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The Burnside Ring of the Infinite Cyclic Group and Its Relations to the Necklace Algebra, λ-Rings, and the Universal Ring of Witt Vectors

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TO THE MEMORY OF IRVING REINER

It is shown that well-known product decompositions of formal power series arise from combinatorially defined canonical isomorphisms between the Burnside ring of the infinite cyclic group on the one hand and Grothendieck's ring of formal power series with constant term 1 as well as the universal ring of Witt vectors on the other hand.

1. INTRODUCTION

Let $\mathbf{a} = a(t) = 1 + \sum_{n=1}^{\infty} a_n \cdot t^n \in \mathbf{Z}[[t]]$ be a formal power series with integral coefficients and with constant term 1. It is well known that there exist uniquely determined infinite sequences $\mathbf{b} = (b_1, b_2, ...)$, $\mathbf{q} = (q_1, q_2, ...)$, and $\mathbf{d} = (d_1, d_2, ...)$ of integers such that

$$(1.1) \qquad a(t) = \prod_{n=1}^{\infty} \left(\frac{1}{1-t^n}\right)^{b_n} = \prod_{n=1}^{\infty} \frac{1}{1-q_n t^n} = \exp\left(\int \sum_{n=1}^{\infty} d_n t^{n-1}\, dt\right)$$

and that for any $n \in \mathbf{N} := \{1, 2, 3, ...\}$ the b_n, q_n, and d_n can be computed from the $a_1, ..., a_n$ by evaluating certain uniquely determined universal polynomials $B(x_1, ..., x_n)$, $Q(x_1, ..., x_n)$, and $D(x_1, ..., x_n)$ in $\mathbf{Q}[x_1, ..., x_n]$ at $x_1 = a_1, ..., x_n = a_n$. Moreover any sequence $\mathbf{b} = (b_1, b_2, ...)$ or

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$\mathbf{q}=(q_1, q_2, \ldots)$ of integers can occur that way, while a sequence $\mathbf{d}=(d_1, d_2, \ldots)$ of integers occurs that way if and only if

$$(1.2) \qquad \sum_{i=1}^{n} d_{(i,n)} = \sum_{j \mid n} \varphi\left(\frac{n}{j}\right) d_j \equiv 0 \pmod{n}$$

for all $n \in \mathbf{N}$ (cf. [DS2])—where $(i, n) := \gcd(i, n)$ denotes the greatest common divisor of i and n and φ is the Euler function—or equivalently (cf. [Do]) if and only if

$$(1.3) \qquad \sum_{j \mid n} \mu\left(\frac{n}{j}\right) d_j \equiv 0 \pmod{n}$$

for all $n \in \mathbf{N}$—where μ is the Möbius function.

For specific functions $\mathbf{a}=a(t)$ it has often proved rather useful to rewrite $\mathbf{a}$ from any one such form into another one. The formalism by which such rewriting can be achieved has also been studied and has been related to the universal ring of Witt vectors and the necklace algebra (cf. [C, MR1]). In this paper we want to propose a surprisingly simple combinatorial interpretation of the relations between the various parameter systems $(a_1, a_2, \ldots)$, $(b_1, b_2, \ldots)$, $(q_1, q_2, \ldots)$, and $(d_1, d_2, \ldots)$ associated with a given $\mathbf{a}=a(t)$.

More precisely, let $\mathbf{C}$ denote the (multiplicatively written) infinite cyclic group with neutral element 1. A set X on which $\mathbf{C}$ is acting from the left will be called a *cyclic set*—as suggested by G.-C. Rota. The orbit of an element of the cyclic set X is a finite or an infinite cycle. Since cycles of equal length are isomorphic as cyclic sets we may denote the cycle of length n, $n \in \mathbf{N}_\infty := \mathbf{N} \cup \{\infty\}$, by $\mathbf{C}(n)$. Note that every cyclic set decomposes uniquely into a disjoint union of cycles.

A cyclic set X containing no infinite cycles will be called an *almost finite* cyclic set if in addition for every integer n there are only finitely many cycles contained in X which have length n. Since the disjoint union $X_1 \sqcup X_2$ and the cartesian product $X_1 \times X_2$ of two almost finite cyclic sets X_1 and X_2 again are almost finite cyclic sets, we may consider the Burnside–Grothendieck ring $\hat{\Omega}(\mathbf{C})$ (of isomorphism classes) of almost finite cyclic sets. Often we shall identify an almost finite cyclic set X with the element $[X]$ in $\hat{\Omega}(\mathbf{C})$ represented by it.

Note that the Burnside–Grothendieck ring $\Omega(\mathbf{C})$ of finite cyclic sets is a proper subring of $\hat{\Omega}(\mathbf{C})$. We shall see in a moment that $\hat{\Omega}(\mathbf{C})$ is a complete topological ring with respect to a canonical topology which can be defined on it and that $\Omega(\mathbf{C})$ is a dense subring of $\hat{\Omega}(\mathbf{C})$ with respect to that topology.

If one associates to an almost finite cyclic set X the number $\varphi_{\mathbf{C}^n}(X)$ of those elements of X which are invariant under the operation of the unique subgroup $\mathbf{C}^n$ of $\mathbf{C}$ which has the index n in $\mathbf{C}$, then the map $X \mapsto \varphi_{\mathbf{C}^n}(X)$ extends to a ring homomorphism $\varphi_{\mathbf{C}^n}: \hat{\Omega}(\mathbf{C}) \to \mathbf{Z}$. The family of ring homomorphisms $\varphi_{\mathbf{C}^n}$ provides us with another ring homomorphism

$$(1.4) \qquad \hat{\varphi} := \prod_{n \in \mathbf{N}} \varphi_{\mathbf{C}^n}: \hat{\Omega}(\mathbf{C}) \to \mathrm{gh}(\mathbf{C}) := \mathbf{Z}^{\mathbf{N}} \qquad \text{with} \quad (\hat{\varphi}(x))(n) := \varphi_{\mathbf{C}^n}(x),$$

where the *ghost ring* $\mathrm{gh}(\mathbf{C})$—well known from the context of Witt vectors (cf. [L])—is defined to be the ring of all maps $a: \mathbf{N} \to \mathbf{Z}$ with addition and multiplication defined componentwise. It follows easily from arguments already well known to Burnside that $\hat{\varphi}: \hat{\Omega}(\mathbf{C}) \to \mathrm{gh}(\mathbf{C})$ is injective. It is also well known (cf. [Do, DS2, Dr3, tD]) that $\hat{\varphi}$ is *not* surjective and that $d \in \mathrm{gh}(\mathbf{C})$ is in the image of $\hat{\varphi}$ if and only if

$$(1.5) \qquad \sum_{i=1}^{n} d(\gcd(i, n)) = \sum_{i \mid n} \varphi\left(\frac{n}{i}\right) \cdot d(i) \equiv 0 \qquad (\mathrm{mod}\ n)$$

or equivalently if and only if

$$(1.6) \qquad \sum_{i \mid n} \mu\left(\frac{n}{i}\right) \cdot d(i) \equiv 0 \qquad (\mathrm{mod}\ n)$$

for all $n \in \mathbf{N}$. In particular, $\hat{\Omega}(\mathbf{C})$ is a complete topological ring with respect to the coarsest topology on $\hat{\Omega}(\mathbf{C})$ for which all the maps $\varphi_{\mathbf{C}^n}$ from $\hat{\Omega}(\mathbf{C})$ into the discrete ring $\mathbf{Z}$ are continuous and the image of $\hat{\Omega}(\mathbf{C})$ under the continuous injection $\hat{\varphi}: \hat{\Omega}(\mathbf{C}) \to \mathrm{gh}(\mathbf{C})$ is a closed subring of $\mathrm{gh}(\mathbf{C})$ if $\mathrm{gh}(\mathbf{C}) \cong \prod_{n=1}^{\infty} \mathbf{Z}$ is provided with the product topology. Of course, it is this topology, for which $\Omega(\mathbf{C})$ is a dense subring of $\hat{\Omega}(\mathbf{C})$ (i.e., $\hat{\Omega}(\mathbf{C})$ could also have been defined as the completion of $\Omega(\mathbf{C})$, if $\Omega(\mathbf{C})$ is provided with the coarsest topology, for which all the maps $\varphi_{\mathbf{C}^n}: \Omega(\mathbf{C}) \to \mathbf{Z}$ are continuous). We prefer the definition in terms of almost finite cyclic sets, given above, since it does not need any topological considerations.

The present paper will mainly be concerned with the commutative diagram

$$(1.7) \qquad \begin{array}{ccccc} & & Nr(\mathbf{Z}) & & \\ & & \big\downarrow{\scriptstyle \text{interpretation}} & & \\ \mathbf{W}(\mathbf{Z}) & \xrightarrow{\ \tau\ } & \hat{\Omega}(\mathbf{C}) & \xrightarrow{\ s_t\ } & \Lambda(\mathbf{Z}) \\ {\scriptstyle \Phi}\big\downarrow & & {\scriptstyle \hat{\varphi}}\big\downarrow & & \big\downarrow{\scriptstyle L_t} \\ \prod_{\mathbf{N}} \mathbf{Z} & \xrightarrow{\text{obvious}} & \mathrm{gh}(\mathbf{C}) & \xrightarrow{\text{identification}} & t\mathbf{Z}[[t]] \end{array}$$

where the upper horizontal arrows τ and s_t are combinatorially defined canonical isomorphisms and the vertical arrow *interpretation* as well as the lower horizontal arrows *obvious* and *identification* are canonical isomorphisms resulting from inspection. As usual $\mathbf{W}(\mathbf{Z})$ denotes the universal ring of Witt vectors and $\Lambda(\mathbf{Z}) = 1 + t\mathbf{Z}[[t]]$ is the multiplicative group of formal power series with integer coefficients and constant term 1 which can be considered as the additive group of a commutative ring whose multiplication has been introduced by A. Grothendieck in terms of certain universal polynomials (cf. [Gr]). The ring homormorphism Φ has been defined by P. Cartier in [C] and the functorial mapping $L_{\mathbf{Z}}$ is given by logarithmic derivative (cf. [E, Ga]), i.e., by

$$a(t) \mapsto L_{\mathbf{Z}}(a(t)) := t \cdot \frac{d}{dt} \log a(t) = t \cdot \frac{a'(t)}{a(t)}. \tag{1.8}$$

Moreover, the composition $\mathbf{W}(\mathbf{Z}) \xrightarrow{s_t \circ \tau} \Lambda(\mathbf{Z})$ coincides with the isomorphism E: $\mathbf{W}(\mathbf{Z}) \to \Lambda(\mathbf{Z})$ which has been defined by P. Cartier in [C] in a purely formal way.

It will turn out that the above diagram is closely related to the rewriting procedures for formal power series mentioned above. More precisely, for a sequence $\mathbf{b} = (b_1, b_2, ...)$ of integers let

$$X(\mathbf{b}) := \sum_{n=1}^{\infty} b_n \cdot \mathbf{C}(n) \in \hat{\Omega}(\mathbf{C}) \tag{1.9}$$

denote the almost finite (and in case $b_n < 0$ for some n only „virtual") cyclic set with exactly b_n cycles of length n. In other words, if $I_+ := \{n \in \mathbf{N} \mid b_n > 0\}$ and $I_- := \{n \in \mathbf{N} \mid b_n < 0\}$, then $X(\mathbf{b})$ is the formal difference of the two almost finite cyclic sets

$$X_+(b) := \bigsqcup_{n \in I_+} b_n \cdot \mathbf{C}(n) \tag{1.10}$$

$$X_-(\mathbf{b}) := \bigsqcup_{n \in I_-} (-b_n) \cdot \mathbf{C}(n), \tag{1.11}$$

considered as elements in $\hat{\Omega}(\mathbf{C})$, where—as usual—for an integer $b \in \mathbf{N}$ and a cyclic set X the product $b \cdot X$ denotes the disjoint union of b copies of X.

Recall that the *necklace algebra* $Nr(\mathbf{Z})$ with integer coefficients as defined by N. Metropolis and G.-C. Rota in [MR1] is the set $\mathbf{Z}^{\mathbf{N}}$ of infinite sequences of integers with addition defined componentwise while the nth component of the product of two sequences $\mathbf{b} = (b_1, b_2, ...)$ and $\mathbf{b}' = (b'_1, b'_2, ...)$ is given by

$$(\mathbf{b} \cdot \mathbf{b}')_n := \sum_{[i,j]=n} (i, j) \cdot b_i b'_j, \tag{1.12}$$

where—as usual—$[i, j] := \mathrm{lcm}(i, j)$ is the least common multiple of the integers i. It will be easy to deduce from this definition our first theorem:

THEOREM 1. *The interpretation map*

$$\begin{aligned} \mathbf{Z}^{\mathbf{N}} &\to \hat{\Omega}(\mathbf{C}) \\ \mathbf{b} &\mapsto X(\mathbf{b}) \end{aligned}$$

defines a ring isomorphism

$$\mathrm{itp} = \textit{interpretation}\colon Nr(\mathbf{Z}) \to \hat{\Omega}(\mathbf{C}).$$

It was this interpretation of the necklace algebra in terms of cyclic sets which started our investigations in this field.

