Van der Waals phase transition in the framework of holography

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ABSTRACT

Phase structure of the quintessence Reissner–Nordström–AdS black hole is probed by the nonlocal observables such as holographic entanglement entropy and two point correlation function. Our result shows that, as the case of the thermal entropy, both the observables exhibit the Van der Waals-like phase transition. To reinforce this conclusion, we further check the equal area law for the first order phase transition and critical exponent of the heat capacity for the second order phase transition. We also discuss the effect of the state parameter on the phase structure of the nonlocal observables.

1. Introduction

Investigation on the thermodynamic phase transition of a black hole is always a hot topic in black hole physics. On one hand, it is helpful for us to understand the nature of some quantities of black holes such as entropy. On the other hand, it also may shed light on the understanding of the relation between gravity and thermodynamics. Until now there are many works to study the phase transition of black holes. The AdS space time is the most popular background for it exhibits more abundant phase structures comparing with its counterparts. The particular phase transition in AdS space time is the Hawking–Page phase transition between the AdS black hole and thermal gas [1], which is interpreted as the confinement/deconfinement phase transition in the dual gauge field theory [2]. Another interesting phenomenon for a charged AdS black hole is that it exhibits the Van der Waals-like phase transition in the $T - S$ plane [3]. Namely the black holes endowed with different charges have different phase structures. As the charge increases from small to large, the hole will undergo first order phase transition and second order phase transition successively before it reaches to a stable phase. Recently in the extended phase space, where the negative cosmological constant is treated as the pressure while its conjugate acts as the thermodynamical volume, the Van der Waals-like phase transition is reconstructed in the $P - V$ plane [4–10]. It was stressed that there is a duality between the Van der Waals-like phase transition in $P - V$ plane and $T - S$ plane similar to the T-duality of string theory [11].

In this paper, we intend to study the Van der Waals-like phase transition in the framework of holography. Our work is based on the previous study in [12] where the phase structure of entanglement entropy is studied in a fixed charge ensemble and chemical potential ensemble. They found that the phase structure of entanglement entropy was similar to that of the thermal entropy for a charged black hole. They also studied the critical behavior of the heat capacity in the neighborhood of the critical point and found that the critical exponent was the same as the one of the thermal entropy for the second order phase transition of holographic entanglement entropy. In light of these interesting results, this work was extended to the extended phase space later, where the cosmological constant was treated as a thermodynamical variable. It was found that the entanglement entropy has the similar phase structure as that of the black hole entropy too [13]. In [14], Nguyen investigated exclusively the equal area law of holographic entanglement entropy and found that, as the case of thermal entropy, the equal area law holds for the entanglement entropy regardless of the size of the entangling region. Very recently [15] investigated entanglement entropy for a quantum system with infinite volume, and [16] investigated entanglement entropy in the bulk with Weyl correction, both of their result showed that there is a Van der Waals-like phase transition in the entanglement entropy-temperature plane.

Note that in all the works mentioned above, the authors considered only the phase structure of entanglement entropy in the field theory. In this paper, we will further study the phase structure of two point correlation function besides that of the entanglement entropy. The two point correlation function is also a nonlocal observable and to some extent it has the similar properties as the

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entanglement entropy. For example, both of them can probe the non-equilibrium thermalization behavior [17–22], superconductor phase transition [23–30], and cosmological singularity [31,32]. In this paper, we intend to explore whether it exhibits the Van der Waals-like phase transition as the entanglement entropy.

