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# On a Class of Complete Polynomial Vector Fields in the Plane

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The class $\mathscr{C}_n$ of polynomial vector fields of degree $n$ which define global flows in $\mathbb{R}^2$ is characterized, provided the zeros of the vector fields at infinity are simple. Some analytical properties of $\mathscr{C}_n$ are also established. © 1986 Academic Press, Inc.

## 1. INTRODUCTION

Let

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$$

denote a polynomial vector field of degree $n$ in the plane $\mathbb{R}^2$, i.e., $P = \sum a_{ij}x^iy^j$, $Q = \sum b_{ij}x^iy^j$ for $0 \leqslant i+j \leqslant n$ with the vector $(a_{ij}; b_{ij}) \in \mathbb{R}^N$, $N = (n+1)(n+2)$. The maximal integral curve $\phi_t(p)$ of $X$ with initial point $p$ is defined on a maximal time interval $I(p)$. If $I(p) = \mathbb{R}$, $\phi_t(p)$ is called *complete*. If every integral curve of $X$ is complete, $X$ is called *complete*. This paper is concerned with the problem of deciding the completeness of $X$ by means of elementary operations with its coefficients $(a_{ij}; b_{ij})$.

The uniqueness theorem for ordinary differential equations implies that $X$ is incomplete if and only if there is an integral curve which escapes to

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infinity in finite time. Thus, our anallysis will be based on an understanding of the behavior of $X$ at infinity.

It is standard procedure in the study of polynomial vector fields to study their behavior at infinity by extension to the Poincaré sphere, the crucial point being that this extension procedure when applied to a polynomial vector field $X$ results in a new vector field $\tilde{X}$ which is analytic on the sphere. This vector field $\tilde{X}$ has the equator of the sphere (which corresponds to the line at infinity) as an invariant set which will either be a periodic orbit or contain a certain number of zeros of $\tilde{X}$ connected by paths which stay in the equator. As we shall see in Section 2 the integral curves of $\tilde{X}$ near the equator of the sphere are reparametrizations of integral curves of $X$ in the finite plane. After taking this reparametrization into account the completeness of $X$ is determined by an analysis of the orbits of $\tilde{X}$ which have limit points on the equator of the sphere. This analysis falls into three cases: (i) the consideration of orbits which have a zero of $\tilde{X}$ on the equator as their $\alpha$ or $\omega$ limit set, (ii) the consideration of orbits which have a nonstationary orbit of $\tilde{X}$ on the equator as their $\alpha$ or $\omega$ limit set, and (iii) the consideration of orbits which have the equator as their $\alpha$ or $\omega$ limit set when $\tilde{X}$ has zeros on the equator. In this paper we will characterize the complete polynomial vector fields of degree $n$ which belong to the class $\mathcal{S}_n$ of those polynomial vector fields $\tilde{X}$ of degree $n$ such that all zeros of $\tilde{X}$ restricted to the equator are simple. For $X \in \mathcal{S}_n$ only cases (i) and (ii) can occur. However, using elimination theory it can be shown that the set of vector fields not in $\mathcal{S}_n$ is a meager closed semi-algebraic subset of the space $\mathfrak{X}_n \simeq \mathbb{R}^N$ of all polynomial vector fields of degree $n$ in which $X$ is identified with the $N$-tuple of its coefficients.

The elementary analytic structure of the set $\mathcal{C}_n$ of complete polynomial vector fields of degree $n$ is also established. In fact, we show that $\mathcal{S}_n \cap \mathcal{C}_n$ is the union of regular analytic submanifolds. In some cases, formulated in Theorem 3.6 and the remarks following Theorem 4.3, the codimensions of these submanifolds are determined.

We conclude this introduction with a few comments on the background for this work, but we make no attempt to give a complete survey of the previous work in this area. The obstructions to completeness and the analytical patterns through which the solutions of differential equations diverge to infinity near finite extremes of their maximal domains of definition (movable singularities) have been studied by Painlevé [P], Hille [Hi], and Smith [S]. The conditions for completeness, or incompleteness, for polynomial vector fields with a periodic orbit at infinity (in our case (ii)) certainly concerned Smith [S, p. 314]. More recently Bass and Meisters [B–M] characterized, using algebraic methods, a class of vector fields on the plane with complete flows. These are the vector fields whose flows $\phi_t(x, y)$ are, for fixed $t$, polynomial in the initial point $(x, y) \in \mathbb{R}^2$.

Finally, the relationship between movable singularities in the complex domain and the integrability and chaotic behavior of the Lorentz equations in $\mathbb{R}^3$ has been studied by Tabor and Weiss [T–W].

## 2. Reparametrization and Completeness

For the polynomial vector field

$$X = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}$$

of degree $n$ we consider the extension $\tilde{X}$ to the Poincaré sphere. This extension is made via central projection by considering the plane $\mathbb{R}^2$ as the tangent space at the north pole of the unit sphere in $\mathbb{R}^3$. Following Andronov *et al.* [$A_1$, p. 219–228] (see also [6]) the upper hemisphere is covered by four coordinate charts which corresponds in pairs to the coordinate changes $x = v/z$, $y = 1/z$ or $x = 1/z$, $y = u/z$. The first coordinate change gives the two open half planes in $\mathbb{R}^2$ when the $x$-axis is removed and the second gives the open half planes when the $y$-axis is removed. We concentrate on the first coordinate change and note that in the $(v, z)$-coordinates, our vector field becomes

$$\left[zP\left(\frac{v}{z}, \frac{1}{z}\right) - vz\, Q\left(\frac{v}{z}, \frac{1}{z}\right)\right]\frac{\partial}{\partial v} - z^2 Q\left(\frac{v}{z}, \frac{1}{z}\right)\frac{\partial}{\partial z}.$$

After multiplication by the scale function $f(v, z) = z^{n-1}$ we obtain the local representation of $\tilde{X}$, the Poincaré extension, as the *polynomial* vector field

$$[P^*(v, z) - v\, Q^*(v, z)]\frac{\partial}{\partial v} - z\, Q^*(v, z)\frac{\partial}{\partial z}$$

in the $(v, z)$ coordinate charts. Here the circle at infinity is the $v$-axis, which is clearly an invariant set of $\tilde{X}$ since $\dot{z} = -z\, Q^*(v, z)$. For the other charts given by the coordinate change $x = 1/z$, $y = u/z$ we obtain the local representative from the vector field

$$\left[zQ\left(\frac{1}{z}, \frac{u}{z}\right) - uz\, P\left(\frac{1}{z}, \frac{u}{z}\right)\right]\frac{\partial}{\partial u} - z^2 P\left(\frac{1}{z}, \frac{u}{z}\right)\frac{\partial}{\partial z},$$

which after multiplication by $f$ gives

$$[Q^{**}(u, z) - u\, P^{**}(u, z)]\frac{\partial}{\partial u} - z\, P^{**}(u, z)\frac{\partial}{\partial z}.$$

Since the Poincaré sphere is compact all integral curves of $\tilde{X}$ are complete. Let $\gamma$ denote an integral of $X$ which has a limit set at infinity. Since $\gamma$ is, after a reparametrization, an integral curve of $\tilde{X}$ which lies in the finite part of the plane we must study the effect of the scale function $f(v, z) = z^{n-1}$ in order to determine the completeness of $\gamma$. To this end we digress to give an account of the general situation which occurs under reparametrization.

