ vertices. Consider a coloring of K_n with the colors $1, 2, 3, 4$. We use the same notations as in the proof of Lemma 2.2 (recalling that in case $r = 2$ the sets U are singletons). If $b_v \geqslant 3$ for all $v = 1, \ldots, n$ then at least one color has $s_i \geqslant 3n/4 > 3n/5$. Assume, therefore, that there exist vertices with $b_v = 2$. (If there exists a vertex with $b_v = 1$ then the unique color i incident with v has $s_i = n$.) Let $X \subset \{1, \ldots, n\}$ be the subset of vertices with $b_v = 2$. Each $v \in X$ is associated with the unique pair of colors incident with v. Notice that if v is associated with (i, j) then no $v' \in X$ is associated with (k, l) where $\{k, l\} \cap \{i, j\} = \emptyset$. Thus, there are only at most three types of associations. It follows that there are two (not necessarily mutually exclusive) cases. Either there exists a color i such that each $v \in X$ is incident with i, or there exists a color j such that no $v \in X$ is incident with j. Consider the first case. If $|X| \geqslant 3n/5$ then $s_i \geqslant 3n/5$ and we are done. Otherwise, $s_1 + s_2 + s_3 + s_4 \geqslant 3(n - |X|) + 2|X| = 3n - |X| \geqslant 12n/5$. Thus, some s_i has $s_i \geqslant 3n/5$. Consider the second case. Then $s_1 + s_2 + s_3 + s_4 - s_j \geqslant 2n$. Hence, some $s_i (i \neq j)$ has $s_i \geqslant 2n/3$. $\quad \square$

The coloring of K_5 with four colors described above yielding $z_4 \leqslant 3/5$ is somewhat "reducible". Indeed, if we allow weights on the vertices of K_n then the following coloring of K_4 with four colors is more efficient: Color the edge $(i, 4)$ with color i for $i = 1, 2, 3$. Color the triangle $(1, 2, 3)$ with color 4. Assign the weight $2/5$ to vertex 4, and the weight $1/5$ to each of the vertices $1, 2, 3$. Thus, the sum of weights of the vertices incident with color i is $3/5$ for all $i = 1, 2, 3, 4$, as expected. Indeed, any c-coloring of a weighted K_n (with rational weights) can be transformed to a non-weighted c-coloring of a larger $K_{n'}$ where each vertex of K_n is "blown-up" proportionally to its weigh (an edge of $K_{n'}$ connecting two vertices corresponding to the same blown-up vertex of K_n can be colored with any arbitrary color that is incident with that vertex of K_n).

The proof for the case $k = 5$ is more complicated. For the upper bound, color K_9 as follows: Two colors induce each a copy of K_5. The two copies of K_5 share one vertex. The remaining 16 yet uncolored edges form a $K_{4,4}$. $K_{4,4}$ can be decomposed into two $K_{2,3}$ and one $K_{1,4}$. Thus, we have a 5-coloring of K_9 where each color is incident with 5 vertices. This shows that $z_5 \leqslant 5/9$. Again, we cannot match this upper bound with Lemma 2.2. We therefore show:

Proposition 2.8. $z_5 \geqslant 5/9$.