We choose the quintessence Reissner–Nordström–AdS black hole as the gravity background. Quintessence dark energy model is an important model that can explain the acceleration expansion of our universe. The black hole is a crucial component of the universe, so it will be interesting to exploring the effect of quintessence dark energy on the properties of black holes. Some attempts about this topic have been done until now. In [33,34], the quasinormal modes and Hawking radiation of the black holes surrounded by quintessence have been studied. In [37], the phase structure of the quintessence Reissner–Nordström–AdS black hole has been investigated in the extended phase space. Especially recently, there are also some works to study the quintessence AdS black hole in the framework of holography. [35] discussed effect of the quintessence dark energy on the formation of the superconductor. [36] studied the influence of quintessence dark energy on the non-equilibrium thermalization. At present, although the dual field theoretical interpretation about the quintessence is still unclear, with the further investigations, one can get better understanding. With this motivation, we will study the phase structure of the quintessence Reissner–Nordström–AdS black hole

This paper is organized as follows. In the next section, we will discuss the thermal entropy phase transition in the space time dominated by the quintessence dark energy. We mainly concentrate on phase transition in $T - S$ plane in a fixed charge ensemble, which is shown to be the Van der Waals-like phase transition. In Section 3, we study phase transition of the holographic entanglement entropy and two point correlation function respectively and find that both of them exhibit the Van der Waals-like phase transition. Particularly for each observable, the equal area law is checked and the critical exponent of the heat capacity is obtained. The last section is devoted to discussions and conclusions.

2. Thermodynamics of the quintessence Reissner–Nordström–AdS black hole

2.1. Quintessence Reissner–Nordström–AdS black hole

For a space time dominated by the quintessence dark energy with energy–momentum tensor

$$T_t^t = \rho_q(r), \tag{1}$$

$$T_j^j = 3\rho_q(r)\omega[-(1+3B)\frac{r^i r_j}{r_n r^n} + B\delta_i^j], \tag{2}$$

where ρ_q is dark energy density, ω is state parameter, and B is an arbitrary parameter depending on the internal structure of quintessence, the charged AdS black hole solution can be written as [38]

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3}$$

where

$$f(r) = 1 - \frac{a}{r^{3\omega+1}} + \frac{r^2}{l^2} - \frac{2M}{r} + \frac{Q^2}{r^2}, \tag{4}$$

in which l is the AdS radius which will be set to 1 during the numerics, a is the normalization factor which relates to the density of quintessence with the relation

$$\rho_q = -\frac{a}{2}\frac{3\omega}{r^{3(\omega+1)}}. \tag{5}$$

In cosmology, it is well known that for quintessence dark energy $-1 < \omega < -1/3$, while for phantom dark energy $\omega < -1$. Mathematically, from Eq. (4), we know that for different values of state parameter ω, the dark energy has different effect on the space time. For $\omega = -1$, the dark energy affects the AdS radius that is related to the cosmological constant. While for $\omega = -1/3$, the dark energy affects the curvature k of the space time.

From Eq. (4), we can get the Hawking temperature of this space time

$$T_{bh} = \frac{f'(r)}{4\pi}\Big|_{r_+} = \frac{r_+\left(3a\omega r_+^{-3\omega} + 3r_+^3/l^2 + r_+\right) - Q^2}{4\pi r_+^3}, \tag{6}$$

which is regarded as the temperature of the dual conformal field theory according to AdS/CFT duality. In addition, according to the entropy area relation, we also can get the entropy of the black hole

$$S = \pi r_+^2, \tag{7}$$

in above equations r_+ is the event horizon of the black hole, which is the largest root of $f(r_+) = 0$.

2.2. Van der Waals-like phase transition of black hole entropy

Substituting (7) into (6) and eliminating the parameter r_+, we can get the relation between the temperature T_{bh} and entropy S of the quintessence Reissner–Nordström–AdS black hole, that is

$$T_{bh} = -\frac{S^{-\frac{3\omega}{2}-\frac{3}{2}}\left(-3al^2\sqrt{S}\pi^{\frac{3\omega}{2}+\frac{3}{2}}\omega + \pi^2 l^2 Q^2 S^{\frac{3\omega}{2}} - \pi l^2 S^{\frac{3\omega}{2}+1} - 3S^{\frac{3\omega}{2}+2}\right)}{4\pi^{3/2}l^2}. \tag{8}$$