The next results provide combinatorial interpretations of more formally defined isomorphisms between $Nr(\mathbf{Z})$, $\Lambda(\mathbf{Z})$, and $\mathbf{W}(\mathbf{Z})$, studied in [MR1]. Here we have

THEOREM 2. *If for the sequence $\mathbf{b} = (b_1, b_2, ...)$ of integers the (virtual) cyclic set $X(\mathbf{b})$ is defined as above and if*

$$d_n = \varphi_{\mathbf{C}^n}(X(\mathbf{b})) = \varphi_{\mathbf{C}^n}(X_+(\mathbf{b})) - \varphi_{\mathbf{C}^n}(X_-(\mathbf{b}))$$

for $n = 1, 2, ...,$ i.e., if

$$\mathbf{d} = (d_1, d_2, ...) \in \mathrm{gh}(\mathbf{C}) = \mathbf{Z}^{\mathbf{N}}$$

is the image of $X(\mathbf{b})$ under the map

$$\hat{\varphi} := \prod_{n \in \mathbf{N}} \varphi_{\mathbf{C}^n}\colon \hat{\Omega}(\mathbf{C}) \to \mathrm{gh}(\mathbf{C})$$

as defined above, then the sequence $\mathbf{d}$ is related to the sequence $\mathbf{b}$ by

$$\prod_{n=1}^{\infty} \left(\frac{1}{1-t^n}\right)^{b_n} = \exp\left(\int \sum_{n=1}^{\infty} d_n t^{n-1}\, dt\right).$$

Moreover the composition $Nr(\mathbf{Z}) \xrightarrow{\mathrm{itp}} \hat{\Omega}(\mathbf{C}) \xrightarrow{\hat{\varphi}} \mathrm{gh}(\mathbf{C})$ coincides modulo the obvious identification of the ghost ring and $\prod_{\mathbf{N}} \mathbf{Z}$ with the map

$$\mathrm{gh}\colon Nr(\mathbf{Z}) \to \prod_{\mathbf{N}} \mathbf{Z}$$

$$\mathbf{b} = (b_1, b_2, ...) \mapsto \mathrm{gh}(\mathbf{b}) = \hat{\mathbf{b}}, \qquad \textit{where} \quad (\hat{\mathbf{b}})_n := \sum_{i \mid n} i \cdot b_i.$$

Similarly let $S^n(X)$ denote the nth symmetric power of the cyclic set X, i.e., the set of all maps $g: X \to \mathbf{N}_0 := \mathbf{N} \cup \{0\}$ with finite support and with $\sum_{x \in X} g(x) = n$, which is a cyclic set via the canonical action of $\mathbf{C}$ on $S^n(X)$ given by

$$(1.13) \qquad \mathbf{C} \times S^n(X) \to S^n(X)$$
$$(z, g) \mapsto (z \cdot g: X \to \mathbf{N}_0) \qquad \text{with } z \cdot g \text{ defined by } (z \cdot g)(x) = g(z^{-1}x).$$

Note that $S^n(X)$ is almost finite if X is almost finite and that

$$(1.14) \qquad S^n(X_1 \sqcup X_2) \cong \bigsqcup_{i+j=n} S^i(X_1) \times S^j(X_2)$$

(see, for instance, [DS2]). On the level of the Burnside ring, the relation (1.14) can be interpreted as follows: associate with any almost finite cyclic set X the formal power series

$$(1.15) \qquad s_t(X) := 1 + \sum_{n=1}^{\infty} \varphi_{\mathbf{C}}(S^n(X)) \cdot t^n$$
$$= 1 + \varphi_{\mathbf{C}}(S^1(X)) \cdot t + \varphi_{\mathbf{C}}(S^2(X)) \cdot t^2 + \cdots \in \Lambda(\mathbf{Z}).$$

Then

$$(1.16) \qquad s_t(X_1 \sqcup X_2) = s_t(X_1) \cdot s_t(X_2).$$

Hence, the map s_t extends naturally to an additive/multiplicative homomorphism, also denoted by s_t from $\hat{\Omega}(\mathbf{C})$ into $\Lambda(X)$ satisfying

$$(1.17) \qquad s_t(x_1 + x_2) = s_t(x_1) \cdot s_t(x_2).$$

We claim

THEOREM 3. *If $\Lambda(\mathbf{Z}) = 1 + t\mathbf{Z}[[t]]$ is provided with Grothendieck's ring structure, then*

$$s_t: \hat{\Omega}(\mathbf{C}) \to \Lambda(\mathbf{Z})$$

becomes a ring isomorphism. Moreover, if $\mathbf{a} = (a_1, a_2, ...)$ and $\mathbf{b} = (b_1, b_2, ...)$ are two sequences of integers as above, then

$$s_t(X(\mathbf{b})) = 1 + \sum_{n=1}^{\infty} a_n \cdot t^n,$$

i.e.,

$$\varphi_{\mathbf{C}}(S^n(X(\mathbf{b}))) = a_n,$$

if and only if

$$\prod_{n=1}^{\infty}\left(\frac{1}{1-t^n}\right)^{b_n}=1+\sum_{n=1}^{\infty} a_n\cdot t^n.$$

In particular, the isomorphism s_t maps the cycle $\mathbf{C}(n)$ of length n onto the series $1/(1-t^n)=1+t^n+t^{2n}+\cdots$.

Finally, the canonical isomorphism $\tau\colon \mathbf{W}(\mathbf{Z})\to\hat{\Omega}(\mathbf{C})$ will be defined in such a way that for a given (universal) Witt vector $\mathbf{q}=(q_1, q_2, \ldots)\in \mathbf{W}(\mathbf{Z})$—i.e., for any infinite sequence $\mathbf{q}=(q_1, q_2, \ldots)$ of integers, interpreted as an element of the universal ring $\mathbf{W}(\mathbf{Z})$ of Witt vectors according to [B, C], or [L]—one has

$$\prod_{n=1}^{\infty}\frac{1}{1-q_n t^n}=\prod_{n=1}^{\infty}\left(\frac{1}{1-t^n}\right)^{b_n} \tag{1.18}$$

if and only if

$$\tau(\mathbf{q})=X(\mathbf{b}).$$

To define τ consider for any $q_1\in\mathbf{N}_0$ the set $q_1^{(\mathbf{C})}$ of *congruence maps* from $\mathbf{C}$ into the finite set $\underline{q_1}:=\{1, 2, \ldots, q_1\}$, i.e., of such maps $g\colon \mathbf{C}\to\underline{q_1}$ for which there exists some integer $n\in\mathbf{N}$ such that $g(z_1)=g(z_2)$ for all $z_1, z_2\in\mathbf{C}$ for which $z_1z_2^{-1}\in\mathbf{C}^n=\{z^n\mid z\in\mathbf{C}\}$. Note that $q_1^{(\mathbf{C})}$ is an almost finite cyclic set with respect to the canonical $\mathbf{C}$-action on $q_1^{(\mathbf{C})}$ given by

$$\begin{gathered}\mathbf{C}\times q_1^{(\mathbf{C})}\to q_1^{(\mathbf{C})}\\ (z,g)\mapsto(z\cdot g\colon \mathbf{C}\to q_1)\qquad\text{with } (z\cdot g)(z')=g(z^{-1}z')\end{gathered} \tag{1.19}$$

and that moreover

$$(q_1\cdot q_1')^{(\mathbf{C})}\cong q_1^{(\mathbf{C})}\times q_1'^{(\mathbf{C})}. \tag{1.20}$$

One easily checks that

$$\varphi_{\mathbf{C}^n}(q_1^{(\mathbf{C})})=\#(\underline{q_1}^{\mathbf{C}/\mathbf{C}^n})=\#(\underline{q_1}^{\mathbf{C}(n)})=q_1^n \tag{1.21}$$

and that the map

$$\begin{gathered}\mathbf{N}\to\hat{\Omega}(\mathbf{C})\\ q_1\mapsto q_1^{(\mathbf{C})}\end{gathered} \tag{1.22}$$

has a canonical extension $\mathbf{Z} \to \hat{\Omega}(\mathbf{C})$ (also denoted by $q_1 \mapsto q_1^{(\mathbf{C})}$) such that

$$\varphi_{\mathbf{C}^n}(q_1^{(\mathbf{C})}) = q_1^n \tag{1.23}$$

for all $q_1 \in \mathbf{Z}$.

Obviously

$$\hat{\varphi}(q_1^{(\mathbf{C})}) = (q_1, q_1^2, ..., q_1^n, ...) \tag{1.24}$$

and therefore

$$\begin{aligned} s_t(q_1^{(\mathbf{C})}) &= \exp\left(\int \sum_{n=1}^{\infty} q_1^n t^{n-1}\, dt\right) \\ &= \frac{1}{1 - q_1 t} = 1 + \sum_{n=1}^{\infty} q_1^n \cdot t^n. \end{aligned} \tag{1.25}$$

To extend the map $\mathbf{Z} \to \hat{\Omega}(\mathbf{C})$: $q_1 \mapsto q_1^{(\mathbf{C})}$ to a map from $\mathbf{W}(\mathbf{Z})$ into $\hat{\Omega}(\mathbf{C})$ one has to observe that for any almost finite cyclic set X and any integer $n \in \mathbf{N}$ one has another almost finite cyclic set $\mathrm{ind}_n X$ defined by *induction* with respect to the nth power map $\sigma_n \colon \mathbf{C} \to \mathbf{C} \colon z \mapsto z^n$. To define $\mathrm{ind}_n X$ one considers the cartesian product $\mathbf{C} \times X$ as a cyclic set relative to the $\mathbf{C}$-action

$$\begin{aligned} \mathbf{C} \times (\mathbf{C} \times X) &\to \mathbf{C} \times X \\ (u, (z, x)) &\mapsto (zu^{-n}, ux). \end{aligned} \tag{1.26}$$

Then $\mathrm{ind}_n X$ is defined as the set of $\mathbf{C}$-orbits $\|z, x\| := \mathbf{C} \cdot (z, x)$ in $\mathbf{C} \times X$ with respect to this action. The group $\mathbf{C}$ acts again on this orbit space via

$$\begin{aligned} \mathbf{C} \times \mathrm{ind}_n X &\to \mathrm{ind}_n X \\ (z', \|z, x\|) &\mapsto \|z'z, x\|. \end{aligned} \tag{1.27}$$

On can show that ind_n is well-defined and additive, i.e., that

$$\mathrm{ind}_n(X_1 \sqcup X_2) \cong \mathrm{ind}_n X_1 \sqcup \mathrm{ind}_n X_2 \tag{1.28}$$

and that

$$\mathrm{ind}_n(\mathbf{C}(i)) \cong \mathbf{C}(ni), \tag{1.29}$$

which shows that these two properties could also have been used to define $\mathrm{ind}_n X$ in a purely formal way. In particular ind_n induces an additive map, also denoted by ind_n from $\hat{\Omega}(\mathbf{C})$ into itself such that $X(\mathbf{b})$ is mapped onto $X(\mathbf{b}')$, where

$$\mathbf{b}'(j)=\begin{cases} b(j/n) & \text{if } n \text{ divides } j \\ 0 & \text{otherwise.}\end{cases} \tag{1.30}$$

One can also show that

$$\varphi_{\mathbf{C}^m}(\mathrm{ind}_n X)=\begin{cases} n\cdot\varphi_{\mathbf{C}^{m/n}}(X) & \text{if } n \text{ divides } m \\ 0 & \text{otherwise}\end{cases} \tag{1.31}$$

and therefore

$$\varphi_{\mathbf{C}^m}(\mathrm{ind}_n q_1^{(\mathbf{C})})=\begin{cases} n\cdot q_n^{m/n} & \text{if } n \text{ divides } m \\ 0 & \text{otherwise.}\end{cases} \tag{1.32}$$

Combining all this one can define a (well defined!) map from the set $\mathbf{W}(\mathbf{Z}):=\{\mathbf{q}=(q_1,q_2,\ldots)\mid q_i\in\mathbf{Z}\}$ into $\hat{\Omega}(\mathbf{C})$, namely the map

$$\begin{aligned} \tau\colon \mathbf{W}(\mathbf{Z}) &\to \hat{\Omega}(\mathbf{C}) \\ \mathbf{q}=(q_1,q_2,\ldots) &\mapsto \sum_{n=1}^{\infty}\mathrm{ind}_n(q_n^{(\mathbf{C})}) \end{aligned} \tag{1.33}$$

which satisfies

$$\varphi_{\mathbf{C}^m}(\tau(\mathbf{q}))=\sum_{d\mid m} d\cdot q_d^{m/d}, \tag{1.34}$$

and one has

$$\prod_{n=1}^{\infty}\frac{1}{1-q_n t^n}=\prod_{n=1}^{\infty}\left(\frac{1}{1-t^n}\right)^{b_n}$$

if and only if

$$\tau(\mathbf{q})=X(\mathbf{b}). \tag{1.35}$$

In other words, we claim:

THEOREM 4. *For a sequence $\mathbf{q}=(q_1,q_2,\ldots)$ of integers let $\tau(\mathbf{q})$ denote the well defined* (!) *virtual cyclic set $\sum_{n=1}^{\infty}\mathrm{ind}_n(q_n^{(\mathbf{C})})\in\hat{\Omega}(\mathbf{C})$. Then $\tau\colon \mathbf{Z}^{\mathbf{N}}\to\hat{\Omega}(\mathbf{C})$ is a ring isomorphism, if $\mathbf{Z}^{\mathbf{N}}$ is considered as the universal ring of Witt vectors $\mathbf{W}(\mathbf{Z})$ with coefficients in $\mathbf{Z}$ as defined in* [C] or [L]. *Moreover, for* *a given sequence* $\mathbf{b} = (b_1, b_2, \ldots)$ *of integers, as above, we have* $\tau(\mathbf{q}) = X(\mathbf{b})$ *if and only if*

$$\prod_{n=1}^{\infty} \frac{1}{1-q_n t^n} = \prod_{n=1}^{\infty} \left(\frac{1}{1-t^n}\right)^{b_n}.$$

As a consequence we get in view of (1.34), (1.5), and (1.6) (or (1.2) and (1.31)) the following

COROLLARY. *Given a sequence* $\mathbf{d} = (d_1, d_2, \ldots)$ *of integers, there exists a sequence* $\mathbf{q} = (q_1, q_2, \ldots)$ *of integers with*

$$d_m = \sum_{i \mid m} i \cdot q_i^{m/i}$$

for all $m = 1, 2, \ldots$ *if and only if*

$$\sum_{i=1}^{m} d_{(i,m)} = \sum_{i \mid m} \varphi\left(\frac{m}{i}\right) \cdot d_i \equiv 0 \qquad (\mathrm{mod}\; m)$$

for all $m = 1, 2, \ldots$ *if and only if*

$$\sum_{i \mid m} \mu\left(\frac{m}{i}\right) \cdot d_i \equiv 0 \qquad (\mathrm{mod}\; m)$$

for all $m = 1, 2, \ldots$.

Remark. As shown in [DS2], the isomorphism $\tau\colon \mathbf{W}(\mathbf{Z}) \to \hat{\Omega}(\mathbf{C})$ can be used to introduce the concept of Witt vectors and to prove the essential fact that addition and multiplication in $\mathbf{Z}^{\mathbf{N}} = \mathbf{W}(\mathbf{Z})$ are described by universal polynomials with integral coefficients and therefore induce a canonical ring structure on $\mathbf{A}^{\mathbf{N}} = \mathbf{W}(\mathbf{A})$ for every commutative ring $\mathbf{A}$ satisfying all the functorial properties described in [C].

Another obvious, but worthwile observation concerning almost finite cyclic sets is that for any such X the cyclic set $\mathrm{res}_n X$ which—as a set—coincides with X while $\mathbf{C}$ acts on $\mathrm{res}_n X$ via nth powers, i.e., via

(1.36)
$$\mathbf{C} \times \mathrm{res}_n X \to \mathrm{res}_n X$$
$$(z, x) \mapsto z^n x,$$

is also almost finite. Moreover the obvious isomorphisms

(1.37)
$$\mathrm{res}_n(X_1 \sqcup X_2) \cong \mathrm{res}_n X_1 \sqcup \mathrm{res}_n X_2$$

and

(1.38) $$\mathrm{res}_n(X_1 \times X_2) \cong \mathrm{res}_n X_1 \times \mathrm{res}_n X_2$$

show that res_n induces a ring endomorphism, also denoted by res_n, of $\hat{\Omega}(\mathbf{C})$. The following theorem relates restriction and induction to Frobenius and Verschiebung operators, well known in the context of Witt vectors, and provides combinatorial proofs of the usual identities for Frobenius and Verschiebung.

THEOREM 5. (1) $\mathrm{res}_n \mathbf{C}(m) \cong (n, m) \cdot \mathbf{C}([n, m]/n)$.

(2) $\mathrm{ind}_n \mathbf{C}(m) \cong \mathbf{C}(nm)$.

(3) $\mathrm{Ker}(\mathrm{res}_n) = \{x \in \hat{\Omega}(\mathbf{C}) \mid \varphi_{\mathbf{C}^m}(x) = 0$ *for all multiples* m *of* $n\}$.

(4) $\mathrm{Im}(\mathrm{ind}_n) = \{x \in \hat{\Omega}(\mathbf{C}) \mid \varphi_{\mathbf{C}^m}(x) = 0$ *for all* m *not divided by* $n\}$.