Proof. We need to show that in any coloring of a complete graph K_n with at most five colors, at least one color is incident with at least $5n/9$ vertices. Consider a coloring of K_n with the colors $1, 2, 3, 4, 5$. We use the same notations as in the proof of Lemma 2.2. If some vertex has $b_v = 1$ we are done. Assume, therefore all vertices have $b_v \geqslant 2$. If at least $7n/9$ vertices have $b_v > 2$ then $s_1 + s_2 + s_3 + s_4 + s_5 \geqslant 3n - 2n/9$ which implies that some i has $s_i \geqslant 5n/9$ as required. Thus, let $X \subset \{1, \ldots, n\}$ be the subset of vertices with $b_v = 2$, and we may assume $|X| \geqslant 2n/9$. As in Proposition 2.7, each $v \in X$ is associated with the unique pair of colors incident with v, and if v is associated with (i,j) then no $v' \in X$ is associated with (k,l) where $\{k,l\} \cap \{i,j\} = \emptyset$. Thus, either there exists a color i such that each $v \in X$ is incident with i, or there exist three colors i, j, k such that only the associations $(i,j), (i,k), (j,k)$ are valid associations for the vertices of A. Consider the second case. In this case, each $u \notin X$ is incident with at least two of the colors $\{i,j,k\}$. To see this, notice that if u is not incident with, say, both i and j then the edge connecting u to $v \in A$ where v is associated with (i,j), cannot be colored. We therefore have $s_i + s_j + s_k \geqslant 2n$ so at least one of the colors is incident with at least $2n/3 > 5n/9$ vertices. Now consider the first case. Since color i is incident with all vertices of X we may assume $|X| \leqslant 5n/9$, otherwise we are done. Let s be the number of additional colors, other than i, incident with vertices of X. Without loss of generality, assume $i = 1$ and assume all other colors incident with vertices of X are $2, \ldots, s+1$. Clearly $1 \leqslant s \leqslant 4$. Let Y be the subset of vertices not in X and which are incident with color 1. Thus, color 1 is incident with precisely $|Y| + |X|$ vertices, and colors $2, \ldots, s+1$ are all incident with at least all vertices of $v(K_n) \backslash (X \cup Y)$, namely to at least $n - |X| - |Y|$ vertices. Assume first that $s \geqslant 2$. Each vertex of Y is incident with at least three colors, one of which is color 1, so it is also incident with at least $s - 2$ of the colors $2, \ldots, s+1$. It follows that $s_2 + \cdots + s_{s+1} \geqslant s(n - |X| - |Y|) + (s-2)|Y| + |X|$. Thus, some s_i is incident with at least $n - |X| - 2|Y|/s + |X|/s$ vertices. It suffices to show that $\max\{|Y| + |X|, n - |X| - 2|Y|/s + |X|/s\} \geqslant 5n/9$. Since the maximum is minimized when both terms are equal, and this happens when $|Y| = ns/(s+2) - |X|(2s-1)/(s+2)$ it suffices to show that $ns/(s+2) - |X|(s-3)/(s+2) \geqslant 5n/9$. For $s = 2$ this holds since $|X| \geqslant 2n/9$ and thus $n/2 + |X|/4 \geqslant 5n/9$. For $s = 3$ this holds since $3n/5 > 5n/9$. For $s = 4$ this holds since $|X| \leqslant 5n/9$ and thus $2n/3 - |X|/6 > 5n/9$. Finally, assume $s = 1$. In this case colors 1 and 2 are incident with all vertices of K_n and therefore $s_1 + s_2 \geqslant n + |X|$. Thus, one of them is incident with at least $n/2 + |X|/2 > 5n/9$ since $|X| \geqslant 2n/9$. □

The reader may notice the significant added complexity to the proof of Proposition 2.8 as opposed to Proposition 2.7. It is plausible that with an increasing amount of effort one may determine z_k or $z_{k,r}$ for every specific k and r with an appropriate "ad hoc" proof.

A non-symmetric construction in the hypergraph ($r = 3$) case that yields an exact result is the following:

Proposition 2.9. $z_{6,3} = 2/3$.

Proof. The lower bound follows from Lemma 2.2 by taking $r = 3, k = 6$ and $d = 4$ and recalling that $z_{3,2} = z_3 = 2/3$. For the upper bound, color K_6^3 with six colors as follows: Color 1 appears in $(1, 2, 3), (1, 2, 4), (1, 3, 4), (2, 3, 4)$. Color 2 appears in $(1, 2, 5), (1, 2, 6), (1, 5, 6), (2, 5, 6)$. Color 3 appears in $(1, 3, 5), (1, 3, 6), (3, 5, 6)$. Color 4 appears in $(2, 3, 5), (2, 4, 5), (3, 4, 5)$. Color 5 appears in $(1, 4, 5), (1, 4, 6), (4, 5, 6)$. Color 6 appears in $(2, 3, 6), (2, 4, 6), (3, 4, 6)$. Every color is incident with four vertices hence $z_{6,3} \leqslant 4/6 = 2/3$. $\square$