Based on this relation, we will study the Van der Waals-like phase transition in the $T - S$ plane. In fact, it has been shown that there also exist Van der Waals-like phase transitions in the $P - V$ plane, $T - S$ plane, and $\Phi - Q$ plane for a charged black hole in the extended phase space, in which the cosmological constant is treated as a thermal variable. To compare with the phase transition of entanglement entropy and some other nonlocal observables directly, we focus on only the phase transition of thermal entropy in the $T - S$ plane in this paper. We will also pay attention to the affect of both ω and a on the phase structure of the black hole. We take $\omega = -1, -2/3$ and $a = 0.5/2, 1.1/2$ as examples.

In order to understand the phase transition in the $T - S$ plane, we should first find the critical charge, which is determined by the following equations

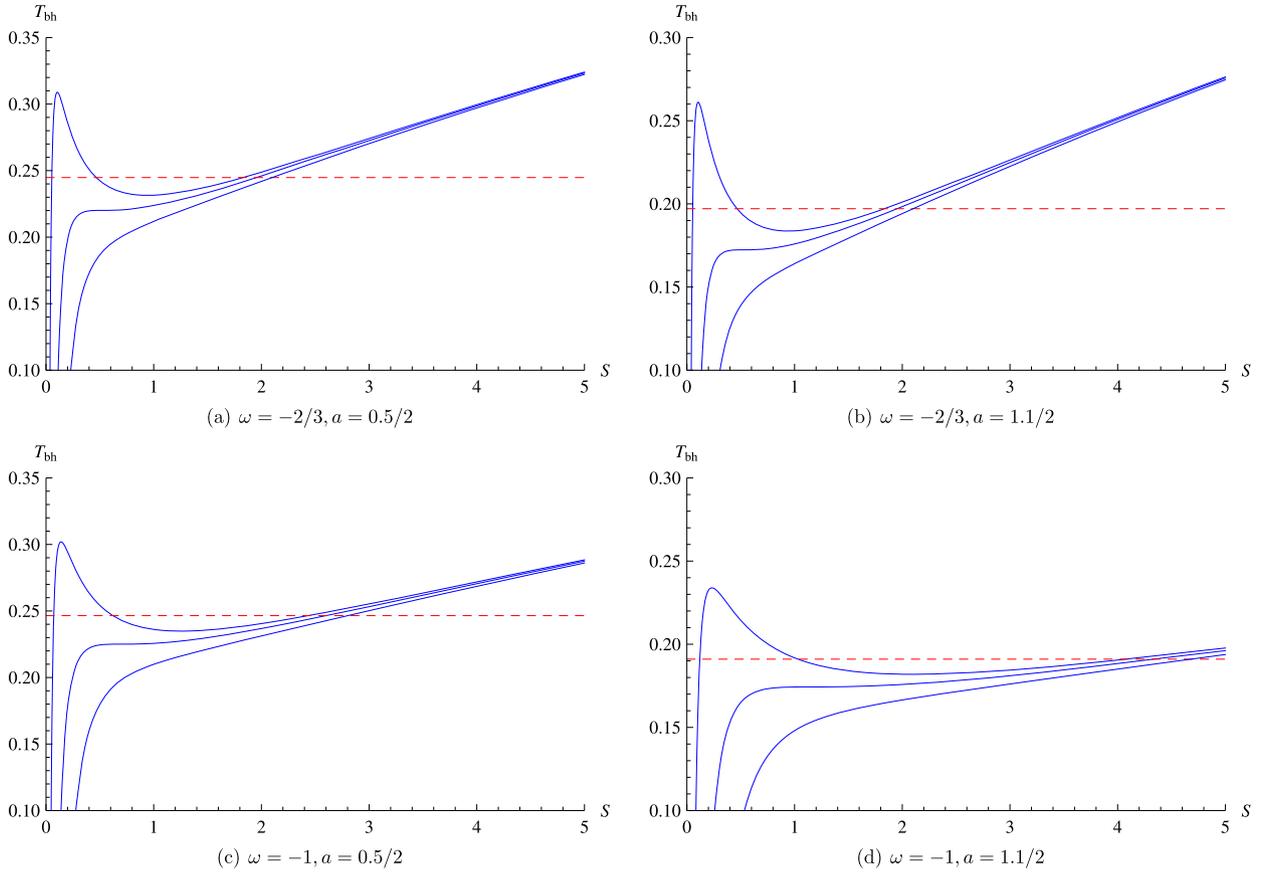$$(\frac{\partial T_{bh}}{\partial S})_Q = (\frac{\partial^2 T_{bh}}{\partial S^2})_Q = 0. \tag{9}$$

