Journal of Algebra 353 (2012) 93–120

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Cohomology theories based on flats ☆

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A R T I C L E   I N F O

Article history:
Received 13 January 2011
Available online 28 December 2011
Communicated by J.T. Stafford

MSC:
18E30
18E35
16E40
16E05
14F25

Keywords:
Totally acyclic complex
Homotopy category
Tate cohomology
Complete cohomology
Semi-separated noetherian scheme

A B S T R A C T

Let A be an associative ring with identity, $\mathbf{K}(\mathrm{Flat}\,A)$ the homotopy category of flat modules and $\mathbf{K}_{\mathrm{p}}(\mathrm{Flat}\,A)$ the full subcategory of pure complexes. The quotient category $\mathbf{K}(\mathrm{Flat}\,A)/\mathbf{K}_{\mathrm{p}}(\mathrm{Flat}\,A)$, called here the pure derived category of flats, was introduced by Neeman. In this category flat resolutions are unique up to homotopy and so can be used to compute cohomology. We develop theories of Tate and complete cohomology in the pure derived category of flats. These theories extend naturally to sheaves over semi-separated noetherian schemes, where there are not always enough projectives, but we do have enough flats. As applications we characterize rings with finite sfli and schemes which are locally Gorenstein.

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☆ The authors are supported in part by a grant from IPM (Nos. 89130215 and 89130217).

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doi:10.1016/j.jalgebra.2011.12.008

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1. Introduction

There is a long history of studying algebraic structures using homological methods. The particular cohomology theories of interest to us in this paper are Tate cohomology and complete cohomology, which have been developed by many people, see for example [Fa77,B86,BC92,Goi92,Mi94,N98]. These theories have also been studied in the context of representation theory of algebras, see [B86,Kra05, Ch08,Be09].

As is typical in homological algebra, these theories are developed using projective and/or injective objects, in order to have some kind of lifting property and hence uniqueness of resolutions up to homotopy. Although in many situations it is more natural to work with *flat* modules rather than projectives, at first it seems futile to build a cohomology theory out of flats, since flat resolutions are not unique up to homotopy. One approach is to work with a more rigid kind of flat resolution, possibly due to a series of papers by Enochs and his collaborators proving that the flat modules form a covering class [E84,EGR98,BEE01]. A natural advantage of building a cohomology theory out of flat modules is that it can be more naturally extended to the setting of quasi-coherent sheaves on schemes, where there are very few projectives [EE05,EEGO04]. In this paper we define Tate cohomology and complete cohomology for both rings and schemes using flat modules; the technical formalism making this possible is the pure derived category of flat modules. We would like to point out that even the reader who does not care about these particular cohomology theories may find some of our techniques involving cotorsion modules interesting, as we expect that these techniques may be helpful in applying the setting of the pure derived category to other problems.

Let A be an associative ring with identity, and let all modules be left modules. In his study of the homotopy category $\mathbf{K}(\operatorname{Proj} A)$ of projective A-modules as a subcategory of the homotopy category $\mathbf{K}(\operatorname{Flat} A)$ of flat modules, Neeman discovered an interesting equivalence [Nee08, §8]

$$\mathbf{K}(\operatorname{Proj} A) \to \mathbf{K}(\operatorname{Flat} A) \to \frac{\mathbf{K}(\operatorname{Flat} A)}{\mathbf{K}_{\mathrm{p}}(\operatorname{Flat} A)}$$

of triangulated categories. Here $\mathbf{K}_{\mathrm{p}}(\operatorname{Flat} A)$ denotes the full subcategory of $\mathbf{K}(\operatorname{Flat} A)$ consisting of pure complexes, those complexes that remain exact after tensoring with any A^{op}-module. As a consequence of this equivalence flat resolutions are unique in the quotient category $\mathbf{K}(\operatorname{Flat} A)/\mathbf{K}_{\mathrm{p}}(\operatorname{Flat} A)$, which we call the pure derived category of flat modules and denote $\mathbf{D}(\operatorname{Flat} A)$. This uniqueness provides us with the setting for defining our theories of Tate and complete cohomology.

The origin of the notion of *totally acyclic complexes* goes back to the 1950s, when Tate discovered that the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, when G is a finite group, can be considered as a syzygy of an acyclic complex T of projective $\mathbb{Z}G$-modules. This complex T has the property that it remains exact after applying the functor $\operatorname{Hom}_{\mathbb{Z}G}(\ ,\mathbb{Z}G)$. Such a complex is actually unique up to homotopy, and can therefore be used to define a cohomology theory, now known as Tate cohomology.

To generalize the theory to any group, Mislin [Mi94], Benson and Carlson [BC92] and Vogel [Goi92] independently developed a cohomology theory now known as complete cohomology, see 4.10 below for the history of the theory. In full generality, a totally acyclic complex of projective A-modules is an acyclic complex of projective modules which is exact with respect to the functor $\operatorname{Hom}_A(\ , P)$, for any projective A-module P. Let $\mathbf{K}_{\mathrm{tac}}(\operatorname{Proj} A)$ denote the full subcategory of $\mathbf{K}(\operatorname{Proj} A)$ consisting of totally acyclic complexes. A *complete projective resolution* of a complex M of A-modules is a totally acyclic complex T together with a morphism $\nu : T \to P_M$, where P_M is a projective resolution of M, such that ν^i is bijective for all $i \ll 0$.

Once we have a complete resolution for M, we can compute Tate cohomology. The existence of a complete projective resolution for M is known in a few cases. For example, in case the Gorenstein projective dimension of M is finite, such resolutions exist. Also it is shown in [Jor07] that, when the inclusion functor $I : \mathbf{K}_{\text{tac}}(\text{Proj}\, A) \to \mathbf{K}(\text{Proj}\, A)$ has a right adjoint I_ρ, for any A-module M the map $I_\rho(P_M) \to P_M$ can be considered as the best approximation to M by a complete projective resolution, in the sense that when M admits a complete resolution $T \to P_M$, then $I_\rho(P_M) \cong T$ in $\mathbf{K}(\text{Proj}\, A)$. This, together with [Jor05], are important results in the direction of obtaining Krause's results in [Kra05] for projectives instead of injectives (at least over rings). A natural question arises: is it possible to find a variation of these theories in the category of flats? Our answer in the affirmative is given in Section 4.

The paper is structured as follows: In Section 2 we collect the definitions and results that we need throughout the paper, including the basic properties of the pure derived category of flat modules.

In Section 3, we study the notion of total acyclicity in the category $\mathbf{D}(\text{Flat}\, A)$, which we call **F**-total acyclicity. We define finiteness of an **F**-Gorenstein flat dimension for any complex. We show that the **F**-Gorenstein flat dimension of a homologically bounded below complex X is finite if and only if one of the syzygies of a flat resolution of X is a syzygy of an **F**-totally acyclic complex. Examples are provided in which **F**-totally acyclic complexes exist and/or the **F**-Gorenstein flat dimension is finite.

Section 4 is devoted to the study of **F**-Tate cohomology. We establish several properties of this theory, discuss its connection with the previous notions of Tate cohomology, investigate the rigidity of the theory and show that its vanishing characterizes the flat dimension of complexes.

To any pair of complexes of flat modules X and Y we assign a subcomplex of the Hom complex consisting of homotopically bounded below morphisms using which we introduce a complete cohomology theory in $\mathbf{D}(\text{Flat}\, A)$. Section 5 contains our main results on this theory. Under certain circumstances it is compatible with the **F**-Tate cohomology groups. Moreover, its vanishing characterizes the flat dimension of complexes. As an application of our results for (non-commutative) rings we provide a characterization of rings with finite sfli, where sfli is the supremum of the flat dimension of injective modules. This invariant has connections with other (co)homological invariants of groups.

In Section 6, we explain how one can get a satisfactory version of Tate cohomology in the category of quasi-coherent sheaves over a semi-separated noetherian scheme X. In particular, this theory can be used to characterize locally Gorenstein schemes.

2. Preliminaries

In this paper A denotes an associative ring with identity and by default all modules are left A-modules. If we say that X is a complex, we mean that it is a complex of A-modules, that is, a sequence of (left) A-modules X^i and A-linear maps $\partial_X^i : X^i \to X^{i+1}$, $i \in \mathbb{Z}$, such that $\partial\partial = 0$. Modules are viewed as complexes concentrated in degree zero. For any integer i, $\Sigma^i X$ denotes the complex X shifted i degrees to the left. For any complex X by the (left) soft truncation of X at n, denoted $X_{\leqslant n}$, we mean the complex $\cdots \to X^{n-2} \to X^{n-1} \to \text{Ker}\, \partial_X^{n-1} \to 0$. By the (left) hard truncation of X at n, denoted ${}_hX_{\leqslant n}$, we mean the complex $\cdots \to X^{n-2} \to X^{n-1} \to X^n \to 0$. Right truncations, $X_{\geqslant n}$ and ${}_hX_{\geqslant n}$ are defined similarly. We define $\inf(X)$ to be the infimum of all integers ℓ such that $H^\ell(X) \neq 0$, with $\inf(X) = \infty$ if X is homologically trivial, and $-\infty$ if X is not homologically bounded below. The category of A-complexes and chain maps is denoted by $\mathcal{C}(A)$.

The homotopy category $\mathbf{K}(A)$ has as objects the complexes in A and the morphisms are the homotopy equivalences of morphisms in $\mathcal{C}(A)$. Let $\mathcal{X}$ be a class of A-modules. We denote by $\mathbf{K}(\mathcal{X})$ the homotopy category of complexes over $\mathcal{X}$, which is a triangulated subcategory of $\mathbf{K}(A)$.

2.1. Orthogonality. Let $\mathcal{S}$ be a triangulated subcategory of $\mathcal{T}$. The left and right orthogonal of $\mathcal{S}$ in $\mathcal{T}$ are defined, respectively, by

$$^{\perp}\mathcal{S} = \{X \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(X, S) = 0, \text{ for all } S \in \mathcal{S}\},$$
$$\mathcal{S}^{\perp} = \{X \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(S, X) = 0, \text{ for all } S \in \mathcal{S}\}.$$

2.2. Localization sequences. Let $\mathcal{S}$ be a localizing subcategory of $\mathcal{T}$, that is, $\mathcal{S}$ is a triangulated subcategory of $\mathcal{T}$ and is closed under arbitrary small coproducts. It is a standard fact in the theory of

Bousfield localization (see e.g. [Nee01, Theorem 9.1.13]) that the inclusion $\mathcal{S} \to \mathcal{T}$ has a right adjoint if and only if for any object X in $\mathcal{T}$, there exists a triangle $X' \to X \to X'' \rightsquigarrow$ in $\mathcal{T}$ with $X' \in \mathcal{S}$ and $X'' \in \mathcal{S}^{\perp}$. This triangle is unique up to isomorphism and the right adjoint of $\mathcal{S} \to \mathcal{T}$ maps X to X'. In the language of Verdier [Ver96, §II.2] the above situation can be stated by saying that $\mathcal{S} \to \mathcal{T} \to \mathcal{T}/\mathcal{S}$ is a localization sequence.

2.3. Resolution of complexes. Following Spaltenstein [Sp88], a complex P of A-modules is called **K**-projective if the functor $\operatorname{Hom}_A(P, \ )$ preserves acyclicity of complexes. It is well known that any bounded above complex of projectives is **K**-projective. A **K**-*projective resolution* of a complex X is a quasi-isomorphism $P \to X$, with P **K**-projective. If, moreover, P is a complex of projectives, then the resolution P is called a *projective resolution of* X. **K**-injective complexes, **K**-injective resolutions and injective resolutions are defined similarly. Such resolutions exist thanks to [Sp88]. Projective (resp. injective) resolutions are called semiprojective (resp. semiinjective) by Avramov et al., see [AFH03].

A complex F is called **K**-*flat* if the functor $- \otimes_A F$ preserves acyclicity. Any bounded above complex of flat modules is **K**-flat. Moreover, the class of **K**-flat complexes form a localizing subcategory of $\mathbf{K}(A)$, that is closed under direct limits and homotopy colimits. A quasi-isomorphism $F \to X$, with F **K**-flat, is called a **K**-*flat resolution* of a complex X. The existence of **K**-flat resolutions for any complex is guaranteed by [Sp88, §5]. By a *flat resolution* of a complex we mean a **K**-flat resolution by flat modules; these resolutions are also called semiflat resolutions [AFH03].

2.4. Proper flat resolution. Let M be an A-module. A flat resolution

$$\cdots \to F^{-2} \xrightarrow{\partial^{-2}} F^{-1} \xrightarrow{\partial^{-1}} F^{0} \xrightarrow{\partial^{0}} 0$$

of M is called proper if for any $i \leqslant 0$, $F^i \to \operatorname{Im} \partial^i$ is a flat precover. It is easy to see that a flat resolution of M is proper if and only if it remains exact after applying the functor $\operatorname{Hom}_A(F, \ )$, for any flat module F. It is known that the class of flat modules is (pre)covering and hence any A-module M admits a proper flat resolution, see [BEE01].

2.1. Pure derived category of flats

A complex X of A-modules is called pure if for any right A-module M, the tensored complex $M \otimes_A X$ is acyclic. Clearly, a pure complex is acyclic. Let $\mathbf{K}_{\mathrm{p}}(\operatorname{Flat} A)$ denote the homotopy category of all pure complexes of flat modules. It is a triangulated subcategory of $\mathbf{K}(\operatorname{Flat} A)$. It is proved by Neeman [Nee08, Theorem 8.6, Corollary 9.4] that the objects of $\mathbf{K}_{\mathrm{p}}(\operatorname{Flat} A)$ are acyclic complexes with flat kernels, or equivalently, acyclic **K**-flat complexes. Moreover, by [Nee08, Theorem 8.6], $\mathbf{K}_{\mathrm{p}}(\operatorname{Flat} A)$ is the right orthogonal of $\mathbf{K}(\operatorname{Proj} A)$ in $\mathbf{K}(\operatorname{Flat} A)$, that is, $\mathbf{K}_{\mathrm{p}}(\operatorname{Flat} A) = \mathbf{K}(\operatorname{Proj} A)^{\perp}$. He also establishes the existence of a recollement

$$\mathbf{K}_{\mathrm{p}}(\operatorname{Flat} A) \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \mathbf{K}(\operatorname{Flat} A) \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \mathbf{K}(\operatorname{Proj} A),$$

which implies the following equivalence

$$\mathbf{K}(\operatorname{Proj} A) \xrightarrow{\text{in}} \mathbf{K}(\operatorname{Flat} A) \xrightarrow{\text{can}} \frac{\mathbf{K}(\operatorname{Flat} A)}{\mathbf{K}_{\mathrm{p}}(\operatorname{Flat} A)}$$

of triangulated categories. We denote the quotient $\mathbf{K}(\operatorname{Flat} A)/\mathbf{K}_{\mathrm{p}}(\operatorname{Flat} A)$ by $\mathbf{D}(\operatorname{Flat} A)$, and refer to it as 'pure derived category of flat modules'. It is a triangulated category with coproducts.

2.1.1. The above recollement, in particular, implies that the inclusion $j : \mathbf{K}(\operatorname{Proj} A) \to \mathbf{K}(\operatorname{Flat} A)$ has a right adjoint $j_\rho : \mathbf{K}(\operatorname{Flat} A) \to \mathbf{K}(\operatorname{Proj} A)$, see [Nee08, Proposition 8.1]. This means that any complex F in $\mathbf{K}(\operatorname{Flat} A)$ fits into a triangle $P \to F \to L \rightsquigarrow$, in which $P \in \mathbf{K}(\operatorname{Proj} A)$ and $L \in \mathbf{K}(\operatorname{Proj} A)^{\perp} = \mathbf{K}_{\mathrm{p}}(\operatorname{Flat} A)$.