The upper and lower bounds for z_k in case $k = 8, 9, 10, 11$ appearing in Table 1 are obtained as follows: For $k = 8, z_8 \leqslant z_7 = 3/7$, and $z_8 \geqslant 3/8$ by selecting $d = 3$ in Lemma 2.2. For $k = 9, z_9 \leqslant 2/5$ since K_4^- (the graph obtained from K_4 be deleting an edge) decomposes K_{10} into 9 copies (see [2]). $z_9 \geqslant 1/3$ by selecting $d = 4$ in Lemma 2.2. $z_{10} \leqslant z_9 \leqslant 2/5$. $z_{10} \geqslant 1/3$ by selecting $d = 4$ in Lemma 2.2. $z_{11} \leqslant 4/11$ since K_4^- decomposes K_{11} into 11 copies (see [2]). $z_{11} \geqslant 1/3$ by selecting $d = 4$ in Lemma 2.2.

Finally, infinitely many non-trivial upper bounds in the case $r = 3$ are obtained by using Möbius designs. For q a prime power there exist the Möbius designs $3 - (q^s, q + 1, 1)$ (cf. [2]). Hence, K_{q+1}^3 decomposes $K_{q^s+1}^3$ and hence, for $s = 2$ we have that for $k = (q^2 + 1)q, z(k, 3) \leqslant (q + 1)/(q^2 + 1)$. In particular, $z(10, 3) \leqslant 3/5$. This can be compared with the lower bound of $z(10, 3) \geqslant 5/9$ that follows from Lemma 2.2 for $r = 3, k = 10$ and $d = 6$, and using Proposition 2.8 which states that $z_{5,2} = z_5 = 5/9$.

3. Proof of the main results

We start with the lower bound in Theorem 1.3 as its proof will also be used to prove one direction in Theorem 1.1.

3.1. A lower bound for $f(n, k, r)$

We shall require the following lemma, whose proof is an immediate consequence of a theorem of Baranyai [1].

Lemma 3.1. Let $r \geqslant 2$ and $t \geqslant 1$ be positive integers. There exists $n_0 = n_0(r, t)$ such that for all $n \geqslant n_0, K_n^r$ contains t disjoint maximum matchings (a maximum matching is a set of $\lfloor n/r \rfloor$ independent edges). $\square$

It is well-known that $n_0(2, t) = t + 1$, since the complete graph on $t + 1$ vertices decomposes into t perfect matchings in case t is odd, and $t + 1 > t$ maximum matchings in case t is even. For $r \geqslant 3$ the theorem of Baranyai states that if n is a multiple of r then K_n^r decomposes into perfect matchings, and hence precisely $t = (n - 1)!/((r - 1)!(n - r)!)$ disjoint perfect matchings. Thus, $n_0(r, t) \approx (t(r - 1)!)^{1/(r-1)}$.

We need to prove that $f(n, k, r) \geqslant n\left(\frac{1}{r} - \frac{z_{k,r}}{r}\right)(1 + o_n(1))$. Let $\varepsilon > 0$. Let $t = t(k, r, \varepsilon)$ be the smallest integer such that K_t^r has a coloring C using at most k colors, such that each color is incident with at most $t(z_{k,r} + \varepsilon)$ vertices. Notice that in some cases it is possible to have $\varepsilon = 0$. In Section 2 we have shown, e.g., that for $r = 2$ and $k = 3, 4, 5, 6, 7, 12, 13$ we can take $\varepsilon = 0$ (in other words, z_k is *realized* in these cases). As an example, for $r = 2$ and $k = 3$ we can pick $\varepsilon = 0$ and $t = 3$.

Lemma 3.2. *Suppose n satisfies*

$$\left\lfloor \frac{n}{t} \right\rfloor \geqslant n_0\left(r, k - \left\lceil \frac{1}{z_{k,r} + \varepsilon} \right\rceil\right),$$

then $f(n, k, r) \geqslant \left\lfloor \frac{n}{rt} \right\rfloor \left\lceil t(1 - z_{k,r} - \varepsilon) \right\rceil + 1$.

Proof. Assume the vertices of K_t^r are $\{1, \ldots, t\}$ and assume the colors of C are $\{1, \ldots, k\}$. Let $C(e)$ denote the color of edge e (notice that it is possible that C is not onto $\{1, \ldots, k\}$ as c may use less than k colors).