Inserting (8) into (9), we find for $\omega = -1$, the critical charge, critical entropy and critical temperature can be expressed as

$$Q_c = \frac{l}{6\sqrt{1-al^2}}, \tag{10}$$

$$S_c = \frac{\pi l^2}{6\left(1-al^2\right)}, \tag{11}$$

$$T_c = \frac{0.259899a^2l^4 - 0.519798al^2 + 0.259899}{\sqrt{\frac{l^2}{1-al^2}}\left(1-al^2\right)^2}, \tag{12}$$

and for $\omega = -2/3$, these quantities are

Fig. 1. Relations between the entropy and temperature for different ω and a. The red dashed lines correspond to the locations of first order phase transition.

$$Q_c = \frac{l}{6}, \tag{13}$$

$$S_c = \frac{\pi l^2}{6}, \tag{14}$$

$$T_c = \frac{2\sqrt{6} - 3al}{6\pi l}. \tag{15}$$

All these critical values are useful for us to study the critical behavior of heat capacity near the critical point next.

With (8), we now plot the isocharges for different ω and a, which are shown in Fig. 1. In (a) and (b) of Fig. 1, the curves from top to down correspond to the isocharges for the case $Q = 0.6/6$, $1/6$, $1.4/6$. In (c) and (d), these curves correspond to $Q = \frac{0.6}{6\sqrt{1-a}}$, $\frac{1}{6\sqrt{1-a}}$, $\frac{1.4}{6\sqrt{1-a}}$. It is obvious that for a fixed ω and a, the phase structure of the quintessence Reissner–Nordström–AdS black hole is similar to that of the Van der Waals phase transition. That is, for the small charge, which corresponds to the top curves in each graph in Fig. 1, there is an unstable hole interpolating between the stable small hole and stable large hole. The small stable hole will jump to the large stable hole as the temperature of the hole is larger than the critical temperature $T_\star$. As the charge increases to the critical charge, the smallest hole and the largest hole merge into one and squeeze out the unstable phase. So there is an inflection point in the middle curves for each graphic in Fig. 1. According to the definition of the specific heat capacity

$$C_Q = T_{bh} \frac{\partial S}{\partial T_{bh}} \mid_Q, \tag{16}$$

we know that the heat capacity is divergent at the inflection point and the phase transition near there is second order. As the charge

exceeds the critical charge, the black hole is stable always, which correspond to the lowest curves in each graph in Fig. 1. In addition, from Fig. 1, we can also observe how ω and a affect the phase structure. From the top curves in (b) and (d), we know that as ω decreases, the unstable stage is longer. So the large ω promotes the hole to reach the stable stage. From the top curves in (c) and (d), we know that the large a delays the hole to reach the stable stage.