Let $X$ denote a smooth vector field on a manifold $M$ (we do not assume $M$ is compact) and let $\phi_t$ denote the flow of $X$. Given a smooth function $f\colon M \to \mathbb{R}^+$ consider the vector field $fX$ and its flow $\psi_t$. Since $fX$ and $X$ have the same integral curves as point sets there is a function $\rho\colon J \times M \to \mathbb{R}$, where $J$ is some interval containing zero such that $\psi_t(x) = \phi_{\rho(t,x)}(x)$ for all $x \in M$. It follows easily that the reparametrization function $\rho$ satsfies the initial value problem

$$\frac{\partial \rho}{dt} = f(\phi_{\rho(t,x)}(x))$$
$$\rho(0, x) = 0$$

for each $x \in M$.

Let $\gamma$ denote an integral curve of $X$ and let $\sigma$ denote the corresponding integral curve of $fX$ with $\gamma(0) = x = \sigma(0)$.

PROPOSITION 2.1. *The integral curve $\gamma$ is defined on $[0, T)$ if and only if $\sigma$ is defined on $[0, t_\infty)$ where $t_\infty = \int_0^T [f(\gamma(s))]^{-1}\, ds$. In particular, if $\gamma$ is complete then $\sigma$ is complete if and only if $\int_0^\infty [f(\gamma(s))]^{-1}\, ds = \infty$.*

*Proof.* In our notation we have $\psi_t = \phi_{\rho(t,x)}(x)$ and $\gamma(t) = \phi_t(x)$ for some $x \in M$. Set $\rho(t) = \rho(t, x)$ so that $\sigma(t) = \psi_t(x) = \phi_{\rho(t)}(x)$. Define

$$B(t) = \int_0^t [f(\gamma(s))]^{-1}\, ds$$

and recall that

$$\rho'(t) = f(\phi_{\rho(t)}(x)) = f(\gamma(\rho(t))).$$

This differential equation is equivalent to the equation

$$\frac{d}{dt}[B(\rho(t))] = 1,$$

from which we obtain the solution

$$B(\rho(t)) = t + B(\rho(0)).$$

But, since $\rho(0) = 0$ we have $B(\rho(t)) = t$.

By the definition of $\rho$ we have $\lim_{t \to t_\infty} \rho(t) = T$. Thus

$$t_\infty = \lim_{t \to t_\infty} t = \lim_{t \to t_\infty} B(\rho(t)) = \lim_{t \to T} B(t) = \int_0^T [f(\gamma(s))]^{-1} \, ds. \quad \blacksquare$$

EXAMPLE 2.2. The vector field $-x(\partial/\partial x)$ on the line which gives the scalar ODE $\dot{x} = -x$ generates the flow $\phi_t(x) = e^{-t}x$. For $x > 0$ consider $f(x) = x^{-n}$, $n \geqslant 1$, on the manifold $M = (0, \infty)$. The vector field $-f(x)\, x(\partial/\partial x)$ which gives the scalar ODE $\dot{x} = -f(x)\, x$ generates the flow $\Psi_t(x) = (x^n - nt)^{1/n}$ and $\Psi_t(x) = \phi_{\rho(t)}(x)$, where $\rho(t) = \ln x(x^n - nt)^{-1/n}$. Moreover, it is evident by inspection that $\Psi_t(x)$ is incomplete with $t_\infty = x^n/n$. Of course, $x^n/n = \int_0^\infty x^n e^{-nt} dt$.

Returning to the general situation where $X$ is a polynomial vector field of degree $n$ we let $\gamma$ denote an integral curve of $\tilde{X}$ defined near the circle at infinity and let $\sigma$ denote the corresponding integral curve of $X$. In local coordinates $\gamma(t) = (v(t), z(t))$ (or $(u(t), z(t))$). Restating Proposition (2.1) we have

PROPOSITION 2.3. *The integral curve $\sigma$ of $X$ is complete if and only if* $\int_0^\infty z(t)^{n-1} dt = \infty$.

*Proof.* We have $X = g\tilde{X}$, where $g(v, z) = z^{1-n}$ so $\sigma(t)$ is complete by Proposition 2.1 if and only if

$$\int_0^\infty [g(\gamma(t))]^{-1} \, dt = \int_0^\infty z(t)^{n-1} \, dt = \infty. \quad \blacksquare$$

*Remark.* This result is sufficient to treat the completeness of orbits in the finite plane which have a zero of $\tilde{X}$ at infinity as their limit set. In Section 3 we deal with the case of periodic orbits at infinity. There it will be necessary to consider a global chart around the circle at infinity with another scale function $(\cos^{n-1}\phi)$ to which Proposition 2.1 also applies to give a completeness result analogous to Proposition 2.3.

Proposition 2.3 and the last remark provide an answer to the completeness problem; a polynomial vector field is complete if and only if each integral curve of $\tilde{X}$ satisfies $\int_0^\infty (z(t))^{n-1} \, dt = \infty$. This result can be regarded as an initial step towards our goal. However, we must free ourselves from the dependence of our result on the individual integral curves and furnish simple conditions which can be verified in terms of the coefficients of the vector field which imply the divergence of the above integral for every orbit $\gamma$. This will be done in the following two sections provided the zeros at infinity are simple, i.e., for vector fields in $\mathscr{S}_n$.

## 3. Periodic Orbits at Infinity

In this section we treat the case when our polynomial vector field

$$X = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}$$

of degree $n$ ($n$ is odd of course) has no zeros at infinity. Since the circle at infinity will then be a periodic orbit for the Poincaré extension of $X$ to the sphere it is convenient to choose a coordinate system which is global near the equator of the Poincaré sphere. The obvious way to do this is to choose spherical coordinates. Since we are considering central projection to the tangent plane at the north pole $(x, y, z) = (0, 0, 1)$ of the unit sphere all points on the plane satisfy $z = 1$ so the spherical coordinates

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

satisfy (since $\rho = (\cos\phi)^{-1}$)

$$x = \frac{\sin\phi}{\cos\phi}\cos\theta$$
$$y = \frac{\sin\phi}{\cos\phi}\sin\theta$$

with

$$\phi = \cos^{-1}\left(\frac{1}{p}\right) = \cos^{-1}((x^2 + y^2 + 1)^{-1/2})$$
$$\theta = \tan^{-1}(y/x).$$

The vector field $X$ is expressed in $(\phi, \theta)$-coordinates on the Poincaré sphere by the ODE

$$\begin{aligned}\dot\phi &= \cos^2\phi \left[\cos\theta P\left(\frac{\sin\phi}{\cos\phi}\cos\theta, \frac{\sin\phi}{\cos\phi}\sin\theta\right)\right.\\ &\quad \left. + \sin\theta Q\left(\frac{\sin\phi}{\cos\phi}\cos\theta, \frac{\sin\phi}{\cos\phi}\sin\theta\right)\right]\\ \dot\theta &= \frac{\cos\phi}{\sin\phi}\left[\cos\theta Q\left(\frac{\sin\phi}{\cos\phi}\cos\theta, \frac{\sin\phi}{\cos\phi}\sin\theta\right)\right.\\ &\quad \left. - \sin\theta P\left(\frac{\sin\phi}{\cos\phi}\cos\theta, \frac{\sin\phi}{\cos\phi}\sin\theta\right)\right],\end{aligned} \tag{1}$$

which, after the reparametrization accomplished by multiplying each equation by $\cos^{n-1}\phi$, gives the analytic Poincaré extension $\tilde{X}$ near the equator of the sphere in spherical coordinates as the ODE

$$\dot{\phi} = \cos^{n+1}\phi[\cos\theta P + \sin\theta Q]$$

$$\dot{\theta} = \frac{\cos^n\phi}{\sin\phi}[\cos\theta Q - \sin\theta P].$$

Thinking of the coordinates $(\phi, \theta)$ as "polar coordinates" in the $(\phi, \theta)$-plane our assumption that there are no zeros of $X$ at infinity is equivalent to saying that in the $(\phi, \theta)$-plane there is a periodic orbit $\Gamma$ at distance $\phi = \pi/2$ from the origin; i.e., the periodic orbit is a circle.