Partition the vertices of K_n^r into t equitable parts $A_1, \ldots, A_t$. The cardinality of each part is either $\lceil n/t \rceil$ or $\lfloor n/t \rfloor$ and by the assumption in the statement of the lemma, the parts are nonempty.

We now show how to color each edge of K_n^r. Let f be an arbitrary edge, and let $U = \{i: A_i \cap f \neq \emptyset\}$. Clearly $1 \leqslant |U| \leqslant r$. Consider first the case $|U| > 1$. In this case, let $e \in K_t^r$ be an edge such that $U \subset e$. In this case, we color f with the color $C(e)$. Consider next the case $|U| = 1$. In this case f is completely within some part A_i. Hence, it remains to show how to color edges that are completely within some A_i. Let $C_i \subset \{1, \ldots, k\}$ be the subset of colors not appearing in any edge of K_t^r that contains the vertex i. We claim that $|C_i| \leqslant k - \left\lceil \frac{1}{z_{k,r} + \varepsilon} \right\rceil$. To see this, notice that if there were less than $\frac{1}{z_{k,r} + \varepsilon}$ colors incident with i then at least one color would have been incident with more than $t(z_{k,r} + \varepsilon)$ vertices, contradicting the assumption. We claim that we can find in A_i a set of $|C_i|$ disjoint maximum matchings. Indeed this follows from Lemma 3.1, and by the assumption in the current lemma that states that $|A_i| \geqslant \lfloor n/t \rfloor \geqslant n_0\left(r, k - \left\lceil \frac{1}{z_{k,r} + \varepsilon} \right\rceil\right)$. Fixing $|C_i|$ disjoint maximum matchings in A_i we now color each maximum matching with a distinct color of C_i. It remains to show how to color the edges completely within A_i that do not belong to any of the selected maximum matchings. In this case, we can color them with any color of $C \backslash C_i$ (does not matter which one).

Now consider any color $c \in C$. If c is not used at all in the coloring of K_t^r then the subhypergraph of K_n^r induced by the edges colored c is composed of isolated edges, exactly $\lfloor |A_i|/r \rfloor$ isolated edges from each A_i. Therefore, this subhypergraph has at least $t \lfloor n/(rt) \rfloor$ components. If c is used as a color of at least one edge of K_t^r then the subhypergraph induced by the edges colored c in K_n^r has some large components consisting of all the s sets A_i such that the vertex i of K_t^r is incident with an edge colored c, and isolated edges, exactly $\lfloor |A_i|/r \rfloor$ isolated edges from each of the $t - s$ sets A_i where vertex i of K_t^r is not incident with any edge colored c. Thus, this

subhypergraph has at least $(t - s)\lfloor n/(rt) \rfloor + 1$ components. Since $s \leqslant (z_{k,r} + \varepsilon)t$ our construction shows that $f(n, k, r) \geqslant \lfloor \frac{n}{rt} \rfloor \lceil t(1 - z_{k,r} - \varepsilon) \rceil + 1$. $\quad\square$

3.2. Proof of Theorem 1.1

Trivially $f(3, 3) = 1$. $f(4, 3) = 2$ as seen by coloring K_4 with three colors each inducing a perfect matching. $f(5, 3) = 2$ as seen by coloring K_5 with a red triangle and a red edge vertex-disjoint with the triangle, and coloring the remaining 6 edges with three blue edges forming a $K_{1,2}$ and a $K_{1,1}$ vertex disjoint with each other, and three green edges that are now also forced to induce a $K_{1,2}$ and a $K_{1,1}$ vertex disjoint with each other. Thus, we assume $n \geqslant 6$.

For the lower bound we use Lemma 3.2. Recall that $z_3 = 2/3$ and K_3 realizes z_3 with $\varepsilon = 0$ by giving each edge of K_3 a distinct color. The condition in Lemma 3.2 is satisfied for all $n \geqslant 6$ since $n_0(2, 1) = 2$, and hence we get $f(n, 3) \geqslant \lfloor n/6 \rfloor + 1$.

