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On classes of analytic functions defined by convolution with incomplete beta functions

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Abstract

Carlson and Shaffer [SIAM J. Math. Anal. 15 (1984) 737–745] defined a convolution operator $L(a,c)$ on the class A of analytic functions involving an incomplete beta function $\phi(a,c;z)$ as $L(a,c)f = \phi(a,c) \star f$. We use this operator to introduce certain classes of analytic functions in the unit disk and study their properties including some inclusion results, coefficient and radius problems. It is shown that these classes are closed under convolution with convex functions.

Keywords: Convolution; Convex; Close-to-convex; Starlike functions; Integral operator

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f\colon f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in the unit disk $E = \{z\colon |z| < 1\}$. The class $\mathcal{A}$ is closed under the Hadamard product or convolution

$$(f \star g)(z) = \sum_{n=0}^{\infty} a_n b_n z^{n+1} \quad (a_0 = b_0 = 1),$$

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where

$$f(z)=\sum_{n=0}^{\infty}a_n z^{n+1},\qquad g(z)=\sum_{n=0}^{\infty}b_n z^{n+1}.$$

In particular we consider convolution with an incomplete beta function $\phi(a,c)$ related to the Gauss hypergeometric function ${}_2F_1(1,a;c,z)$ by

$$\phi(a,c;z)=z\,{}_2F_1(1,a;c,z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}z^{n+1}$$
$$|z|<1,\ c\neq 0,-1-2,\ldots,$$

where $(a)_n$ denotes the Pochhamer symbol (or the shifted factorial) and in terms of the gamma function Γ is defined by

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=\begin{cases}1, & n=0,\ a\neq 0,\\ a(a+1)\ldots(a+n-1), & n\in N=\{1,2,\ldots\}.\end{cases}$$

Also, if $\operatorname{Re} c>\operatorname{Re} a>0$, then

$$\phi(a,c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}e^{tz}\,dt=\int_0^1 d\mu(t),$$

where

$$\mu(t)=\left[\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\right]t^{a-1}(1-t)^{c-a-1}$$

is a probability measure on $[0,1]$. In fact

$$\int_0^1 d\mu(t)=\left[\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\right]B(a,c-a)=1,$$

where B is the beta function.

Note that $\phi(a,1;z)=z/(1-z)^a$ and $\phi(2,1;z)$ is the Koebe function.

Carlson and Shaffer [1] defined a convolution operator on $\mathcal{A}$ involving an incomplete beta function as $L(a,c)f=\phi(a,c)\star f$, $f\in\mathcal{A}$. If $a,c\neq 0,-1,-2,\ldots,$ application of the root test shows that the infinite series for $L(a,c)f$ has the same radius of convergence as that for f because $\lim_{n\to\infty}[(a)_n/(c)_n]^{1/n}=1$. Hence $L(a,c)$ maps $\mathcal{A}$ into itself. $L(a,a)$ is the identity and for $a\neq 0,-1,-2,\ldots,$ $L(a,c)$ has a continuous inverse $L(c,a)$. Also we note that $L(a,c)$ provides a convenient representation of differentiation and integration. If $g(z)=zf'(z)$, then $g=L(2,1)f$ and $f=L(1,2)g$.

We shall assume, unless otherwise stated, that $a\neq 0,-1,-2,\ldots,$ and $c\neq 0,-1,-2,\ldots.$

Let P_k be the class of analytic functions p defined in E by $p(z)=1+c_1z+c_2z^2+\cdots$ and with representation

$$p(z) = \frac{1}{2}\int_{-\pi}^{\pi} \frac{1+ze^{-it}}{1-ze^{-it}}\,dt\mu(t), \tag{1.1}$$

where $\mu(t)$ is a function with bounded variation on $[-\pi, \pi]$ and it satisfies the conditions

$$\int_{-\pi}^{\pi} d\mu(t) = 2, \qquad \int_{-\pi}^{\pi} \left|d\mu(t)\right| \leqslant k. \tag{1.2}$$

We note that $k \geqslant 2$ and $P_2 = P$ is the class of analytic functions p with positive real part in E with $p(0) = 1$. The class P_k was introduced in [9]. From the integral representation (1.1), (1.2), it can be seen that $p \in P_k$ if and only if there are analytic functions $p_1, p_2 \in P$ such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)p_2(z), \quad \text{see [9].}$$

We can define the class $P_k(\lambda)$, $0 \leqslant \lambda < 1$, as follows:

An analytic function p, with $p(0) = 1$, belongs to $P_k(\lambda)$ if and only if there exists p_1, p_2 $(p_i(0) = 1, i = 1, 2)$ such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)p_2(z),$$

where $\operatorname{Re} p_i(z) > \lambda$, $i = 1, 2$. We note that $P_2(\lambda) = P(\lambda)$.