(5) *If the Frobenius operators* $f_n \colon Nr(\mathbf{Z}) \to Nr(\mathbf{Z})$ *and the Verschiebung operators* $v_n \colon Nr(\mathbf{Z}) \to Nr(\mathbf{Z})$ *are defined according to* [MR1] *by*

$$(b_1, b_2, \ldots) \mapsto \left(\sum_{[n,i]=n} (n, i) b_i, \sum_{[n,i]=2n} (n, i) b_i, \ldots \right)$$

and

$$(b_1, b_2, \ldots) \mapsto (\underbrace{0, \ldots, 0}_{n-1 \text{ times}}, b_1, \underbrace{0, \ldots, 0}_{n-1 \text{ times}}, b_2, \underbrace{0, \ldots, 0}_{n-1 \text{ times}}, b_3, \ldots),$$

respectively, then the diagrams

$$\begin{array}{ccc} Nr(\mathbf{Z}) & \xrightarrow{\text{interpretation}} & \hat{\Omega}(\mathbf{C}) \\ {\scriptstyle f_r}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_r} \\ Nr(\mathbf{Z}) & \xrightarrow{\text{interpretation}} & \hat{\Omega}(\mathbf{C}) \end{array} \qquad \begin{array}{ccc} Nr(\mathbf{Z}) & \xrightarrow{\text{interpretation}} & \hat{\Omega}(\mathbf{C}) \\ {\scriptstyle v_r}\downarrow & & \downarrow{\scriptstyle \mathrm{ind}_r} \\ Nr(\mathbf{Z}) & \xrightarrow{\text{interpretation}} & \hat{\Omega}(\mathbf{C}) \end{array}$$

are commutative.

(6) $\mathrm{res}_r \circ \mathrm{res}_s = \mathrm{res}_{rs}$ *and therefore* $f_r \circ f_s = f_{rs}$.

(7) $\mathrm{ind}_r \circ \mathrm{ind}_s = \mathrm{ind}_{rs}$ *and therefore* $v_r \circ v_s = v_{rs}$.

(8) *Frobenius reciprocity for restriction and induction*:

$$\mathrm{ind}_r(\mathrm{res}_r(x) \cdot y) = x \cdot \mathrm{ind}_r(y) \qquad \textit{for all} \quad x, y \in \hat{\Omega}(\mathbf{C})$$

and therefore

$$v_r(f_r(x) \cdot y) = x \cdot v_r(y) \qquad \textit{for all} \quad x, y \in Nr(\mathbf{Z}).$$

(9) *Mackey's subgroup formula for restriction and induction*:

$$\mathrm{res}_r \circ \mathrm{ind}_s = (r, s)\, \mathrm{ind}_{[r,s]/r} \circ \mathrm{res}_{[r,s]/s}$$

and therefore

$$f_r \circ v_s = (r, s)\, v_{[r,s]/r} \circ f_{[r,s]/s}.$$

As observed already by J. F. Adams [A], the fact that

$$S^n(X_1 \sqcup X_2) \cong \coprod_{i+j=n} S^i(X_1) \times S^j(X_2) \tag{1.14}$$

can be used to define additive endomorphisms ψ^n of $\hat{\Omega}(\mathbf{C})$ ($n = 1, 2, \ldots$) such that for any (virtual) cyclic set X the identity

$$t \cdot \frac{d}{dt} \log\left(\sum_{n=1}^{\infty} S_n(X) \cdot t^n\right) = \frac{\sum_{n=1}^{\infty} n \cdot S^n(X) \cdot t^n}{1 + \sum_{n=1}^{\infty} S^n(X) \cdot t^n} \tag{1.39}$$
$$= \psi^1(X) \cdot t + \psi^2(X) \cdot t^2 + \cdots$$

holds in the ring $\hat{\Omega}(\mathbf{C})[[t]]$ of formal power series with coefficients in $\hat{\Omega}(\mathbf{C})$. The Adams operations ψ^n have already proved useful in many situations. In our context we have:

THEOREM 6. *The* n*th Adams operation* $\psi^n: \hat{\Omega}(\mathbf{C}) \to \hat{\Omega}(\mathbf{C})$ *coincides with the restriction map* $\mathrm{res}_n: \hat{\Omega}(\mathbf{C}) \to \hat{\Omega}(\mathbf{C})$. *Hence in particular,* ψ^n *is multiplicative and* $\psi^n \circ \psi^m = \psi^{nm}$ *holds for all* $n, m \in \mathbf{N}$, *i.e.,* $\hat{\Omega}(\mathbf{C})$ *is a so-called* special λ-ring *with respect to the λ-ring structure defined by symmetric powers of cyclic sets.*

Remark. While for any group symmetric powers provide a λ-ring structure for the associated Burnside ring (cf. [DS2]), these λ-rings turn out to be special λ-rings if and only if the group is cyclic. (Cf. [S] for the case of finite groups. The argument given there generalizes immediately to the infinite case.)

Finally, we will show directly, i.e., without using the ghost ring as above (cf. (1.25)), that the number of C-invariant elements of the nth symmetric power of the cyclic set $q^{(\mathbf{C})}$ equals q^n. This provides a combinatorial proof of the so-called cyclotomic identity (cf. [MR2])

$$\frac{1}{1 - qt} = \prod_{n=1}^{\infty} \left(\frac{1}{1 - t^n}\right)^{M(q,n)}, \tag{1.40}$$

where $M(q, n)$ denotes the number of n-cycles in $q^{(\mathbf{C})}$. We will show that this identity is a simple consequence of a more general and more precise result characterizing the cyclic sets $q^{(\mathbf{C})}$ (virtual if $q<0$) among all (virtual) cyclic sets. For this purpose define for any element $x \in \hat{\Omega}(\mathbf{C})$ the formal power series

$$S_t(X) := 1 + \sum_{n=1}^{\infty} S^n(X) \cdot t^n \tag{1.41}$$

in Grothendieck's ring $\Lambda(\hat{\Omega}(\mathbf{C}))$ of formal power series with coefficients in $\hat{\Omega}(\mathbf{C})$ and constant term 1. Then we have

THEOREM 7. *A (virtual) almost finite syclic set $x \in \hat{\Omega}(\mathbf{C})$ is of the form $q^{(\mathbf{C})}$ for some $q \in \mathbf{N}_0$ $(q \in \mathbf{Z})$ if and only if for every $n \in \mathbf{N}$ the two (virtual) almost finite cyclic sets x^n and $S^n(x)$ are isomorphic, i.e., if and only if*

$$S_t(x) = 1 + \sum_{n=1}^{\infty} S^n(x) \cdot t^n = 1 + \sum_{n=1}^{\infty} x^n t^n = \frac{1}{1-xt}$$

holds in $\Lambda(\hat{\Omega}(\mathbf{C}))$, in which case the decomposition

$$x = q^{(\mathbf{C})} = \sum_{n=1}^{\infty} M(q,n) \cdot \mathbf{C}(n)$$

leads to

$$\frac{1}{1-q^{(\mathbf{C})}t} = S_t(q^{(\mathbf{C})}) = \prod_{n=1}^{\infty} S_t(\mathbf{C}(n))^{M(q,n)}.$$

In particular, applying $\varphi_{\mathbf{C}}$ to the coefficients of both sides yields

$$\frac{1}{1-qt} = \prod_{n=1}^{\infty} \left(\frac{1}{1-t^n}\right)^{M(q,n)}.$$

We want to point out that Gauss already used the cyclotomic identity for the enumeration of the number of irreducible polynomials of degree n and leading coefficient 1 over the finite field $\mathbf{F}_q$ with q elements (cf. [Ga]). A combinatorial interpretation of his reasoning may be given in terms of cyclic sets: Consider the algebraic closure $\bar{\mathbf{F}}_q$ of the field $\mathbf{F}_q$. It becomes an almost finite cyclic set via the Frobenius automorphism $F\colon z \mapsto z^q$, of $\bar{\mathbf{F}}_q$. The $\mathbf{C}^n$-invariant elements of $\bar{\mathbf{F}}_q$ are just the elements of the unique (!) extension field $\mathbf{F}_{q^n}$ of $\mathbf{F}_q$, which has degree n over $\mathbf{F}_q$; hence their number equals q^n—in other words, $\bar{\mathbf{F}}_q$ and $q^{(\mathbf{C})}$ are isomorphic as cyclic sets. Moreover two elements in $\bar{\mathbf{F}}_q$ lie in the same cycle of length n if and only if their minimal polynomials coincide and have degree n. Hence the number $M(q,n)$ of cycles of length n in $q^{(\mathbf{C})} \cong \bar{\mathbf{F}}_q$ coincides indeed with the number of irreducible polynomials of degree n.

This leads one to consider more generally for an arbitrary algebraic variety V over the finite field $\mathbf{F}_q$ the set $V(\bar{\mathbf{F}}_q)$ of points over the algebraic closure $\bar{\mathbf{F}}_q$. As observed above in case V is the affine line, this set is an almost finite cyclic set via the Frobenius automorphism $x \mapsto x^q$ of $\bar{\mathbf{F}}_q$ and it follows from the above considerations that the infinite power series $s_t(V(\bar{\mathbf{F}}_q))$ coincides with André Weil's zeta function $Z_V(t)$ of the variety V (cf. [We], [DS1]).

We also want to point out that the symmetric powers of almost finite cyclic sets are closely related to partitions. Recall that—as defined above—an element $\lambda \in S^n(X)$ is a mapping $\lambda: X \to \mathbf{N}_0$ with finite support and with $\sum_{x \in X} \lambda(x) = n$. Such a map is $\mathbf{C}$-invariant if and only if it is constant on the cycles contained in X, so in particular it has to be zero on cycles of length greater than n. Consequently if $X = X(\mathbf{b})$ for some sequence $\mathbf{b} = (b_1, b_2, ...)$ of non-negative integers then a $\mathbf{C}$-invariant element $\lambda \in S^n(X)$ defines a partition $\boldsymbol{\lambda} = (1^{\lambda_1} 2^{\lambda_2} \cdots n^{\lambda_n})$ of the integer n, given by $\lambda^k := (1/k) \cdot \sum_{x \in X_k} \lambda(x)$, where $X_k \subset X$ denotes the union of all k-cycles of X—say $\underline{b_k} \times \mathbf{C}(n) \cong X_k$ with trivial $\mathbf{C}$-action on $\underline{b_k}$—together with n functions $f_k: \underline{b_k} \to \mathbf{N}_0$ $(k = 1, ..., n)$ satisfying

$$(1.42) \qquad \sum_{j=1}^{b_k} f_k(j) = \lambda_k,$$

(well!) defined by $f_k(j) := \lambda(x)$ if $x \in X$ is in the jth k-cycle of X. If we call such an object $(\boldsymbol{\lambda}; f_1, ..., f_n)$ satisfying (1.42) a $\mathbf{b}$-partition of n, then $\varphi_{\mathbf{C}}(S^n(X(\mathbf{b})))$ equals the number $p_{\mathbf{b}}(n)$ of $\mathbf{b}$-partitions of n, i.e., we have

$$(1.43) \qquad \varphi_{\mathbf{C}}(S^n(X(\mathbf{b}))) = p_{\mathbf{b}}(n) = \sum_{\boldsymbol{\lambda} = (1^{\lambda_1} 2^{\lambda_2} \cdots n^{\lambda_n})} \sum_{k=1}^{n} \binom{\lambda_k + b_k - 1}{\lambda_k},$$

where the summation is taken over all partitions $\boldsymbol{\lambda}$ of n. Obviously the above considerations imply for the generating function for $\mathbf{b}$-partitions:

$$(1.44) \qquad 1 + \sum_{n=1}^{\infty} p_{\mathbf{b}}(n) t^n = \prod_{n=1}^{\infty} \left(\frac{1}{1 - t^n}\right)^{b_n}.$$

In particular in case $\mathbf{b} = (1, 1, ...)$ the $\mathbf{b}$-partitions are just ordinary partitions, so one gets once again Euler's formula (cf. [E])

$$(1.45) \qquad 1 + \sum_{n=1}^{\infty} p(n) t^n = \prod_{n=1}^{\infty} \frac{1}{1 - t^n}.$$

Our paper is organized as follows. In Section 2 we collect and prove some generalities on cyclic sets, where the concept of a group acting on a set is the organizing principle. This leads to the definition of the Burnside ring $\hat{\Omega}(\mathbf{C})$ of almost finite cyclic sets and to a proof of Theorem 1. Next we prove Theorem 5 in Section 3, Theorem 2, Theorem 3, and Theorem 6 in Section 4, and Theorem 4 in Section 5. In the final Section 6 we prove Theorem 7 and discuss some related material.

2. Some Generalities Concerning Cyclic Sets

(2.1) As explained already above, a *cyclic set* X is understood to be just a set X together with a group action of the (multiplicatively written) infinite cyclic group $\mathbf{C}$ from the left on this set. In this section we want to collect the basic facts concerning cyclic sets. Our discussion will be very short since we are treating here (in a way) a very special case of the more general situation considered already in [DS2].

(2.2) For any $r \in \mathbf{N}$ let

(2.2.1)
$$\begin{aligned} \sigma_r \colon \mathbf{C} &\to \mathbf{C} \\ z &\mapsto z^r \end{aligned}$$

denote the rth power map which maps every element of $\mathbf{C}$ onto its rth power. This map is an injective homomorphism and its image is the unique subgroup $\mathbf{C}^r$ of $\mathbf{C}$ which has index r in $\mathbf{C}$. Note that every subgroup of $\mathbf{C}$, except the trivial subgroup $\{1\}$—which we may interpret as the unique subgroup $\mathbf{C}^\infty$ of infinite index in $\mathbf{C}$—is of this type so that we may parametrize these subgroups by the set $\mathbf{N}$ of positive integers—and hence the set of all subgroups by the set $\mathbf{N}_\infty := \mathbf{N} \cup \{\infty\}$.

Since the *stabilizer group* $\mathbf{C}_x := \{z \in \mathbf{C} \mid zx = x\}$ of an element x in a cyclic set X is uniquely determined by its index in $\mathbf{C}$—it may be finite or infinite—the isomorphism class of its *orbit* $\mathbf{C} \cdot x := \{zx \mid z \in \mathbf{C}\}$ is uniquely determined by its cardinality. In other words, the orbit of an element of a cyclic set is either infinite—and hence isomorphic to the *infinite cycle* $\mathbf{C}(\infty) := \mathbf{C}/\{1\} = \mathbf{C}/\mathbf{C}^\infty$—or finite and isomorphic to one of the finite coset spaces $\mathbf{C}(n) := \mathbf{C}/\mathbf{C}^n$; such an orbit will be called a *cycle of length n*.

(2.3) Recall that disjoint unions and cartesian products of cyclic sets again are cyclic sets in a natural way and that the set Y^X of all maps from the cyclic set X into the cyclic set Y becomes a cyclic set if—as usual—one defines for a map $f\colon X \to Y$ and an element $z \in \mathbf{C}$ the map $z \cdot f$ by $(z \cdot f)(x) := zf(z^{-1}x)$ for all $x \in X$. Note that the set $(Y^X)^{\mathbf{C}}$ of $\mathbf{C}$-invariant elements in Y^X coincides with the set $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ of $\mathbf{C}$-mappings from X to Y, i.e., those maps $f\colon X \to Y$ with $f(zx) = zf(x)$ for all $z \in \mathbf{C}$ and $x \in X$.

Note also that for any cyclic sets X, X_1, X_2, Y, Y_1, Y_2 one has canonical **C**-isomorphisms

(2.3.1) $$(Y_1 \times Y_2)^X \cong Y_1^X \times Y_2^X$$

(2.3.2) $$Y^{X_1 \sqcup X_2} \cong Y^{X_1} \times Y^{X_2}$$

and

(2.3.3) $$(Y^{X_1})^{X_2} \cong Y^{X_1 \times X_2}.$$

(2.4) Since the orbits of two elements of a cyclic set are either disjoint or coincide, a cyclic set decomposes uniquely into a disjoint union of cycles.