Recall that for our completeness problem it is crucial to know if the integral of the scale function giving the reparametrization over the $t$ interval $[0, \infty)$ diverges. In the present context this is the question of the divergence of the integral $\int_0^\infty \cos^{n-1}\phi(t)\,dt$. Usually the integral cannot be computed directly. However, the convergence or divergence of the integral is determined by the power series expansion of the Poincaré return map (which *is* computable) on a transverse section $\Sigma$ to $\Gamma$. We will now consider this relationship in detail for a general flow in the plane with a circular periodic orbit.

Let $\Gamma$ denote a circle in the plane which is a closed orbit for the flow $\phi_t$. Define for $x$ near $\Gamma$, $z(t) = d(\phi_t(x), \Gamma)$, which is the distance along a radial line from $\phi_t(x)$ to $\Gamma$. Our basic assumption will be that $\Gamma$ is asymptotically stable from the inside so that $\lim_{t\to\infty} z(t) = 0$. For our completeness problem we wish to know when $\int_0^\infty z(t)^{n-1}dt$ diverges. The purpose of this section is to relate the convergence properties of this integral to the power series representation of the Poincaré map $\rho$ on a radial section $\Sigma$ which we always take to have the standard coordinates obtained as a rigid motion of a coordinate axis of $\mathbb{R}^2$ with $\Gamma \cap \Sigma$ the origin of the coordinate system and the positive coordinates on the inside of $\Gamma$.

Since $\rho$ is analytic and $\rho(0) = 0$ it follows that $\rho(x) = c_1 x + c_2 x + \cdots$. If we assume that $\Gamma$ is not only asymptotically stable but also a hyperbolic attractor then $0 < c_1 < 1$ and using the usual hyperbolic estimates it is easy to see that $z(t) \leqslant Ke^{-kt}$ for some positive constants $K, k$ so that $\int_0^\infty z(t)^{n-1}dt \leqslant \int_0^\infty K^n e^{-knt}dt < \infty$ independent of $n > 1$. It is not quite so clear what happens when $\Gamma$ is not hyperbolic, i.e., when $\rho(x) = x - cx^k + x^{k+1}\alpha(x)$ for $\alpha(x)$ some analytic function. Our goal is to show that, in fact, $\int_0^\infty z(t)^{n-1}dt$ converges if and only if $n-1 \geqslant k$. Thus, in particular, our orbit $\phi_t(x)$ will be complete if and only if $n-1 < k$. Our first lemma shows that $k$ is independent of the choice of (radial) section.

LEMMA 3.1. *Let $\Sigma$ and $\Lambda$ be radial sections of $\Gamma$ with our standard choice* *of coordinates. Let* $\rho: \Sigma \to \Sigma$ *and* $\lambda: \Lambda \to \Lambda$ *be the Poincaré maps and let* $f: \Sigma \to \Lambda$ *be the section map. Define* $a_1 = f'(0)$. *If* $\rho(x) = x - cx^k + x^{k+1}\alpha(x)$, *then* $\lambda(x) = x - (c/a_1^{k-1})x^k + x^{k+1}\beta(x)$ *for some analytical function* $\beta$.

*Proof.* Clearly $f(x) = \sum_{n=1}^{\infty} a_n x^n$ is analytic and so is $\lambda(x) = \sum_{n=1}^{\infty} \lambda_k x^k$. Also, since $f$ is invertible $a_1 = f'(0) \neq 0$. We have $\lambda \circ f = f \circ \rho$ so $\lambda_1(\sum_{n=1}^{\infty} a_n x^n) + \lambda_2(\sum_{n=1}^{\infty} a_n x^n)^2 + \cdots = a_1(x - cx^k + x^{k+1}\alpha(x)) + a_2(x - cx^k + x^{k+1}\alpha(x))^2 + \cdots$. Since $\lambda_1 a_1 = a_1$ we have $\lambda_1 = 1$. After rearranging the right-hand side of the power series equality we obtain $\sum_{n=1}^{\infty} a_n x^n + \lambda_2(\sum_{n=1}^{\infty} a_n x^n)^2 + \lambda_3(\sum_{n=1}^{\infty} a_n x^n)^3 + \cdots = \sum_{n=1}^{\infty} a_n x^n - a_1 cx^k + \cdots$, where all terms not appearing on the right-hand side of equality have degree larger than $k$. It follows that $\lambda_2, ..., \lambda_{k-1} = 0$ and $\lambda_k a_1^k = a_1 c$. Thus $\lambda(x) = x - (c/a_1^{k-1})\, x^k + x^{k+1}\beta(x)$. ∎

Using the notation developed above we are now prepared to state the first theorem of this section.

THEOREM 3.2. *If* $\rho(x) = x - cx^k + x^{k+1}\alpha(x)$, *then* $\int_0^{\infty} z(t)^{n-1} dt$ *converges when* $n - 1 \geqslant k$ *and diverges when* $0 \leqslant n - 1 < k$.

The idea of the proof of the theorem is to compare the rate of convergence of $z(t)$ to $\Gamma$ with the rate of convergence of an orbit of $\dot{x} = -x^k$ with $x(0) > 0$ to the origin. With this application in mind we have the following lemma.

LEMMA 3.3. *For each integer* $k > 1$ *consider the scalar ODE* $\dot{x} = -x^k$, $x > 0$, *and let* $\phi_t$ *denote the associated flow. Then,*

(i) $\phi_t(x) = (x^{1-k} - (1-k)\, t)^{(1/1-k)}$,

(ii) *for fixed* $t > 0$ *and* $0 < x < 1$ *there is an analytic function* $\beta(x)$ *so that* $\phi_t(x) = x - tx^k + x^{2k-1}\beta(x)$,

(iii) $\lim_{t \to \infty} \phi_t(x) = 0$ *for all* $x > 0$ *and the convergence is monotone*,

(iv) $\sum_{n=1}^{\infty} (\phi_{nt}(x))^p = \infty$ *for* $0 \leqslant p \leqslant k$, *and*

(v) $\sum_{n=1}^{\infty} (\phi_{nt}(x))^p < \infty$ *for* $p \geqslant k$.