By S, K, $S^\star$ and C, we denote the subclasses of $\mathcal{A}$ which consist of univalent, close-to-convex, starlike and convex functions, respectively. It is well known that, for $f \in C$, $f/z \in P(1/2)$. For $\alpha \geqslant 0$, an analytic function $f \in \mathcal{A}$ is in class Q_α of alpha-quasi-convex function if and only if $\{(1-\alpha)f + \alpha z f'\} \in K$ for $z \in E$. When $\alpha = 0$, we obtain the class $C^\star$ of quasi-convex functions which was first introduced and studied in [5,7].

We now define the following

Definition 1.1. Let $f \in \mathcal{A}$. Then $f \in C(a, c)$ if and only if $L(a, c)f \in C$, $z \in E$. Also $f \in K(\lambda)$, $0 \leqslant \lambda < 1$, if there exists $g \in C$ such that $f'/g' \in P(\lambda)$, $f \in K(\lambda; a, c)$ if and only if $L(a, c)f \in K(\lambda)$ for $z \in E$.

Definition 1.2. Let $f \in \mathcal{A}$. Then $f \in Q_\alpha(\lambda; a, c)$ if and only if $\{(1-\alpha)f + \alpha z f'\} \in K(\lambda; a, c)$ for $\alpha \geqslant 0$, $0 \leqslant \lambda < 1$ and $z \in E$.

Definition 1.3. Let $f \in \mathcal{A}$. Then, for $\alpha \geqslant 0$, $0 \leqslant \lambda < 1$, $f \in K_k(\lambda; a, c)$ if and only if there exists $g \in C$ such that

$$\left\{\frac{(L(a, c)f)'}{(L(a, c)g)'}\right\} \in P_k(\lambda), \quad z \in E.$$

Also $f \in C_k^\star(\lambda; a, c)$ if and only if $zf' \in K_k(\lambda; a, c)$.

Definition 1.4. Let $f \in \mathcal{A}$. Then $f \in Q_\alpha(k, \lambda; a, c)$ if and only if $\{(1-\alpha)f + \alpha z f'\} \in K_k(\lambda; a, c)$ for $z \in E$, $\alpha \geqslant 0$, $0 \leqslant \lambda < 1$.

2. Preliminaries results

Lemma 2.1 [4]. *If $c \neq 0$ and a, c are real and satisfy $a > N(c)$, where*

$$N(c) = \begin{cases} |c| + \frac{1}{2}, & \text{if } |c| \geqslant \frac{1}{3}, \\ \frac{3c^2}{2} + \frac{1}{3}, & \text{if } |c| < \frac{1}{3}, \end{cases} \tag{2.1}$$

then $\phi(c, a; z)$ is convex in E.

From Herglotz representation (1.1), (1.2) for $k = 2$, we have the following

Lemma 2.2. *If $p(z)$ is analytic in E, $p(0) = 1$ and $\operatorname{Re} p(z) > 1/2$, $z \in E$, then for any function F, analytic in E, the function $p \star F$ takes values in the convex hull of the image of E under F.*

Lemma 2.3 [11]. *Let ψ be convex and g be starlike in E. Then, for F analytic in E with $F(0) = 1$, $\psi \star Fg/\psi \star g$ is contained in the convex hull of $F(E)$.*

Lemma 2.4 [2]. *Let w be analytic in E and satisfy $w(0) = 0$ and $|w(z)| < 1$ for $z \in E$. Then, if $|w|$ assumes its maximum value on the circle $|z| = r$ at a point z_0, we can write $z_0 w'(z_0) = k w(z_0)$, where k is real and $k \geqslant 1$.*