(2.4.1) DEFINITION. Let X be a cyclic set. X will be called *almost finite* if the following two conditions are satisfied:

(1) Every cycle in X has finite length.

(2) For every $n \in \mathbf{N}$ there are only finitely many cycles contained in X which have length n.

If X is an almost finite cyclic set and if $b_n(X)$ denotes the number of cycles contained in X which have length n, then X is isomorphic—as a cyclic set— to the disjoint union $\bigsqcup_{n \in \mathbf{N}} b_n(X) \cdot \mathbf{C}(n)$, where $b_n(X) \cdot \mathbf{C}(n)$ denotes the disjoint union of $b_n(X)$ copies of the n-cycle $\mathbf{C}(n)$. Note that two almost finite cyclic sets X and Y are isomorphic if and only if $b_n(X) = b_n(Y)$ for all $n \in \mathbf{N}$. Since disjoint unions and cartesian products of almost finite cyclic sets are almost finite again, these operations induce an addition and a multiplication on the set of isomorphism classes of almost finite cyclic sets, providing this set with the structure of a commutative half ring. The associated Grothendieck ring is called the (completed) *Burnside ring* of the infinite cyclic group and will be denoted by $\hat{\Omega}(\mathbf{C})$. It is well known and goes back at least to Burnside (cf. (2.6.2) below) that the set of isomorphism classes of almost finite cyclic sets is mapped injectively into $\hat{\Omega}(\mathbf{C})$. Usually we shall not distinguish between an almost finite cyclic set and its image in $\hat{\Omega}(\mathbf{C})$, so that $\mathbf{C}(n)$ denotes at the same time the cycle $\mathbf{C}/\mathbf{C}^n$ of length n, its isomorphism class, and its image in $\hat{\Omega}(\mathbf{C})$. Obviously every element in $\hat{\Omega}(\mathbf{C})$ can be represented uniquely as an infinite linear combination $\sum_{n \in \mathbf{N}} b_n \cdot \mathbf{C}(n)$ with integer coefficients b_n. More generally, if for every $n \in \mathbf{N}$ and every $b \in \mathbf{Z}$ an element $x_{(n,b)} \in \hat{\Omega}(\mathbf{C})$ of the form

$$x_{(n,b)} = b \cdot \mathbf{C}(n) + b' \cdot \mathbf{C}(n+1) + b'' \cdot \mathbf{C}(n+2) + \cdots$$

(i.e., with $b_m(x_{(n,b)}) = 0$ for $m < n$ and $b_n(x_{(n,b)}) = b$ and with arbitrarily given $b', b'', ...$) is specified, then the family $x = (x_{(n,b)})_{(n,b) \in \mathbf{N} \times \mathbf{Z}}$ of elements in $\hat{\Omega}(\mathbf{C})$ defines a canonical bijection

$$(2.4.2) \qquad \beta_x : \mathbf{Z}^{\mathbf{N}} \to \hat{\Omega}(\mathbf{C})$$

$$(b_n)_{n \in \mathbf{N}} \mapsto \sum_{n \in \mathbf{N}} x_{(n, b_n)},$$

an observation which will be useful for us in various instances.

(2.5) In order to determine the multiplicative structure of $\hat{\Omega}(\mathbf{C})$ one has to calculate the product of any two cycles. Since every element in $\mathbf{C}(r) \times \mathbf{C}(s)$ has the stabilizer group $\mathbf{C}^r \cap \mathbf{C}^s$ and since $\mathbf{C}^r \cap \mathbf{C}^s = \mathbf{C}^{[r,s]}$, we have necessarily

$$(2.5.1) \qquad \mathbf{C}(r) \times \mathbf{C}(s) \cong n_{r,s} \cdot \mathbf{C}([r, s])$$

for some integer $n_{r,s} = b_{[r,s]}(\mathbf{C}(r) \times \mathbf{C}(s))$. Moreover, counting cardinalities and using the identity $r \cdot s = (r, s)[r, s]$—where, as usual, (r, s) denotes the greatest common divisor of r and s—one immediately gets

$$(2.5.2) \qquad n_{r,s} = (r, s),$$

i.e.,

$$(2.5.3) \qquad \mathbf{C}(r) \times \mathbf{C}(s) \cong (r, s) \cdot \mathbf{C}([r, s]).$$

This observation leads quickly to

(2.5.4) THEOREM 1. *Let $Nr(\mathbf{Z}) := \{(b_r)_{r \in \mathbf{N}} \mid b_r \in \mathbf{Z}\}$ denote the necklace algebra, as defined in* [MR1], *i.e., put*

$$(b_r)_{r \in \mathbf{N}} + (b'_r)_{r \in \mathbf{N}} := (b_r + b'_r)_{r \in \mathbf{N}}$$

and

$$(b_r)_{r \in \mathbf{N}} \cdot (b'_r)_{r \in \mathbf{N}} := \left(\sum_{[s,t] = r} (s, t) \cdot b_s b'_t \right)_{r \in \mathbf{N}}$$

for $(b_r)_{r \in \mathbf{N}}, (b'_r)_{r \in \mathbf{N}}$ in $Nr(\mathbf{Z})$. Then the interpretation map

$$\mathrm{itp} = \textit{interpretation}: Nr(\mathbf{Z}) \to \hat{\Omega}(\mathbf{C})$$

$$(b_r)_{r \in \mathbf{N}} \mapsto \sum_{r \in \mathbf{N}} b_r \cdot \mathbf{C}(r)$$

defines a canonical isomorphism between $Nr(\mathbf{Z})$ and $\hat{\Omega}(\mathbf{C})$.

(2.6) There is another way to characterize almost finite cyclic sets up to isomorphism. Consider for a cyclic set X and a positive integer n the set $X^{\mathbf{C}^n}$ of elements of X which are invariant under the action of the subgroup $\mathbf{C}^n$ of $\mathbf{C}$ and let $\varphi_{\mathbf{C}^n}(X) := \#(X^{\mathbf{C}^n})$ denote its cardinality. Note that

$$(\mathbf{C}(s))^{\mathbf{C}^n} = \{z \cdot \mathbf{C}^s \subset \mathbf{C} \mid \mathbf{C}^n \subset z\mathbf{C}^s z^{-1} = \mathbf{C}^s\}$$
$$= \begin{cases} \mathbf{C}(s) & \text{if } s \text{ divides } n \\ 0 & \text{otherwise.} \end{cases}$$

Therefore for a cycle of length s one has

(2.6.1) $$\varphi_{\mathbf{C}^n}(\mathbf{C}(s)) = \begin{cases} s & \text{if } s \text{ divides } n \\ 0 & \text{otherwise.} \end{cases}$$

Note also that a cyclic set X is almost finite if and only if every cycle in X has finite length and $\varphi_{\mathbf{C}^n}(X)$ is finite for all $n \in \mathbf{N}$.

Since

$$\varphi_{\mathbf{C}^n}(X \sqcup Y) = \varphi_{\mathbf{C}^n}(X) + \varphi_{\mathbf{C}^n}(Y)$$

and

$$\varphi_{\mathbf{C}^n}(X \times Y) = \varphi_{\mathbf{C}^n}(X) \cdot \varphi_{\mathbf{C}^n}(Y),$$

the map $X \mapsto \varphi_{\mathbf{C}^n}(X)$ can be extended uniquely to a ring homomorphism $\varphi_{\mathbf{C}^n} : \hat{\Omega}(\mathbf{C}) \to \mathbf{Z}$. Using an appropriate recursion argument one shows easily:

(2.6.2) Proposition (Burnside). *Two almost finite cyclic sets X and Y are isomorphic if and only if $\varphi_{\mathbf{C}^n}(X) = \varphi_{\mathbf{C}^n}(Y)$ for all $n \in \mathbf{N}$.*

This result implies immediately that the family $\varphi_{\mathbf{C}^n}$ of ring homomorphisms provides us with an injective ring homomorphism

(2.6.3) $$\hat{\varphi} := \prod_{n \in \mathbf{N}} \varphi_{\mathbf{C}^n} : \hat{\Omega}(\mathbf{C}) \to \mathrm{gh}(\mathbf{C}) := \mathbf{Z}^{\mathbf{N}} \quad \text{with} \quad (\hat{\varphi}(x))(n) := \varphi_{\mathbf{C}^n}(x)$$

of the Burnside ring into the ghost ring (cf. the Introduction, (1.4)).

If $x = \sum_{s=1}^{\infty} b_s \cdot \mathbf{C}(s)$ one has by (2.6.1)

(2.6.4) $$\hat{\varphi}(x)(n) = \varphi_{\mathbf{C}^n}(x) = \sum_{j \mid n} b_j \cdot j.$$

Comparing this with the ghost ring embedding

$$\mathrm{gh}: Nr(\mathbf{Z}) \to \prod_{\mathbf{N}} \mathbf{Z}$$
$$(b_n)_{n \in \mathbf{N}} \mapsto \Big(\sum_{s \mid n} s \cdot b_s\Big)_{n \in \mathbf{N}},$$

as defined in [MR1] one has

(2.6.5) PROPOSITION. *The diagram*

$$\begin{array}{ccc} Nr(\mathbf{Z}) & \xrightarrow{\text{interpretation}} & \hat{\Omega}(\mathbf{C}) \\ {\scriptstyle \mathrm{gh}}\downarrow & & \downarrow{\scriptstyle \hat{\varphi}} \\ \prod_{\mathbf{N}} \mathbf{Z} & \xrightarrow{\text{identification}} & \mathrm{gh}(\mathbf{C}) \end{array}$$

is commutative.

(2.7) *Remark.* As has already been mentioned in the Introduction (cf. (1.5), (1.6)) the injection $\hat{\varphi}\colon \hat{\Omega}(\mathbf{C}) \to \mathrm{gh}(\mathbf{C})$ of the Burnside ring into the ghost ring is not surjective. However, the image of $\hat{\varphi}$ can be characterized by canonically arising congruences. If $X = \sum_{k \in \mathbf{N}} b_k \cdot \mathbf{C}(k)$ then

(2.7.1) $$\varphi_{\mathbf{C}^n}(X) = \sum_{j \mid n} j \cdot b_j.$$

If $\mu\colon \mathbf{N} \to \mathbf{Z}$ denotes the Möbius function—as usual—then Möbius inversion (cf. [HW]) implies

(2.7.2) $$n \cdot b_n = \sum_{j \mid n} \mu\left(\frac{n}{j}\right) \cdot \varphi_{\mathbf{C}^j}(X)$$

and hence

(2.7.3) $$\sum_{j \mid n} \mu\left(\frac{n}{j}\right) \cdot \varphi_{\mathbf{C}^j}(X) \equiv 0 \qquad (\mathrm{mod}\ n) \text{ for every } n \in \mathbf{N}.$$

Dold shows in [Do] that an element $\mathbf{d} \in \mathrm{gh}(\mathbf{C})$ is in the image of $\hat{\varphi}$ if and only if

(2.7.4) $$\sum_{j \mid n} \mu\left(\frac{n}{j}\right) \mathbf{d}(j) \equiv 0 \qquad (\mathrm{mod}\ n) \text{ for every } n \in \mathbf{N}.$$

These congruences are equivalent to the congruences

(2.7.5) $$\sum_{j=1}^{n} \mathbf{d}(\gcd(j, n)) = \sum_{j \mid n} \varphi\left(\frac{n}{j}\right) \mathbf{d}(j) \equiv 0 \qquad (\mathrm{mod}\ n) \text{ for every } n \in \mathbf{N},$$

which arise in a natural way in the theory of Burnside rings of arbitrary groups (cf. [D3]). The well-known identity

$$\sum_{j \mid n} \varphi(j) = n$$

for the Euler function and its Möbius inverse provide a means to go from one system of congruences to the other equivalent system.

(2.8) *Remark.* For an almost finite cyclic set X it is in general not useful to consider its cardinality, since

$$\# X = \#(X^{\mathbf{C}^\infty}) = \varphi_{\mathbf{C}^\infty}(X) = \varphi_{\{1\}}(X)$$

may be infinite. Bur for finite cyclic sets the map $X \mapsto \# X$ provides a ring homomorphism $\varphi_{\mathbf{C}^\infty}: \Omega(\mathbf{C}) \to \mathbf{Z}$ of the Burnside–Grothendieck ring $\Omega(\mathbf{C}) \subset \hat{\Omega}(\mathbf{C})$ of finite cyclic sets. It obviously does not extend naturally to $\hat{\Omega}(\mathbf{C})$.

3. Restriction and Induction

(3.1) The rth power homomorphisms $\sigma_r: \mathbf{C} \to \mathbf{C}, z \mapsto z^r, r \in \mathbf{N}$, induce restriction and induction functors from the category of almost finite cyclic sets into itself. Restriction res_r with respect to σ_r is defined by assigning to a cyclic set X the cyclic set $\mathrm{res}_r X$ which has X as underlying set and where the $\mathbf{C}$-operation is defined via rth powers, i.e.,

$$(z, x) \mapsto z^r x. \tag{3.1.1}$$

Note that

$$\mathrm{res}_r(X_1 \sqcup X_2) \cong \mathrm{res}_r(X_1) \sqcup \mathrm{res}_r(X_2)$$

and

$$\mathrm{res}_r(X_1 \times X_2) \cong \mathrm{res}_r(X_1) \times \mathrm{res}_r(X_2).$$

Hence res_r is uniquely determined by its values on cycles:

$$\mathrm{res}_r\, \mathbf{C}(s) \cong (r, s)\, \mathbf{C}([r, s]/r)$$

$$\mathrm{res}_r \mathbf{C}(\infty) \cong r\, \mathbf{C}(\infty).$$

The first isomorphism is verified by noting that $z \in \mathbf{C}$ acts trivially on $\mathrm{res}_r\, \mathbf{C}(s)$, if and only if $z^r \in \mathbf{C}^s$ if and only if $z^r \in \mathbf{C}^s \cap \mathbf{C}^r = \mathbf{C}^{[r,s]}$, i.e., if and only if $z \in \mathbf{C}^{[r,s]/r}$. Hence $\mathrm{res}_r\, \mathbf{C}(s) \cong k\mathbf{C}([r, s]/r)$ for some integer k and so, comparing cardinalities, one gets $k \cdot [r, s]/r = s$ and therefore $k = rs/[r, s] = (r, s)$. To show the second isomorphism note that the cosets of $\mathbf{C}$ modulo $\mathbf{C}^r$ are the $\mathbf{C}$-orbits of $\mathrm{res}_r\, \mathbf{C}(\infty)$. If one agrees that $(r, \infty) = r$ and $[r, \infty] = \infty$ one may collect both formulas in one:

$$\mathrm{res}_r\, \mathbf{C}(s) \cong (r, s)\, \mathbf{C}([r, s]/r). \tag{3.1.2}$$