*Proof.* Statement (i) is proved by differentiation, statement (ii) is proved by an easy calculation after expanding the function $H(x) = x^{1/1-k}$ about the point $x = 1$, and statement (iii) follows from (i). To prove (iv) and (v) just observe that $\phi_{st}(x)$ interpolates $\phi_{nt}(x)$ and that $\phi_{st}(x)$ converges monotonically to zero as $s \to \infty$ so we can apply the integral test to the series $\sum_{n=1}^{\infty} (\phi_{nt}(x))^p$ using the interpolating comparison function $(\phi_{st}(x))^p$. We compute

$$\int_1^\infty (\phi_{st}(x))^p\,ds = -\frac{1}{t}\int_1^\infty (\phi_{st}(x))^{p-k}(\phi_{st}(x))^k(-t)\,ds$$
$$= \frac{1}{t}\int_0^{\phi_t(x)} u^{p-k}du.$$

It now follows from the computation of the last integral that $\sum_{n=1}^\infty (\phi_{nt}(x))^p$ converges for $p \geqslant k$ and diverges for $0 \leqslant p < k$. ∎

*Proof of Theorem.* Fix $x$, which we may as well assume is a point on the section $\Sigma$, let $T$ denote the period of $\Gamma$, and let $T_l$ denote the time of the $l$th return of $x$ to $\Sigma$ under the action of the flow $\phi_t$. For each $t$, $z(t)$ belongs to some radial section $\Lambda$ of $\Gamma$ for which the *Poincaré* map $\lambda\colon \Lambda \to \Lambda$ has power series expansion (in our choice of coordinates) $\lambda(z) = z - (c/a_1^{k-1})\,z^k + \cdots$, where $a_1 = f'(0)$ for $f$ the section map from $\Sigma$ to $\Lambda$. It can be shown (cf. Andronov *et al.* [$A_2$], p. 291) that $f'(0)$ depends analytically on the choice of radial section. Since the radial sections are parametrized by their intersections with the circle $\Gamma$ it follows from the fact that $a_1(\theta) > 0$ for each $\theta$ that there is a number $\tau$ such that $0 < \tau < \min\{c/a_1(\theta)^{k-1}) : 0 \leqslant \theta < 2\pi\}$. We consider the scalar ODE $\dot{z} = -z^k$ and define its time $\tau$ map $g(z) = \Psi_\tau(z)$, where $\Psi_\tau$ is the flow of the ODE. By the lemma $g(z) = z - \tau z^k + \cdots$ and we conclude that the Poincaré map $\lambda$ at any radial section $\Lambda$ satisfies $\lambda(z) < g(z)$ for all $z \in \Lambda$ sufficiently near $\Gamma$. Since $g(z)$ is monotone increasing near $z = 0$ it follows by induction that $\lambda^l(z) < g^l(z)$ for $l = 1, 2, 3, \ldots$.

Assume $n - 1 \geqslant k$. We compute

$$\int_0^\infty z(t)^{n-1}dt = \sum_{l=0}^\infty \int_{T_l}^{T_{l+1}} z(t)^{n-1}dt = \sum_{l=0}^\infty \frac{T_{l+1}-T_l}{T}\int_0^T [z(\sigma_l(s))]^{n-1}ds,$$

where $\sigma_l(s) = T_l + ((T_{l+1} - T_l)/T)\,s$. Consider the orbit segment $\{\phi_t(x)\colon 0 \leqslant t < T_1\}$ and note that there is a reparametrization $t = \delta(s)$ so that $0 \leqslant s < T$ and $\phi_{\delta(s)}(x)$ lies on the same radial section as $\phi_{\sigma_l(s)}(\phi_{T_l}(x))$ does. It follows that $z(\sigma_l(s)) < g^l(z(\delta(s)))$ and that

$$\int_0^\infty z(t)^{n-1}dt < \sum_{l=0}^\infty \frac{T_{l+1}-T_l}{T}\int_0^T [g^l(z(\delta(s)))]^{n-1}ds.$$

Define $A = \min\{z(\delta(s))\colon 0 \leqslant s < T\}$ and $B = \max\{z(\delta(s))\colon 0 \leqslant s < T\}$. Then, using the lemma we have the estimate

$$g^l(z(\delta(s))) = \frac{z(\delta(s))}{[1 + (k-1)\,l\tau(z(\delta(s)))^{k-1}]^{1/(k-1)}}$$
$$\leqslant \frac{B}{[1 + (k-1)\,l\tau\, A^{k-1}]^{1/(k-1)}},$$

from which it follows that

$$\int_0^\infty z(t)^{n-1}dt < \sum_{l=0}^{\infty} (T_{l+1} - T_l) \frac{B^{n-1}}{[1+(k-1)\, l\tau\, A^{k-1}]^{(n-1)/(k-1)}}.$$

Since $n-1 \geqslant k$ we have $(n-1)/(k-1) > 1$. So, taking into account the fact that $\lim_{n\to\infty} T_{l+1} - T_l = T$ it follows that the integral converges when $n-1 \geqslant k$ as required.

Under the assumption that $0 \leqslant n-1 < k$ the proof proceeds in the same way except we choose $\infty > \tau > \max\{c/a_1(\theta)^{k-1}\colon\ 0 \leqslant \theta < 2\pi\}$ so that $z(\sigma_l(s)) > g^l(z(\delta(s)))$ and we obtain the estimate

$$\int_0^\infty z(t)^{n-1}dt > \sum_{l=0}^{\infty} (T_{l+1} - T_l) \frac{A^{n-1}}{[1+(k-1)\, l\tau\, B^{k-1}]^{(n-1)/(k-1)}}.$$

Since now $n-1 \leqslant k-1$ we have $(n-1)/(k-1) \leqslant 1$ and the sum diverges. ▮

Following Andronov *et al.* [$A_1$] we define a periodic orbit $\Gamma$ of a vector field on a twodimensional manifold to have *multiciplicity* $k$ for $1 \leqslant k \leqslant \infty$ if for $d(x) = \rho(x) - x$, where $\rho$ is the Poincaré map on a section transverse to $\Gamma$ and $\Gamma$ crosses the section at coordinate $x = 0$, we have $d'(0) = d''(0) = \cdots = d^{(k-1)}(0) = 0$ but $d^{(k)}(0) \neq 0$. In addition, we say $\Gamma$ has *multiciplity* $\infty$ if $d^{(j)}(0) = 0$ for $j = 1, 2, 3, \ldots$.

*Remark.* For an arbitrary twodimensional manifold $M$ let $\Gamma$ be a periodic orbit with multiplicity $k$, perhaps $k = \infty$, of a vector field $X$ on $M$ and let $F$ denote a smooth function defined in a neighborhood of $\Gamma$ such that $F(p) = 0$ and $dF(p) \neq 0$ for all $p \in \Gamma$. If $\phi_t(x)$ is an orbit of $X$, $x \notin \Gamma$, whose $\omega$ limit set is $\Gamma$ then $\int_0^\infty F(\phi_t(x))^{n-1} dt = \infty$ if and only if $k > n-1$.