For the first order phase transition in Fig. 1, we will check whether Maxwell's equal area law holds, which states

$$A_1 \equiv \int_{S_1}^{S_3} T_{bh}(S, Q)dS = T_\star(S_3 - S_1) \equiv A_3. \tag{17}$$

Obviously, to check this equation, we should first find the value of the critical temperature $T_\star$. Usually there are two different ways to get it. On one hand, one can construct an equation that produces the temperature with the supposition that the equal area law is true. On the other hand, one can find the horizontal coordinate of the junction of the swallowtail structure in the $F - T$ plane, where $F = M - TS$ is the Helmholtz free energy. Here our goal is to check the equal area law and the second method thus is more appropriate. The relations between F and T for different ω and a are plotted in Fig. 2. We can see that there is always a swallowtail structure in each graph, which corresponds to the unstable stage of the first order phase transition in Fig. 1. Now we take $\omega = -2/3$, $a = 0.5/2$ as an example to show how to check the equal area law. From (a) in Fig. 2, we find $T_\star = 0.2448$. Substituting this temperature into (8), we get the smallest and largest values of the entropy, $S_1 = 0.210442$, $S_3 = 2.93091$. With these values, we find A_1 and
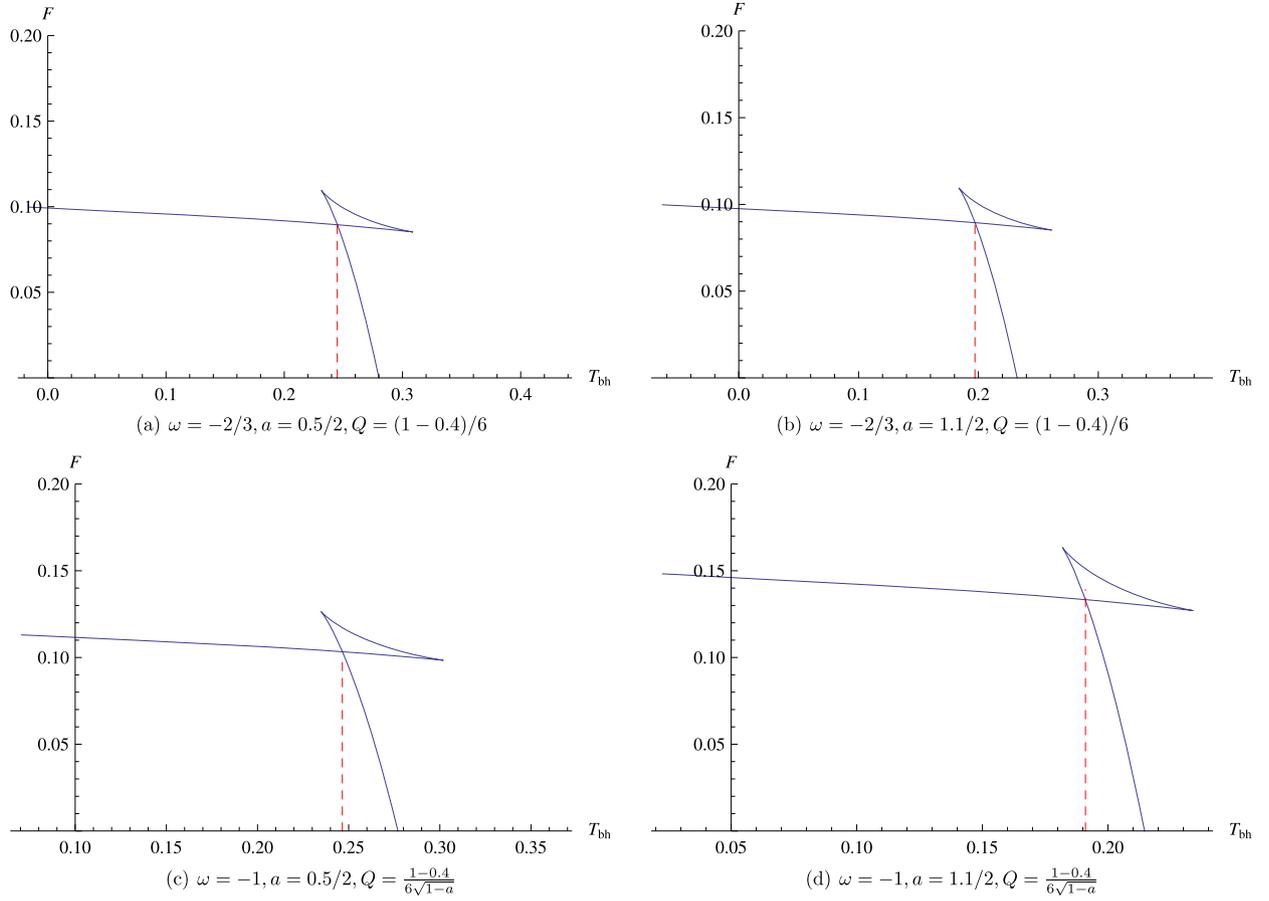
Fig. 2. Relations between the free energy and temperature for different ω and a. The intersection point between the red dashed line and horizontal coordinate in each graph is the first order phase transition temperature.