If X is an almost finite cyclic set, then obviously $\operatorname{res}_r X$ is almost finite, too. Hence res_r defines a ring homomorphism $\hat{\Omega}(\mathbf{C}) \to \hat{\Omega}(\mathbf{C})$ also denoted by res_r, and by (3.1.2) we have

$$
\begin{aligned}
(3.1.3) \qquad \varphi_{\mathbf{C}^s}(\operatorname{res}_r X) &= \#\{x \in X \mid z^r x = x \text{ for all } z \in \mathbf{C}^s\} \\
&= \#\{x \in X \mid z^{rs} x = x \text{ for all } z \in \mathbf{C}\} \\
&= \varphi_{\mathbf{C}^{rs}}(X),
\end{aligned}
$$

and this fully determines the restriction homorphism. Note also that (3.1.3) implies the useful formula

$$
(3.1.4) \qquad \varphi_{\mathbf{C}^r}(X) = \varphi_{\mathbf{C}}(\operatorname{res}_r X).
$$

(3.2) Now recall the definition of the Frobenius operators f_r, $r \in \mathbf{N}$, as defined in [MR1] for the necklace algebra by

$$
\begin{aligned}
(3.2.1) \qquad f_r \colon Nr(\mathbf{Z}) &\to Nr(\mathbf{Z}) \\
(b_s)_{s \in \mathbf{N}} &\mapsto \Bigl(\sum_{[r,t]=rs} (r,s) b_t \Bigr)_{t \in \mathbf{N}}
\end{aligned}
$$

and for the ghost ring (as usual in the context of Witt vectors, cf. [L, B])

$$
\begin{aligned}
(3.2.2) \qquad f_r \colon \operatorname{gh}(\mathbf{C}) &\to \operatorname{gh}(\mathbf{C}) \\
(d_s)_{s \in \mathbf{N}} &\mapsto (d_{rs})_{s \in \mathbf{N}}.
\end{aligned}
$$

Comparing (3.1.2) with (3.2.1) and (3.1.3) with (3.2.2) one gets

(3.2.3) PROPOSITION. *For every $r \in \mathbf{N}$ one has commutative diagrams*

$$
\begin{tikzcd}
Nr(\mathbf{Z}) \arrow[r, "\mathrm{itp}"] \arrow[d, "f_r"'] \arrow[dr, phantom, "\operatorname{res}_r"] & \hat{\Omega}(\mathbf{C}) \\
Nr(\mathbf{Z}) \arrow[r, "\mathrm{itp}"] & \hat{\Omega}(\mathbf{C})
\end{tikzcd}
\qquad
\begin{tikzcd}
Nr(\mathbf{Z}) \arrow[r, "\mathrm{gh}"] \arrow[d, "f_r"'] & \operatorname{gh}(\mathbf{C}) \arrow[d, "f_r"] \\
Nt(\mathbf{Z}) \arrow[r, "\mathrm{gh}"] & \operatorname{gh}(\mathbf{C})
\end{tikzcd}
\qquad
\begin{tikzcd}
\hat{\Omega}(\mathbf{C}) \arrow[r, "\hat{\varphi}"] \arrow[d, "\operatorname{res}_r"] & \operatorname{gh}(\mathbf{C}) \arrow[d, "f_r"] \\
\hat{\Omega}(\mathbf{C}) \arrow[r, "\hat{\varphi}"] & \operatorname{gh}(\mathbf{C})
\end{tikzcd}
$$

(3.3) In order to define the induction functor with respect to the rth power map σ_r—which will be the left adjoint of the restriction functor res_r—one considers for an almost finite cyclic set X the cartesian product $\mathbf{C} \times X$. This becomes a cyclic set by the mapping

$$
\begin{aligned}
(3.3.1) \qquad \mathbf{C} \times (\mathbf{C} \times X) &\to \mathbf{C} \times X \\
(u, (z, x)) &\mapsto (zu^{-r}, ux).
\end{aligned}
$$

Let $\operatorname{ind}_r X$ denote the set of $\mathbf{C}$-orbits of $\mathbf{C} \times X$ with respect to this operation. $\mathbf{C}$ acts again on this set by left multiplication, i.e., if $\|z, x\|$ denotes the image of (z, x) in $\operatorname{ind}_r X$ under the canonical projection onto the orbit space, then one defines

$$(3.3.2) \qquad z' \, \|z, x\| := \|z'z, x\|.$$

Obviously, this operation is well defined: if $\|z_1, x_1\| = \|z, x\|$ then one has $z_1 = zu^{-r}$ and $x_1 = ux$ for an appropriate $u \in \mathbf{C}$ and therefore $\|z'z_1, x_1\| = \|z'z, x\|$.

Clearly, $\operatorname{ind}_r X$ is an almost finite cyclic set and one has

$$(3.3.3) \qquad \operatorname{ind}_r(X_1 \sqcup X_2) \cong \operatorname{ind}_r(X_1) \sqcup \operatorname{ind}_r(X_2).$$

Therefore ind_r induces an additive homomorphism $\hat{\Omega}(\mathbf{C})$—$\hat{\Omega}(\mathbf{C})$, also denoted by ind_r. Again ind_r is completely determined by its values on the finite cycles. One has

$$(3.3.4) \qquad \operatorname{ind}_r \mathbf{C}(j) \cong \mathbf{C}(rj).$$

Proof. Note first that $\mathbf{C}$ operates transitively on $\operatorname{ind}_r \mathbf{C}(j)$, since for $\|z, w\mathbf{C}^j\|$ and $\|z_1, w_1\mathbf{C}^j\|$ in $\operatorname{ind}_r \mathbf{C}(j)$ one has

$$((z_1 z^{-1})(w w_1^{-1})^{-r}) \, \|z, w\mathbf{C}^j\| = \|z_1, w_1 \mathbf{C}^j\|.$$

Furthermore if $z' \, \|z, w\mathbf{C}^j\| = \|z, w\mathbf{C}^j\|$, then for some $u \in \mathbf{C}$ one has $(z'zu^{-r}, uw\mathbf{C}^j) = (z, w\mathbf{C}^j)$, which implies $u \in \mathbf{C}^j$ and therefore $z' = u^r \in \mathbf{C}^{rj}$. ∎

By (3.3.4) we have

$$(3.3.5) \qquad \varphi_{\mathbf{C}^s}(\operatorname{ind}_r \mathbf{C}(j)) = \varphi_{\mathbf{C}^s}(\mathbf{C}(rj))$$
$$= \begin{cases} rj & \text{if } jr \text{ divides } s \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} r \cdot \varphi_{\mathbf{C}^{s/r}}(\mathbf{C}(j)) & \text{if } r \text{ divides } s \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, by linearity, we have

$$(3.3.6) \qquad \varphi_{\mathbf{C}^s}(\operatorname{ind}_r x) = \begin{cases} r \cdot \varphi_{\mathbf{C}^{s/r}}(x) & \text{if } r \text{ divides } s \\ 0 & \text{otherwise.} \end{cases}$$
$$= \varphi_{\mathbf{C}^s}(\mathbf{C}(r)) \cdot \varphi_{\mathbf{C}^{s/r}}(x).$$

Note that the first factor of the last expression is zero if $s/r \notin \mathbf{N}$.

(3.4) Now recall the definition of the Verschiebung operators, defined in [MR1] for the necklace algebra by

$$(3.4.1) \qquad v_r\colon Nr(\mathbf{Z}) \to Nr(\mathbf{Z})$$
$$(b_s)_{s\in\mathbf{N}} \mapsto (b_{s/r})_{s\in\mathbf{N}} \qquad \text{with} \quad b_{s/r} := 0 \text{ if } s/r \notin \mathbf{N}$$

and for the ghost ring by

$$(3.4.2) \qquad v_r\colon \mathrm{gh}(\mathbf{C}) \to \mathrm{gh}(\mathbf{C})$$
$$(d_s)_{s\in\mathbf{N}} \mapsto (r\cdot d_{s/r})_{s\in\mathbf{N}} \qquad \text{with} \quad d_{s/r} := 0 \text{ if } s/r \notin \mathbf{N}.$$

Comparing (3.3.4) with (3.4.1) and (3.3.6) with (3.4.2) one gets

(3.4.3) PROPOSITION. *For every $r\in\mathbf{N}$ one has commutative diagrams*

$$\begin{tikzcd} Nr(\mathbf{Z}) \arrow[r, "\mathrm{itp}"] \arrow[d, "v_r"'] & \hat{\Omega}(\mathbf{C}) \arrow[d, "\mathrm{ind}_r"'] \\ Nr(\mathbf{Z}) \arrow[r, "\mathrm{itp}"] & \hat{\Omega}(\mathbf{C}) \end{tikzcd} \qquad \begin{tikzcd} Nr(\mathbf{Z}) \arrow[r, "\mathrm{gh}"] \arrow[d, "v_r"'] & \mathrm{gh}(\mathbf{C}) \arrow[d, "v_r"] \\ Nr(\mathbf{Z}) \arrow[r, "\mathrm{gh}"] & \mathrm{gh}(\mathbf{C}) \end{tikzcd} \qquad \begin{tikzcd} \hat{\Omega}(\mathbf{C}) \arrow[r, "\hat{\varphi}"] \arrow[d, "\mathrm{ind}_r"] & \mathrm{gh}(\mathbf{C}) \arrow[d, "v_r"] \\ \hat{\Omega}(\mathbf{C}) \arrow[r, "\hat{\varphi}"] & \mathrm{gh}(\mathbf{C}) \end{tikzcd}$$

(3.5) Hence we are now in a position to use the large variety of canonical isomorphisms describing the interplay between restriction and induction functors—which result from functoriality, adjointness, Frobenius reciprocity, and Mackey's subgroup theorem (cf. [DS2])—to derive the well-known identities for the Frobenius and Verschiebung operators and to provide combinatorial interpretations for them. More precisely one has

(3.5.1) PROPOSITION. (1) $\mathrm{res}_r \circ \mathrm{res}_s = \mathrm{res}_{rs}$ *and therefore* $f_r \circ f_s = f_{rs}$.

(2) $\mathrm{ind}_r \circ \mathrm{ind}_s = \mathrm{ind}_{rs}$ *and therefore* $v_r \circ v_s = v_{rs}$.

(3) $\mathrm{ind}_r(\mathrm{res}_r\, x\cdot y) = \mathrm{ind}_r\, x\cdot y$ *and therefore* $v_r(f_r(x)\cdot y) = v_r(x)\cdot y$ (*Frobenius reciprocity*).

(4) $\mathrm{res}_r \circ \mathrm{ind}_s = (r, s)\, \mathrm{ind}_{[r,s]/r} \cdot \mathrm{res}_{[r,s]/s}$ *and therefore* $f_r \circ v_s = (r, s)\, v_{[r,s]/r} \cdot f_{[r,s]/s}$ (*Mackey's subgroup formula*).

Note that by the above considerations we have proved Theorem 5 except for the statements (3) and (4). To prove these note that by (3.2.3) (resp. (3.4.3)) one has $\hat{\varphi} \circ \mathrm{res}_n = f_n \circ \hat{\varphi}$ (resp. $\hat{\varphi} \circ \mathrm{ind}_n = v_n \circ \hat{\varphi}$). Since $\hat{\varphi}$ is injective, an element $x \in \hat{\Omega}(\mathbf{C})$ lies in $\mathrm{Ker}(\mathrm{res}_n)$ (resp. in $\mathrm{Im}(\mathrm{ind}_n)$) if and only if $\hat{\varphi}(x)$ is in $\mathrm{Ker}(f_n)$ (resp. $\mathrm{Im}(v_n)$). The definition of Frobenius and Verschiebung in the ghost ring yields (3) and (4).

4. Symmetric Powers of Cyclic Sets

(4.1) We now want to study symmetric powers of almost finite cyclic sets on the level of Burnside rings (cf. also [DS1]). For a cyclic set X its *symmetric algebra*—or, rather, its symmetric monoid—$S(X)$ is defined to be the set of all maps from X to $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$ vanishing almost everywhere, i.e., of those maps $f: X \to \mathbf{N}_0$, for which $\operatorname{supp}(f) := \{x \in X \mid f(x) \neq 0\}$ is finite, or, in other words, the free commutative monoid generated by X. $\mathbf{C}$ acts on $S(X)$ in the usual way by

(4.1.1) $$\begin{aligned} \mathbf{C} \times S(X) &\to S(X) \\ (z, f) &\mapsto z \cdot f, \qquad \text{where} \quad (z \cdot f)(x) := f(z^{-1}x). \end{aligned}$$

Note that evidently

(4.1.2) $$S(X_1 \sqcup X_2) \cong S(X_1) \times S(X_2)$$

for any two cyclic sets X_1 and X_2.

(4.2) It is easy to see that for an almost finite cyclic set X the cyclic set $S(X)$ contains only cycles of finite length. Moreover, for every $n \in \mathbf{N}_0$ the $\mathbf{C}$-subset

(4.2.1) $$S^n(X) := \left\{ f \in S(X) \,\middle|\, \sum_{x \in X} f(x) = n \right\}$$

of $S(X)$, the *nth symmetric power* of X, is an almost finite cyclic set. Obviously, the stabilizer group $\mathbf{C}_f$ contains $\bigcap_{x \in \operatorname{supp}(f)} \mathbf{C}_x$ for every $f \in S(X)$, so every cycle in $S(X)$ has finite length. So, by using restriction (cf. (3.1.4)), it is enough to show that $(S^n(X))^{\mathbf{C}}$ is finite. But $f \in S(X)$ is $\mathbf{C}$-invariant if and only if it is constant on the cycles of X. Hence $f \in (S^n(X))^{\mathbf{C}}$ can be non-zero only on the finite number of those cycles of X which have length n at most. Since in addition one has $f(x) \leqslant n$ for all $f \in S^n(X)$, the set $(S^n(X))^{\mathbf{C}}$ must be finite.

The same argument shows in particular that for $X = \mathbf{C}(k)$ one has (cf. [S])

(4.2.2) $$\varphi_{\mathbf{C}}(S^n(\mathbf{C}(k))) = \begin{cases} 1 & \text{if } k \text{ divides } n \\ 0 & \text{otherwise.} \end{cases}$$

It follows from (4.1.2) that for the disjoint union $X_1 \sqcup X_2$ of any two almost finite cyclic sets X_1 and X_2 one has

(4.2.3) $$S^n(X_1 \sqcup X_2) \cong \bigsqcup_{i+j=n} S^i(X_1) \times S^j(X_2).$$

(4.3) Note that there is another way of describing symmetric powers of cyclic sets. To see this consider the canonical projection

$$p: X^n \to S^n(X)$$

of the nth cartesian power $X^n = X \times \cdots \times X$ of the cyclic set X onto its nth symmetric power. It is defined by mapping an element $x = (x_1, ..., x_n)$ of X^n to the map $p(x): X \to \mathbf{N}_0$ which maps an element y in X onto the number of those indices $i \in \underline{n} = \{1, ..., n\}$ for which $x_i = y$, i.e., $p(x)(y) = \#\{i \in \underline{n} \mid x_i = y\}$. Obviously p is a surjective $\mathbf{C}$-map which is compatible with the canonical action of the symmetric group Σ_n on X^n. It factors over the set $\Sigma_n \backslash X^n$ of Σ_n-orbits of X^n and therefore induces a $\mathbf{C}$-map

(4.3.1) $$\pi: \Sigma_n \backslash X^n \to S^n(X),$$

which is well known to be a $\mathbf{C}$-isomorphism.

(4.4) Let us now consider the formal power series

(4.4.1) $$S_t(X) := 1 + \sum_{n=1}^{\infty} S^n(X) \cdot t^n$$

as an element in the ring $\hat{\Omega}(\mathbf{C})[[t]]$ of formal power series with coefficients in $\hat{\Omega}(\mathbf{C})$. Since by (4.2.3) one has

(4.4.2) $$S_t(X_1 \sqcup X_2) = S_t(X_1) \cdot (X_2)$$

it follows that the map $X \mapsto S_t(X)$ induces a homomorphism

(4.4.3) $$S_t: \hat{\Omega}(\mathbf{C}) \to \Lambda(\hat{\Omega}(\mathbf{C}))$$

of the additive group of the Burnside ring into the multiplicative group of formal power series with constant term 1 and coefficients in $\hat{\Omega}(\mathbf{C})$.