In fact this reduces to the case studied above in which $\Gamma$ is a circle in the plane and $F$ is $z$, the radial distance to $\Gamma$. Clearly, it is enough to construct a diffeomorphism $D$ which maps the level curves of $F$ into concentric circles in $\mathbb{R}^2$ so that the radial distance between two of them is given by the difference in $F$ level. To obtain $D$ choose a diffeomorphism $f$ from a circle $C$ in $\mathbb{R}^2$ onto $\Gamma$, let $G_t$ be the flow of the vector field $(1/|\nabla F|^2)\,\nabla F$, where the gradient $\nabla$ is taken with respect to some Riemannian metric on $M$, and define $D^{-1}$ by mapping a point $p$ on the normal ray through $q \in C$ to $G_{|p-q|}(f(q))$. Since $(F \circ G_t)' \equiv 1$ it follows that $|F(D^{-1}(p_1)) - F(D^{-1}(p_2))| = |\,|p_1| - |p_2|\,|$. The idea for this construction of $D$ is due to Milnor [M].

Returning to our original considerations we assume

$$X = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}$$

is a polynomial vector field of degree $n = 2m+1$ with no stationary points at infinity. Thus, the equator $\Gamma$ of the Poincaré sphere is a periodic orbit for the Poincaré extension $\tilde{X}$ of $X$. Let $\rho$ denote the Poincaré map on a (radial) section transverse to $\Gamma$ and notice that in the present case the multiplicity $k = 2l+1$ of $\Gamma$ must be odd due to the fact that $\tilde{X}$ is invariant under the antipodal map of the sphere and, hence, covers a vector field in the projective plane [G]. Clearly, $\Gamma$ has multiplicity 1 if and only if $\Gamma$ is hyperbolic, multiplicity $k$, $1 < k < \infty$, if and only if $\rho(x) = x - cx^k + \cdots$, and multiplicity $\infty$ if and only if $\Gamma$ is contained in a collar neighborhood of itself where every orbit is periodic. The results of this section, particularly Theorem 3.2 and the remark after its proof, applied to the function $F = \cos\phi$ in Eq. (2) give the following completeness theorem.

THEOREM 3.4. *Let the equator of the Poincaré sphere be a periodic orbit of multiplicity $k$ for the extension $\tilde{X}$ of the polynomial vector field $X$ of degree $n$, then $X$ is complete if and only if $k > n-1$.*

The Poincaré map $\rho$ has a power series expansion which is computable by successive integrations of trigonometric functions and exponentials. However, in practice these computations are too formidable to be done by hand. A general theory for such computations is presented by Andronov *et al.* [$A_2$, Chap. 10] and by Urabe [U], where the formulae for the derivatives are given in rectangular coordinates. Since we are using spherical coordinates it is usually easier to compute directly in the natural coordinates.

Rather than repeating the general theory for our case we are content to observe a few facts which will allow us to draw our conclusions about the manifold structure of the class of complete vector fields which have a periodic orbit at infinity.

LEMMA 3.5. *Suppose $X \in \mathfrak{X}_n$ has Poincaré extension $\tilde{X}$ with a periodic orbit of multiplicity larger than $2r-1$. If $H: \mathbb{R}^r \to \mathfrak{X}_n$ is the family of vector fields defined by*

$$H(\lambda_1, \ldots, \lambda_r) = X + \left(\sum_{i=1}^{r} \lambda_i (x^2+y^2)^{m-(i-1)}\right)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$$

*then*

$$\text{(i)} \quad \frac{\partial \rho^{(2i-1)}}{\partial \lambda_i}(0, 0, \ldots, 0) \neq 0 \qquad \textit{for} \quad i = 1, \ldots, r$$

$$\text{(ii)} \quad \frac{\partial \rho^{(2j-1)}}{\partial \lambda_i}(0, 0, \ldots, 0) = 0 \qquad \textit{for} \quad j < i,$$

*where $\rho^{(l)}(x, \lambda_1,..., \lambda_r)$ denotes the $l$th derivative of the Poincaré return map $\rho(x, \lambda_1,..., \lambda_r)$ of $H(\lambda_1,..., \lambda_r)$ on a segment transverse to the equator.*

*Proof.* Notice that in spherical coordinates $(\phi, \theta)$ the vector field $H(\lambda_1,..., \lambda_r)$ expressed as an ODE has the form

$$\dot{\phi} = F(\phi, \theta) + \sum_{i=1}^{r} \lambda_i (\cos\phi)^{2i-1}$$
$$\dot{\theta} = G(\phi, \theta).$$

The proofs of (i) and (ii) now follow with obvious similarity to the proof of Lemma 1 in [$A_2$, p. 274]. Part (ii) also follows from the observation that the $l$th derivative of the return map for the orbit at infinity depends only on the homogeneous terms of degree greater than or equal to $n-l+1$. ∎

THEOREM 3.6. *Let $n = 2m + 1$. The class $\mathscr{C}_{n,0}$ of complete polynomial vector fields of degree $n$ with no zeros at infinity is a regular analytic submanifold of codimension $m$ in the space $\mathfrak{X}_n \simeq \mathbb{R}^N$, $N = (n+1)(n+2)$, of all polynomial vector fields of degree $n$ in the plane.*

*Proof.* The theorem follows directly from Lemma 3.5 and Theorem 3.4. In Lemma 3.5 take $r = m$. Let $\mathscr{H}^m$ denote the image of $H$: $\mathbb{R}^m \to \mathfrak{X}_n$ and $\mathscr{K}$ its orthogonal complement. The set $\mathscr{C}_{n,0}$ is given in a small neighborhood $\mathscr{V}$ of $X$ by the common zeros of $\ln \rho^{(1)}$, $\rho^{(3)},..., \rho^{(m)}$, which by Lemma 3.5 (i) and (ii) satisfy the hypothesis of the implicit function theorem at $X$. Actually, by choosing $\mathscr{V}$ sufficiently small, $\mathscr{V} \cap \mathscr{C}_{n,0}$ can be realized as the graph of an analytic mapping from $\mathscr{K}$ to $\mathscr{H}^m$. ∎

Below we give an example of a polynomial vector field with a multiple limit cycle at infinity. To this end we review some of the expression for the derivatives of the return map. The vector field $\tilde{X}$ on the Poincaré sphere as we have seen leads to an ODE of form $\dot{\phi} = F(\phi, \theta)$, $\dot{\theta} = G(\phi, \theta)$ with periodic orbit $\Gamma$ the circle at infinity given by $\phi = \pi/2$. Suppose $\Sigma$ is the transverse section given by $\theta = 0$ (the projection of the $x$-axis to the $(\phi, \theta)$-plane) and consider a solution $\phi(\theta, x)$ of $d\phi/d\theta = F(\phi, \theta)/G(\phi, \theta) = S(\phi, \theta)$ which satisfies the initial condition $\phi(0, x) = x$ for each $x \in \Sigma$. Near $\phi = \pi/2$ the return map $\rho$ is defined and, of course, $\rho(x) = \phi(2\pi, x)$. In these coordinates $x = 1$ is the coordinate of the intersection of $\Gamma$ and $\Sigma$. To apply our theory we need to compute $\rho^{(k)}(1)$.

The equation of variation

$$\frac{d}{d\theta}\left(\frac{\partial\phi}{\partial x}(\theta, x)\right) = \frac{\partial S}{\partial\phi}(\phi(\theta, x), \theta)\frac{\partial\phi}{\partial x}(\theta, x)$$

is easily solved for $\partial\phi/\partial x$ so we obtain

$$\rho'(x) = \frac{\partial \phi}{\partial x}(\theta, x)|_{\theta=2\pi} = \exp \int_0^{2\pi} \frac{\partial S}{\partial \phi}(\phi(\theta, x), \theta)\, d\theta,$$

which implies

$$\rho'(1) = \exp \int_0^{2\pi} \frac{\partial S}{\partial \phi}\left(\frac{\pi}{2}, \theta\right) d\theta.$$

Of course, if $\rho'(1) \neq 1$ then $\Gamma$ is hyperbolic and in our context we will have incompleteness unless $n = 1$.