It remains to show the upper bound. Consider a coloring of K_n with the three colors $1, 2, 3$. Let A_i be the subset of vertices in a maximum cardinality component induced by color i. Put $|A_i| = a_i$ and assume $a_1 \geqslant a_2 \geqslant a_3$. Put $D = A_1 \cap A_2$ and $d = |D|$. Put $M = V \backslash (A_1 \cup A_2)$ where V is the set of vertices of K_n, and $m = |M|$. If $a_1 \geqslant 2n/3$ then we are done since the subgraph induced by color 1 has at most $\lfloor 1 + (n - 2n/3)/2 \rfloor = 1 + \lfloor n/6 \rfloor$ components. Hence we assume $a_i < 2n/3$ for $i = 1, 2, 3$. Consider the following cases

1. $d = 0$. In this case every edge between A_1 and A_2 is colored 3. Thus, $a_3 \geqslant a_1 + a_2$, a contradiction since the a_i cannot be zero.
2. $d > 0, m = 0$, and $d = a_1$ or $d = a_2$ (or both). In this case one of the colors 1 or 2 (or both) induces a connected spanning subgraph of K_n.
3. $d > 0, m = 0, d < a_1, d < a_2$. In this case all edges between $A_1 \backslash D$ and $A_2 \backslash D$ are colored 3. Thus, $2n/3 > a_3 \geqslant (a_1 - d) + (a_2 - d)$. On the other hand $(a_1 - d) + (a_2 - d) + d = n$. Thus, $d > n/3$. It follows that $a_1 + a_2 = n + d > 4n/3$. Hence, $a_1 > 2n/3$, a contradiction.
4. $d > 0, m > 0, a_1 > d$ and $a_2 > d$. As in the previous case all edges between $A_1 \backslash D$ and $A_2 \backslash D$ are colored 3. Also all edges between M and D are colored 3. Thus, the subgraph induced by 3 has at most two components and $2 \leqslant \lfloor n/6 \rfloor + 1$.
5. $d > 0, m > 0, a_2 = d$. In this case all edges between M and D are colored 3. Thus, $a_3 \geqslant d + m > d = a_2$, a contradiction to the assumption $a_3 \leqslant a_2$. $\quad\square$

3.3. Proof of Theorem 1.2

1. If n is even then K_n decomposes into $n - 1$ perfect matchings. Thus, $f(n, n - 1) \geqslant n/2$. If n is odd then K_n decomposes into $(n - 1)/2$ Hamiltonian cycles. Each cycle further decomposes into a matching of size $(n - 1)/2$ and the remaining $(n + 1)/2$ edges form a subgraph with $(n - 1)/2$ components (one component is a path with three vertices and the others are independent edges). Thus, $f(n, n - 1) \geqslant (n - 1)/2 = \lfloor n/2 \rfloor$. In both the even and odd cases we always have $f(n, n - 1) \leqslant \lfloor \binom{n}{2}/(n - 1) \rfloor = \lfloor n/2 \rfloor$. Thus, $f(n, n - 1) = \lfloor n/2 \rfloor$.

2. If n is odd then the chromatic index of K_n is n. Thus, K_n decomposes into n matchings. This forces each matching to be of size $(n-1)/2$. Thus, $f(n,n) \geqslant (n-1)/2$. If n is even then $n+1$ is odd and we have $f(n+1,n) = n/2$ by the previous case. Deleting the additional vertex causes each induced subgraph of a color to loose at most one component. Thus, $f(n,n) \geqslant n/2 - 1$. In both cases we always have $f(n,n) \geqslant \lfloor (n-1)/2 \rfloor$. Since, trivially, $f(n,n) \leqslant \lfloor \binom{n}{2}/n \rfloor = \lfloor (n-1)/2 \rfloor$ we have shown $f(n,n) = \lfloor (n-1)/2 \rfloor$.

3. Assume $k \geqslant n-1$ and assume $t = n(n-1)/(2k)$ is an integer. Thus, $2t \leqslant n$ and it is well known that in this case tK_2 decomposes K_n [5,3]. Thus, $f(n,k) \geqslant t$. The other direction follows from the trivial fact that $f(n,k) \leqslant \binom{n}{2}/k = t$.