Combining S_t with the multiplicative map

(4.4.4) $$\Lambda(\varphi_{\mathbf{C}}): \Lambda(\hat{\Omega}(\mathbf{C})) \to \Lambda(\mathbf{Z})$$

$$1 + \sum_{n=1}^{\infty} x_n t^n \mapsto 1 + \sum_{n=1}^{\infty} \varphi_{\mathbf{C}}(x_n) t^n$$

resulting—by functoriality—from the homomorphism $\varphi_{\mathbf{C}}: \hat{\Omega}(\mathbf{C}) \to \mathbf{Z}$, we get a map

(4.4.5) $$s_t := \Lambda(\varphi_{\mathbf{C}}) \circ S_t: \hat{\Omega}(\mathbf{C}) \to \Lambda(\mathbf{Z}).$$

We claim:

(4.4.6) THEOREM 3. *The homomorphism* $s_t\colon \hat{\Omega}(\mathbf{C}) \to \Lambda(\mathbf{Z})$ *is an isomorphism of abelian groups. It becomes an isomorphism of rings if* $\Lambda(\mathbf{Z})$ *is supplied—up to a sign—with the ring structure defined by A. Grothendieck in* [GR]. *Moreover*

$$s_t(X(\mathbf{b})) = 1 + \sum_{n=1}^{\infty} a_n \cdot t^n$$

for some sequence $\mathbf{b} = (b_1, b_2, ...)$ *of integers if and only if*

$$\prod_{n=1}^{\infty} \left(\frac{1}{1-t^n}\right)^{b_n} = 1 + \sum_{n=1}^{\infty} a_n \cdot t^n.$$

Proof. By (4.2.2) we see that

$$\begin{aligned} s_t(\mathbf{C}(k)) &= \Lambda(\varphi_{\mathbf{C}}) \circ S_t(\mathbf{C}(k)) \\ &= 1 + \sum_{n=1}^{\infty} \varphi_{\mathbf{C}}(S^n(\mathbf{C}(k))) \cdot t^n \\ &= 1 + \sum_{n=1}^{\infty} t^{nk} \\ &= \frac{1}{1-t^k}. \end{aligned}$$

Since the family $\{1/(1-t^k) \mid k \in \mathbf{N}\}$ is a topological $\mathbf{Z}$-basis of $\Lambda(\mathbf{Z})$ it follows immediately that s_t is an isomorphism from the additive group of $\hat{\Omega}(\mathbf{C})$ onto the multiplicative group $\Lambda(\mathbf{Z})$ and that

$$s_t(X(\mathbf{b})) = \prod_{n=1}^{\infty} s_t(\mathbf{C}(n))^{b_n} = \prod_{n=1}^{\infty} \left(\frac{1}{1-t^n}\right)^{b_n}.$$

In order to show the remaining part of the theorem we have to consider the *logarithmic derivative*

$$\begin{aligned} L_A\colon \Lambda(A) &\to tA[[t]] \\ a(t) &\mapsto t\frac{d}{dt}\log a(t) = t \cdot \frac{a'(t)}{a(t)} \end{aligned}$$

which is defined for any commutative ring A with unit element 1 and which provides a natural homomorphism from the multiplicative group $\Lambda(A)$ of formal power series with constant term 1 and coefficients in A into the additive group $tA[[t]]$ of formal power series with constant term 0. Defining—with Hadamard—the product of two power series in $tA[[t]]$

componentwise, i.e., identifying $tA[[t]]$ with the product ring $\prod_{\mathbf{N}} A = A^{\mathbf{N}}$ via the *obvious identification* $(d_1, d_2, ...) \mapsto \sum_{n=1}^{\infty} d_n t^n$, then Grothendieck's ring structure can be characterized as the unique ring structure on $\Lambda(A)$ for which the logarithmic derivative becomes a natural ring homomorphism $L_A: \Lambda(A) \to tA[[t]]$ (cf. [AT]). Specializing this to the ring $\mathbf{Z}$ of integers and using the obvious identification of $\mathrm{gh}(\mathbf{C})$ with $t\mathbf{Z}[[t]]$ as above, the second half of the above theorem will evidently be implied by the following proposition:

(4.4.7) PROPOSITION. *The diagram*

$$\begin{array}{ccc} \hat{\Omega}(\mathbf{C}) & \xrightarrow{\quad s_t \quad} & \Lambda(\mathbf{Z}) \\ {\scriptstyle \hat{\varphi}}\downarrow & & \downarrow{\scriptstyle L_{\mathbf{Z}}} \\ \mathrm{gh}(\mathbf{C}) & \xrightarrow[\text{identification}]{\text{obvious}} & t\mathbf{Z}[[t]] \end{array}$$

is commutative.

Proof. One has

$$\begin{aligned} L_{\mathbf{Z}} \circ s_t(\mathbf{C}(k)) &= L_{\mathbf{Z}}\left(\frac{1}{1-t^k}\right) \\ &= t \cdot \frac{d}{dt} \log \frac{1}{1-t^k} \\ &= \frac{kt^k}{1-t^k} \\ &= kt^k + kt^{2k} + kt^{3k} + \cdots . \end{aligned}$$

On the other hand, by (2.6.1) one has

$$\begin{aligned} \varphi(\mathbf{C}(k)) &= (\varphi_{\mathbf{C}^n}(\mathbf{C}(k)))_{n \in \mathbf{N}} \\ &= (\underbrace{0, ..., 0}_{k-1 \text{ times}}, k, \underbrace{0, ..., 0}_{k-1 \text{ times}}, k, \underbrace{0, ..., 0}_{k-1 \text{ times}}, ...). \end{aligned}$$

Hence modulo the obvious identification the maps $L_{\mathbf{Z}} \circ s_t$ and $\hat{\varphi}$ agree on a topological basis of $\hat{\Omega}(\mathbf{C})$ and therefore coincide. ∎

Note that Proposition (4.4.7) together with the results of Section 2 provides as well a proof of Theorem 2.

(4.5) *Remark.* Dold already established in [Do] an isomorphism

$$\mathscr{L}_t : \hat{\Omega}(\mathbf{C}) \to \Lambda(\mathbf{Z}),$$

of abelian groups by associating to an almost finite cyclic set X its Lefschetz power series $\mathscr{L}_t(X)$, which is formally defined in such a way that

$$-t \cdot \frac{d}{dt} \log \mathscr{L}_t(X) = \sum_{n=1}^{\infty} \varphi_{\mathbf{C}^n}(X) \cdot t^n.$$

Obviously this implies that $\mathscr{L}_t(X) = 1/s_t(X)$. Note that originally Grothendieck's ring structure on $\Lambda(\mathbf{Z})$ had been defined in such a way that $\mathscr{L}_t$—which is related to exterior powers—rather than s_t becomes a ring homomorphism. Since in the context of group actions on sets—rather than on vector spaces—there is no completely satisfying definition of exterior powers (cf. [S]), we preferred to change Grothendieck's ring structure on $\Lambda(\mathbf{Z})$ slightly, hereby avoiding lots of unnecessary *minus* signs in our formulas (compare [C] and [B]).

(4.6) Now recall that combining

$$S_t \colon \hat{\Omega}(\mathbf{C}) \to \Lambda(\hat{\Omega}(\mathbf{C}))$$

with

$$L_{\hat{\Omega}(\mathbf{C})} \colon \Lambda(\hat{\Omega}(\mathbf{C})) \to t\hat{\Omega}(\mathbf{C})[[t]]$$
$$a(t) \mapsto t \cdot \frac{d}{dt} \log a(t) = t \cdot \frac{a'(t)}{a(t)}$$

one gets the associated *Adams operations* $\psi^r \colon \hat{\Omega}(\mathbf{C}) \to \hat{\Omega}(\mathbf{C})$, defined implicitly by

$$L_{\hat{\Omega}(\mathbf{C})} \circ S_t(x) =: \sum_{r=1}^{\infty} \psi^r(x) \cdot t^r. \tag{4.6.1}$$

We claim

(4.6.2) Theorem 2. *The maps $\psi^r \colon \hat{\Omega}(\mathbf{C}) \to \hat{\Omega}(\mathbf{C})$ coincide with the maps* $\mathrm{res}_r \colon \hat{\Omega}(\mathbf{C}) \to \hat{\Omega}(\mathbf{C})$, *i.e., for every $x \in \hat{\Omega}(\mathbf{C})$ one has*

$$\sum_{r=1}^{\infty} \mathrm{res}_r\, x \cdot t^r = L_{\hat{\Omega}(\mathbf{C})} \circ S_t(x).$$

Therefore the ψ^r are ring homomorphisms satisfying $\psi^r \circ \psi^s = \psi^{rs} = \psi^s \circ \psi^r$ and hence, according to [AT], *$\hat{\Omega}(\mathbf{C})$ is a special λ-ring with respect to the λ-structure, defined by $S_t \colon \hat{\Omega}(\mathbf{C}) \to \Lambda(\hat{\Omega}(\mathbf{C}))$.*

Proof. It is enough to show that

$$(4.6.3)\qquad \sum_{r=1}^{\infty} \varphi_{\mathbf{C}^j}(\mathrm{res}_r\, \mathbf{C}(s)) \cdot t^r = L_{\mathbf{Z}}\Big(1 + \sum_{r=1}^{\infty} \varphi_{\mathbf{C}^j}(S^r(\mathbf{C}(s))) \cdot t^r\Big)$$

for all $j, s \in \mathbf{N}$. But

$$(4.6.4)\qquad \varphi_{\mathbf{C}^j}(\mathrm{res}_r\, \mathbf{C}(s)) = \varphi_{\mathbf{C}^{jr}}(\mathbf{C}(s)) = \begin{cases} s & \text{if } s \text{ divides } jr \\ 0 & \text{otherwise} \end{cases}$$

and therefore

$$(4.6.5)\qquad \sum_{r=1}^{\infty} \varphi_{\mathbf{C}^j}(\mathrm{res}_r\, \mathbf{C}(s)) \cdot t^r = \sum_{r=1}^{\infty} s \cdot t^{rs/(j,s)},$$

while

$$\begin{aligned}(4.6.6)\quad 1 + \sum_{r=1}^{\infty} \varphi_{\mathbf{C}^j}(S^r(\mathbf{C}(s))) \cdot t^r &= 1 + \sum_{r=1}^{\infty} \varphi_{\mathbf{C}}(\mathrm{res}_j\, S^r(\mathbf{C}(s))) \cdot t^r \\ &= \Lambda(\varphi_{\mathbf{C}})\Big(1 + \sum_{r=1}^{\infty} S^r(\mathrm{res}_j\, \mathbf{C}(s)) \cdot t^r\Big) \\ &= s_t(\mathrm{res}_j\, \mathbf{C}(s)) \\ &= s_t((j, s)\, \mathbf{C}(s/(j, s))) \\ &= (s_t(\mathbf{C}(s/(j, s))))^{(j,s)} \\ &= \left(\frac{1}{1 - t^{s/(j,s)}}\right)^{(j,s)}\end{aligned}$$

and therefore

$$\begin{aligned}(4.6.7)\quad L_{\mathbf{Z}}\Big(1 + \sum_{r=1}^{\infty} \varphi_{\mathbf{C}^j}(S^r(\mathbf{C}(s))) \cdot t^r\Big) &= t \cdot \frac{d}{dt} \log\left(\left(\frac{1}{1 - t^{s/(j,s)}}\right)^{(j,s)}\right) \\ &= (j, s) \cdot t \cdot \frac{d}{dt} \log\left(\frac{1}{1 - t^{s/(j,s)}}\right) \\ &= (j, s) \cdot \frac{s/(j, s) \cdot t^{s/(j,s)}}{1 - t^{s/(j,s)}} \\ &= \frac{s \cdot t^{s/(j,s)}}{1 - t^{s/(j,s)}} \\ &= \sum_{r=1}^{\infty} s \cdot t^{rs/(j,s)}. \quad \blacksquare\end{aligned}$$

(4.7) Finally, observe that, since $\mathrm{ind}_r\, \mathbf{C}(n) = \mathbf{C}(rn)$, one has

$$s_t(\mathrm{ind}_r\, \mathbf{C}(n)) = \frac{1}{1 - t^{nr}}.$$

This implies for an arbitrary x in $\hat{\Omega}(\mathbf{C})$ that

(4.7.1) $$s_t(\mathrm{ind}_r\, x) = s_{t^r}(x).$$

Moreover, since $\mathrm{res}_r\, \mathbf{C}(n) = (n, r)\, \mathbf{C}([n, r]/r)$, one has

(4.7.2) $$s_t(\mathrm{res}_r\, \mathbf{C}(n)) = \left(\frac{1}{1 - t^{[n,r]/r}}\right)^{(r,n)}.$$

If one defines maps

$$v_r\colon \Lambda(\mathbf{Z}) \to \Lambda(\mathbf{Z}) \qquad \text{and} \qquad f_r\colon \Lambda(\mathbf{Z}) \to \Lambda(\mathbf{Z})$$

by $v_r(a(t)) = a(t^r)$ and $f_r(a(t))$ as usual (cf. [C]), one has the commutative diagrams

(4.7.3)
$$\begin{array}{ccccccc}
\hat{\Omega}(\mathbf{C}) & \xrightarrow{s_t} & \Lambda(\mathbf{Z}) & \qquad & \hat{\Omega}(\mathbf{C}) & \xrightarrow{s_t} & \Lambda(\mathbf{Z}) \\
\downarrow{\scriptstyle \mathrm{ind}_r} & & \downarrow{\scriptstyle v_r} & & \downarrow{\scriptstyle \mathrm{res}_r} & & \downarrow{\scriptstyle f_r} \\
\hat{\Omega}(\mathbf{C}) & \xrightarrow{s_t} & \Lambda(\mathbf{Z}) & & \hat{\Omega}(\mathbf{C}) & \xrightarrow{s_t} & \Lambda(\mathbf{Z})
\end{array}$$

Note that the formula for $f_r(a(t))$ is simple only for series of the form $a(t) = 1/(1 - a_s t^s)$; here one has

(4.7.4) $$f_r\left(\frac{1}{1 - a_s t^s}\right) = \left(\frac{1}{1 - a_s^{[r,s]/s} \cdot t^{[r,s]/r}}\right)^{(r,s)}.$$

For the general case see [C, B] or do the calculations in the ghost ring and then go back to $\Lambda(\mathbf{Z})$ by integration and exponentiation.

5. Witt Vectors

(5.1) Recall from (2.3) that for two cyclic sets X and Y the set Y^X of maps from X to Y is again a cyclic set in a canonical way. But note also that for two almost finite cyclic sets X and Y the cyclic set Y^X is in general not an almost finite cyclic set. However, there is an interesting situation where new almost finite cyclic sets arise by exponentiation. We claim:

(5.1.1) Definition and Lemma. *If X and Y are cyclic sets and if $u\colon X \to Y$ is a map, then u will be called a* congruence map *if there is a*

$\mathbf{C}$-map $p\colon X \to F$ into a finite cyclic set F and a map $\hat{u}\colon F \to Y$ such that $u = \hat{u} \circ p$, i.e., if u can be factored via a $\mathbf{C}$-map over a finite cyclic set. Denote by $Y^{(X)}$ the cyclic subset of Y^X which consists of all congruence maps from X to Y. If Y is finite and if the cyclic set X contains only finitely many cycles—of finite or infinite length—then $Y^{(X)}$ is an almost finite cyclic set.