We state as a remark the following interesting fact, which follows from direct calculation of $(\partial S/\partial \phi)(\frac{\pi}{2}, \theta)$.

*Remark.* For the vector field

$$P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y},$$

$\rho'(1)$ depends only on the highest-order terms of $P$ and $Q$. In fact, if the vector field has degree $n$

$$\rho'(1) = \exp\left[ -\int_0^{2\pi} \frac{\cos\theta\, P_n(\cos\theta, \sin\theta) + \sin\theta\, Q_n(\cos\theta, \sin\theta)}{\cos\theta\, Q_n(\cos\theta, \sin\theta) - \sin\theta\, P_n(\cos\theta, \sin\theta)}\, d\theta \right],$$

where $P_n$ and $Q_n$ denote the homogeneous polynomial of $n$th-degree terms of $P$ and $Q$.

Higher-order derivatives are computed from the integral formula for $\rho'$. We compute $\rho''$ as follows:

$$\rho''(x) = \rho'(x)\left[\int_0^{2\pi} \frac{\partial^2 S}{\partial \phi^2}(\phi(\theta, x), \theta) \exp\left(\int_0^{\theta} \frac{\partial S}{\partial \phi}(\phi(\sigma, x), \sigma)\, d\sigma\right) d\theta\right].$$

Thus, if $\rho'(1) = 1$ (the only case of interest here)

$$\rho''(x) = \int_0^{2\pi} \frac{\partial^2 S}{\partial \phi^2}\left(\frac{\pi}{2}, \theta\right) \exp\left(\int_0^{\theta} \frac{\partial S}{\partial \phi}\left(\frac{\pi}{2}, \sigma\right) d\sigma\right) d\theta$$

and, as we have seen, $\rho''(1) = 0$ always. The formulae for higher-order derivatives are much more complicated; $\rho'''(x)$ is in [LL] as follows:

$$\rho'''(x) = E(x, 2\pi)\left[\frac{3}{2}\left(\int_0^{2\pi} D(x, \sigma)\, d\sigma\right)^2 + \int_0^{2\pi} (E(x, \sigma))^2 \frac{\partial^3 S}{\partial \phi^3}(\phi(\sigma, x), \sigma)\, d\sigma\right],$$

where

$$E(x, \sigma) = \exp \int_0^\sigma \frac{\partial S}{\partial \phi}(\phi(\theta, x), \theta)\, d\theta$$

and

$$D(x, \sigma) = E(x, \sigma) \frac{\partial^2 S}{\partial \phi^2} (\phi(\theta, x), \theta).$$

As before when $\rho'(1) = 1$, $\rho'''(1)$ simplifies somewhat to

$$\rho'''(1) = \frac{3}{2}\left(\int_0^{2\pi} D(x, \sigma)\, d\sigma\right)^2 + \int_0^{2\pi} (E(x, \sigma))^2 \frac{\partial^3 S}{\partial \phi^3}\left(\frac{\pi}{2}, \sigma\right) d\sigma.$$

EXAMPLE 3.7. Let

$$X = (x + y - y^3)\frac{\partial}{\partial x} + (-x + y + x^3)\frac{\partial}{\partial y}.$$

We claim $X$ is complete. This will follow from Theorem 3.4 if the first non-zero derivative of $d(x) = \rho(x) - x$ at $x = 1$ is at least the third. If suffices to show $\rho'(1) = 1$ since $\rho''(1) = 0$ automatically.

In this example

$$\dot{\phi} = \sin \phi \cos^3 \phi + \cos \phi \sin^3 \phi(\sin \phi \cos^3 \phi - \sin^3 \theta \cos \theta)$$
$$\dot{\theta} = \sin^2 \phi(\cos^4 \theta + \sin^4 \theta) - \cos^2 \phi.$$

We compute

$$\frac{\partial S}{\partial \phi}\left(\frac{\pi}{2}, \theta\right) = \frac{\sin^3 \theta \cos \theta - \sin \theta \cos^3 \theta}{\cos^4 \theta + \sin^4 \theta}$$

and

$$\rho'(1) = \exp\left(\frac{1}{4}\int_0^{2\pi} \frac{f'(\theta)}{f(\theta)}\, d\theta\right),$$

where $f(\theta) = \cos^4 \theta + \sin^4 \theta$. Thus,

$$\rho'(1) = \exp\left(\frac{1}{4}(\ln f(2\pi) - \ln f(0))\right) = 1.$$

The reader may be amused by computing $\rho'''(1) = 0$. However, for this example there is an ad hoc method which also show it is complete. Let $\sigma = \frac{1}{4}(x^4 + y^4)$. Then $\dot{\sigma} = 4\sigma + x^3y - xy^3$. Since $x^3y - xy^3 \leqslant x^4 + y^4$ we have $\dot{\sigma} \leqslant 8\sigma$ and, therefore, $\sigma(t) \leqslant \sigma(0)\, e^{8t}$. It follows that for any finite time an orbit $(x(t), y(t))$ of $X$ is bounded and hence complete.

## 4. Stationary Points at Infinity

In this section we consider integral curves of $\tilde{X}$ whose $\omega$ limit set is a zero of $\tilde{X}$ on the circle at infinity. If an integral curve has as an $\alpha$ limit point a zero of $\tilde{X}$ on the circle at infinity we could consider the vector field $-X$. Thus, there is no loss of generality if only $\omega$ limit points are considered. Since the circle at infinity is invariant a zero of $\tilde{X}$ has a scheme made up of hyperbolic, elliptic, and parabolic sectors. In particular, we do not have any orbits which spiral around a zero of $\tilde{X}$ on the circle at infinity. It is a standard fact [$A_1$, Sect. 9, 20] that in this case any integral curve which has a zero of the *analytic* vector field $\tilde{X}$ as an $\omega$ limit set must approach that zero along a definite direction; i.e., if $\gamma$ is such an orbit then as $t \to \infty$ its tangent line $\mathbb{R}\dot{\gamma}(t)$ has a limit.

The next theorem is the first step towards understanding incompleteness of integral curves tending to zeros at infinity.

Theorem 4.1. *If $\tilde{X}$ has a hyperbolic zero on the circle at infinity then $X$ is not complete* (*unless $n = 1$*).

*Proof.* With no loss of generality assume the hyperbolic zero is the origin in $(v, z)$-coordinates and that it is either a saddle or a sink so that at least one orbit in the finite plane satisfies $\lim_{t \to \infty} \gamma(t) = (0, 0)$. The usual hyperbolic estimates imply that near the origin $\gamma$ satisfies an inequality of form

$$|\gamma(t)|_{\max} \leqslant K e^{-ct} |\gamma(0)|_{\max}$$

where $K$ and $c$ are positive constants. If $\gamma(t) = (v(t), z(t))$, we may assume $z(t) > 0$, and it follows that

$$z(t) = K e^{-ct} |\gamma(0)|_{\max} .$$

Thus, when $n > 1$,

$$\int_0^\infty z(t)^{n-1} dt \leqslant K |\gamma(0)|_{\max} \int_0^\infty e^{-nct} dt < \infty$$

and $X$ is incomplete by Proposition 2.3. ∎

*Remarks.* (1) This result also follows from more precise results in [S, p. 309], which describe the various analytical patterns of the divergence of incomplete orbits under the hyperbolicity condition on the zeros at infinity. (2) The proof of Theorem 4.1 also shows that a vector field $X$ (of degree $n > 1$) is not complete if $\tilde{X}$ has a zero which is only hyperbolic in the direction normal to the equator.