Table 1
Check of the equal area law in the $T - S$ plane for different ω and a.

	$w = -2/3$	$w = -1$
$a = 0.5/2$	$T_\star = 0.2448$	$T_\star = 0.2465$
$a = 1.1/2$	$T_\star = 0.1971$	$T_\star = 0.19103$
$a = 0.5/2$	$S_1 = 0.053875 \mid S_3 = 1.82594$	$S_1 = 0.071847 \mid S_3 = 2.43723$
$a = 1.1/2$	$S_1 = 0.053886 \mid S_3 = 1.82799$	$S_1 = 0.119813 \mid S_3 = 4.07542$
$a = 0.5/2$	$A_1 = 0.4340 \mid A_3 = 0.4338$	$A_1 = 0.5832 \mid A_3 = 0.5831$
$a = 1.1/2$	$A_1 = 0.4398 \mid A_3 = 0.3497$	$A_1 = 0.7555 \mid A_3 = 0.7557$

A_3 in (17) equal 0.4340, and 0.4338 respectively. Adopting the similar strategy, we can get A_1 and A_3 for other ω and a, which are shown in Table 1.

It is obvious that A_1 equals A_3 roughly for different ω and a, so the equal area law holds. In other words, though ω and a affect the phase structure of thermal entropy, it does not break the equal area law.

For the second order phase transition, we are interested in the critical exponent associated with the heat capacity defined in (16). We take $\omega = -2/3$ as an example. Near the critical point, writing the entropy as $S = S_c + \delta$ and expanding the temperature in small δ, we find

$$T_{bh} - T_c = \frac{((35\pi^2 l^2 Q^2)/S_c - 5\pi l^2 + 3S_c)}{64\pi^{3/2} l^2 S_c^{7/2}} (S - S_c)^3, \qquad (18)$$

in which we have used (9). With the definition of the heat capacity, we find further $C_Q \sim (T_{bh} - T_c)^{-2/3}$, namely the critical

exponent is $-2/3$, which is the same as the one from the mean field theory. In addition taking logarithm to (18), we find there is always a linear relation

$$\log | T_{bh} - T_c | = 3 \log | S - S_c | + \text{constant}, \qquad (19)$$

with 3 the slope. Next, we will employ this relation to check the critical exponent of heat capacity in entanglement entropy-temperature plane as well as two point correlation function-temperature plane.