Proof. Note first that $Y^{(X)} = Y^X$ if both X and Y are finite and that $Y^{(X)} \cong Y^{(X_1)} \times Y^{(X_2)}$ if $X = X_1 \sqcup X_2$. Hence the lemma will follow if we can show that $Y^{(\mathbf{C}(\infty))}$ is an almost finite cyclic set for any finite cyclic set Y. To this end note that if $u\colon \mathbf{C}(\infty) \to Y$ is a congruence map and $u = \hat{u} \circ p$ a factorization of u, then for $z \in \mathbf{C}$ and $x \in \mathbf{C}(\infty)$ one has

$$\begin{aligned}(5.1.2)\qquad (z \cdot u)(x) &= zu(z^{-1}x) = z(\hat{u} \circ p)(z^{-1}x) = z\hat{u}(z^{-1}p(x)) \\ &= ((z \cdot \hat{u}) \circ p)(x)\end{aligned}$$

and hence

$$(5.1.3)\qquad z \cdot u = (z \cdot \hat{u}) \circ p.$$

Since $\hat{u}$ is a map between finite cyclic sets, its $\mathbf{C}$-orbit is finite, so the $\mathbf{C}$-orbit of u is finite, too.

Furthermore one has

$$\begin{aligned}(5.1.4)\qquad \varphi_{\mathbf{C}^n}(Y^{(\mathbf{C}(\infty))}) &= \varphi_{\mathbf{C}}(\mathrm{res}_n(Y^{(\mathbf{C}(\infty))})) \\ &= \varphi_{\mathbf{C}}((\mathrm{res}_n\, Y)^{(\mathrm{res}_n \mathbf{C}(\infty))}) \\ &= \#\ \mathrm{Hom}_{\mathbf{C}}(\mathrm{res}_n\, \mathbf{C}(\infty), \mathrm{res}_n\, Y) \\ &= \#\ \mathrm{Hom}_{\mathbf{C}}(n \cdot \mathbf{C}(\infty), \mathrm{res}_n\, Y) \\ &= (\#\ \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}(\infty), \mathrm{res}_n\, Y))^n \\ &= \varphi_{\{1\}}(\mathrm{res}_n\, Y)^n \\ &= (\#\ Y)^n.\end{aligned}$$

Hence $\varphi_{\mathbf{C}^n}(Y^{(\mathbf{C}(\infty))})$ is finite for every $n \in \mathbf{N}$ and this implies that $Y^{(\mathbf{C}(\infty))}$ is an almost finite cyclic set. ∎

(5.2) *Remark.* If $X = \bigsqcup_{k \in \mathbf{N}_\infty} x_k \cdot \mathbf{C}(k)$ and $Y = \bigsqcup_{l \in \mathbf{N}} y_l \cdot \mathbf{C}(l)$, where for both cyclic sets only finitely many of the coefficients are non-zero positive integers, then we claim

$$\begin{aligned}(5.2.1)\qquad \varphi_{\mathbf{C}^n}(Y^{(X)}) &= \prod_{k \in \mathbf{N}_\infty} \Big(\sum_{l \in \mathbf{N}} y_l \cdot \varphi_{\mathbf{C}^{[n,k]}}(\mathbf{C}(l))\Big)^{(n,k) \cdot x_k} \\ &= \prod_{k \in \mathbf{N}_\infty} \Big(\sum_{\substack{l \in \mathbf{N} \\ l \mid [n,k]}} l \cdot y_l\Big)^{(n,k) \cdot x_k},\end{aligned}$$

where—with the conventions of (3.1.2)—$[n,k] := \infty$ and $(n,k) := n$ if $k=\infty$. In particular $\varphi_{\mathbf{C}^n}(Y^{(X)})$ is a polynomial with integer coefficients in the coordinates y_l of the finite cyclic set Y.

Proof. One has

$$\begin{aligned}
\varphi_{\mathbf{C}^n}(Y^{(X)}) &= \varphi_{\mathbf{C}}(\mathrm{res}_n(Y^{(X)}))\\
&= \varphi_{\mathbf{C}}(\mathrm{res}_n\, Y^{(\mathrm{res}_n X)})\\
&= \#\, \mathrm{Hom}_{\mathbf{C}}(\mathrm{res}_n X, \mathrm{res}_n Y)\\
&= \#\, \mathrm{Hom}_{\mathbf{C}}\Big(\mathrm{res}_n\Big(\bigsqcup_{k\in\mathbf{N}_\infty} x_k\mathbf{C}(k)\Big), \mathrm{res}_n Y\Big)\\
&= \prod_{k\in\mathbf{N}_\infty} \#\, \mathrm{Hom}_{\mathbf{C}}(x^k\, \mathrm{res}_n \mathbf{C}(k), \mathrm{res}_n Y)\\
&= \prod_{k\in\mathbf{N}_\infty} \#\, \mathrm{Hom}_{\mathbf{C}}(x_k\cdot(n,k)\,\mathbf{C}([n,k]/n), \mathrm{res}_n Y)\\
&= \prod_{k\in\mathbf{N}_\infty} (\varphi_{\mathbf{C}^{[n,k]/n}}(\mathrm{res}_n Y))^{(n,k)\cdot x_k}\\
&= \prod_{k\in\mathbf{N}_\infty} (\varphi_{\mathbf{C}^{[n,k]}}(Y))^{(n,k)\cdot x_k}\\
&= \prod_{k\in\mathbf{N}_\infty} \Big(\sum_{l\in\mathbf{N}} y_l\,\varphi_{\mathbf{C}^{[n,k]}}(\mathbf{C}(l))\Big)^{(n,k)\cdot x_k}\\
&= \prod_{k\in\mathbf{N}_\infty} \Big(\sum_{\substack{l\in\mathbf{N}\\ l\mid[n,k]}} l\cdot y_l\Big)^{(n,k)\cdot x_k}. \quad \blacksquare
\end{aligned}$$

(5.3) The construction $Y\mapsto Y^{(X)}$ extends to a map between Grothendieck rings, i.e., we have

(5.3.1) THEOREM. *Let X be a cyclic set with only finitely many* **C**-*orbits. Then there exists a well defined map $\tau^X\colon \Omega(\mathbf{C})\to\hat{\Omega}(\mathbf{C})$ from the Burnside ring $\Omega(\mathbf{C})$ of finite cyclic sets into the Burnside ring of almost finite cyclic sets such that if $X=\bigsqcup_{k\in\mathbf{N}_\infty} x_k\cdot\mathbf{C}(k)$ and $y=\bigsqcup_{l\in\mathbf{N}} y_l\cdot\mathbf{C}(l)\in\Omega(\mathbf{C})$, one has*

$$\varphi_{\mathbf{C}^n}\circ\tau^X(y) = \prod_{k\in\mathbf{N}_\infty}\Big(\sum_{\substack{l\in\mathbf{N}\\ l\mid[n,k]}} l\cdot y_l\Big)^{(n,k)\cdot x_k}.$$

Proof. Consider for $y=\bigsqcup_{l\in\mathbf{N}} y_l\cdot\mathbf{C}(l)$ in $\Omega(\mathbf{C})$ and $X=\bigsqcup_{k\in\mathbf{N}_\infty} x_k\cdot\mathbf{C}(k)$ the element $\boldsymbol{\eta}=(\eta_n)_{n\in\mathbf{N}}\in\mathrm{gh}(\mathbf{C})$ with

$$\eta_n := \prod_{k\in\mathbf{N}_\infty}\Big(\sum_{\substack{l\in\mathbf{N}\\ l\mid[n,k]}} l\cdot y_l\Big)^{(n,k)\cdot x_k}. \tag{5.3.2}$$

We will show that $\boldsymbol{\eta}$ is the image of an element $\hat{\Omega}(\mathbf{C})$ under $\hat{\varphi}$. This will be done by showing that $\boldsymbol{\eta}$ satisfies the congruences (2.7.4). If Y is an actual and not only a virtual finite cyclic set then the element $\boldsymbol{\eta} = (\eta_n)_{n \in \mathbf{N}}$ is by (5.2.1) contained in the image of $\hat{\Omega}(\mathbf{C})$ in the ghost ring and hence the congruences (2.7.4) are satisfied for the components η_n of $\boldsymbol{\eta}$, i.e., one has

$$\sum_{j \mid n} \mu\left(\frac{n}{j}\right) \cdot \eta_j \equiv 0 \pmod{n} \tag{5.3.3}$$

for all $n \in \mathbf{N}$. But since $\eta_n = \eta_n(y_l)$ is an integral polynomial in the coordinates y_l of y, (5.3.3) must hold for all $y_l \in \mathbf{Z}$. It follows that $\boldsymbol{\eta}$ is contained in the image of $\hat{\Omega}(\mathbf{C})$ in $\mathrm{gh}(\mathbf{C})$, whether y is an actual or a virtual finite cyclic set. ▮

(5.3.4) THEOREM. *With the notations of Theorem* (5.3.1) *one has*

$$\tau^X(y \cdot y') = \tau^X(y) \cdot \tau^X(y')$$

for $y, y' \in \Omega(\mathbf{C})$. *Similarly if* $X = X_1 \sqcup X_2$ *one has*

$$\tau^X(y) = \tau^{X_1}(y) \cdot \tau^{X_2}(y)$$

for all $y \in \Omega(\mathbf{C})$. *Moreover, if X is finite, then $\tau^X(y)$ is already contained in* $\Omega(\mathbf{C}) \subset \hat{\Omega}(\mathbf{C})$, *so for any other cyclic set X' with only finitely many* $\mathbf{C}$*-orbits one may consider* $\tau^{X'}(\tau^X(y))$ *and one has*

$$\tau^{X'}(\tau^X(y)) = \tau^{X' \times X}(y).$$

Proof. If Y is an actual finite cyclic set the proof is obvious and it can be carried over to virtual cyclic sets in a routine manner. ▮

(5.3.5) *Remark.* Note the particularly simple form of the components of the image of the element $\tau^{\mathbf{C}(\infty)}(q_1 \cdot \mathbf{C}(1)) =: q_1^{(\mathbf{C})}$ in the ghost ring. By (5.3.2) we have

$$\varphi_{\mathbf{C}^n}(q_1^{(\mathbf{C})}) = q_1^n. \tag{5.3.6}$$

Hence one has

$$L_{\mathbf{Z}}(s_t(q_1^{(\mathbf{C})})) = \frac{q_1 t}{1 - q_1 t} \tag{5.3.7}$$

and

$$s_t(q_1^{(\mathbf{C})}) = \frac{1}{1 - q_1 t}. \tag{5.3.8}$$

On the other hand, (5.3.6) implies that

$$b_1(q_1^{(\mathbf{C})}) = \varphi_{\mathbf{C}}(q_1^{(\mathbf{C})}) = q_1.$$

Hence $b_m(\operatorname{ind}_n q^{(\mathbf{C})}) = 0$ for $m < n$ and $b_n(\operatorname{ind}_n q^{(\mathbf{C})}) = q$, so (2.4.2) implies

(5.3.9) THEOREM. *For any $x \in \hat{\Omega}(\mathbf{C})$ there exists a unique sequence $\mathbf{q} = (q_n)_{n \in \mathbf{N}} \in \mathbf{Z}^{\mathbf{N}}$ such that*

$$x = \tau(\mathbf{q}) := \sum_{n=1}^{\infty} \operatorname{ind}_n q_n^{(\mathbf{C})},$$

i.e., the Teichmüller map

$$\tau \colon \mathbf{Z}^{\mathbf{N}} \to \hat{\Omega}(\mathbf{C})$$
$$\mathbf{q} \mapsto \tau(\mathbf{q})$$

is bijective.

Note that (5.3.8) together with (4.7.1) allows one to determine the series $s_t(\tau(\mathbf{q}))$. Indeed one has

$$s_t(\tau(\mathbf{q})) = \prod_{n=1}^{\infty} s_t(\operatorname{ind}_n q_n^{(\mathbf{C})}) = \prod_{n=1}^{\infty} s_t{}^n(q_n^{(\mathbf{C})})$$
$$= \prod_{n=1}^{\infty} \frac{1}{1 - q_n t^n},$$

In other words, $X = X(\mathbf{b}) \in \hat{\Omega}(\mathbf{C})$ is the image of $\mathbf{q} = (q_n)_{n \in \mathbf{N}}$ with respect to τ according to Theorem (5.3.9) if and only if

$$\prod_{n=1}^{\infty} \left(\frac{1}{1 - t^n} \right)^{b_n} = \prod_{n=1}^{\infty} \frac{1}{1 - q_n t^n}. \tag{5.3.10}$$

(5.4) Moreover we have for the components of the image of $\tau(\mathbf{q})$ in the ghost ring

$$\varphi_{\mathbf{C}^r}(\tau(\mathbf{q})) = \varphi_{\mathbf{C}^r}\left(\sum_{n=1}^{\infty} \operatorname{ind}_n q_n^{(\mathbf{C})} \right)$$
$$= \sum_{n=1}^{\infty} \varphi_{\mathbf{C}^r}(\mathbf{C}(n)) \cdot \varphi_{\mathbf{C}^{r/n}}(q_n^{(\mathbf{C})})$$
$$= \sum_{n=1}^{\infty} \varphi_{\mathbf{C}^r}(\mathbf{C}(n)) \cdot q_n^{r/n}$$

and since $\varphi_{\mathbf{C}^r}(\mathbf{C}(n))=0$ unless n divides r we have

(5.4.1) $$\varphi_{\mathbf{C}^r}(\tau(\mathbf{q}))=\sum_{n\mid r} n\cdot q_n^{r/n}.$$

Hence if one considers according to E. Witt and P. Cartier (cf. [L, C]) the well (!) defined ring structure on the set $\mathbf{W}(\mathbf{Z}):=\mathbf{Z}^{\mathbf{N}}$ of Witt vectors for which all the maps

$$\begin{aligned} \Phi_r\colon \mathbf{W}(\mathbf{Z}) &\to \mathbf{Z} \\ (q_n)_{n\in\mathbf{N}} &\mapsto \sum_{n\mid r} n\cdot q_n^{r/n} \end{aligned}$$

are ring homomorphisms, we get

(5.4.2) THEOREM. *The Teichmüller map*

$$\begin{aligned} \tau\colon \mathbf{W}(\mathbf{Z}) &\to \hat{\Omega}(\mathbf{C}) \\ \mathbf{q} &\mapsto \tau(\mathbf{q}) \end{aligned}$$

is an isomorphism of rings.