Now we consider stationary points at infinity which are hyperbolic only along the equator. These points are called *semi-hyperbolic at infinity.* First we review the essential properties of such points for a general vector field.

A zero $p$ of a $C^\infty$ vector field

$$V = A(x, y)\frac{\partial}{\partial x} + B(x, y)\frac{\partial}{\partial y}$$

is called *semi-hyperbolic* if $DV(p)$ has two eigenvalues and exactly one of them vanishes. The *multiplicity* of the semi-hyperbolic zero $p$ is the multiplicity of the intersection of the curves given by $A(x, y) = 0$ and $B(x, y) = 0$ at $p$. If coordinates are chosen such that $p = (0, 0)$ and $\partial/\partial x$ is an eigenvector associated with the nonzero eigenvalue of $DV(p)$, the multiplicity can be evaluated as follows: Let $x = a(y)$ be the $C^\infty$ curve defined implicitly by $A(a(y), y) = 0$, with $a(0) = 0$. The multiplicity $m$ of the semi-hyperbolic point $p$ is the order of 0 as a zero of the function $b(y) = B(a(y), y)$, i.e., the least integer $m$ such that $b(y) = b_m y^m + \cdots$, with $b_m \neq 0$. If no such $m$ exists we say the zero has multiplicity $\infty$.

Notice that the multiplicity depends only on the formal power series of $A$ and $B$ at $p = (0, 0)$ and that the multiplicity can be computed in terms of the coefficients of these series by purely algebraic operations. Actually, for a zero with $\partial/\partial x$ as the nonzero eigenvector, the coefficients $b_i$ of the formal power series of $b$, $\sum_{i=1}^{\infty} b_i y^i$, are rational functions of the coefficients of the $i$th degree polynomials obtained by truncation of the power series of $A$ and $B$ at $(0, 0)$. With this notation a semi-hyperbolic point of multiplicity greater than or equal to $k$ is characterized by the equations $b_1 = 0$, $b_2 = 0, \ldots, b_{k-1} = 0$.

The main properties of semi-hyperbolic zeros of analytic vector fields are listed below (cf. [$A_2$, H-P-S]). Let $p$ be the semi-hyperbolic zero of $V$ and let $v_1$ be a nonzero eigenvector of $DV(p)$ associated with the nonzero eigenvalue $\lambda_1$ and let $v_0$ be a nonzero eigenvector associated with $\lambda_0$ the zero eigenvalue.

(a) There is a unique analytic invariant curve $W^s$ through the point $p$ and tangent to $v_1$ called the strong invariant manifold.

(b) For any $k > 0$ there is a $C^k$ invariant curve $W^c$ through $p$ tangent to $v_0$ called a center manifold. The curve $W^c$ in general is not unique.

(c) If $p$ has infinite multiplicity $W^c$ is analytic with each point of $W^c$ a zero of $V$ which is itself semi-hyperbolic with strong invariant manifold normal to $W^c$.

(d) If the multiplicity is finite there is a change of variables which maps the strong invariant manifold into the $x$-axis and the center manifold into the $y$-axis. In the case we will deal with below the strong invariant manifold is already in the equatorial axis and the change of variables which maps the center manifold to the orthogonal axis can be taken so as to preserve the orthogonal coordinate. After the change of variables the vector field $V$ can be written in the new coordinates as

$$V = -x[a + \bar{A}(x, y)] \frac{\partial}{\partial x} - y^k [b + y\bar{B}(x, y)] \frac{\partial}{\partial y}$$

with $\bar{A}(0, 0) = 0$. Also, an orbit which approaches $p$ but does not lie in $W^s$ is asymptotically tangent to $W^c$ with infinite asymptotic order of contact.

(e) Figure 1 illustrates the essentially different possibilities for semi-hyperbolic zeros of analytic vector fields. The double arrows indicate hyperbolicity.

THEOREM 4.2. *Let $\tilde{X}$ restricted to the equator of the Poincaré sphere*

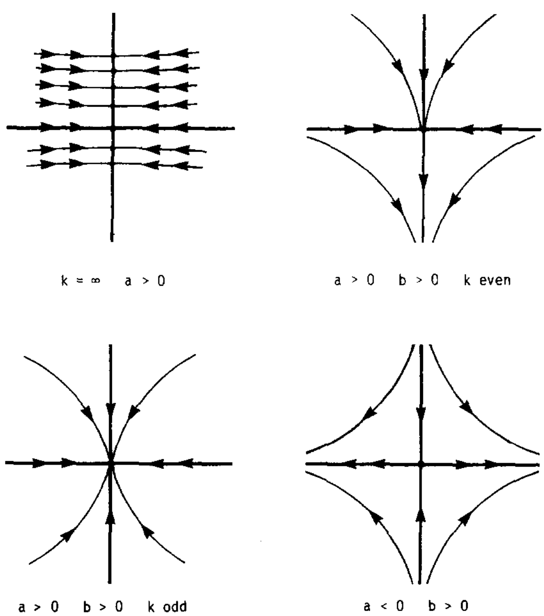


FIGURE 1

*have a simple zero p which is a semi-hyperbolic zero of multiplicity k of* $\tilde{X}$ *(as a vector field on the sphere). An integral curve* $\sigma$ *of X which approaches infinity so that the direction of its tangent line tends to p is complete if and only if* $k > n-1$.

*Proof.* By a linear change of coordinates in the plane we may assume that $p$ lies at the origin of the $(v, z)$ coordinate chart on the sphere and that $\partial/\partial z$ is an eigenvector associated with the zero eigenvalue of $D\tilde{X}(p)$. For the curve $\gamma(t) = (v(t), z(t))$ corresponding to $\sigma$ we must show $\int_0^\infty z(t)^{n-1} dt = \infty$ if and only if $k > n-1$.

The fact that the direction of $\sigma$ tends to $p$ implies that the multiplicity of $p$ is finite, by (c) above. Also, by a change of coordinates which leaves the $z$ coordinate invariant, we can assume that $\tilde{X}$ has the form given in (d) with $x$ replaced by $v$ and $y$ replaced by $z$.