We now prove the following

Lemma 2.5. *Let $p_0 \in P$ and $\{[(1-\alpha) + \alpha p_0]p + \alpha z p'\} \in P$ for $z \in E$. Then $p \in P$, $z \in E$.*

Proof. Since $p_0 \in P$, $z \in E$, we can write

$$p_0(z) = \frac{1 - w_1(z)}{1 + w_1(z)}, \quad \text{with } w_1(0) = 0,\ \left|w_1(z)\right| < 1.$$

Let $p(z) = \frac{1-w(z)}{1+w(z)}$. Then

$$\begin{aligned} &(1-\alpha)p + \alpha[p_0 p + zp'] \\ &\quad = (1-\alpha)\frac{1 - w(z)}{1 + w(z)} + \alpha\left[\frac{1 - w(z)}{1 + w(z)} \cdot \frac{1 - w_1(z)}{1 + w_1(z)} + \frac{2\alpha w'(z)}{(1 + w(z))^2}\right]. \end{aligned} \tag{2.2}$$

Suppose that, for $z_0 \in E$, $\max_{|z| \leqslant |z_0|} |w(z)| = |w(z_0)| = 1$ $(w(z_0) \neq -1)$. Then it follows from Lemma 2.4 that

$$z_0 w'(z_0) = k w(z_0), \quad k \text{ real and } k \geqslant 1.$$

Setting $w(z_0) = e^{i\theta_0}$ and $w_1(z_0) = r_0 e^{i\phi}$ in (2.2), we obtain

$$\begin{aligned} &\left\{(1-\alpha) + \alpha p_0(z_0)\right\} p(z_0) + \alpha z_0 p'(z_0) \\ &\quad = \left\{(1-\alpha) + \alpha\left(\frac{1 - r_0 e^{i\phi}}{1 + r_0 e^{i\phi}}\right)\left(\frac{1 - e^{i\theta_0}}{1 + e^{i\theta_0}}\right) - \frac{2\alpha k e^{i\theta_0}}{(1 + e^{i\theta_0})^2}\right\} \\ &\quad = \left\{\frac{-2(1-\alpha) i \sin\theta_0}{|1 + e^{i\theta_0}|^2} - \frac{2\alpha i \sin\theta_0 (1 - r_0^2 - 2 i r_0 \sin\phi)}{|1 + e^{i\theta_0}|^2 |1 + r_0 e^{i\phi_0}|^2} - \frac{4k(1 + \cos\theta_0)}{|1 + e^{i\theta_0}|^4}\right\}. \end{aligned}$$

Therefore

$$\operatorname{Re}\big\{\big[(1-\alpha)+\alpha p_0(z_0)\big]p(z_0)+\alpha z_0 p'(z_0)\big\}$$
$$=\frac{-4\alpha r_0\sin\theta_0\sin\phi}{|1+e^{i\theta_0}|^2|1+r_0e^{i\phi}|^2}-\frac{4k(1+\cos\theta_0)}{|1+e^{i\theta_0}|^4}.$$

Hence, if $\phi=0$, then

$$\operatorname{Re}\big\{\big[(1-\alpha)+\alpha p_0(z_0)\big]p(z_0)+\alpha z_0 p'(z_0)\big\}<0,$$

which is a contradiction to our hypothesis. Thus $|w(z)|<1$ and hence we conclude that $\operatorname{Re}\{p(z)\}>0$ for $z\in E$. □

Lemma 2.6 [8]. *Let $\alpha\geqslant 0$ and $D\in S^\star$, N be analytic in E and $N(0)=D(0)=0$, $N'(0)=D'(0)=1$. Let, for $z\in E$,*

$$\left\{(1-\alpha)\frac{N(z)}{D(z)}+\alpha\frac{N'}{D'}\right\}\in P_k,$$

then $N(z)/D(z)\in P_k$ for $z\in E$.