Note that by the above considerations we have proved Theorem 4. Note also that in addition our approach provides a new proof for the fact that the ring structure on $\mathbf{W}(\mathbf{Z})$ is well defined, i.e., that for $\mathbf{q}=(q_n)_{n\in\mathbf{N}}$ and $\bar{\mathbf{q}}=(\bar{q}_n)_{n\in\mathbf{N}}$ in $\mathbf{W}(\mathbf{Z})$ the sequences $\mathbf{s}=(s_n)_{n\in\mathbf{N}}$, $\mathbf{d}=(d_n)_{n\in\mathbf{N}}$, and $\mathbf{p}=(p_n)_{n\in\mathbf{N}}$, defined recursively by

(5.4.3) $$s_r:=\frac{1}{r}\Bigg(\sum_{n\mid r} nq_n^{r/n}+\sum_{n\mid r} n\bar{q}_n^{r/n}-\sum_{\substack{n\mid r\\ n\neq r}} ns_n^{r/n}\Bigg),$$

(5.4.4) $$d_r:=\frac{1}{r}\Bigg(\sum_{n\mid r} nq_n^{r/n}-\sum_{n\mid r} n\bar{q}_n^{r/n}-\sum_{\substack{n\mid r\\ n\neq r}} nd_n^{r/n}\Bigg),$$

and

(5.4.5) $$p_r:=\frac{1}{r}\Bigg(\Bigg(\sum_{n\mid r} nq_n^{r/n}\Bigg)\cdot\Bigg(\sum_{n\mid r} n\bar{q}_n^{r/n}\Bigg)-\sum_{\substack{n\mid r\\ n\neq r}} nm_n^{r/n}\Bigg),$$

are integer valued. As mentioned already in the Introduction, it is shown in [DS2] in a much more general context that one can use our interpretation of Witt vectors in the context of Burnside rings to prove the important and much deeper fact—due to E. Witt— that (5.4.3), (5.4.4), and (5.4.5) define s_r, d_r, and p_r as *integral* polynomials in $q_1, \ldots, q_r$ and $\bar{q}_1, \ldots, \bar{q}_r$.

(5.5) Theorem (5.4.2) has many interesting applications. To mention just one, note that the Teichmüller map $\tau: \mathbf{W}(\mathbf{Z}) \to \hat{\Omega}(\mathbf{C})$ transports Frobenius and Verschiebung operators, which are defined for Witt vectors in the usual way, into restriction and induction maps on the Burnside ring. More precisely, one has commutative diagrams

$$\begin{array}{ccc} \mathbf{W}(\mathbf{Z}) & \xrightarrow{r} & \hat{\Omega}(\mathbf{C}) \\ {\scriptstyle f_r}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_r} \\ \mathbf{W}(\mathbf{Z}) & \xrightarrow{r} & \hat{\Omega}(\mathbf{C}) \end{array} \qquad \begin{array}{ccc} \mathbf{W}(\mathbf{Z}) & \xrightarrow{r} & \hat{\Omega}(\mathbf{C}) \\ {\scriptstyle v_r}\downarrow & & \downarrow{\scriptstyle \mathrm{ind}_r} \\ \mathbf{W}(\mathbf{Z}) & \xrightarrow{r} & \hat{\Omega}(\mathbf{C}) \end{array}$$

for all $n \in \mathbf{N}$.

Note also that Cartier's formally defined map $E: \mathbf{W}(\mathbf{Z}) \to \Lambda(\mathbf{Z})$ coincides with the composition $s_t \circ \tau: \mathbf{W}(\mathbf{Z}) \to \Lambda(\mathbf{Z})$.

6. The Cyclotomic Identity

In this section we want to study in more detail the (virtual) almost finite cyclic sets $q^{(\mathbf{C})}$, $q \in \mathbf{Z}$.

(6.1) We first determine the coefficients of the canonical decomposition

$$q^{(\mathbf{C})} = \sum_{k=1}^{\infty} M(q, k) \cdot \mathbf{C}(k) \tag{6.1.1}$$

of $q^{(\mathbf{C})}$ into cycles. Applying $\varphi_{\mathbf{C}^n}$ to (6.1.1) provides

$$q^n = \sum_{k \mid n} k \cdot M(q, k) \tag{6.1.2}$$

and hence, by Möbius inversion (cf. [HW]),

$$n \cdot M(q, n) = \sum_{k \mid n} \mu\left(\frac{n}{k}\right) \cdot q^k. \tag{6.1.3}$$

Therefore the coefficients $M(q, n)$ are integer valued polynomials in q with rational coefficients, the so-called necklace polynomials (cf. [MR1]).

(6.2) *Remark.* N. Metropolis and G.-C. Rota established a number of identities for these necklace polynomials. These identities are almost immediate consequences of the exponential construction (cf. (5.3.4)), which provides

$$(q \cdot q')^{(\mathbf{C})} = q^{(\mathbf{C})} \cdot q'^{(\mathbf{C})}, \tag{6.2.1}$$

the identity

$$\operatorname{res}_r q^{(\mathbf{C})} = (q^r)^{(\mathbf{C})}, \tag{6.2.2}$$

and the interplay of restriction and induction. Evaluating both sides of (6.2.1) leads to

$$M(q \cdot q', n) = \sum_{[i,j]=n} (i, j) \cdot M(q, i) \cdot M(q', j) \tag{6.2.3}$$

while evaluation of both sides of (6.2.2) leads to

$$M(q^r, n) = \sum_{\substack{j \in \mathbf{N} \\ [j,r]/r = n}} (j, r)\, M(q, j). \tag{6.2.4}$$

A twofold application of Frobenius reciprocity yields

$$\operatorname{ind}_r x \cdot \operatorname{ind}_s y = (r, s) \operatorname{ind}_{[r,s]}(\operatorname{res}_{[r,s]/r} x \cdot \operatorname{res}_{[r,s]/s} y), \tag{6.2.5}$$

a formula which is also well known in the context of Witt vectors (cf. [B]). Specializing to $x = q^{(\mathbf{C})}$ and $y = q'^{(\mathbf{C})}$ leads to

$$\operatorname{ind}_r q^{(\mathbf{C})} \cdot \operatorname{ind}_s q'^{(\mathbf{C})} = (r, s) \cdot \operatorname{ind}_{[r,s]}(q^{[r,s]/r} \cdot q'^{[r,s]/s})^{(\mathbf{C})} \tag{6.2.6}$$

and evaluation of both sides provides

$$(r, s)\, M(q^{[r,s]/r} \cdot q'^{[r,s]/s}, n) = \sum_{\substack{i,j \in \mathbf{N} \\ [ir, js]/[r,s] = n}} (ir, js)\, M(q, i)\, M(q', j). \tag{6.2.7}$$

(6.3) We already know that $s_t(q^{(\mathbf{C})}) = 1/(1 - q^t)$. Hence the decomposition of $q^{(\mathbf{C})}$ into cycles provides

$$\frac{1}{1 - q^t} = \prod_{n=1}^{\infty} \left(\frac{1}{1 - t^n}\right)^{M(q,n)}, \tag{6.3.1}$$

the so-called *cyclotomic identity*, which has been useful for several different enumeration problems (cf. [Ga, Mo, Wi2, Hi, MR1]).

Note that up to now we have given only a formal proof for this identity, using the logarithmic derivative. But, to complement the work of N. Metropolis and G.-C. Rota, who succeeded to give a set theoretic derivation of the cyclotomic identity without using ghosts (cf. [MR2]), we want to show directly that

$$\varphi_{\mathbf{C}}(S^n(q^{(\mathbf{C})})) = q^n \tag{6.3.2}$$

thereby establishing another combinatorial proof of the cyclotomic identity.

If we make use of the canonical representation (4.3.1) of the symmetric powers as quotients of cartesian powers modulo the operation of the symmetric group and the fact that

$$S^n(q^{(\mathbf{C})}) = \Sigma_n \backslash (q^{(\mathbf{C})})^n \cong \Sigma_n \backslash (\underline{q}^n)^{(\mathbf{C})}$$

then Eq. (6.3.2) is an immediate consequence of the following lemma, applied with respect to $\Sigma := \Sigma_n$ and its canonical action on $T := \underline{q}^n$.

(6.3.3) LEMMA. *If T is a finite Σ-set (for an arbitrary group Σ), then the number of* $\mathbf{C}$*-invariant elements of the orbit space $\Sigma \backslash T^{(\mathbf{C})}$ is equal to the number of elements of T.*

Proof. It is remarkable that it does not seem to be possible to construct a canonical bijection between T and the set $(\Sigma \backslash T^{(\mathbf{C})})^{\mathbf{C}}$ of $\mathbf{C}$-invariant Σ-orbits in $T^{(\mathbf{C})}$. Instead (cf. also [MR2]) we construct a canonical bijection between $\Sigma \times T$ and $\Sigma \times (\Sigma \backslash T^{(\mathbf{C})})^{\mathbf{C}}$.

Since T is finite, we may assme Σ to be finite too. One verifies easily that the Σ-orbit $\Sigma \cdot h$ of an element h in $T^{(\mathbf{C})}$ is $\mathbf{C}$-invariant if and only if $\mathbf{C} \cdot h \subset \Sigma \cdot h$. Hence, if g denotes a generator of $\mathbf{C}$, then $\Sigma \cdot h$ is a $\mathbf{C}$-invariant orbit in $T^{(\mathbf{C})}$ if and only if there exists an element $\sigma \in \Sigma$ such that $g^{-1}h = \sigma h$. This is equivalent with

$$(6.3.4) \qquad g^{-1}h(g^i) = h(g^{i+1}) = \sigma h(g^i) \qquad \text{for all} \quad i \in \mathbf{Z}$$

and consequently with

$$(6.3.5) \qquad h(g^i) = \sigma^i h(1) \qquad \text{for all} \quad i \in \mathbf{Z},$$

where of course 1 denotes the neutral element of the group $\mathbf{C}$. So we see that h is completely determined by the pair $(\sigma, h(1)) \in \Sigma \times T$. Hence, if we define the mapping

$$(6.3.6) \qquad \begin{aligned} &H: \Sigma \times T \to T^{(\mathbf{C})} \\ &(\sigma, t) \mapsto H_{(\sigma,t)} \qquad \text{with} \quad H_{(\sigma,t)}(g^i) := \sigma^i t \end{aligned}$$

then the composition of H with the quotient map $T^{(\mathbf{C})} \to \Sigma \backslash T^{(\mathbf{C})}$ provides us with a surjection of $\Sigma \times T$ onto the subset of $\mathbf{C}$-invariant elements in $\Sigma \backslash T^{(\mathbf{C})}$. We claim that the fibres of this map all have cardinality $\# \Sigma$, which will prove the lemma.

If (σ, t) and (τ, t') are elements of the fibre of H over the element $h \in T^{(\mathbf{C})}$, i.e., if $H_{(\sigma,t)} = h = H_{(\tau,t')}$, then

$$(6.3.7) \qquad \sigma^i t = \tau^i t' \qquad \text{for all} \quad i \in \mathbf{Z}.$$

In particular one has $t=t'$, and in view of the following lemma, (6.3.7) is equivant to

$$\sigma^{-1}\tau \in \bigcap_{i\in\mathbf{Z}} \Sigma_{h(g^i)} = \Sigma_h. \tag{6.3.8}$$

(6.3.9) LEMMA. *Let T be a Σ-set, let σ and τ be elements of Σ, and let t be an element of T. Then the following conditions are equivalent*:

(1) $\sigma^i t = \tau^i t$ *for all* $i \in \mathbf{Z}$,

(2) $\sigma^{-1}\tau \in \bigcap_{i\in\mathbf{Z}} \Sigma_{h(g^i)}$.

Before proving Lemma (6.3.9) we note that indeed, since there are $\#(\Sigma/\Sigma_h)$ elements in the Σ-orbit of h, there are $\#\Sigma_h \cdot \#(\Sigma/\Sigma_h) = \#\Sigma$ elements in $\Sigma \times T$ which are mapped onto the element $\Sigma \cdot h$ in $(\Sigma\backslash T^{(\mathbf{C})})^{\mathbf{C}}$. Hence

$$\Sigma \times T \to \Sigma \times (\Sigma\backslash T^{(\mathbf{C})})^{\mathbf{C}}$$
$$(\sigma, t) \mapsto (\sigma, \Sigma \cdot H_{(\sigma,t)})$$

is the canonical bijection mentioned in the beginning.

To prove Lemma (6.3.9) assume first that $\sigma^i t = \tau^i t$ for all $i \in \mathbf{Z}$. Then $\sigma\sigma^i t = \tau\tau^i t = \tau\sigma^i t$ and therefore $\sigma^{-1}\tau \in \Sigma_{\sigma^i t} = \Sigma_{h(g^i)}$ for all $i \in \mathbf{Z}$. Vice versa, if $\sigma^{-1}\tau \in \bigcap_{i\in\mathbf{Z}} \Sigma_{\sigma^i t}$, then using induction with respect to $|i|$ we may assume that $\sigma^i t = \tau^i t$ for some $i \in \mathbf{Z}$ to conclude that also

$$(\tau^{i\pm 1}) \cdot t = \tau^{\pm 1}(\tau^i t) = \tau^{\pm 1}(\sigma^i t) = (\sigma^{i\pm 1}) \cdot t. \tag{6.3.10}$$

Equation (6.3.2) allows one to generalize the cyclotomic identity from an identity in $\Lambda(\mathbf{Z})$ to an identity in $\Lambda(\hat{\Omega}(\mathbf{C}))$: Since

$$\begin{aligned} \varphi_{\mathbf{C}^r}(S^n(q^{(\mathbf{C})})) &= \varphi_{\mathbf{C}}(\mathrm{res}_r\, S^n(q^{(\mathbf{C})})) = \varphi_{\mathbf{C}}(S^n(\mathrm{res}_r\, q^{(\mathbf{C})})) \\ &= \varphi_{\mathbf{C}}(S^n((q^r)^{(\mathbf{C})})) = q^{nr} \end{aligned} \tag{6.3.11}$$

we have the following consequence of (6.3.2):

(6.3.12) COROLLARY. *An element $x \in \hat{\Omega}(\mathbf{C})$ is of the form $x = q^{(\mathbf{C})}$ for some $q \in \mathbf{Z}$ if and only if its nth symmetric power $S^n(x)$ coincides with its nth power x^n for all $n \in \mathbf{N}$.*

Proof. The above formula (6.3.11) implies that for $x = q^{(\mathbf{C})}$ one has

$\hat{\varphi}(S^n(x)) = \hat{\varphi}(x^n)$ and therefore $S^n(x) = x^n$ for all $n \in \mathbf{N}$. Vice versa, if $S^n(x) = x^n$ for all $n \in \mathbf{N}$ then

$$\begin{aligned} s_t(x) &= 1 + \sum_{n=1}^{\infty} \varphi_{\mathbf{C}}(S^n(x)) \cdot t^n = 1 + \sum_{n=1}^{\infty} \varphi_{\mathbf{C}}(x^n) \cdot t^n \\ &= 1 + \sum_{n=1}^{\infty} (\varphi_{\mathbf{C}}(x))^n \cdot t^n = \frac{1}{1 - \varphi_{\mathbf{C}}(x) \cdot t} \\ &= s_t((\varphi_{\mathbf{C}}(x))^{(\mathbf{C})}). \end{aligned}$$

Therefore by the injectivity of s_t we have $x = (\varphi_{\mathbf{C}}(x))^{(\mathbf{C})}$. ∎

The result (6.3.12) implies in turn

(6.3.13) $$S_t(q^{(\mathbf{C})}) = \frac{1}{1 - q^{(\mathbf{C})} \cdot t}$$

and therefore one has as another consequence:

(6.3.14) COROLLARY. *The cyclotomic identity*

$$\frac{1}{1 - q^{(\mathbf{C})} \cdot t} = \prod_{n=1}^{\infty} (S_t(\mathbf{C}(n)))^{M(q,n)}$$

holds as an identity in $\Lambda(\hat{\Omega}(\mathbf{C}))$.

Note that by the above discussion we have proved Theorem 7.

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