Let $a > 0$ and $b > 0$; the other cases are similar. Define $b_+$ (resp. $b_-$) to be the maximum (resp. the minimum) of $b + z\bar{B}(v, z)$ in a closed neighborhood $N$ of $(0, 0)$ in which $\gamma$ can be assumed to be contained. Let $z_+(t)$ (resp. $z_-(t)$) be the solution of $\dot{z} = -z^k b_+$ (resp. $\dot{z} = -z^k b_-$) with initial condition $z_\pm(0) = z_0 = z(0)$. Since

$$z_\pm(t) = [z_0^{1-k} - (1-k)\, b_\pm t]^{-1/(k-1)},$$

$$\int_0^\infty z_\pm(t)^{n-1}(t)\, dt = \int_0^\infty [z^{1-k} - (1-k)\, b_\pm t]^{-(n-1)/(k-1)} dt.$$

This integral converges if $n > k$ and diverges if $k > n-1$. From this fact and the inequality $z_+(t) \leqslant z(t) \leqslant z_-(t)$ the result follows. ∎

Recall $\mathscr{S}_n$ is the class of polynomial vector fields $X$ in $\mathfrak{X}_n$ for which $\tilde{X}$ restricted to the equator has only simple zeros. We denote by $\mathscr{S}_{n,j}$ the class of vector fields in $\mathscr{S}_n$ with exactly $2j$ zeros at infinity. We also define $\mathscr{C}_{n,j} = \mathscr{C}_n \cap \mathscr{S}_{n,j}$, where, of course, $\mathscr{C}_n$ is the class of complete vector fields in $\mathfrak{X}_n$. In Theorems 3.4 and 3.6, $\mathscr{C}_{n,0}$ was characterized and its codimension as an analytic submanifold of $\mathfrak{X}_n$ was determined. We now characterize $\mathscr{C}_{n,j}$.

THEOREM 4.3. (a) $X \in \mathscr{C}_{n,j}$ *if and only if all* $2j$ *zeros of* $\tilde{X}$ *at infinity have multiplicity greater than or equal to n.*

(b) *If* $n = 2m+1$ *(resp.* $n = 2m$*),* $\mathscr{C}_{n,j}$ *is empty for* $j = 1, 3, \ldots, 2m+1$ *and for* $2m+2$ *(resp. for* $j = 2, 4, \ldots, 2m$ *and for* $2m+1$*).*

*Proof.* Part (a) follows from Theorem 4.2. To prove (b) for $n = 2m+1$ we observe that if $j = 1, 3, \ldots, 2m+1$ at least one of the zeros of $\tilde{X}$ restricted to the circle at infinity is not simple so we really only need to prove $\mathscr{C}_{n,2m+2}$ is empty.

For $P_n, Q_n$ the $n$th degree homogeneous parts of $X$, $yP_n - xQ_n$, and $xP_n + yQ_n$ must have $j = n+1$ real linear factors in common. Otherwise some of the zeros at infinity would be hyperbolic in the direction normal to the equator and $X$ would be incomplete. Actually, the expression for $b_1$ at a zero at infinity is proportional to $xP_n + yQ_n$ evaluated at the zero. Therefore, we could write

$$yP_n - xQ_n = \prod_{i=1}^{j} (y - \alpha_i x) A_{n-j+1}$$

and

$$xP_n + yQ_n = \prod_{i=1}^{j} (y - \alpha_i x) B_{n-j+1},$$

where $A_{n-j+1}$, $B_{n-j+1}$ are polynomials of degree $n - j + 1$. Hence

$$(x^2 + y^2) P_n = \prod_{i=1}^{j} (y - \alpha_i x)(yA_{n-j+1} + xB_{n-j+1})$$

and

$$(x^2 + y^2) Q_n = \prod_{i=1}^{j} (y - \alpha_i x)(yB_{n-j+1} - xA_{n-j+1}),$$

which is a contradiction when $j = n+1$ because $x^2 + y^2$ is not a factor of the right-hand sides.

The proof of the case $n = 2m$ is similar. ∎

In what follows we glimpse into the analytic structure of $\mathscr{C}_{n,j}$, when $\mathscr{C}_{n,j}$ is nonempty. The set $\mathscr{C}_{n,1}$ is locally defined as a subset of the open semi-algebraic set $\mathscr{S}_n$ by the zeros of the analytic functions $b_i$, $i = 1, 2, ..., n-1$, for the functions $b_i$ defined when we introduced semi-hyperbolic stationary points. From this observation and the results of Lojasievich [L] it follows that $\mathscr{C}_{n,1}$ is the union of disjoint regular analytic submanifolds which are the strata of a Whitney stratified set. In addition, it seems that the functions $b_i$ are independent on $\mathscr{C}_{n,1}$; i.e., their Jacobian matrix as functions defined in $\mathfrak{X}_n$ has rank $n-1$. If true, this would imply that $\mathscr{C}_{n,1}$ consists of a single regular analytic submanifold of codimension $n-1$ in $\mathfrak{X}_n$. However, the computations required to prove this appear to be quite involved and have not been fully carried out. For $\mathscr{C}_{n,j}$ the same argument applied to the $(n-1)j$ analytic functions $b_i$ associated with the $j$ pairs of zeros at infinity in $j$ groups of $n-1$ corresponding to each pair of zeros would show that $\mathscr{C}_{n,j}$ is the union of disjoint regular analytic submanifolds.

We conclude by showing that $\mathscr{C}_{n,j}$ is a regular analytic submanifold of codimension $(n-1)j$ in $\mathfrak{X}_n$ for the cases $\mathscr{C}_{2,1}$, $\mathscr{C}_{3,2}$, and $\mathscr{C}_{4,1}$. For this, we assume with no loss of generality that the zeros of $X \in \mathscr{C}_{n,1}$ (resp. $\mathscr{C}_{n,2}$) at infinity are in the direction of the $x$-axis (resp. the $x$- and $y$-axes). Also, we define for $\lambda = (\lambda_1, ..., \lambda_{n-1})$ the $(n-1)$-parameter family of vector fields

$$X_\lambda = X + \left( \sum_{i=1}^{n-1} \lambda_i x^{n-i+1} \right) \frac{\partial}{\partial x}.$$

For $n = 2, 4$ it can be verified that $b_i(\lambda) = b_i(X_\lambda) = \lambda_i + \tilde{b}_i$, where the $\tilde{b}_i$ are independent of $\lambda$. Thus, the Jacobian matrix of the $b_i$ evaluated at $X$ has rank 1 in case $n = 2$ since the vector field $x^2(\partial/\partial x)$ has nonzero image and has rank 3 in case $n = 4$ since the vector fields

$$x^4 \frac{\partial}{\partial x}, \qquad x^3 \frac{\partial}{\partial x}, \qquad x^2 \frac{\partial}{\partial x}$$

span a three-dimensional subspace of $\mathfrak{X}_4$ which has a three-dimensional image. It follows that the $b_i$ are independent in these cases. (For larger values of $n$ the $\tilde{b}_i(\lambda)$ depend on the $\lambda_i$ in a complicated way for which it is not obvious that the $b_i$ are independent.)

For $j = 2$, consider the $2(n-1)$-parameter family of vector fields

$$X_{\mu,\lambda} = X + \left( \sum_{i=1}^{n-1} \lambda_i x^{n-i+1} \right) \frac{\partial}{\partial x} + \left( \sum_{i=1}^{n-1} \mu_i y^{n-i+1} \right) \frac{\partial}{\partial y}.$$

Then, in case $n = 3$, one can verify that

$$b_i^{(1)}(\lambda, \mu) = \lambda_i + \tilde{b}_i^{(1)} \qquad \text{and} \qquad b_i^{(2)}(\lambda, \mu) = \mu_i + \tilde{b}_i^{(2)},$$

where the upper indices refer to one pair of zeros at infinity and, as above, if follows that the $b_i$ are independent.

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