4. Suppose $t \geqslant 1$ and rt divides n. In particular, r divides n, and by the result of Baranyai [1], K_n^r decomposes into $(n-1)!/((r-1)!(n-r)!)$ perfect matchings. Each perfect matching has cardinality n/r and is further decomposed into t matchings, each consisting of $n/(rt)$ edges. Hence, for $k = t(n-1)!/((r-1)!(n-r)!)$ we have $f(n,k,r) \geqslant n/(rt)$. Trivially, $f(n,k,r) \leqslant \binom{n}{r}/k = n/(rt)$.

5. Consider the line graph L of K_n^r. L has $N = \binom{n}{r}$ vertices and is regular of degree $s = \binom{n}{r} - \binom{n-r}{r} - 1$. By the theorem of Hajnal and Szemerédi [4], for all $k' > s$, L has a k' equipartite coloring, namely, a vertex-coloring with k' colors where each color class has either $\lceil N/k' \rceil$ or $\lfloor N/k' \rfloor$ elements. Using $k = k'$ and translating this back to the original hypergraph K_n^r, we have a k-edge coloring of K_n^r such that each color induces a matching of cardinality at least $\lfloor \binom{n}{r}/k \rfloor$. Thus, $f(n,k,r) \geqslant \lfloor \binom{n}{r}/k \rfloor$. The other direction is trivial. $\quad\square$

3.4. Proof of Theorem 1.3

In case $k \geqslant r+1$, the lower bound of Theorem 1.3 is shown in Lemma 3.2. For the upper bound, notice that in a k-edge coloring of K_n^r at least one color appears in at least $\binom{n}{r}/k$ edges. Fix such a color, and let $s_1, \dots, s_t$ denote the cardinalities of the components in the subhypergraph induced by this color. Clearly, $s_i \geqslant r$ for $i = 1, \dots, t$ and

$$\binom{n-r(t-1)}{r} + (t-1) \geqslant \binom{s_1 + \cdots + s_t - r(t-1)}{r} + (t-1)$$

$$\geqslant \sum_{i=1}^{t} \binom{s_i}{r} \geqslant \binom{n}{r}\frac{1}{k}. \tag{1}$$

It follows that $n - r(t-1) \geqslant (1 + o_n(1))n/k^{1/r}$ and hence $t \leqslant (n/r - n/(rk^{1/r}))(1 + o_n(1))$. In case $k \leqslant r, f(n,k,r) = 1$ is shown in Lemma 2.1. $\quad\square$

It is not difficult to show that in case $r = 2$, the negation of the error term in the upper bound of Theorem 1.3 is at least $\Theta(1/k^{3/2})$, by considering the maximal cardinality components induced by each color, and showing that either there is a color with a "huge" component, or otherwise there must be (at least) two such maximal components (belonging to two distinct colors) that intersect in $\Theta(n)$

vertices, and thus at least one of these maximal components is far from being a clique (misses $\Theta(n^2)$ edges from being a clique), and hence the convexity argument in inequality (1) cannot be exploited to its extreme. However, this improvement is negligible. Even for $k = 4$ we could only improve the constant from 0.25 to (a little less than) 0.243. That is, $\limsup f(n.4)/4 \leqslant 0.243$.

3.5. The bipartite analog

Let C be a k-edge coloring of K_r where color i is incident with s_i vertices. We show how to construct a coloring C' of $K_{r,r}$ where color i is incident with $2s_i$ vertices. Assume the vertices of one partite class are labeled $\{a_1, \ldots, a_r\}$ and the vertices of the other partite class are labeled $\{b_1, \ldots, b_r\}$. For $i \neq j$ color (a_i, b_j) with the same color as the edge (i, j) of K_r. Color (a_i, b_i) with any color incident with vertex i in K_r. The obtained coloring has the desired property. Now, this construction together with the same "blow up" argument as in Lemma 3.2 yields the following:

$$f_k(K_{n,n}) \geqslant n(1 - z_k)(1 + o_n(1)).$$

An upper bound density argument similar to the one in Theorem 1.3 gives $f_k(K_{n,n}) \leqslant n(1 - 1/\sqrt{k})(1 + o_n(1))$. Recalling that $z_k = 1/\sqrt{k} + O(1/k)$ we get that the upper and lower bounds for $f_k(K_{n,n})$ are very close.

Acknowledgments

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