Therefore F is isomorphic to P in $\mathbf{D}(\operatorname{Flat} A)$, as L belongs to $\mathbf{K}_{\mathrm{p}}(\operatorname{Flat} A)$. So any complex of flats can be replaced in $\mathbf{D}(\operatorname{Flat} A)$ by a complex of projectives.

Remark 2.1.2. (1) Note that $\mathbf{K}_{\mathrm{p}}(\operatorname{Flat} A)$ and $\mathbf{K}(\operatorname{Flat} A)$ can be generalized to schemes. This is one of the most advantages features of working in $\mathbf{D}(\operatorname{Flat} A)$ rather than $\mathbf{K}(\operatorname{Proj} A)$. In view of this fact, Murfet [Mu07] studied the homotopy category $\mathbf{K}(\operatorname{Flat} X)/\mathbf{K}_{\mathrm{p}}(\operatorname{Flat} X)$, in which X is a noetherian semi-separated scheme. He called this quotient, the mock homotopy category of projectives, denoted $\mathbf{K}_{\mathrm{m}}(\operatorname{Proj} X)$.

(2) Let $Q : \mathbf{K}(\operatorname{Flat} A) \to \mathbf{D}(\operatorname{Flat} A) = \mathbf{K}(\operatorname{Flat} A)/\mathbf{K}_{\mathrm{p}}(\operatorname{Flat} A)$ be the Verdier quotient. Clearly the canonical functor $\mathbf{K}(\operatorname{Flat} A) \to \mathbf{D}(A)$ vanishes on the objects of $\mathbf{K}_{\mathrm{p}}(\operatorname{Flat} A)$ and so there exists a unique functor $U : \mathbf{D}(\operatorname{Flat} A) \to \mathbf{D}(A)$, completing the diagram

$$\begin{array}{ccc} \mathbf{K}(\operatorname{Flat} A) & \xrightarrow{\;Q\;} & \mathbf{D}(\operatorname{Flat} A) \\ & \searrow & \Big\downarrow{\scriptstyle U} \\ & & \mathbf{D}(A). \end{array}$$

The kernel of U is the full subcategory of $\mathbf{D}(\operatorname{Flat} A)$ consisting of acyclic complexes. We denote it by $\mathbf{D}_{\mathrm{ac}}(\operatorname{Flat} A)$. By [Mu07, Theorem 5.5], the sequence

$$\mathbf{D}_{\mathrm{ac}}(\operatorname{Flat} A) \xrightarrow{\text{inc}} \mathbf{D}(\operatorname{Flat} A) \to \mathbf{D}(A)$$

induces a recollement

$$\mathbf{D}_{\mathrm{ac}}(\operatorname{Flat} A) \;\begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix}\; \mathbf{D}(\operatorname{Flat} A) \;\begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix}\; \mathbf{D}(A).$$

In particular, the functor U has a left adjoint $U_\lambda : \mathbf{D}(A) \to \mathbf{D}(\operatorname{Flat} A)$ that, by [Mu07, Remark 5.9(iii)], takes flat resolutions. That is, for a complex X, $U_\lambda(X)$ is isomorphic to a flat resolution of X in $\mathbf{D}(\operatorname{Flat} A)$.

(3) In $\mathbf{D}(\operatorname{Flat} A)$ flat resolutions are unique up to isomorphism and are functorial, [Mu07, Remark 3.7]. So, in particular, they can be used to define cohomology. This is another advantage of working in $\mathbf{D}(\operatorname{Flat} A)$ rather than $\mathbf{K}(\operatorname{Flat} A)$, where flat resolutions are not unique.

(4) The class of pure complexes was first introduced and studied in [EGR98], where these complexes were called *flat*. Some characterizations of such complexes are given in [EGR98, Theorem 2.4].

(5) Let P and C be complexes in $\mathbf{K}(\operatorname{Flat} A)$. Using the formalism of localization functors, it is easy to see [Nee01, Lemma 9.1.5] that the canonical map

$$\operatorname{Hom}_{\mathbf{K}(\operatorname{Flat} A)}(P, C) \to \operatorname{Hom}_{\mathbf{D}(\operatorname{Flat} A)}(P, C),$$

is an isomorphism if either P is a complex of projectives or C is a bounded below complex of cotorsion flat A-modules. Recall that an A-module C is called cotorsion if $\operatorname{Ext}^1_A(F, C) = 0$, for all flat A-modules F. Throughout the paper, we shall use this fact without any further mention. The class of all cotorsion flat modules will be denoted by $\operatorname{Cof} A$.

3. Total acyclicity and homological dimensions

In this section, using a notion of total acyclicity in the pure derived category of flats, we introduce the class of **F**-Gorenstein flat modules and an associated homological dimension. We compare this dimension with other related dimensions and show (Theorem 3.15) that this dimension can be computed using certain flat resolutions. Finally, we present some explicit examples.

3.1. Totally acyclic complexes. A complex P of projectives is called totally acyclic if it is acyclic and remains acyclic after applying the functor $\mathrm{Hom}_A(\ ,Q)$, for any projective module Q. We let $\mathbf{K}_{\mathrm{tac}}(\mathrm{Proj}\,A)$ denotes the full subcategory of $\mathbf{K}(\mathrm{Proj}\,A)$ consisting of totally acyclic complexes of projectives. So

$$\mathbf{K}_{\mathrm{tac}}(\mathrm{Proj}\,A)=\mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}\,A)\cap{}^{\perp}(\mathrm{Proj}\,A),$$

where the orthogonal is taken in $\mathbf{K}(\mathrm{Proj}\,A)$. An A-module G is called Gorenstein projective if it is a syzygy of a totally acyclic complex of projectives, see [EJ95].

3.2. F-totally acyclic complexes. A complex F of flat modules is called **F**-totally acyclic if it belongs to

$$\mathbf{D}_{\mathrm{ac}}(\mathrm{Flat}\,A)\cap{}^{\perp}(\mathrm{Flat}\,A),$$

where the orthogonal is taken in $\mathbf{D}(\mathrm{Flat}\,A)$. The full subcategory of $\mathbf{D}(\mathrm{Flat}\,A)$ consisting of **F**-totally acyclic complexes is denoted by $\mathbf{D}_{\mathrm{tac}}(\mathrm{Flat}\,A)$. An A-module G is called **F**-Gorenstein flat if it is a syzygy of an **F**-totally acyclic complex.

Clearly, any flat module is **F**-Gorenstein flat. In Example 3.1.3, we present an example of an **F**-Gorenstein flat module which is neither of finite flat dimension nor of finite Gorenstein projective dimension.

Remark 3.3. (1) Let us remind the notion of Gorenstein flat modules, see [EJ00]. A complete flat resolution of modules is an exact sequence of flat R-modules

$$\mathbf{F}_\bullet=\cdots\to F_{i+1}\to F_i\to F_{i-1}\to\cdots$$

such that $E\otimes_R\mathbf{F}_\bullet$ is exact for any injective right R-module E. An R-module M is called Gorenstein flat if it is a syzygy of a complete flat resolution, i.e. it has the form $M=\mathrm{Ker}(F_i\to F_{i-1})$ for some integer i. It is clear that every flat R-module is Gorenstein flat.

(2) The name '**F**-totally acyclic' has been used in a recent joint paper of the second and Daniel Murfet. They work over a commutative noetherian ring and have used this name for acyclic complexes of flat modules that remain acyclic after applying the functor $I\otimes_A -$, for any injective A-module I [MS09, Definition 4.1]. Theorem 4.18 of [MS09] shows that over noetherian rings both definitions are the same. Hence our choice of nomenclature is compatible with the name used there.

Definition 3.4. Let X be a complex and let P_X (resp. F_X) be a projective (resp. flat) resolution of X.

(1) A *complete projective resolution* of X is a triangle

$$T_X\to P_X\to Z_X\rightsquigarrow$$

in $\mathbf{K}(\mathrm{Proj}\,A)$, such that $T_X\in\mathbf{K}_{\mathrm{tac}}(\mathrm{Proj}\,A)$ and $Z_X\in\mathbf{K}_{\mathrm{tac}}(\mathrm{Proj}\,A)^{\perp}$. If furthermore X is homologically bounded below, we say that the Gorenstein projective dimension of X is less than or equal to n, a fixed integer, denoted $\mathrm{Gpd}_A X\leqslant n$, if $-n\leqslant\inf X$ and for all $i<-n$, $\mathrm{Coker}\,\partial^i_{Z_X}$ is projective. If no integer n exists with $\mathrm{Gpd}_A X\leqslant n$, then we define $\mathrm{Gpd}_A X=\infty$.

(2) An **F**-*complete flat resolution* of X is a triangle

$$T_X\to F_X\to Z_X\rightsquigarrow$$

in $\mathbf{D}(\mathrm{Flat}\,A)$, such that $T_X\in\mathbf{D}_{\mathrm{tac}}(\mathrm{Flat}\,A)$ and $Z_X\in\mathbf{D}_{\mathrm{tac}}(\mathrm{Flat}\,A)^{\perp}$. If furthermore X is homologically bounded below, we say that the **F**-Gorenstein flat dimension of X is less than or equal to n, a fixed integer, denoted $\mathbf{F}\text{-}\mathrm{Gfd}_A X\leqslant n$, if $-n\leqslant\inf X$ and for all $i<-n$, $\mathrm{Coker}\,\partial^i_{Z_X}$ is flat. If no integer n exists with $\mathbf{F}\text{-}\mathrm{Gfd}_A X\leqslant n$, then we define $\mathbf{F}\text{-}\mathrm{Gfd}_A X=\infty$.

It is easy to see that the notion $\mathrm{Gpd}_A X < n$, for a complex X, is well defined. Our next proposition guarantees that for X the notion $\mathbf{F}\text{-}\mathrm{Gfd}_A X < n$ also makes sense. We preface the proposition with a well-known lemma.

Lemma 3.5. *Let F be a homologically bounded below complex of flats and n be a fixed integer with $n \leqslant \inf F$. Then $\operatorname{Ker} \partial_F^i$ is flat for all $i < n$ if and only if $\mathrm{H}^i \operatorname{Hom}_A(N, F) = 0$, for all $i < n$ and all finitely presented A-modules N.*

Proof. Both directions follow immediately by applying Cohen's Theorem, see e.g. [R09, Lemma 3.70]. □

Proposition 3.6. *Consider the triangle $F \to F' \to F'' \rightsquigarrow$ in $\mathbf{D}(\mathrm{Flat}\, A)$ of homologically bounded below complexes and let n be an integer with $n \leqslant \min\{\inf F, \inf F', \inf F''\}$. Assume that any two of F, F' and F'' have flat kernels in degrees less than n. Then so has the complex.*

Proof. We assume that F and F'' have flat kernels in degrees less than n and show that F' also has this property. By 2.1.1, there exist triangles

$$P \to F \to E \rightsquigarrow, \qquad P' \to F' \to E' \rightsquigarrow \quad \text{and} \quad P'' \to F'' \to E'' \rightsquigarrow$$

in $\mathbf{K}(\mathrm{Flat}\, A)$ such that P, P' and P'' are complexes of projectives and E, E' and E'' are acyclic complexes with flat kernels. Lemma 3.5 implies that P and P'' also have flat kernels in degrees less than n. Now consider the diagram

$$\begin{array}{ccccccc}
P & \longrightarrow & P' & \longrightarrow & P'' & \rightsquigarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
F & \longrightarrow & F' & \longrightarrow & F'' & \rightsquigarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
E & \longrightarrow & E' & \longrightarrow & E'' & \rightsquigarrow & \\
\wr\!\downarrow & & \wr\!\downarrow & & \wr\!\downarrow & &
\end{array}$$

of triangles. Applying Lemma 3.5 to the triangle in the first row, we learn that the modules $\operatorname{Ker} \partial_{P'}^i$ are flat for $i < n$. Another application of Lemma 3.5, this time to the second column, completes the proof. □

Remark 3.7. There is a notion of Gorenstein projective dimension for complexes introduced by Veliche. By [V06, 3.1], a complex X is of finite Gorenstein projective dimension if there exists a totally acyclic complex T_X of projectives and a morphism $\nu : T_X \to P_X$ such that $\nu^i : T_X^i \to P_X^i$ is bijective for all $i \ll 0$, where P_X is a projective resolution of X. In this case, $\nu : T_X \to P_X$ is called a complete projective resolution of X. She showed that the Gorenstein projective dimension of X is finite if and only if X admits a complete projective resolution.

Now assume that X is a complex of finite Gorenstein projective dimension in the sense of Veliche. By taking the mapping cone of ν, we get the triangle $T_X \to P_X \to Z_X \rightsquigarrow$ such that Z_X is a bounded below complex of projectives. So $Z_X \in \mathbf{K}_{\mathrm{tac}}(\mathrm{Proj}\, A)^{\perp}$. Therefore, X is of finite Gorenstein projective dimension in our sense; see [Jor07, Lemma 3.6]. But, as we will see, for a complex X it may happen that the complete projective resolution in our sense exists, without any assumption on the finiteness of the Gorenstein projective dimension of X.

Proposition 3.8. *In $\mathbf{D}(\mathrm{Flat}\, A)$, any $\mathbf{F}$-complete flat resolution of a complex X is isomorphic to an $\mathbf{F}$-complete flat resolution whose terms are all projectives.*

Proof. Let $T_X \to F_X \to Z_X \rightsquigarrow$ be an **F**-complete flat resolution of X. In the notation of 2.1.1, we have a diagram

$$\begin{array}{ccccccc}
jj_\rho(T_X) & \longrightarrow & jj_\rho(F_X) & \longrightarrow & jj_\rho(Z_X) & \rightsquigarrow \\
\downarrow{\scriptstyle \xi_{T_X}} & & \downarrow{\scriptstyle \xi_{F_X}} & & \downarrow{\scriptstyle \xi_{Z_X}} & \\
T_X & \longrightarrow & F_X & \longrightarrow & Z_X & \rightsquigarrow
\end{array}$$

in which $\xi : jj_\rho \to \mathrm{id}$ is the adjunction morphism. But the mapping cones of ξ_{T_X}, ξ_{F_X} and ξ_{Z_X} are all in $\mathbf{K}_\mathrm{p}(\mathrm{Flat}\, A)$. Therefore both triangles are isomorphic in $\mathbf{D}(\mathrm{Flat}\, A)$. □

Proposition 3.9. *Consider an exact triangle $X' \to X \to X'' \rightsquigarrow$ in $\mathbf{D}(A)$. If any two of the complexes X', X and X'' admit* **F***-complete flat resolutions, then so does the third.*

Proof. This follows by a standard argument in the theory of Bousfield localization. It is enough to consider the case where both X' and X have **F**-complete flat resolutions. So we have triangles in $\mathbf{D}(\mathrm{Flat}\, A)$

$$T_{X'} \to F_{X'} \to Z_{X'} \rightsquigarrow \quad \text{and} \quad T_X \to F_X \to Z_X \rightsquigarrow$$

in which $T_{X'}, T_X \in \mathbf{D}_{\mathrm{tac}}(\mathrm{Flat}\, A)$ and $Z_{X'}, Z_X \in \mathbf{D}_{\mathrm{tac}}(\mathrm{Flat}\, A)^\perp$. Since $Z_X \in \mathbf{D}_{\mathrm{tac}}(\mathrm{Flat}\, A)^\perp$, we may deduce that there exists a morphism $T_{X'} \to T_X$

$$\begin{array}{ccccccc}
T_{X'} & \longrightarrow & F_{X'} & \longrightarrow & Z_{X'} & \rightsquigarrow \\
\downarrow & & \downarrow & & & \\
T_X & \longrightarrow & F_X & \longrightarrow & Z_X & \rightsquigarrow
\end{array}$$

that commutes the square on the left. We complete it to a morphism of triangles

$$\begin{array}{ccccccc}
T_{X'} & \longrightarrow & F_{X'} & \longrightarrow & Z_{X'} & \rightsquigarrow \\
\downarrow & & \downarrow & & \downarrow & \\
T_X & \longrightarrow & F_X & \longrightarrow & Z_X & \rightsquigarrow .
\end{array}$$

But the mapping cone of $T_{X'} \to T_X$ belongs to $\mathbf{D}_{\mathrm{tac}}(\mathrm{Flat}\, A)$ and the diagram

$$\begin{array}{ccccccc}
T_{X'} & \longrightarrow & F_{X'} & \longrightarrow & Z_{X'} & \rightsquigarrow \\
\downarrow & & \downarrow & & \downarrow & \\
T_X & \longrightarrow & F_X & \longrightarrow & Z_X & \rightsquigarrow \\
\downarrow & & \downarrow & & \downarrow & \\
T_{X''} & \longrightarrow & F_{X''} & \longrightarrow & Z_{X''} & \rightsquigarrow \\
\wr\downarrow & & \wr\downarrow & & \wr\downarrow &
\end{array}$$

implies that $Z_{X''} \in \mathbf{D}_{\mathrm{tac}}(\mathrm{Flat}\, A)^\perp$. Hence X'' also admits an **F**-complete flat resolution. □

In view of Proposition 3.6 and the functoriality of flat resolutions this implies:

Corollary 3.10. *Let $X' \to X \to X'' \rightsquigarrow$ be an exact triangle in $\mathbf{D}(A)$. If any two of the complexes X', X and X'' have finite* **F***-Gorenstein flat dimension, then so does the third.*

Towards the end of this section we prove that finiteness of **F**-Gorenstein flat dimension can be determined via a flat resolution, see Theorem 3.15 below. To this end, we need some lemmas and propositions.