3. Main results

Theorem 3.1. *$f\in Q_\alpha(k,\lambda;a,c)$ if and only if there exists $F\in K_k(\lambda;a,c)$ such that for $z\in E$, $\alpha>0$,*

$$f(z)=\frac{1}{\alpha}z^{1-1/\alpha}\int_0^z \xi^{1/\alpha-2}F(\xi)\,d\xi. \tag{3.1}$$

The proof follows directly from Definition 1.4. In fact, using convolution, we can write (3.1) as

$$L(a,c)f=k\star F,\qquad F\in K_k(\lambda;a,c),$$

and

$$k(z)=\frac{1}{\alpha}z^{1-1/\alpha}\int_0^z\frac{t^{1/\alpha-1}}{1-t}\,dt,$$

where $k(z)$ is convex; see [10].

Theorem 3.2. *Let $f\in Q_\alpha(k,\lambda;a,c)$. Then $f\in Q_0(k,\lambda;a,c)\equiv K_k(\lambda;a,c)$.*

Proof. Set

$$\frac{F'(z)}{G'(z)}=(1-\lambda)p(z)+\lambda,$$

where

$$F = L(a,c)f, \quad G = L(a,c)g \in C, \quad p(0) = 1,$$

and $p(z)$ is analytic in E. We want to show that $p \in P_k$.

Now

$$\frac{1}{1-\lambda}\left\{\left[(1-\alpha)\frac{F'(z)}{G'(z)} + \alpha\frac{(zF'(z))'}{G'(z)}\right] - \lambda\right\} = \left\{\left[\alpha p_0(z) + (1-\alpha)\right]p(z) + \alpha z p'(z)\right\}$$

with $(p_0(z) = (zG'(z))'/G'(z) \in P)$.

Writing

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)p_2(z)$$

and using Lemma 2.5 with the fact that $f \in Q_\alpha(k,\lambda;a,c)$, we conclude that $\operatorname{Re} p_i(z) > 0$ for $z \in E$, $i = 1, 2$, and this proves our result. □

Theorem 3.3. *For* $0 \leqslant \beta < \alpha$, $Q_\alpha(k,\lambda;a,c) \subset Q_\beta(k,\lambda;a,c)$.

Proof. For $\beta = 0$, the proof is immediate. Therefore we let $\beta > 0$ and $f \in Q_\alpha(k,\lambda;a,c)$. Then there exists two functions $h_1, h_2 \in P_k(\lambda)$ such that

$$(1-\alpha)\frac{(L(a,c)f(z))'}{(L(a,c)g(z))'} + \alpha\frac{(z(L(a,c)f(z))')'}{(L(a,c)g(z))'} = h_1(z)$$

and

$$\frac{(L(a,c)f(z))'}{(L(a,c)g(z))'} = h_2(z), \quad g \in C(a,c).$$

Hence

$$(1-\beta)\frac{(L(a,c)f(z))'}{(L(a,c)g(z))'} + \beta\frac{(z(L(a,c)f(z))')'}{(L(a,c)g(z))'} = \frac{\beta}{\alpha}h_1(z) + \left(1 - \frac{\beta}{\alpha}\right)h_2(z). \tag{3.2}$$

Since the class $P_k(\lambda)$ is a convex set, see [6], it follows that

$$H = \left\{\frac{\beta}{\alpha}h_1 + \left(1 - \frac{\beta}{\alpha}\right)h_2\right\} \in P_k(\lambda), \quad z \in E,$$

and this proves the result. □

Theorem 3.4. *Let* $a > N(c)$ *where* $N(c)$ *is given by* (2.1). *Then, for* $a \neq 0, -1, -2, \ldots,$ *we have*

(i) $Q_\alpha(k,\lambda;d,c) \subset Q_\alpha(k,\lambda;d,a)$,

(ii) $Q_\alpha(k,\lambda;a,d) \subset Q_\alpha(k,\lambda;c,d)$.

Proof. We prove (i) and the proof of (ii) is similar. Let $f \in Q_\alpha(k,\lambda;d,c)$. Then, for $g \in C(d,c)$,

$$
\begin{aligned}
&(1-\alpha)\frac{(\phi(d,a)\star f)'}{(\phi(d,a)\star g)'}+\alpha\frac{(z(\phi(d,a)\star f)')'}{(\phi(d,a)\star g)'}\\
&\quad=(1-\alpha)\frac{[\phi(d,c)\star\phi(c,a)\star f]'}{[\phi(d,c)\star\phi(c,a)\star g]'}+\alpha\frac{[z(\phi(d,c)\star\phi(c,a)\star f)']'}{[\phi(d,c)\star(c,a)\star g]'}\\
&\quad=(1-\alpha)\left[\frac{\phi(c,a)\star\frac{N}{D}(\phi(d,c)\star g)'}{\phi(c,a)\star(\phi(d,c)\star g)'}\right]+\alpha\left[\frac{\phi(c,a)\star\frac{N'}{D'}(\phi(d,c)\star zg')}{\phi(c,a)\star(\phi(d,c)\star zg')}\right], \qquad (3.3)
\end{aligned}
$$

where $N(z)=(\phi(d,c)\star f)'$, $D(z)=(\phi(d,c)\star g)'\in S^\star$, and $\phi(c,a;z)\in C$ by Lemma 2.1. Now, using Lemma 2.6, we see that the right-hand side of (3.3) is contained in the convex hull of the image of E under $[(1-\alpha)\frac{N}{D}+\alpha\frac{N'}{D'}]$, and this gives us the desired result. □

Theorem 3.5. *Let $f\in K_k(\lambda;a,c)$. Then $f\in Q_\alpha(k,\lambda;a,c)$ for*

$$
|z|<r_\alpha=\frac{1}{2\alpha+\sqrt{4\alpha^2-2\alpha+1}}. \qquad (3.4)
$$

The value of r_α is best possible.

Proof. Proceeding on the similar lines as in Theorem 3.2, we obtain

$$
\begin{aligned}
&\frac{1}{1-\lambda}\left[\left\{(1-\alpha)\frac{(L(a,c)f(z))'}{(L(a,c)g(z))'}+\alpha\frac{(z(L(a,c)f(z))')'}{(L(a,c)g(z))'}\right\}-\lambda\right]\\
&\quad=\left(\frac{k}{4}+\frac{1}{2}\right)\left\{\left[(1-\alpha)+\alpha p_0(z)\right]p_1(z)+\alpha z p_1'(z)\right\}\\
&\qquad-\left(\frac{k}{4}-\frac{1}{2}\right)\left\{\left[(1-\alpha)+\alpha z p_0(z)\right]p_2(z)+\alpha z p_2'\right\}, \qquad (3.5)
\end{aligned}
$$

where $g\in C(a,c)$, $p_0,\ p_i\in P$, $i=1,2$.

Now it is well known that for $p\in P$,

$$
\begin{aligned}
&\frac{1-r}{1+r}\leqslant \operatorname{Re}p(z)\leqslant\left|p(z)\right|\leqslant\frac{1+r}{1-r},\\
&\left|p'(z)\right|\leqslant\frac{2\operatorname{Re}p(z)}{1-r^2}, \qquad \text{see [3].}
\end{aligned}
$$

We use these bounds to have

$$
\operatorname{Re}\left\{\left[(1-\alpha)+\alpha p_0(z)\right]p_i(z)+\alpha z p_i'(z)\right\}\geqslant 0, \qquad \text{for } |z|<r_\alpha,
$$

where r_α is given by (3.4). This implies that the right-hand side of (3.5) belongs to P_k for $|z|<r_\alpha$ and therefore $f\in Q_\alpha(k,\lambda;a,c)$ for $|z|<r_\alpha$.

The function $f_0\in Q_\alpha(k,\lambda;a,c)$ shows that the value of r_α is best possible and is given as follows.

Let

$$
F_0=L(a,c)f_0, \qquad G_0=L(a,c)g_0=\frac{z}{1-z},
$$

and

$$(1-\alpha)\frac{F_0'(z)}{G_0'(z)}+\alpha\frac{(zF_0'(z))'}{G_0'(z)}=(1-\lambda)H_0(z)+\lambda,$$

where

$$H_0(z)=\left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+z}{1-z}-\left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-z}{1+z}. \qquad \square \tag{3.6}$$

Theorem 3.6. *Let $f\in Q_\alpha(k,\lambda;a,c)$ and let $L(a,c)f(z)=z+\sum_{n=2}^{\infty}a_nz^n$. Then*

$$|a_2|\leqslant\frac{c[k(1-\lambda)+2]}{2a(1+\alpha)}.$$

The function f_0 defined by (3.6) *shows that this bound is sharp.*

Proof. Let $g\in C(a,c)$ and let $L(a,c)g(z)=z+\sum_{n=2}^{\infty}b_nz^n$ and $h(z)=1+\sum_{n=1}^{\infty}c_nz^n$, where $h\in P_k(\lambda)$. It is well known that $|b_n|\leqslant 1$, and $|c_n|\leqslant k(1-\lambda)$ for all n.