Lemma 3.11. *Let F be a homologically bounded below complex of flats and t be an integer with $t \leqslant \inf F$. Set $K^i = \operatorname{Ker} \partial_F^i$. Let $n < t-1$ and $G \xrightarrow{\varepsilon} K^{n+1}$ be a proper flat resolution of K^{n+1} which is constructed by taking flat covers and kernels, repeatedly. Let F' denote the complex*

$$\cdots \to G^{-2} \xrightarrow{\partial_G^{-2}} G^{-1} \xrightarrow{\partial_G^{-1}} G^0 \xrightarrow{\iota\varepsilon} F^{n+1} \xrightarrow{\partial_F^{n+1}} F^{n+2} \to \cdots,$$

where $\iota : K^{n+1} \to F^{n+1}$ is the inclusion. Then $F' \cong F$ in $\mathbf{D}(\operatorname{Flat} A)$.

Proof. Since G is a proper flat resolution of K^{n+1}, there is a morphism $\varphi : {}_hF_{\leqslant n} \to G$, lifting the identity morphism on K^{n+1}. So we get a morphism of complexes

$$\begin{array}{ccccccccccccc}
\cdots \longrightarrow & F^{n-3} & \longrightarrow & F^{n-2} & \longrightarrow & F^{n-1} & \longrightarrow & F^{n} & \longrightarrow & F^{n+1} & \longrightarrow & F^{n+2} & \longrightarrow \cdots \\
 & \downarrow{\scriptstyle \varphi^{n-3}} & & \downarrow{\scriptstyle \varphi^{n-2}} & & \downarrow{\scriptstyle \varphi^{n-1}} & & \downarrow{\scriptstyle \varphi^{n}} & & \| & & \| & \\
\cdots \longrightarrow & G^{-3} & \longrightarrow & G^{-2} & \longrightarrow & G^{-1} & \longrightarrow & G^{0} & \longrightarrow & F^{n+1} & \longrightarrow & F^{n+2} & \longrightarrow \cdots .
\end{array}$$

It is easy to see that its mapping cone is obtained by pasting the mapping cones of the following two diagrams

$$\begin{array}{cccccccccc}
\cdots \longrightarrow & F^{n-3} & \longrightarrow & F^{n-2} & \longrightarrow & F^{n-1} & \longrightarrow & F^{n} & \longrightarrow & 0 \\
 & \downarrow{\scriptstyle \varphi^{n-3}} & & \downarrow{\scriptstyle \varphi^{n-2}} & & \downarrow{\scriptstyle \varphi^{n-1}} & & \downarrow{\scriptstyle \varphi^{n}} & & \\
\cdots \longrightarrow & G^{-3} & \longrightarrow & G^{-2} & \longrightarrow & G^{-1} & \longrightarrow & G^{0} & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{cccccccc}
0 \longrightarrow & G^{0} & \longrightarrow & F^{n+1} & \longrightarrow & F^{n+2} & \longrightarrow \cdots \\
 & \| & & \| & & \| & \\
0 \longrightarrow & G^{0} & \longrightarrow & F^{n+1} & \longrightarrow & F^{n+2} & \longrightarrow \cdots .
\end{array}$$

The mapping cone of the first diagram is pure because both rows are flat resolutions of K^{n+1}. Moreover, it is easy to see that the mapping cone of the second diagram is pure exact. Therefore the mapping cone of the main diagram is pure and so belongs to $\mathbf{K}_{\mathrm{p}}(\operatorname{Flat} A)$. This means that considering in $\mathbf{D}(\operatorname{Flat} A)$, F' is isomorphic to F. $\square$

Recall that the kernel of a flat cover is cotorsion, and so it follows from the lemma that for any homologically bounded below complex F in $\mathbf{D}(\operatorname{Flat} A)$ and any integer $t \leqslant \inf F$, we may assume that F^i and also $\operatorname{Ker} \partial_F^i$, for all $i < t-2$, are cotorsion. We use this fact in the proof of the next proposition.

Definition 3.12. Two modules L and K are called *flat equivalent* if there exist flat modules F and F' such that $L \oplus F \cong K \oplus F'$.

Proposition 3.13. *Let $F \xrightarrow{f} G \xrightarrow{g} E \rightsquigarrow$ be a triangle in $\mathbf{D}(\operatorname{Flat} A)$ in which F and G are homologically bounded below and E has flat kernels in degrees small enough. Then there exists a triangle $F' \xrightarrow{f'} G' \xrightarrow{g'} E' \rightsquigarrow$ which is isomorphic to the above triangle in $\mathbf{D}(\operatorname{Flat} A)$ such that $\operatorname{Ker} \partial_{F'}^i$ and $\operatorname{Ker} \partial_{G'}^i$ are flat equivalent, for all $i \ll 0$.*

Proof. Without loss of generality, we may assume that F is a complex of projectives and that f is a chain map. Let $t \leqslant \inf G$. The above lemma allows us to replace G with a complex G' such that $\operatorname{Ker}\partial^i_{G'}$ is cotorsion, for all $i < t-2$. Let f_1 be the composition of f with the map $G \to G'$ of the lemma, and let E_1 be the mapping cone of f_1. In the following commutative diagram π denotes the truncation map, and E_2 the mapping cone of πf_1 (on the level of chain maps)

$$\begin{array}{ccccc} F & \xrightarrow{f_1} & G' & \longrightarrow & E_1 \rightsquigarrow \\ \| & & \downarrow{\scriptstyle\pi} & & \downarrow \\ F & \xrightarrow{\pi f_1} & {}_hG'_{\leqslant t-3} & \longrightarrow & E_2 \rightsquigarrow . \end{array}$$

In view of the fact that $\operatorname{Ker}\partial^i_{E_1}$ is flat in small degrees, we may deduce that $\operatorname{Ker}\partial^i_{E_2}$ is flat for all $i \ll 0$.

We claim that ${}_hG'_{\leqslant t-3}$ belongs to $\mathbf{K}_{\mathrm{p}}(\operatorname{Flat} A)^{\perp}$. To see this, consider the complex

$$G'' : \cdots \to G^{t-5} \to G^{t-4} \xrightarrow{\partial^{t-4}} G^{t-3} \to \operatorname{Coker}\partial^{t-4} \to 0,$$

which is an acyclic complex of cotorsion modules with cotorsion kernels. It is clear that for any complex $B \in \mathbf{K}_{\mathrm{p}}(\operatorname{Flat} A)$, $\operatorname{Hom}_{\mathbf{K}(A)}(B, \operatorname{Coker}\partial^{t-4})$ and $\operatorname{Hom}_{\mathbf{K}(A)}(B, G'')$ both vanish. So one can deduce the claim from the associated triangle.

Up to isomorphism in $\mathbf{D}(\operatorname{Flat} A)$, F can be replaced by a complex F' whose kernels in small degrees are cotorsion. Moreover the composite $F' \cong F \to {}_hG'_{\leqslant t-3}$ in $\mathbf{D}(\operatorname{Flat} A)$ can, by the observations we have just made, be represented by a chain map. Hence we get a triangle

$$F' \to {}_hG'_{\leqslant t-3} \to E_3 \rightsquigarrow,$$

in which $\operatorname{Ker}\partial^i_{E_3}$ is flat for all $i \ll 0$.

Now choose an integer s small enough that both $F'_{\leqslant s}$ and $G'_{\leqslant s}$ are acyclic, F'^i, G'^i and $L'^i = \operatorname{Ker}\partial^i_{F'}$ are cotorsion for $i < s$ and $\operatorname{Ker}\partial^i_{E_3}$ is flat for all $i < s$. Consider the diagram (with $K'^i = \operatorname{Ker}\partial^i_{G'}$)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L'^i & \longrightarrow & F'^i & \longrightarrow & F'^{i+1} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K'^i & \longrightarrow & G'^i & \longrightarrow & G'^{i+1} & \longrightarrow & \cdots \end{array}$$

where $i < s-1$, to get the mapping cone sequence

$$0 \to L'^i \to F'^i \oplus K'^i \to F'^{i+1} \oplus G'^i \xrightarrow{\varphi^i} \cdots.$$

This gives us the short exact sequence $0 \to L'^i \to F'^i \oplus K'^i \to \operatorname{Ker}\varphi^i \to 0$, which is split, because L'^i is cotorsion and $\operatorname{Ker}\varphi^i$, which is a kernel of E_3, is flat. So L'^i and K'^i are flat equivalent.

Finally, note that the triangle $F' \xrightarrow{f'} G' \xrightarrow{g'} E' \rightsquigarrow$ is isomorphic to the first triangle and the previous paragraph, implies that the syzygies of F' and G', in small degrees are flat equivalent. $\square$

Lemma 3.14. *Assume that $X \oplus F$ is* **F**-*Gorenstein flat, where F is a flat module. Then X is* **F**-*Gorenstein flat.*

Proof. Let T be an **F**-totally acyclic complex with syzygy $X \oplus F$. So there exists an exact sequence

$$0 \to X \oplus F \to T^0 \to T^1 \to T^2 \to T^3 \to \cdots$$

which is $\operatorname{Hom}_A(\ ,G)$-exact, for any cotorsion flat module G. Now let

$$0 \to F \to C^0 \to C^1 \to C^2 \to C^3 \to \cdots$$

be a resolution of F by cotorsion modules which is constructed step by step using cotorsion preenvelopes of cokernels in the usual way. Note that C^i, for $i \geqslant 0$, is flat and cotorsion; see [ET01, Theorem 10] and [BEE01] for the construction of this resolution. Consider the diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & X \oplus F & \longrightarrow & T^0 & \longrightarrow & T^1 & \longrightarrow & T^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & C^0 & \longrightarrow & C^1 & \longrightarrow & C^2 & \longrightarrow & \cdots.
\end{array}$$

The vertical maps exist because the upper row is $\operatorname{Hom}_A(\ ,\operatorname{Cof} A)$-exact. Clearly the mapping cone of this diagram

$$0 \to X \oplus F \to T^0 \oplus F \to T^1 \oplus C^0 \to T^2 \oplus C^1 \to \cdots$$

is $\operatorname{Hom}_A(\ ,\operatorname{Cof} A)$-exact. Using this, we easily obtain an exact sequence which is moreover $\operatorname{Hom}_A(\ ,\operatorname{Cof} A)$-exact

$$0 \to X \to T^0 \to T^1 \oplus C^0 \to T^2 \oplus C^1 \to \cdots.$$

Now one can paste a projective resolution of X to the first column and use the fact that $\operatorname{Ext}^1_A(X \oplus F, C) = 0$, for any $C \in \operatorname{Cof} A$, to deduce the result. $\square$

Theorem 3.15. *Let X be a homologically bounded below complex of A-modules. Then $\mathbf{F}\text{-}\operatorname{Gfd}_A X$ is finite if and only if there exists a flat resolution F_X of X and an integer $n < \inf X$ such that $\operatorname{Ker} \partial^n_{F_X}$ is $\mathbf{F}$-Gorenstein flat.*

Proof. 'If'. Let t be an integer less than or equal to $\inf X$. Assume that $F_X \to X$ is a flat resolution of X such that $K = \operatorname{Ker} \partial^n_{F_X}$ is a syzygy of an $\mathbf{F}$-totally acyclic complex T_X. We can assume that $n \leqslant t$. Also, without loss of generality, we may assume that $T^i_X = F^i_X$, for all $i < n$. On the other hand, there is an $\mathbf{F}$-totally acyclic complex $\overline{T}_X$ whose terms are projective and fits into a triangle $\overline{T}_X \xrightarrow{f} T_X \to Z_X \rightsquigarrow$ in which Z_X is acyclic with flat kernels. Since $\overline{T}_X$ is a complex of projectives and it is $\mathbf{F}$-totally acyclic, we may deduce that for all $i \geqslant n$, there exist maps $h^i : \overline{T}^i_X \to F^i_X$

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \overline{T}^{n-2}_X & \longrightarrow & \overline{T}^{n-1}_X & \longrightarrow & \overline{T}^{n}_X & \longrightarrow & \overline{T}^{n+1}_X & \longrightarrow & \cdots \\
 & & \downarrow {\scriptstyle f^{n-2}} & & \downarrow {\scriptstyle f^{n-1}} & & \downarrow {\scriptstyle f^{n}} & & \downarrow {\scriptstyle f^{n+1}} & & \\
\cdots & \longrightarrow & F^{n-2}_X & \longrightarrow & F^{n-1}_X & \longrightarrow & T^{n}_X & \longrightarrow & T^{n+1}_X & \longrightarrow & \cdots \\
 & & \| & & \| & & {\scriptstyle h^n} & & & & \\
\cdots & \longrightarrow & F^{n-2}_X & \longrightarrow & F^{n-1}_X & \longrightarrow & F^{n}_X & \longrightarrow & F^{n+1}_X & \longrightarrow & \cdots.
\end{array}$$

So considering these maps and the composition of the identity with the maps f^i, $i < n$, we get a morphism $\overline{T}_X \to F_X$ such that its mapping cone has flat kernels, for all $i < n$. That is $\mathbf{F}\text{-}\operatorname{Gfd}_A X < \infty$.