Now we have

$$(1-\alpha)\left(1+\sum_{n=2}^{\infty}\frac{n(a)_{n-1}}{(c)_{n-1}}a_nz^{n-1}\right)+\alpha\left(1+\sum_{n=2}^{\infty}\frac{n^2(a)_{n-1}}{(c)_{n-1}}a_nz^{n-1}\right)$$
$$=\left(1+\sum_{n=2}^{\infty}nb_nz^{n-1}\right)\left(1+\sum_{n=1}^{\infty}c_nz^n\right),$$

and this gives us

$$\left((1-\alpha)\frac{2a}{c}+4\alpha\frac{a}{c}\right)a_2=c_1+2b_2.$$

Thus we have

$$\left|\frac{2a}{c}(1+\alpha)a_2\right|\leqslant|c_1|+2|b_2|\leqslant k(1-\lambda)+2,$$

and so

$$|a_2|\leqslant\frac{c[k(1-\lambda)+2]}{2a(1+\alpha)}. \qquad \square$$

Theorem 3.7. *Let, for $a>N(c)$ with $N(c)$ given by* (2.1)*, $f\in Q_\alpha(2,\lambda;a,c)$. Then $f(E)$ contains the schlicht disk*

$$|z|\leqslant\frac{a(1+\alpha)}{2a(1+\alpha)+c(2-\lambda)}.$$

Proof. Since $f\in Q_\alpha(2,\lambda;a,c)$, $L(a,c)f$ is univalent in E. Let $f(z)\neq w_0$. Then, if $f_1(z)=\frac{w_0f(z)}{w_0-f(z)}$, $L(a,c)f_1$ is also univalent in E. Let

$$L(a,c)f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

and

$$L(a,c)f(z) = \frac{w_0 L(a,c)f(z)}{w_0 - L(a,c)f(z)} = z + \left(a_2 + \frac{1}{w_0}\right)z^2 + \cdots.$$

Since $|a_2 + 1/w_0| \leqslant 2$, we obtain the required result by using Theorem 3.6 for $k = 2$. □

Theorem 3.8. *For $\alpha \geqslant 1$, $Q_\alpha(k,\lambda;a,c) \subset C^\star(k,\lambda;a,c)$.*

Proof. Let $f \in Q_\alpha(k,\lambda;a,c)$. Then there exists $g \in C(a,c)$ such that

$$\begin{aligned}\frac{z(L(a,c)f(z))')'}{(L(a,c)g(z))'} &= \frac{1}{\alpha}\big[J_\alpha(a,c)\big] + \left(1 - \frac{1}{\alpha}\right)\frac{(L(a,c)f(z))'}{(L(a,c)g(z))'}\\ &= \frac{1}{\alpha}H_1 + \left(1 - \frac{1}{\alpha}\right)H_2,\end{aligned}$$

where

$$J_\alpha(a,c) = (1-\alpha)\frac{(L(a,c)f(z))'}{(L(a,c)g(z))'} + \alpha\frac{(z(L(a,c)f(z))')'}{(L(a,c)g(z))'}$$

and $H_i \in P_k(\lambda)$, $i = 1,2$, by using Definition 1.4 and Theorem 3.2. Now, since $P_k(\lambda)$ is a convex set, we have $\frac{(z(L(a,c)f(z))')'}{(L(a,c)g(z))'} \in P_k(\lambda)$ and this implies that $f \in C^\star(k,\lambda;a,c)$. □

We now show that the classes $Q_\alpha(k,\lambda;a,c)$ are closed under convolution with convex functions.