'Only if'. Since $\mathbf{F}\text{-}\operatorname{Gfd}_A X$ is finite, by definition X admits an $\mathbf{F}$-complete flat resolution. But by Proposition 3.13, we may consider an $\mathbf{F}$-complete flat resolution

$$T_X \to F_X \to Z_X \rightsquigarrow$$

in $\mathbf{D}(\operatorname{Flat} A)$, such that $\operatorname{Ker}\partial^i_{T_X}$ and $\operatorname{Ker}\partial^i_{F_X}$ are flat equivalent, for all $i \ll 0$. Hence there exists a flat A-module G such that $\operatorname{Ker}\partial^i_{F_X} \oplus G$ is a syzygy of an $\mathbf{F}$-totally acyclic complex. The proof now can be completed using Lemma 3.14. $\square$

3.16. Let F_X be a flat resolution of an A-module X. Assume that there exists an integer n such that for all $i < n$, F^i_X and $\operatorname{Ker}\partial^i_{F_X}$ are cotorsion. Then the proofs of Proposition 3.13 and Theorem 3.15, imply that $\mathbf{F}\text{-}\mathrm{Gfd}_A X$ is finite if and only if $\operatorname{Ker}\partial^j_{F_X}$, for some $j < n$ is $\mathbf{F}$-Gorenstein flat. In particular, if F_X is a minimal flat resolution of X, by [EJ00, Lemma 5.3.25], F^i and $\operatorname{Ker}\partial^i_{F_X}$, for all $i < 0$, are cotorsion and hence $\mathbf{F}\text{-}\mathrm{Gfd}_A X$ is finite if and only if there exists an integer j such that $\operatorname{Ker}\partial^j_{F_X}$ is $\mathbf{F}$-Gorenstein flat. Recall that a flat resolution F_X of an A-module X is called minimal, if it is constructed using repeated kernels and flat covers (instead of precovers).

3.1. Comparison and examples

We provide some examples of objects admitting $\mathbf{F}$-complete flat resolutions and objects of finite $\mathbf{F}$-Gorenstein flat dimension.

Consider the following inclusion functors of triangulated categories

$$I : \mathbf{K}_{\mathrm{tac}}(\operatorname{Proj} A) \to \mathbf{K}(\operatorname{Proj} A) \quad \text{and} \quad J : \mathbf{D}_{\mathrm{tac}}(\operatorname{Flat} A) \to \mathbf{D}(\operatorname{Flat} A).$$

When we know that one of these functors admits a right adjoint, we may deduce that the corresponding triangle in Definition 3.4 exists, see e.g. [Kra05, Lemma 3.2].

When A is a commutative noetherian ring with a dualizing complex, the existence of a right adjoint for I is proved in [Jor05]. In case A is commutative, noetherian and has finite Krull dimension, the existence of a right adjoint for I is proved in [MS09]. But for an arbitrary ring, the problem of the existence of a right adjoint for I is still open.

When A is commutative and noetherian, the existence of right adjoint for J is proved in [MS09]. This, in turn, implies that, over such rings, any complex of flat A-modules admits an $\mathbf{F}$-complete flat resolution. In fact their result is more general: they prove the existence of adjoint when X is a semi-separated noetherian scheme. But the problem of the existence of a right adjoint for J, for an arbitrary ring, is still open.

Examples 3.1.1.

(1) Assume that A is a ring with the property that every cotorsion flat module has finite injective dimension. With this assumption, one can deduce easily that any acyclic complex of flats is $\mathbf{F}$-totally acyclic. That is $\mathbf{D}_{\mathrm{tac}}(\operatorname{Flat} A) = \mathbf{D}_{\mathrm{ac}}(\operatorname{Flat} A)$. But by [Mu07, Proposition 5.4], the inclusion functor $\mathbf{D}_{\mathrm{ac}}(\operatorname{Flat} A) \to \mathbf{D}(\operatorname{Flat} A)$ admits a right adjoint. Hence J admits a right adjoint.
(2) Assume that A is a ring with the property that every flat module has finite projective dimension. Then $\mathbf{K}_{\mathrm{tac}}(\operatorname{Proj} A)$ is equivalent to the category $\mathbf{D}_{\mathrm{tac}}(\operatorname{Flat} A)$. This, in turn, implies that any Gorenstein projective A-module is $\mathbf{F}$-Gorenstein flat. Such rings have already studied by several authors in different settings. For example, it is shown in [BG00], that if A is a ring in which any flat module has finite projective dimension, then for any finite group G, the group ring AG also has the same property.

It is obvious that any flat module is $\mathbf{F}$-Gorenstein flat. In the following, we provide an example of a group G and an AG-module F such that F is $\mathbf{F}$-Gorenstein flat but it is neither of finite flat dimension nor of finite Gorenstein projective dimension.

Lemma 3.1.2. *Let G be a finite group. Then any flat A-module F is an $\mathbf{F}$-Gorenstein flat AG-module, with the trivial action.*

Proof. Consider the short exact sequence

$$0 \to F \xrightarrow{\iota} \mathrm{Hom}_A(AG, F) \to L^0 \to 0,$$

of AG-modules, where ι is the canonical injection. It is known that as an A-sequence, it is split and hence L^0 is flat A-module. Moreover, since G is finite, $AG \otimes_A F \cong \mathrm{Hom}_A(AG, F)$ and hence $\mathrm{Hom}_A(AG, F)$ is flat AG-module. This argument can be repeated this time starting from L^0 to get a short exact sequence

$$0 \to L^0 \to F^1 \to L^1 \to 0,$$

of AG-modules in which F^1 is flat AG-module, L^1 is flat A-module and the sequence is A-split. By splicing these short exact sequences we get a right resolution

$$0 \to F \to F^0 \to F^1 \to F^2 \to \cdots$$

of F by flat AG-modules. Moreover, the resolution is split when considered over A. On the other hand, consider a free resolution of the trivial AG-module A by finitely generated free AG-modules

$$\cdots \to (AG)^{n_1} \to (AG)^{n_0} \to A \to 0.$$

Clearly this sequence is split over A. By applying the functor $- \otimes_A F$ on this sequence we get a resolution of F by flat AG-modules

$$\cdots \to (AG \otimes_A F)^{n_1} \to (AG \otimes_A F)^{n_0} \to F \to 0.$$

So we get an exact sequence

$$\begin{array}{ccccccccccccc} \cdots \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & F_0 & & \longrightarrow & & F^0 & \longrightarrow F^1 \longrightarrow & F^2 & \longrightarrow \cdots \\ & & & & & & \searrow & & \nearrow & & & & \\ & & & & & & & F & & & & & \end{array} \qquad (*)$$

of flat AG-modules which is split over A. We claim that the sequence is $\mathrm{Hom}_G(-, X)$-exact, for any flat AG-module X. Since X is flat as an AG-module, there exists a flat A-module X' such that $X \cong AG \otimes_A X'$ as AG-modules. So we have

$$\begin{aligned} \mathrm{Hom}_G(-, X) &\cong \mathrm{Hom}_G\big(-, AG \otimes_A X'\big) \\ &\cong \mathrm{Hom}_G\big(-, \mathrm{Hom}_A\big(AG, X'\big)\big) \\ &\cong \mathrm{Hom}_A\big(AG \otimes_G -, X'\big) \\ &\cong \mathrm{Hom}_A\big(-, X'\big). \end{aligned}$$

Since the complex $(*)$ is $\mathrm{Hom}_A(-, X')$-exact, the above isomorphisms imply that it is $\mathrm{Hom}_G(-, X)$-exact and hence the sequence is an **F**-complete resolution. Therefore F is **F**-Gorenstein flat. □

Now we have the necessary ingredients to present our example.

Example 3.1.3. Let G be a finite group and F a flat A-module which is not of finite Gorenstein projective dimension. Moreover, assume that $nF \neq F$, where n is the order of an element of G. Then, by Lemma 3.1.2, with the trivial action, F is an **F**-Gorenstein flat AG-module. We claim that as AG-module, it is neither of finite Gorenstein projective dimension nor of finite flat dimension.

If F is of finite Gorenstein projective dimension as AG-module, then it is easily seen that it is of finite Gorenstein projective dimension as an A-module, contradicting our assumption. To complete the claim, we show that F is not a flat AG-module. In fact, we show something stronger: the flat dimension of F over AG is not finite. Since for any subgroup G' of G, $\mathrm{fd}_{AG'} F \leqslant \mathrm{fd}_{AG} F$, we may assume that $G = \langle x \rangle$ is a cyclic group, where the order of x is n. It is known that, in this case, we have a projective resolution of A as AG-module

$$\cdots \xrightarrow{x-1} AG \xrightarrow{t} AG \xrightarrow{x-1} AG \to A \to 0,$$

where t is the norm element $1 + x + x^2 + \cdots + x^{n-1}$ of AG. By applying the functor $- \otimes_A F$ on this sequence we get the sequence

$$\cdots \xrightarrow{x-1} AG \otimes_A F \xrightarrow{t} AG \otimes_A F \xrightarrow{x-1} AG \otimes_A F \to A \otimes_A F \to 0.$$

We know that $A \otimes_A F$, with the diagonal action, is isomorphic to F as an AG-module and $AG \otimes_A F$, with the diagonal action, is a flat AG-module. That is, the above sequence is a flat resolution of F as AG-module. Now, by applying the functor $- \otimes_{AG} F$ on the above sequence we get the sequence

$$\cdots \xrightarrow{t} (AG \otimes_A F) \otimes_{AG} F \xrightarrow{x-1} (AG \otimes_A F) \otimes_{AG} F \to (A \otimes_A F) \otimes_{AG} F \to 0.$$

But it is easy to see that the homology of this complex at even degrees is isomorphic to $((AG \otimes_A F) \otimes_{AG} F)/n((AG \otimes_A F) \otimes_{AG} F)$ which is not zero, because $F/nF \neq 0$ and G is finite. Hence $\mathrm{fd}_{AG} F = \infty$.

Example 3.1.4. Let $(A, \mathfrak{m})$ be a noetherian local ring and let $\widehat{A}$ be its $\mathfrak{m}$-adic completion. Assume further that $\widehat{A}$ is not projective as an A-module. Let G be a finite group such that its order is a non-unit in A. By Lemma 3.1.2, $\widehat{A}$ is an **F**-Gorenstein flat AG-module which is not Gorenstein projective and, by the same method used in the above example, $\widehat{A}$ does not have finite flat dimension as AG-module.

4. Tate cohomology based on flat modules

In this section, we introduce a version of Tate cohomology defined using flat modules. The theory is based on Neeman's results in [Nee08] and [Nee10]. For a complex X, P_X (resp. F_X) denotes a projective (resp. flat) resolution of X. Note that by Remark 2.1.2(3), F_X is unique up to isomorphism in $\mathbf{D}(\mathrm{Flat}\, A)$.

4.1. We know that $\mathbf{K}(\mathrm{Proj}\, A)$ is a localizing subcategory of $\mathbf{K}(A)$ and by [Nee08, Corollary 5.10] satisfies Brown representability. So by [Nee01, Theorem 8.4.4], the inclusion $E : \mathbf{K}(\mathrm{Proj}\, A) \to \mathbf{K}(A)$ possesses a right adjoint E_ρ. This, in particular, implies that for an arbitrary complex Y we have a triangle

$$EE_\rho(Y) \to Y \to Z_Y \rightsquigarrow \qquad (*)$$

in $\mathbf{K}(A)$ such that $E_\rho(Y) \in \mathbf{K}(\mathrm{Proj}\, A)$ and $Z_Y \in \mathbf{K}(\mathrm{Proj}\, A)^\perp$. In the case where Y is bounded above $E_\rho(Y)$ is a projective resolution of Y, because in this case for any projective resolution P_Y of Y, the cone of the morphism $P_Y \to Y$ is an acyclic bounded above complex, and hence belongs to $\mathbf{K}(\mathrm{Proj}\, A)^\perp$. We fix the triangle $(*)$ throughout this section.

Now assume that X admits a complete projective resolution. So dual to Krause's definition [Kra05], we may consider the triangle $T_X \to P_X \to Z_X \rightsquigarrow$ in $\mathbf{K}(\operatorname{Proj} A)$ with $T_X \in \mathbf{K}_{\mathrm{tac}}(\operatorname{Proj} A)$ and $Z_X \in \mathbf{K}_{\mathrm{tac}}(\operatorname{Proj} A)^{\perp}$. The ith Tate cohomology group of X and Y, denoted $\widehat{\operatorname{Ext}}^i_A(X,Y)$, is then defined by

$$\widehat{\operatorname{Ext}}^i_A(X,Y) := \operatorname{Hom}_{\mathbf{K}(A)}\big(T_X, \Sigma^i E_\rho(Y)\big).$$

Note that since $Z_Y \in \mathbf{K}(\operatorname{Proj} A)^{\perp}$, it is clear that $\widehat{\operatorname{Ext}}^i_A(X,Y) = \operatorname{Hom}_{\mathbf{K}(A)}(T_X, \Sigma^i Y)$. Moreover, it follows from Remark 3.7 that, when X is a complex of finite Gorenstein projective dimension in the sense of [V06, 3.1], our definition of Tate cohomology is compatible with the one introduced in [V06, §4]. This motivates our next definition.

Definition 4.2. Assume that X admits an $\mathbf{F}$-complete flat resolution, so there exists a triangle $T_X \to F_X \to Z_X \rightsquigarrow$ in $\mathbf{D}(\operatorname{Flat} A)$ with $T_X \in \mathbf{D}_{\mathrm{tac}}(\operatorname{Flat} A)$ and $Z_X \in \mathbf{D}_{\mathrm{tac}}(\operatorname{Flat} A)^{\perp}$. The ith $\mathbf{F}$-Tate cohomology group of X and Y, denoted $\widehat{\operatorname{Ext}}^i_{\mathbf{F}}(X,Y)$, is defined by

$$\widehat{\operatorname{Ext}}^i_{\mathbf{F}}(X,Y) := \operatorname{Hom}_{\mathbf{D}(\operatorname{Flat} A)}\big(T_X, \Sigma^i E_\rho(Y)\big).$$

Theorem 4.3. *The above definition of* $\mathbf{F}$*-Tate cohomology group, is independent of the choice of complete resolutions.*

Proof. Consider another $\mathbf{F}$-complete flat resolution of X, say $T'_X \to F'_X \to Z'_X \rightsquigarrow$. Since flat resolutions are unique in $\mathbf{D}(\operatorname{Flat} A)$, there exists a homotopy equivalence $\mu : F_X \to F'_X$. Since $T_X \in \mathbf{D}_{\mathrm{tac}}(\operatorname{Flat} A)$ and $Z'_X \in \mathbf{D}_{\mathrm{tac}}(\operatorname{Flat} A)^{\perp}$,

$$\operatorname{Hom}_{\mathbf{D}(\operatorname{Flat} A)}\big(T_X, T'_X\big) \cong \operatorname{Hom}_{\mathbf{D}(\operatorname{Flat} A)}\big(T_X, F'_X\big).$$

So we get a unique morphism $\hat{\mu} : T_X \to T'_X$ commuting the square on the left of the following diagram

$$\begin{array}{ccccc} T_X & \longrightarrow & F_X & \longrightarrow & Z_X \rightsquigarrow \\ \downarrow{\scriptstyle \hat{\mu}} & & \downarrow{\scriptstyle \mu} & & \\ T'_X & \longrightarrow & F'_X & \longrightarrow & Z'_X \rightsquigarrow . \end{array}$$

The uniqueness of $\hat{\mu}$ implies that T_X and T'_X are homotopy equivalence in $\mathbf{D}(\operatorname{Flat} A)$, as desired. □

Note that, by the axioms of a triangulated category, we may deduce that there exists a morphism $\bar{\mu} : Z_X \to Z'_X$ such that $(\hat{\mu}, \mu, \bar{\mu})$ is a morphism of triangles.

Theorem 4.4. *Assume that A is a commutative noetherian ring of finite Krull dimension. Then for any integer i,*

$$\widehat{\operatorname{Ext}}^i_{\mathbf{F}}(X,Y) \cong \widehat{\operatorname{Ext}}^i_A(X,Y).$$

Proof. First note that, since $\dim A < \infty$, both inclusion functors I and J of Section 3.1, have right adjoints and so the corresponding triangles of Definition 3.4 exist. Therefore both Tate cohomology groups are defined. Now consider an $\mathbf{F}$-complete flat resolution $T_X \to F_X \to Z_X \rightsquigarrow$ of X in $\mathbf{D}(\operatorname{Flat} A)$. By Proposition 3.8, we may (and do) assume that all terms of this triangle are complexes of projectives. To prove the theorem, it is enough to show that $T_X \in \mathbf{K}_{\mathrm{tac}}(\operatorname{Proj} A)$ and $Z_X \in \mathbf{K}_{\mathrm{tac}}(\operatorname{Proj} A)^{\perp}$. Since T_X is an $\mathbf{F}$-totally acyclic complex of projectives, by [MS09], it is totally acyclic. Let $L \in \mathbf{K}_{\mathrm{tac}}(\operatorname{Proj} A)$. Since the projective dimension of any flat module is finite, another application of [MS09] implies that

$L \in \mathbf{D}_{\text{tac}}(\text{Flat}\, A)$. So by assumption, $\text{Hom}_{\mathbf{D}(\text{Flat}\, A)}(L, Z_X) = 0$. But since L is a complex of projectives, we may deduce that $\text{Hom}_{\mathbf{K}(\text{Proj}\, A)}(L, Z_X) = 0$. Therefore $Z_X \in \mathbf{K}_{\text{tac}}(\text{Proj}\, A)^{\perp}$, completing the proof. □

The following result shows that, as expected, the flat dimension of complexes can be detected by the vanishing of the **F**-Tate groups.

Theorem 4.5. *Let X be a homologically bounded below complex of finite* **F***-Gorenstein flat dimension. Then the following are equivalent.*

(i) $\text{fd}_A X < \infty$.
(ii) $\widehat{\text{Ext}}^i_{\mathbf{F}}(X, \ ) = 0$, *for some $i \in \mathbb{Z}$.*
(iii) $\widehat{\text{Ext}}^i_{\mathbf{F}}(X, \ ) = 0$, *for all $i \in \mathbb{Z}$.*

If X is bounded above, the above properties are also equivalent to the following.

(iv) $\widehat{\text{Ext}}^0_{\mathbf{F}}(X, X) = 0$.

Proof. (i) $\Rightarrow$ (iii). Consider the triangle $0 \to F_X \to F_X \rightsquigarrow$. Since $\text{fd}_A X$ is finite, F_X is a bounded below complex of flat modules and hence belongs to $\mathbf{D}_{\text{tac}}(\text{Flat}\, A)^{\perp}$. Therefore, this triangle is an **F**-complete flat resolution of X. So (iii) follows.

(iii) $\Rightarrow$ (ii). This is trivial.

(ii) $\Rightarrow$ (i). Consider an **F**-complete flat resolution $T_X \to F_X \to Z_X \rightsquigarrow$ of X. By Proposition 3.8, we may assume that T_X is a complex of projectives. Set $L = \text{Im}\, \partial^i_{T_X}$ and let P_L be a projective resolution of L. By assumption, we have $\text{Hom}_{\mathbf{D}(\text{Flat}\, A)}(T_X, \Sigma^i P_L) = 0$. Therefore $\text{Hom}_{\mathbf{K}(A)}(T_X, \Sigma^i P_L) = 0$, because $T_X \in \mathbf{K}(\text{Proj}\, A)$. Hence $\text{H}^i \text{Hom}_A(T_X, \Sigma^i P_L) = 0$. This implies that the induced complex

$$\text{Hom}_A\big(T_X^{i+1}, L\big) \to \text{Hom}_A\big(T_X^{i}, L\big) \to \text{Hom}_A\big(T_X^{i-1}, L\big)$$

is exact. Hence the morphism

$$\text{Hom}_A\big(T^{i+1}, L\big) \to \text{Hom}_A(L, L)$$

is surjective. So the canonical injection $\ell : L \to T^{i+1}$ splits. Therefore L and hence $\text{Ker}\, \partial^i_{T_X}$ are projectives. Hence $\text{Ker}\, \partial^j_{T_X}$ is projective for all $j \leqslant i$. But since $\mathbf{F}\text{-Gfd}_A X < \infty$, $\text{Ker}\, \partial^j_{Z_X}$ is flat, for all $j \ll 0$. Therefore, by Proposition 3.6, we may deduce that $\text{Ker}\, \partial^j_{F_X}$ is flat, for all $j \ll 0$. Hence $\text{fd}_A X < \infty$.

Now assume that X is bounded above. To complete the proof, we have to establish the implication (iv) $\Rightarrow$ (i). Since $\widehat{\text{Ext}}^0_{\mathbf{F}}(X, X) = 0$, by definition, we have $\text{Hom}_{\mathbf{D}(\text{Flat}\, A)}(T_X, E_\rho(X)) = 0$. But since X is bounded above and we are in $\mathbf{D}(\text{Flat}\, A)$, we may replace $E_\rho(X)$ with F_X and hence deduce that $\text{Hom}_{\mathbf{D}(\text{Flat}\, A)}(T_X, F_X) = 0$. On the other hand, Z_X in the triangle $T_X \to F_X \to Z_X \rightsquigarrow$ belongs to $\mathbf{D}_{\text{tac}}(\text{Flat}\, A)^{\perp}$. So we get $\text{Hom}_{\mathbf{D}(\text{Flat}\, A)}(T_X, T_X) = 0$. Therefore $F_X = Z_X$ in $\mathbf{D}(\text{Flat}\, A)$. But finiteness of the **F**-Gorenstein flat dimension of X means, by definition, that $\text{fd}_A Z_X$ is finite. Hence, by Proposition 3.6, $\text{fd}_A F_X$ and therefore $\text{fd}_A X$ is finite. □

Corollary 4.6. *Let X be a bounded above, homologically bounded below, complex with both finite* **F***-Gorenstein flat dimension and finite injective dimension. Then its flat dimension is finite.*

Proof. We show that $\widehat{\text{Ext}}^0_{\mathbf{F}}(X, X) = 0$, and the claim follows from the theorem. In view of our fixed notations, $\widehat{\text{Ext}}^0_{\mathbf{F}}(X, X) = \text{Hom}_{\mathbf{D}(\text{Flat}\, A)}(T_X, E_\rho(X))$. But we can consider T_X to be a complex of projectives. So $\text{Hom}_{\mathbf{D}(\text{Flat}\, A)}(T_X, E_\rho(X)) = \text{Hom}_{\mathbf{K}(A)}(T_X, E_\rho(X))$. But since X is bounded above, by 4.1, we may replace $E_\rho(X)$ by P_X with P_X bounded above. So L in the triangle $P_X \to X \to L \rightsquigarrow$, is a bounded above complex which is acyclic. So $\mathrm{Hom}_{\mathbf{K}(A)}(T_X, L)$ vanishes. This implies that even we can replace P_X by X. This we do. On the other hand, consider the triangle $X \to I_X \to L' \rightsquigarrow$, in which I_X is an injective resolution of X. Since $\mathrm{id}_A X < \infty$, I_X is bounded above. So L' is an acyclic bounded above complex. Therefore, $\mathrm{Hom}_{\mathbf{K}(A)}(T_X, L') = 0$. This implies that $\mathrm{Hom}_{\mathbf{K}(A)}(T_X, X) = \mathrm{Hom}_{\mathbf{K}(A)}(T_X, I_X)$. Since $\mathbf{F}\text{-}\mathrm{Gd}_A X < \infty$, X is homologically bounded below. Hence we may assume that I_X is bounded below. So it preserves acyclicity, that is $\mathrm{Hom}_{\mathbf{K}(A)}(T_X, I_X) = 0$. Therefore $\widehat{\mathrm{Ext}}^0_{\mathbf{F}}(X, X) = 0$, as claimed. □

Proposition 4.7. *For complexes X and Y in $\mathbf{K}(A)$,*

$$\mathrm{Hom}_{\mathbf{D}(\mathrm{Flat}\,A)}\big(F_X, \Sigma^i E_\rho(Y)\big) \cong \mathrm{Ext}^i_A(X, Y),$$

for all $i \in \mathbb{Z}$, where F_X is a flat resolution of X.

Proof. Without loss of generality, we may replace F_X by a projective resolution P_X of X. Therefore

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}(\mathrm{Flat}\,A)}\big(F_X, \Sigma^i E_\rho(Y)\big) &\cong \mathrm{Hom}_{\mathbf{D}(\mathrm{Flat}\,A)}\big(P_X, \Sigma^i E_\rho(Y)\big)\\ &\cong \mathrm{Hom}_{\mathbf{K}(A)}\big(P_X, \Sigma^i Y\big)\\ &\cong \mathrm{Ext}^i_A(X, Y). \qquad \square\end{aligned}$$

4.8. Suppose that X admits an $\mathbf{F}$-complete flat resolution $T_X \to F_X \to Z_X \rightsquigarrow$. By applying the 'Hom' functor on this triangle and using the above theorem, we get an induced morphism on cohomology

$$\eta^i(X, Y) : \mathrm{Ext}^i_A(X, Y) \to \widehat{\mathrm{Ext}}^i_{\mathbf{F}}(X, Y).$$

Theorem 4.9. *Let $n \in \mathbb{Z}$ be a fixed integer. With the above notations, $\mathbf{F}\text{-}\mathrm{Gfd}_A X < n$ if and only if $\eta^i(X, Y)$ is an isomorphism, for any cotorsion module Y and all integers $i > n$.*

Proof. Consider the $\mathbf{F}$-complete flat resolution $T_X \to F_X \to Z_X \rightsquigarrow$ of X. We can assume that all terms in the triangle are projective. Let Y be a cotorsion R-module. It admits a flat resolution F_Y with cotorsion flat terms and with cotorsion kernels. Let F'_Y denote the corresponding augmented flat resolution

$$F'_Y : \cdots \to F_Y^{-2} \xrightarrow{\partial^{-2}} F_Y^{-1} \xrightarrow{\partial^{-1}} F_Y^0 \xrightarrow{\partial^0} Y \to 0.$$

Since F'_Y is an acyclic and bounded above complex of cotorsion modules with cotorsion kernels, the complex $\mathrm{Hom}_A(Z_X, F'_Y)$ is exact. Therefore $\mathrm{H}^i(\mathrm{Hom}_A(Z_X, F_Y)) \simeq \mathrm{H}^i(\mathrm{Hom}_A(Z_X, Y))$. Now assume that $\mathbf{F}\text{-}\mathrm{Gfd}_A X < n$. Hence, for all $i < n$, $\mathrm{H}^i(Z_X) = 0$ and $\mathrm{Ker}\,\partial^i_{Z_X}$ is flat. So, for any cotorsion module Y, $\mathrm{H}^i(\mathrm{Hom}_A(Z_X, Y)) = 0$, for all $i > n$. Therefore, for these degrees, $\mathrm{H}^i(\mathrm{Hom}_A(Z_X, F_Y)) = 0$. This implies easily that the maps $\eta^i(X, Y)$ are isomorphisms for all integers $i > n$. To prove the converse, we should show that $\mathrm{Ker}\,\partial^i_{Z_X}$ is flat for all $i < n$. To this end, we show that $\mathrm{H}^i(N \otimes_A Z_X) = 0$, for all $i < n$, where N is an arbitrary A^{op}-module. By using the adjoint duality of 'Hom' and tensor, it is enough to show that $\mathrm{H}^i(\mathrm{Hom}_A(Z_X, \mathrm{Hom}_A(N, E))) = 0$ for all $i > n$, where E is an injective cogenerator for A. But this follows form our assumption in view of the fact that $\mathrm{Hom}_A(N, E)$ is cotorsion. □

4.10. Tate cohomology theory was initiated by Tate's observation, that the $\mathbb{Z}G$-module $\mathbb{Z}$ with the trivial action, when G is a finite group, admits a complete projective resolution. Farrell [Fa77] generalized the theory to groups of finite virtual cohomological dimension. Mislin [Mi94], Benson and Carlson [BC92] and Vogel (first publish account in [Goi92]), independently, generalized the theory to all groups. Then it was shown that these theories are isomorphic, and complete cohomology is a common name for them. The theory has also been generalized to the setting of unbounded complexes in [AS07]. The theory has been studied also in the context of modules over rings. Auslander and Buchweitz [ABu89] studied maximal Cohen–Macaulay modules over Gorenstein rings. This class of modules is a special case of modules in Auslander's G-class. Using this class, Buchweitz [B86] studied Tate's theory for finitely generated modules over Gorenstein rings. Enochs and collaborators generalized the theory to all modules, and introduced the notion of Gorenstein projective modules, see e.g. [EJ95]. More recent expositions are in [AM02,AS06b,Jor05,Kra05]. An injective version of the theory and comparison with the original one can be found in [N98]. See also [AS06a] for an injective version of the theory in the setting of complexes.

5. Complete cohomology in the pure derived category

In this section we develop a cohomology theory in the pure derived category of flat modules $\mathbf{D}(\mathrm{Flat}\,A)$, which can be computed for any pair of complexes of modules and is compatible with the $\mathbf{F}$-Tate cohomology theory, introduced in the previous section, provided the complex in the first argument is of finite $\mathbf{F}$-Gorenstein flat dimension. We show that the flat dimension of complexes can be determined by this theory. Since its construction is motivated by the theory of Vogel [Goi92], known as complete cohomology, we shall use the same name.

Let $\mathbf{F}$ be the class of all morphisms $f : F \to F'$ that can be completed to a triangle $F \to F' \to F'' \rightsquigarrow$ in $\mathbf{K}(\mathrm{Flat}\,A)$ with $F'' \in \mathbf{K}_{\mathrm{p}}(\mathrm{Flat}\,A)$. Known techniques tell us that $\mathbf{F}$ is a multiplicative system and that the localization of $\mathbf{K}(\mathrm{Flat}\,A)$ in $\mathbf{F}$, i.e. $\mathbf{F}^{-1}\mathbf{K}(\mathrm{Flat}\,A)$, is exactly the pure derived category of flats, i.e. $\mathbf{D}(\mathrm{Flat}\,A)$. We shall use this fact throughout the section.

5.1. Let X and Y be complexes of flat modules and let $\mathrm{Hom}_A(X,Y)$ denote the Hom complex. We consider the subcomplex consisting of all homotopically bounded below morphisms, denoted $\mathrm{Hom}_A(X,Y)_{hbb}$. A morphism f is called homotopically bounded below if there exists morphism s in $\mathbf{F}$ such that sf is null-homotopic in degrees small enough. More precisely, $f \in \mathrm{Hom}_A(X,Y)_n$, for arbitrary integer n, is homotopically bounded below if there exists integer m and morphism $s \in \mathbf{F}$, such that the truncated morphism $(sf)^i_{\leqslant m}$ is null-homotopic. Since $\mathbf{F}$ is a multiplicative system and hence has a calculus of fractions, we may deduce that the terms of the complex $\mathrm{Hom}_A(X,Y)_{hbb}$ form subgroups of the terms of the complex $\mathrm{Hom}_A(X,Y)$. To see this, assume that h and h' are homotopically bounded below morphisms from X to Y and $s : Y \to Z$ and $s' : Y \to Z'$ in $\mathbf{F}$ are so that sh and $s'h'$ are null-homotopic in small enough degrees. Since $\mathbf{F}$ is a multiplicative system, we can find an object W and morphisms $t : Z \to W$ and $t' : Z' \to W$ in $\mathbf{F}$ such that $ts = t's'$. This will imply that $(h + h')$ is a homotopically bounded below morphisms.