Theorem 3.9. *Let ψ be a convex function in E and let $f \in Q_\alpha(k,\lambda;a,c)$. Then $(\psi \star f) \in Q_\alpha(k,\lambda;a,c)$ for $z \in E$.*

Proof. Let $g \in C(a,c)$. Then

$$L(a,c)(\psi \star g) = \phi(a,c) \star (\psi \star g) = \psi \star \big(\phi(a,c) \star g\big) \in C,$$

since $\phi(a,c) \star g \in C$, $\psi \in C$.

Let

$$N(z) = \big(\phi(a,c) \star f\big)' \quad \text{and} \quad D(z) = \big(\phi(a,c) \star g\big)' \in S^\star.$$

Then

$$\frac{(L(a,c)(\psi \star f))'}{(L(a,c)(\psi \star g))'} = \frac{\psi \star \frac{N}{D}[(\phi(a,c) \star g)']}{\psi \star (\phi(a,c) \star g)'},$$

and so is in the convex hull of the image of E under $\frac{N(z)}{D(z)}$. Similarly $\frac{\{z[(L(a,c) \star f)]'\}'}{[L(a,c)(\psi \star g)]'}$ is in the convex hull of the image of E under $\frac{N'(z)}{D'(z)}$. Since $f \in Q_\alpha(k,\lambda;a,c)$, so

$$(1-\alpha)\frac{N(z)}{D(z)} + \alpha\frac{N'(z)}{D'(z)} \in P_k(\lambda)$$

and consequently $(\psi \star f) \in Q_\alpha(k,\lambda;a,c)$ and this proves the result. □

In fact we can prove the following more general result.

Theorem 3.10. *Let g be analytic in E, $g(0) = 0$, $g'(0) = 1$ and satisfy the condition* $\operatorname{Re}\{g(z)/z\} > 1/2$, $z \in E$. *Let $f \in Q_\alpha(k, \lambda; a, c)$. Then $(f \star g) \in Q_\alpha(k, \lambda; a, c)$, $z \in E$.*

Remark 3.1. We cam easily show that the class $Q_\alpha(k, \lambda; a, c)$ is a convex set.

Remark 3.2. Let

$$f_1(z) = \int_0^z \frac{f(t)}{t}\,dt \quad \text{and} \quad f_2(z) = \frac{1+b}{z^b}\int_0^z t^{b-1} f(t)\,dt \quad (\operatorname{Re} b > -1).$$

Then we write

$$f_i(z) = (f \star \psi_i)(z), \quad i = 1, 2,$$

where

$$\psi_1(z) = -\operatorname{Log}(1-z),$$
$$\psi_2(z) = \sum_{n=1}^{\infty} \frac{1+b}{n+b} z^n,$$

and ψ_1, ψ_2 are convex in E. Thus it follows, from Theorem 3.9, that $Q_\alpha(k, \lambda; a, c)$ is invariant under the integral operators f_i, $i = 1, 2$.

Remark 3.3. Let μ_i, $i = 1, 2$, be linear operators defined on the class $\mathcal{A}$ as follows:

$$\mu_1(f) = zf', \qquad \mu_2(f) = \frac{(zf)'}{2}.$$

We can write $\mu_i(f) = (\phi \star f)$, $i = 1, 2$, where

$$\phi_1(z) = \sum_{n=1}^{\infty} n z^n = \frac{z}{(1-z)^2},$$
$$\phi_2(z) = \sum_{n=1}^{\infty} \frac{n+1}{2} z^n = \frac{z - z^2/2}{(1-z)^2},$$

and the radius of convexity $r_c(\phi_1) = 2 - \sqrt{3}$, $r_c(\phi_2) = 1/2$. Therefore, if $f \in Q_\alpha(k, \lambda; a, c)$, then $\mu_1(f) \in Q_\alpha(k, \lambda; a, c)$ for $|z| < 2 - \sqrt{3}$ and $\mu_2(f) \in Q_\alpha(k, \lambda; a, c)$ for $|z| < 1/2$, where we have used Theorem 3.9.

Remark 3.4. If $f \in Q_\alpha(k, \lambda; a, c)$ for $z \in E$, then it follows, with similar arguments, that $f \in Q_1(k, \lambda; a, c)$ for $|z| < r_0 = 2 - \sqrt{3}$.

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