Definition 5.2. Consider the complexes X and Y of A-modules. Let P_X be a projective resolution of X and F_Y be a flat resolution of Y, and let $\mathrm{Hom}_A(P_X, F_Y)_{hbb}$ be the subcomplex of $\mathrm{Hom}_A(P_X, F_Y)$ consisting of all homotopically bounded below morphisms. Consider the quotient complex

$$\mathrm{Hom}_A(P_X, F_Y)/\mathrm{Hom}_A(P_X, F_Y)_{hbb}.$$

For any integer i, the ith $\mathbf{F}$-complete cohomology of X and Y is defined to be the $-i$th cohomology of this quotient complex, denoted $\widetilde{\mathrm{Ext}}^i_{\mathbf{F}}(X,Y)$.

5.3. The exact sequence

$$0 \to \mathrm{Hom}_A(P_X, F_Y)_{hbb} \to \mathrm{Hom}_A(P_X, F_Y) \to \frac{\mathrm{Hom}_A(P_X, F_Y)}{\mathrm{Hom}_A(P_X, F_Y)_{hbb}} \to 0$$

of complexes induces, for any integer i, an exact sequence

$$\overline{\mathrm{Ext}}^i_{\mathbf{F}}(X,Y) \xrightarrow{\varepsilon} \mathrm{Ext}^i_A(X,Y) \xrightarrow{\zeta} \widetilde{\mathrm{Ext}}^i_{\mathbf{F}}(X,Y) \to \overline{\mathrm{Ext}}^{i+1}_{\mathbf{F}}(X,Y)$$

of cohomology groups, in which $\overline{\mathrm{Ext}}^i_{\mathbf{F}}(X, Y)$, for any integer i, is the $-i$th cohomology of the complex $\mathrm{Hom}_A(P_X, F_Y)_{hbb}$.

5.4. We claim that $\overline{\mathrm{Ext}}^i_{\mathbf{F}}(X, Y)$ is independent of the choice of the resolutions in both arguments. Since any two projective resolutions of X are homotopy equivalent, the independence in the first argument follows easily. We shall discuss independence in the second argument. For this, it is enough to compare the groups calculated via a projective resolution P_Y and also via a flat resolution F_Y of Y. There exists an isomorphism $u: P_Y \to F_Y$ in $\mathbf{D}(\mathrm{Flat}\, A)$. Let $f \in \mathrm{Hom}_A(P_X, P_Y)_{hbb}$. We show that the image of f under the induced map by u is homotopically bounded below. By definition, there exists a morphism $s: P_Y \to Z$ in $\mathbf{F}$ such that sf is null-homotopic in small degrees. Since $\mathbf{F}$ is a multiplicative system, there exits $s': F_Y \to W$ in $\mathbf{F}$ and $u': Z \to W$ such that $s'u = u's$. Hence $s'uf = u'sf$ is null-homotopic in small degrees. Therefore $uf \in \mathrm{Hom}_A(P_X, F_Y)_{hbb}$. On the other hand, since u is an isomorphism in $\mathbf{D}(\mathrm{Flat}\, A)$, the complexes $\mathrm{Hom}_A(P_X, P_Y)$ and $\mathrm{Hom}_A(P_X, F_Y)$ are quasi-isomorphic. Therefore, if $g: P_X \to F_Y$ is a cycle, there exists a morphism $f: P_X \to P_Y$ which is a cycle and maps to g under the induced morphism by u, i.e. $uf = g$ in $\mathbf{K}(\mathrm{Flat}\, A)$. So $g - uf$ is null-homotopic. If g is homotopically bounded below, there exists $s \in \mathbf{F}$ such that sg is null-homotopic in small enough degrees. Since $g - uf$ is null-homotopic, it is easy to see that suf is also null-homotopic in small degrees. But $u \in \mathbf{F}$ and $\mathbf{F}$ is a multiplicative system, hence $su \in \mathbf{F}$. Therefore f is homotopically bounded below. This implies that $\overline{\mathrm{Ext}}^i_{\mathbf{F}}(X, Y)$ is well defined.

Therefore $\widetilde{\mathrm{Ext}}^i_{\mathbf{F}}(X, Y)$ is also independent of the choice of resolutions of X and Y.

Theorem 5.5. *Assume that X is a homologically bounded below complex. Then the following are equivalent.*

(i) $\mathrm{fd}_A X < \infty$;
(ii) $\widetilde{\mathrm{Ext}}^i_{\mathbf{F}}(X, \ ) = 0$ *for all integers* i;
(iii) $\widetilde{\mathrm{Ext}}^i_{\mathbf{F}}(\ , X) = 0$ *for all integers* i;
(iv) $\widetilde{\mathrm{Ext}}^0_{\mathbf{F}}(X, X) = 0$.

Proof. (i) $\Rightarrow$ (ii). It is enough for us to prove the claim for $i = 0$. Let Y be an arbitrary complex and $f \in \mathrm{Hom}_A(P_X, P_Y)_0$. Since $\mathrm{fd}_A X < \infty$, there exists an integer $n \in \mathbb{Z}$, such that $(P_X)_{\leqslant n}$ is acyclic with flat kernels, that is, $(P_X)_{\leqslant n}$ belongs to $\mathbf{K}_{\mathrm{p}}(\mathrm{Flat}\, A)$. This implies that for any morphism $f: P_X \to P_Y$, where Y is an arbitrary complex, the morphism of truncation complexes $f_{\leqslant i}: (P_X)_{\leqslant i} \to (P_Y)_{\leqslant i}$ becomes zero in $\mathbf{D}(\mathrm{Flat}\, A)$. Hence there exists $s \in \mathbf{F}$ such that $s(f_{\leqslant i}) = 0$ in $\mathbf{K}(\mathrm{Flat}\, A)$. This exactly means that sf is null-homotopic in degrees small enough and hence f belongs to $\mathrm{Hom}_A(P_X, P_Y)_{hbb}$. So $\widetilde{\mathrm{Ext}}^0_{\mathbf{F}}(X, Y) = 0$.

(ii) $\Rightarrow$ (iv). This is trivial.

(iv) $\Rightarrow$ (i). The vanishing of $\widetilde{\mathrm{Ext}}^0_{\mathbf{F}}(X, X)$ implies that the identity morphism on P_X, a projective resolution of X, should belongs to $\mathrm{Hom}_A(P_X, P_X)_{hbb}$. So there exists a morphism $s: P_X \to Z$ in $\mathbf{F}$ which is null-homotopic in degrees small enough. This means that, for any A-module M, the induced map $\mathrm{H}^i \mathrm{Hom}_A(M, P_X) \to \mathrm{H}^i \mathrm{Hom}_A(M, Z)$ is zero in small degrees. On the other hand, since $s \in \mathbf{F}$, its mapping cone, say Y, is pure and hence by Lemma 3.5, for any finitely presented A-module M, $\mathrm{H}^i \mathrm{Hom}_A(M, Y) = 0$. Therefore $\mathrm{H}^i \mathrm{Hom}_A(M, P_X) = 0$, for any finitely presented A-module M. The result now follows from Lemma 3.5.

The implications (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) follow similarly. □

Lemma 5.6. *Let T and P be complexes of projective A-modules such that T is $\mathbf{F}$-totally acyclic. Then the complex* $\mathrm{Hom}_A(T, P)_{hbb}$ *is acyclic.*

Proof. By dimension shifting, it is enough to show that the complex is acyclic in degree zero. Let $f \in (\mathrm{Hom}_A(T_X, P_Y)_{hbb})_0$ be a cycle. Since f is homotopically bounded below, there exists a morphism $s: P_Y \to Q$ in $\mathbf{F}$, such that sf is null-homotopic in small enough degrees. Let $n \in \mathbb{Z}$ be such that $(s^i f^i)_{i \leqslant n}$ is null-homotopic, i.e. there exist maps $(t^i: T^i_X \to Q^{i-1})_{i \leqslant n+1}$ with the desired property.

Now consider the hard truncation ${}_hQ_{i\geqslant n}$ and construct a morphism $g : T_X \to {}_hQ_{i\geqslant n}$ by defining $g^i = 0$ for $i < n$, $g^n = s^n f^n - \partial_Q^{n-1} t^n$ and $g^i = s^i f^i$, for $i > n$.

Since T_X belongs to $\mathbf{D}_{\mathrm{tac}}(\mathrm{Flat}\, A)$, $\mathrm{Hom}_{\mathbf{D}(\mathrm{Flat}\, A)}(T_X, F) = 0$, for any flat module F. But since T_X is a complex of projectives, $\mathrm{Hom}_{\mathbf{D}(\mathrm{Flat}\, A)}(T_X, F) = \mathrm{Hom}_{\mathbf{K}(A)}(T_X, F)$. Therefore $\mathrm{Hom}_A(T_X, F)$ is acyclic, for any flat A-module F. Standard techniques in hyperhomological algebra says that the same is true for any bounded below complex F of flat modules.

So, since ${}_hQ_{i\geqslant n}$ is a bounded below complex of flats, $g : T_X \to {}_hQ_{i\geqslant n}$ which is a cycle, should be a boundary. Hence it is null-homotopic; i.e. there exist maps $(t^i : T_X^i \to Q^{i-1})_{i>n+1}$ with the desired property. By pasting these two class of t^i's, we may conclude that sf is null-homotopic, that is, $sf = 0$ in $\mathbf{D}(\mathrm{Flat}\, A)$. But, since s is an isomorphism in $\mathbf{D}(\mathrm{Flat}\, A)$, we deduce that $f = 0$ in $\mathbf{D}(\mathrm{Flat}\, A)$. On the other hand, f is a morphism of projectives, so in fact $f = 0$ in $\mathbf{K}(\mathrm{Proj}\, A)$, i.e. f is null-homotopic. This means that f is a boundary. The proof is therefore complete. □

5.7. Let X be a complex admitting an $\mathbf{F}$-complete flat resolution

$$T_X \to F_X \to Z_X \rightsquigarrow,$$

with $T_X \in \mathbf{D}_{\mathrm{tac}}(\mathrm{Flat}\, A)$. By Proposition 3.8, we may assume that all terms of this triangle are complexes of projective A-modules. This we do. Let Y be a bounded above complex. Hence $E_\rho(Y)$ (see (4.1)) is a projective resolution of Y and so we denote it by P_Y. Consider the short exact sequence

$$0 \to \mathrm{Hom}_A(T_X, P_Y)_{hbb} \to \mathrm{Hom}_A(T_X, P_Y) \to \frac{\mathrm{Hom}_A(T_X, P_Y)}{\mathrm{Hom}_A(T_X, P_Y)_{hbb}} \to 0,$$

of complexes and apply the above lemma, to obtain the isomorphisms

$$\widehat{\mathrm{Ext}}^i_{\mathbf{F}}(X, Y) \cong \mathrm{H}^i\left(\frac{\mathrm{Hom}_A(T_X, P_Y)}{\mathrm{Hom}_A(T_X, P_Y)_{hbb}}\right).$$

On the other hand, by applying the functor $\mathrm{Hom}_A(\ , P_Y)$ to the above triangle we get the exact sequence

$$\mathrm{H}^i\left(\frac{(Z_X, P_Y)}{(Z_X, P_Y)_{hbb}}\right) \to \mathrm{H}^i\left(\frac{(F_X, P_Y)}{(F_X, P_Y)_{hbb}}\right) \to \mathrm{H}^i\left(\frac{(T_X, P_Y)}{(T_X, P_Y)_{hbb}}\right) \to \mathrm{H}^{i+1}\left(\frac{(Z_X, P_Y)}{(Z_X, P_Y)_{hbb}}\right),$$

of cohomology groups, where for simplicity we use parentheses instead of Homs.

This in particular implies that for any integer i, there is a morphism from the ith $\mathbf{F}$-complete cohomology group to the ith $\mathbf{F}$-Tate cohomology groups, $\psi^i(X, Y) : \widetilde{\mathrm{Ext}}^i_{\mathbf{F}}(X, Y) \to \widehat{\mathrm{Ext}}^i_{\mathbf{F}}(X, Y)$. Now, following the same argument as in the proof of part (i) ⇔ (ii) of Theorem 5.5, one can deduce that $\mathrm{H}^i(\frac{(Z_X, P_Y)}{(Z_X, P_Y)_{hbb}}) = 0$, for any complex Y if and only if $\mathrm{Ker}\, \partial^i_{Z_X}$ is flat, for all $i \ll 0$. Therefore, it follows that the above induced maps $\psi^i(X,\ )$ are isomorphisms for any integer i if and only if $\mathbf{F}\text{-}\mathrm{Gfd}_A\, X < \infty$.

5.1. *Rings of finite sfli*

Related to the problem of extending the Farrell–Tate cohomology, two homological invariants were assigned to a group Γ by Gedrich and Gruenberg; $\mathrm{spli}\, \Gamma$, the supremum of the projective lengths of the injective Γ-modules, and $\mathrm{silp}\, \Gamma$ the supremum of the injective lengths of the projective Γ-modules [GG87]. It is shown that the finiteness of these invariants for a group Γ, implies the existence of complete cohomological functors, moreover, $\mathrm{silp}\, \Gamma \leqslant \mathrm{spli}\, \Gamma$ with equality if $\mathrm{spli}\, \Gamma$ is finite. The finiteness of these invariants has important geometric consequences, see e.g. [Ta05] and [ASm01]. For a long time it was not known if the finiteness of $\mathrm{silp}\, \Gamma$ implies the finiteness of the $\mathrm{spli}\, \Gamma$. Recently, this was settled in the affirmative by Emmanouil. He proved that for any group Γ,

$\operatorname{silp}\Gamma = \operatorname{spli}\Gamma$, see [Emm10, Corollary 4.5]. While proving his interesting result, he applied a new invariant $\operatorname{sfli}\Gamma$, the supremum of the flat lengths of injective Γ-modules. Also, some recent results of Benson and Goodearl, show that flat and projective modules over group rings have tight connections. For example, they show that when Γ is a finite group, a flat $A\Gamma$-module which is projective as an A-module is necessarily projective over $A\Gamma$ [BG00]. These results emphasize on the effective role of flats in the study of (co)homology of groups. For more research in this direction, see [DTa08, DTb08,ET10].

In this subsection, we provide a characterization of ring A for which the invariant $\operatorname{sfli} A$ is finite in terms of the finiteness of the $\mathbf{F}$-Gorenstein flat dimension of A-modules.

Remark 5.1.1. For a ring A, let $\operatorname{silfc} A$ denote the supremum of the injective length of cotorsion flat modules. Assume that A is a ring with the property that every cotorsion flat module has finite injective dimension. We claim that, in this case, $\operatorname{silfc} A$ is finite. Assume, otherwise, for any positive integer n, we could have a cotorsion flat module of injective dimension greater than n. Take their coproduct, say X, and let E be its cotorsion envelope. So E is cotorsion flat and hence is of finite injective dimension. But for any module C in this direct sum, the inclusion $C \to E$ is a pure monomorphism and hence is split. Therefore C is a summand of E and so $\operatorname{id}_A C \leqslant \operatorname{id}_A E$, which is a contradiction.

Theorem 5.1.2. *Let n be a fixed integer. For an associative ring A, the following are equivalent.*

(i) $\operatorname{sfli} A \leqslant n$ *and* $\operatorname{silfc} A$ *is finite.*
(ii) *Any A-module M admits an* $\mathbf{F}$*-complete flat resolution and the maps* $\psi^i(M, N)$ *are isomorphisms for any A-module N and all* $i > n$.
(iii) *Any A-module M admits an* $\mathbf{F}$*-complete flat resolution and the maps* $\eta^i(M, N)$ *are isomorphisms for any cotorsion A-module N and all* $i > n$.
(iv) *For any A-module M,* $\mathbf{F}\text{-}\operatorname{Gfd}_A M \leqslant n$.

Moreover, when A is commutative and noetherian, the above conditions are equivalent to the following.

(v) $\operatorname{sfli} A \leqslant n$.

Proof. The equivalence of (ii) and (iv) is proved in 5.7 and the equivalence of (iii) and (iv) is proved in Theorem 4.9.

The statement (i) ⇒ (iv) follows by using a similar argument as the one used in [GG87, §4]. In fact one can construct, for any A-module M, a complete resolution with the desired properties.

Now assume the equivalent statements (ii), (iii) and (iv) hold. Let F be a cotorsion flat A-module. The definition of Tate cohomology implies that the Tate cohomology functors $\widehat{\operatorname{Ext}}^i_{\mathbf{F}}(\ ,F)$ are zero. Therefore the fact that η^i is an isomorphism implies that the cohomology groups $\operatorname{Ext}^i_A(\ ,F)$ are zero, for all $i > n$. This implies that $\operatorname{id}_A F \leqslant n$. So $\operatorname{silfc} A < \infty$. Now, let I be an arbitrary injective module. By (iii) we deduce that the Tate cohomology functors $\widehat{\operatorname{Ext}}^i_{\mathbf{F}}(\ ,I)$ are zero, for any integer i. Therefore, by Theorem 4.5, $\operatorname{fd}_A I \leqslant n$. This implies that $\operatorname{sfli} A \leqslant n$. So (i) follows.

Assume that A is commutative and noetherian. The implication (i) ⇒ (v) is trivial For the converse, let $\mathcal{E}$ be an injective cogenerator for A. Then any cotorsion flat A-module F is a summand of $\operatorname{Hom}(\operatorname{Hom}(F,\mathcal{E}),\mathcal{E})$. But $\operatorname{Hom}(F,\mathcal{E})$ is injective and so by our assumption has flat dimension less then or equal to n. Therefore $\operatorname{Hom}(\operatorname{Hom}(F,\mathcal{E}),\mathcal{E})$ has injective dimension less then or equal to n. This completes the proof. □

6. Applications to semi-separated noetherian schemes

In order to extend Tate cohomology to schemes which are not affine, Krause [Kra05] introduced a Tate cohomology theory based on injective sheaves. This raises the question of how to extend the projective side of the theory to arbitrary schemes, the problem being that in general there is no good notion of a projective quasi-coherent sheaf. For example, for a field k, the only projective quasi-coherent sheaf over $\mathbb{P}^1(k)$ is the zero sheaf, [EEGO04, Corollary 2.3]. On the other hand, resolutions by locally free sheaves exist on most schemes, and the category of quasi-coherent sheaves always contains enough flats.

But another problem arises: flat resolutions fail to be unique in the homotopy category. This problem is solved in the PhD thesis of the Daniel Murfet, based on an idea of Amnon Neeman in [Nee08], and this allows us to use flat resolutions to compute (co)homology. We use this fact and develop a Tate cohomology theory in the category of quasi-coherent sheaves over semi-separated noetherian schemes. We present some applications of this theory to characterizing Gorenstein schemes. Throughout the section X will be a semi-separated noetherian scheme and sheaves are all quasi-coherent. Recall that a scheme X is called *semi-separated* if there exists an open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of X such that for all $\lambda \in \Lambda$, U_λ and all the pairwise intersections $U_\lambda \cap U_\gamma$ are affine, see [TT90] and [AJPV08]. Let us recall some general facts:

- The category of quasi-coherent sheaves on X is denoted by $\mathfrak{Qco}\,X$. The internal Hom in the category $\mathfrak{Qco}\,X$ is denoted by $\mathcal{H}om_{\mathrm{qc}}(-,-)$ and all Ext groups are calculated in the abelian category $\mathfrak{Qco}\,X$. Since by [Ha77, II §7], an injective object in $\mathfrak{Qco}\,X$ is just a quasi-coherent sheaf injective in the larger category of all sheaves of $\mathcal{O}_X$-modules, Ext groups are the same in both categories.
- A monomorphism $\mathcal{F} \to \mathcal{F}'$ of (quasi-coherent) sheaves $\mathcal{F}$, $\mathcal{F}'$ is said to be pure if $\mathcal{F} \otimes \mathcal{N} \to \mathcal{F}' \otimes \mathcal{N}$ is a monomorphism for every sheaf $\mathcal{N}$. An acyclic complex $\mathcal{F}$ of sheaves is pure acyclic if $\mathcal{F} \otimes \mathcal{N}$ is acyclic for every sheaf $\mathcal{N}$ or, equivalently, if $Z^n(F) \to F$ is a pure monomorphism for every $n \in \mathbb{Z}$.

Let $\mathrm{Flat}\,X$ denote the class of flat sheaves on X. The full subcategory of the homotopy category $\mathbf{K}(\mathrm{Flat}\,X)$ consisting of pure acyclic complexes is denoted $\mathbf{K}_{\mathrm{p}}(\mathrm{Flat}\,X)$. This is a localizing subcategory [Mu07, Lemma 3.2], and so the Verdier quotient $\mathbf{D}(\mathrm{Flat}\,X) = \mathbf{K}(\mathrm{Flat}\,X)/\mathbf{K}_{\mathrm{p}}(\mathrm{Flat}\,X)$ is a triangulated category. In [MS09] a complex $\mathcal{F}$ of flat sheaves was defined to be **F**-*totally acyclic* if it is acyclic and $\mathcal{F} \otimes \mathcal{I}$ is acyclic for every injective sheaf $\mathcal{I}$. It is equivalent by [MS09, Theorem 4.18] that $\mathcal{F}$ belongs to the intersection

$$\mathbf{D}_{\mathrm{tac}}(\mathrm{Flat}\,X) := \mathbf{D}_{\mathrm{ac}}(\mathrm{Flat}\,X) \cap {}^{\perp}(\mathrm{Flat}\,X)$$

as an object of $\mathbf{D}(\mathrm{Flat}\,X)$, cf. (3.2). Generalizing Definition 3.4 we use Bousfield localization triangles to define complete flat resolutions and the Gorenstein flat dimension. Recall from (2.3) that a flat resolution of a complex means a quasi-isomorphism to the complex from a **K**-flat complex of flat sheaves.

Definition 6.1. Let $\mathcal{X}$ be a complex of sheaves with flat resolution $F_{\mathcal{X}}$. An **F**-*complete flat resolution* of $\mathcal{X}$ is a triangle

$$T_{\mathcal{X}} \to F_{\mathcal{X}} \to Z_{\mathcal{X}} \rightsquigarrow, \tag{6.1}$$

in $\mathbf{D}(\mathrm{Flat}\,X)$, in which $T_{\mathcal{F}} \in \mathbf{D}_{\mathrm{tac}}(\mathrm{Flat}\,X)$ and $Z_{\mathcal{F}} \in \mathbf{D}_{\mathrm{tac}}(\mathrm{Flat}\,X)^{\perp}$. If $\mathcal{X}$ is homologically bounded below, we say that **F**-Gorenstein flat dimension of $\mathcal{X}$ is less than or equal n, a fixed integer, denoted $\mathbf{F}\text{-}\mathrm{Gfd}_X\,\mathcal{X} \leqslant n$ if $-n \leqslant \inf X$ and for all $i < -n$, $\mathrm{Coker}\,\partial^i_{Z_{\mathcal{X}}}$ is flat. If no integer n exists with $\mathbf{F}\text{-}\mathrm{Gfd}_X\,\mathcal{X} \leqslant n$, then we define $\mathbf{F}\text{-}\mathrm{Gfd}_X\,\mathcal{X} = \infty$.

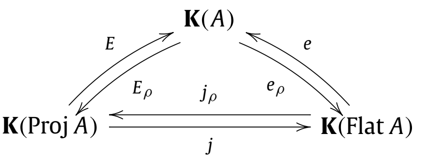
Remark 6.2. If $U \subseteq X$ is an open subset then there is a restriction functor $\mathbf{D}(\mathrm{Flat}\,X) \to \mathbf{D}(\mathrm{Flat}U)$. Since flatness is a local property the direct generalization of Proposition 3.6 holds in $\mathbf{D}(\mathrm{Flat}\,X)$. In particular, it is well defined to speak about the flatness of cokernels of $Z_{\mathcal{X}}$ even though this object is only defined up to isomorphism in $\mathbf{D}(\mathrm{Flat}\,X)$.

It is proved in [MS09, Theorem 4.24] that the inclusion $\mathbf{D}_{\mathrm{tac}}(\mathrm{Flat}\,X) \to \mathbf{D}(\mathrm{Flat}\,X)$ has a right adjoint, from which it follows that any complex of sheaves over X admits an **F**-complete flat resolution. On the other hand, by [Mu07, Theorem A.13], the inclusion functor $\mathfrak{F} : \mathbf{K}(\mathrm{Flat}\,X) \to \mathbf{K}(\mathfrak{Qco}\,X)$ has a right adjoint $\mathfrak{F}_\rho : \mathbf{K}(\mathfrak{Qco}\,X) \to \mathbf{K}(\mathrm{Flat}\,X)$. Based on these facts, for complexes $\mathcal{F}$ and $\mathcal{G}$ of sheaves, the ith **F**-Tate cohomology group of $\mathcal{F}$ and $\mathcal{G}$ over X, denoted $\widehat{\mathrm{Ext}}^i_{\mathbf{F}}(\mathcal{F},\mathcal{G})$, is defined by

$$\widehat{\mathrm{Ext}}^i_{\mathbf{F}}(\mathcal{F},\mathcal{G}) := \mathrm{Hom}_{\mathbf{D}(\mathrm{Flat}\,X)}\big(T_{\mathcal{F}}, \Sigma^i \mathfrak{F}_\rho(\mathcal{G})\big).$$

As in the affine case, it is clear that is a well-defined abelian group.

Remark 6.3. By [Nee10, Theorem 3.2], the inclusion functor $e : \mathbf{K}(\mathrm{Flat}\,A) \to \mathbf{K}(A)$ has a right adjoint e_ρ. The commutativity of the following diagram of inclusion functors and their right adjoints



implies that in affine case, for any complex $Y \in \mathbf{K}(A)$, $E_\rho(Y) = e_\rho(Y)$ in $\mathbf{D}(\mathrm{Flat}\,A)$. Hence the definition of **F**-Tate cohomology here is compatible with the one presented in Definition 4.2 when X is an affine scheme.

6.4. It follows from the proof of Theorem A.13 of [Mu07] that in case $\mathcal{G}$ is a homologically bounded above complex, $\mathfrak{F}_\rho(\mathcal{G})$ is a flat resolution of $\mathcal{G}$.

Remark 6.5. Recall that a sheaf $\mathcal{G}$ is cotorsion if $\mathrm{Ext}^1_X(\mathcal{F},\mathcal{G}) = 0$, for any flat sheaf $\mathcal{F}$. We say that $\mathcal{G}$ is cotorsion flat if it is both cotorsion and flat and denote by $\mathrm{Cof}\,X$ the full subcategory of cotorsion flat sheaves in $\mathfrak{Qco}\,X$ and by $\mathbf{K}(\mathrm{Cof}\,X)$ the corresponding homotopy category. It is clear that $\mathrm{Cof}\,X$ is closed under finite direct sums and direct summands.

For the proof of the next theorem we need the following observation.

Observation 6.6. In [Mu07, Proposition 3.19] it is shown that every sheaf $\mathcal{F}$ is isomorphic in $\mathbf{D}(\mathfrak{Qco}\,X)$ to a bounded above complex of flat sheaves. In case $\mathcal{F}$ is cotorsion, this complex can be chosen so that all its terms and also all its kernels are cotorsion. This follows from [Mu07, Corollary 3.21], where it is shown that flat precovers exist and are epimorphisms. Therefore flat covers exist and are epimorphisms. This means that, if one starts with a cotorsion sheaf and takes its flat cover, that flat cover should be cotorsion with cotorsion kernel. In this way one can construct the desired resolution.

Theorem 6.7. *Let $\mathcal{F}$ be a homologically bounded below complex of finite* **F**-*Gorenstein flat dimension. Then the following are equivalent*:

(i) $\mathrm{fd}_X \mathcal{F} < \infty$.
(ii) $\widehat{\mathrm{Ext}}^i_{\mathbf{F}}(\mathcal{F}, \ ) = 0$, *for some $i \in \mathbb{Z}$.*
(iii) $\widehat{\mathrm{Ext}}^i_{\mathbf{F}}(\mathcal{F}, \ ) = 0$, *for all $i \in \mathbb{Z}$.*

If $\mathcal{F}$ is bounded above, the above properties are also equivalent to the following.

(iv) $\widehat{\mathrm{Ext}}^0_{\mathbf{F}}(\mathcal{F},\mathcal{F}) = 0$.

Proof. Inspecting the proof in the affine case, i.e. Theorem 4.5, we see that only the implication (ii) $\Rightarrow$ (i) needs some explanation. Consider an **F**-complete flat resolution of $\mathcal{F}$, say $T_{\mathcal{F}} \to F_{\mathcal{F}} \to$

$Z_{\mathcal{F}} \rightsquigarrow$. Set $\mathcal{L} = \operatorname{Im} \partial^j_{T_{\mathcal{F}}}$, for j sufficiently small. Using the argument of the proof of Lemma 3.11, we may assume that $\mathcal{L}$ is cotorsion. Hence by Observation 6.6 we may assume that there exists a flat resolution $F_{\mathcal{L}}$ of $\mathcal{L}$ such that all of its terms are cotorsion.

This assumption, in view of 6.4, implies that $\operatorname{Hom}_{\mathbf{D}(\operatorname{Flat} X)}(T_{\mathcal{F}}, \Sigma^i F_{\mathcal{L}}) = 0$. By the argument given in the proof of Proposition 3.13 the complex $F_{\mathcal{L}}$ belongs to $\mathbf{K}_{\mathrm{p}}(\operatorname{Flat} X)^{\perp}$, so we deduce that $\operatorname{Hom}_{\mathbf{K}(\mathfrak{Qco}\, X)}(T_{\mathcal{F}}, \Sigma^i F_{\mathcal{L}}) = 0$. Now one can follow the proof (ii) $\Rightarrow$ (i) in Theorem 4.5 to complete the proof. □

6.8. Tate cohomology using injectives. Let $\mathcal{A}$ be a locally noetherian Grothendieck category and suppose that the derived category $\mathbf{D}(\mathcal{A})$ is compactly generated. A complex E of injective objects is called totally acyclic if it is acyclic and remains acyclic after applying the functor $\operatorname{Hom}_{\mathcal{A}}(I, \ )$, for any injective object I. Let $\mathbf{K}_{\mathrm{tac}}(\operatorname{Inj} \mathcal{A})$ denote the full subcategory of $\mathbf{K}(\operatorname{Inj} \mathcal{A})$ consisting of totally acyclic complexes of injectives. One can see that

$$\mathbf{K}_{\mathrm{tac}}(\operatorname{Inj} \mathcal{A}) = \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} \mathcal{A}) \cap (\operatorname{Inj} \mathcal{A})^{\perp}.$$

Let B be an object of $\mathcal{A}$ with an injective resolution I_B. A *complete injective resolution* of B is a triangle

$$Z_B \to I_B \to T_B \rightsquigarrow, \tag{6.2}$$

in $\mathbf{K}(\operatorname{Inj} \mathcal{A})$ with $Z_B \in {}^{\perp}\mathbf{K}_{\mathrm{tac}}(\operatorname{Inj} \mathcal{A})$ and $T_B \in \mathbf{K}_{\mathrm{tac}}(\operatorname{Inj} \mathcal{A})$. This triangle is unique up to isomorphism. If, moreover, I_B is homologically bounded above, we say that Gorenstein injective dimension of B, denoted $\operatorname{Gid}_{\mathcal{A}} B$, is finite if $\operatorname{Ker} \partial^i_{Z_B}$ is injective, for $i \gg 0$.

It is proved by Krause [Kra05, §7] that the inclusion functor

$$G : \mathbf{K}_{\mathrm{tac}}(\operatorname{Inj} \mathcal{A}) \to \mathbf{K}(\operatorname{Inj} \mathcal{A})$$

has a left adjoint G_{λ}. So one may deduce that any object B admits a complete injective resolution. With such a resolution (6.2) given, we define for any object A in $\mathcal{A}$ the ith injective Tate cohomology group of A and B, denoted $\widehat{\operatorname{ext}}^i_{\mathcal{A}}(A, B)$, by

$$\widehat{\operatorname{ext}}^i_{\mathcal{A}}(A, B) := \operatorname{Hom}_{\mathbf{K}(\mathcal{A})}\big(\Sigma^{-i} A, T_B\big).$$

We refer the reader to [Kra05, §7] for the properties of these Tate cohomology groups. Notice that the category of (quasi-coherent) sheaves over a semi-separated noetherian schemes X is a locally noetherian Grothendieck category, and $\mathbf{D}(\mathfrak{Qco}\, X)$ is compactly generated, so Krause's applies. In this case the Gorenstein injective dimension of a complex $\mathcal{G}$ of quasi-coherent sheaves will be denoted by $\operatorname{Gid}_X \mathcal{G}$, and the Tate cohomology groups are denoted $\widehat{\operatorname{ext}}^i_X(\mathcal{F}, \mathcal{G})$.

The next theorem compares our (flat) Tate cohomology groups with those introduced by Krause, see Theorem 6.12 below.

Lemma 6.9. *Let $\mathfrak{U} = \{U_0, \dots, U_d\}$ be an affine open cover of X. Given a flat sheaf $\mathcal{F}$ we have $\operatorname{Ext}^i_X(\mathcal{F}, -) = 0$ for $i > d + \dim(X)$.*

Proof. Set $e = \dim(X)$ and let $\mathcal{G}$ be a sheaf. Take the Čech resolution

$$0 \to \mathcal{G} \to \mathcal{C}^0(\mathfrak{U}, \mathcal{G}) \to \cdots \to \mathcal{C}^d(\mathfrak{U}, \mathcal{G}) \to 0$$

of $\mathcal{G}$, in which, for any t, $\mathcal{C}^t(\mathfrak{U},\mathcal{G})=\bigoplus_{i_0<\cdots<i_p} f_*(\mathcal{G}|_{U_{i_0,\ldots,i_t}})$, where $f:U_{i_0,\ldots,i_t}\to X$ is the inclusion of the open set $U_{i_0,\ldots,i_t}=U_{i_0}\cap\cdots\cap U_{i_t}$. Observe that for $0\leqslant p\leqslant d$

$$\operatorname{Ext}^i\big(\mathcal{F},\mathcal{C}^p(\mathfrak{U},\mathcal{F})\big)=\bigoplus_{i_0<\cdots<i_p}\operatorname{Ext}^i(\mathcal{F}|_{U_{i_0,\ldots,i_p}},\mathcal{G}|_{U_{i_0,\ldots,i_p}}).$$

So $\operatorname{Ext}^i(\mathcal{F},\mathcal{C}^p(\mathfrak{U},\mathcal{F}))$ vanishes for all $i>e$, because every affine open subset of X has Krull dimension $\leqslant e$, whence flat sheaves over such open subsets have projective dimension $\leqslant e$. From a short exact sequence $0\to\mathcal{K}\to\mathcal{C}^{d-1}(\mathfrak{U},\mathcal{G})\to\mathcal{C}^d(\mathfrak{U},\mathcal{G})\to 0$ and its long exact Ext sequence we deduce that $\operatorname{Ext}^i(\mathcal{F},\mathcal{K})=0$ for $i>e+1$. Continuing in this vein we eventually deduce that $\operatorname{Ext}^i(\mathcal{F},\mathcal{G})=0$ for $i>d+e$ as claimed. □

Proposition 6.10. *Let $\mathcal{F}$ be a complex of sheaves with flat resolution $F_{\mathcal{F}}$ and let $\mathcal{G}$ be a cotorsion sheaf. Then for all $i\in\mathbb{Z}$,*

$$\operatorname{Ext}^i_X(\mathcal{F},\mathcal{G})\cong\operatorname{Hom}_{\mathbf{D}(\operatorname{Flat}X)}\big(F_{\mathcal{F}},\Sigma^i\mathfrak{F}_\rho(\mathcal{G})\big).$$

Proof. By 6.4, $\mathfrak{F}_\rho(\mathcal{G})$ can be taken to be a flat resolution $F_{\mathcal{G}}$ of $\mathcal{G}$. Since the complex $F_{\mathcal{F}}$ is **K**-flat it belongs to the left orthogonal of the subcategory $\mathbf{D}_{\mathrm{ac}}(\operatorname{Flat}X)$ of acyclic complexes in $\mathbf{D}(\operatorname{Flat}X)$, by [Mu07, Proposition 5.2]. But there is a Verdier quotient

$$\mathbf{D}(\operatorname{Flat}X)\to\mathbf{D}(\mathfrak{Qco}\,X)$$

with kernel $\mathbf{D}_{\mathrm{ac}}(\operatorname{Flat}X)$, so it follows that

$$\begin{aligned}\operatorname{Hom}_{\mathbf{D}(\operatorname{Flat}X)}\big(F_{\mathcal{F}},\Sigma^i\mathfrak{F}_\rho(\mathcal{G})\big)&\cong\operatorname{Hom}_{\mathbf{D}(\operatorname{Flat}X)}\big(F_{\mathcal{F}},\Sigma^iF_{\mathcal{G}}\big)\\&\cong\operatorname{Hom}_{\mathbf{D}(\mathfrak{Qco}\,X)}\big(\mathcal{F},\Sigma^i\mathcal{G}\big)\\&=\operatorname{Ext}^i_X(\mathcal{F},\mathcal{G})\end{aligned}$$

as claimed. □

Consider an **F**-complete flat resolution $T_{\mathcal{F}}\to F_{\mathcal{F}}\to Z_{\mathcal{F}}\rightsquigarrow$ of $\mathcal{F}$. Applying the functor $\operatorname{Hom}_{\mathbf{D}(\operatorname{Flat}X)}(\ ,\Sigma^i\mathfrak{F}_\rho(\mathcal{G}))$ we get induced morphisms on cohomology

$$\eta^i(X,Y):\operatorname{Hom}_{\mathbf{D}(\operatorname{Flat}X)}\big(F_{\mathcal{F}},\Sigma^i\mathfrak{F}_\rho(\mathcal{G})\big)\to\widehat{\operatorname{Ext}}^i_{\mathbf{F}}(X,Y).$$

Now assume that $\mathcal{F}$ is a homologically bounded below complex of finite **F**-Gorenstein flat dimension and that $\mathcal{G}$ is a cotorsion sheaf. Using the straightforward generalization of Theorem 3.15 to schemes, there exists a flat resolution $F_{\mathcal{F}}$ of $\mathcal{F}$ and an integer $n<\inf\mathcal{F}$ such that $\operatorname{Ker}\partial^n_{F_{\mathcal{F}}}$ is **F**-Gorenstein flat. Hence we get an **F**-complete flat resolution of $\mathcal{F}$

$$T_{\mathcal{F}}\to F_{\mathcal{F}}\to Z_{\mathcal{F}}\rightsquigarrow$$

in $\mathbf{D}(\operatorname{Flat}X)$, such that for all $i<n$, the terms of $T_{\mathcal{F}}$ and $F_{\mathcal{F}}$ are the same. Moreover, by Observation 6.6, $\mathcal{G}$ admits a flat resolution with cotorsion terms and kernels. Hence by Proposition 6.10 and the same argument applied in Theorem 4.9, we have the following.

Theorem 6.11. *Let $n\in\mathbb{Z}$ be a fixed integer and $\mathcal{F}$ a sheaf. Then* **F**-$\operatorname{Gfd}_X\mathcal{F}<n$ *if and only if* $\eta^i(\mathcal{F},\mathcal{G})$: $\operatorname{Ext}^i_X(\mathcal{F},\mathcal{G})\to\widehat{\operatorname{Ext}}^i_{\mathbf{F}}(\mathcal{F},\mathcal{G})$ *is an isomorphism, for any cotorsion sheaf $\mathcal{G}$ and all integers $i>n$.*

The following theorem is some kind of 'balanced theorem' for Tate cohomologies.

Theorem 6.12. *Let $\mathcal{F}$ be a homologically bounded below complex of sheaves with* $\mathbf{F}\text{-}\mathrm{Gfd}_X\,\mathcal{F} < \infty$ *and $\mathcal{G}$ be a cotorsion sheaf with* $\mathrm{Gid}_X\,\mathcal{G} < \infty$. *Then for any integer i,*

$$\widehat{\mathrm{Ext}}^i_{\mathbf{F}}(\mathcal{F},\mathcal{G}) \cong \widehat{\mathrm{ext}}^i_X(\mathcal{F},\mathcal{G}).$$

Proof. By Theorem 6.11, $\widehat{\mathrm{Ext}}^i_{\mathbf{F}}(\mathcal{F},\mathcal{G}) \cong \mathrm{Ext}^i_X(\mathcal{F},\mathcal{G})$, for all $i \gg 0$. On the other hand, since $\mathrm{Gid}_A\,\mathcal{G} < \infty$, by [Kra05, Proposition 7.10], $\widehat{\mathrm{ext}}^i_X(\mathcal{F},\mathcal{G}) \cong \mathrm{Ext}^i_X(\mathcal{F},\mathcal{G})$, for all $i > \mathrm{Gid}_A\,\mathcal{G}$. So for i big enough, we have $\widehat{\mathrm{Ext}}^i_{\mathbf{F}}(\mathcal{F},\mathcal{G}) \cong \widehat{\mathrm{ext}}^i_X(\mathcal{F},\mathcal{G})$. Now the proof can be extended to any integer i, using the dimension shifting trick. □

Towards the end of this subsection, we study **F**-Tate cohomology groups over locally Gorenstein schemes. Recall that a noetherian scheme X is called locally Gorenstein if its local rings are all Gorenstein rings.

Proposition 6.13. *Suppose X is locally Gorenstein and has finite Krull dimension. Then*:

(i) *Any injective sheaf is of finite flat dimension.*
(ii) *Any flat sheaf is of finite injective dimension.*

Proof. (i) Consider an affine open cover $\mathfrak{U} = \{U_0,\dots,U_d\}$ of X, where $U_i = \mathrm{Spec}(A_i)$ and let $\mathcal{I}$ be an injective sheaf. For any i, set $\widetilde{M_i} = \mathcal{I}|_{U_i}$ and let I_i be the injective envelope of the A_i-module M_i. By the proof of Corollary 3.6 of [Ha77, Chapter III], there exists an injection $\mathcal{I} \to \oplus f_*(\widetilde{I_i})$ of sheaves. Here, for each i, $f: U_i \to X$ denotes the inclusion. Since $\mathcal{I}$ is injective, this injection is split. So it is enough for us to show that, for all $i = 0,\dots,d$, $f_*(\widetilde{I_i})$ has finite flat dimension. To see this, note that since A_i is Gorenstein, $\mathrm{fd}_{A_i} I_i$ is finite, say t. Let $0 \to F_t \to F_{t-1} \to \cdots \to F_1 \to F_0 \to I_i \to 0$ be a flat resolution of I_i. Take $\widetilde{\ }$ and apply f_* on the resulting sequence, give us a flat resolution of $f_*(\widetilde{I_i})$. This completes the proof.

(ii) First assume that $\mathcal{F}$ is a cotorsion flat sheaf. By [MS09, Proposition 3.3] the pure monomorphism $\mathcal{F} \to (\mathcal{F})^+ = \mathcal{H}\mathrm{om}_{\mathrm{qc}}(\mathcal{H}\mathrm{om}_{\mathrm{qc}}(\mathcal{F},\mathcal{E}),\mathcal{E})$ is split, where $\mathcal{E}$ is an injective cogenerator for $\mathfrak{Qco}(X)$. But $\mathcal{H}\mathrm{om}_{\mathrm{qc}}(\mathcal{F},\mathcal{E})$ is injective and hence by part (i) has finite flat dimension. Therefore $\mathrm{id}(\mathcal{F}^+) < \infty$ and hence $\mathrm{id}\,\mathcal{F}$ is finite. Now since $\dim X$ is finite, in view of Lemma 6.9, we may deduce that the pure injective dimension of any flat $\mathcal{O}_X$-module is finite. This means that any flat sheaf $\mathcal{F}$ admits a right resolution of finite length by cotorsion flat sheaves. This implies easily that $\mathrm{id}\,\mathcal{F}$ is finite. □

Our last corollary can be proved in view of the above proposition, Theorems 6.11 and 6.12 and applying the ideas of [AS06a, Theorem 3.2] and [AS07, Proposition 3.3.1]. So we skip the proof.

Corollary 6.14. *Suppose X has finite Krull dimension. The following are equivalent.*

(i) *X is locally Gorenstein.*
(ii) *The injective dimension of any homologically bounded above complex $\mathcal{F}$ of flat sheaves is finite.*
(iii) *The flat dimension of any homologically bounded below complex $\mathcal{G}$ of injective sheaves is finite.*
(iv) *The Gorenstein injective dimension of any homologically bounded above complex $\mathcal{F}$ of sheaves is finite.*
(v) *The* **F**-*Gorenstein flat dimension of any homologically bounded below complex $\mathcal{G}$ of sheaves is finite.*
(vi) *For any homologically bounded below complex $\mathcal{F}$ and any homologically bounded above complex $\mathcal{G}$ of sheaves, we have $\widehat{\mathrm{Ext}}^i_{\mathbf{F}}(\mathcal{F},\mathcal{G}) \cong \widehat{\mathrm{ext}}^i_X(\mathcal{F},\mathcal{G})$, for all integers $i \in \mathbb{Z}$.*
(vii) *The maps $\eta^i(\mathcal{F},\mathcal{G}): \mathrm{Ext}^i_X(\mathcal{F},\mathcal{G}) \to \widehat{\mathrm{Ext}}^i_{\mathbf{F}}(\mathcal{F},\mathcal{G})$ are isomorphisms, for all integers i, any sheaf $\mathcal{F}$ and any cotorsion sheaf $\mathcal{G}$.*

Acknowledgments

The authors would like to thank Daniel Murfet for many ideas, comments and hints that improved our exposition. We owe him for this work. We also would like to thank the referee for her/his comments. The authors thank the Center of Excellence for Mathematics (University of Isfahan). Part of the research of the first author is done while he was visiting at the University of Paderborn. He would like to thank the University of Paderborn, and in particular Professor Helmut Lenzing, for providing a stimulating research environment.

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