3. Van der Waals phase transition in the framework of holography

Having obtained the phase structure of thermal entropy of the quintessence Reissner–Nordström–AdS black hole, we will study the phase structure of the non-local observables such as entanglement entropy and two point correlation function in the filed theory. We intend to explore whether they have the similar phase structure and critical behavior as that of the thermal entropy. As stressed in the introduction, the dual field theoretical interpretation about the quintessence is still unclear, we assume that it would not affect the dual field theory so that the relation between the two point correlation function and length of geodesic as well as holographic entanglement entropy and area of the minimal surface are still valid. The validity of this assumption will be confirmed by our last results.
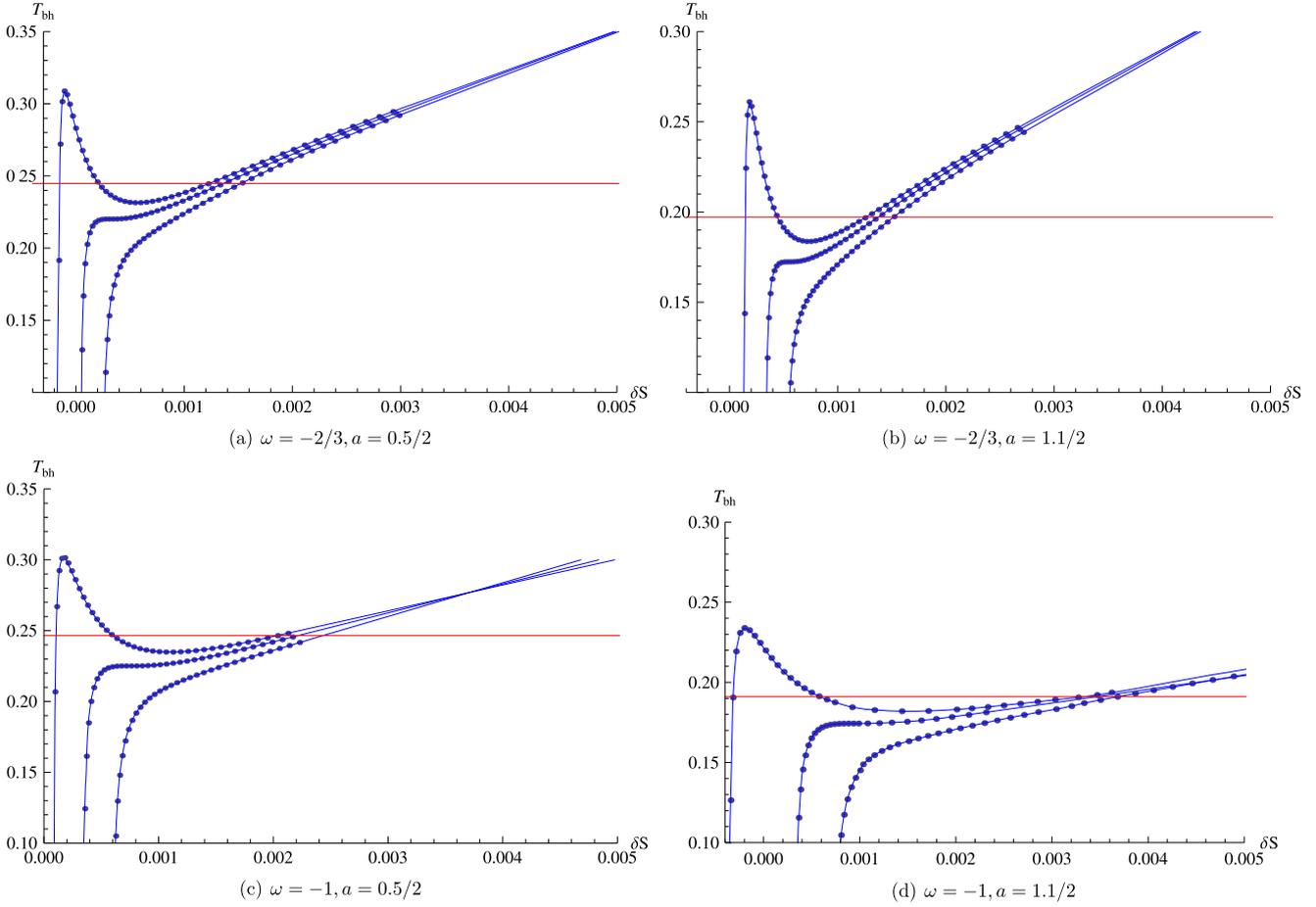
Fig. 3. Relations between the entanglement entropy and temperature for different ω and a. The red solid lines correspond to the locations of first order phase transition.

3.1. Van der Waals Phase transition of entanglement entropy

According to the formula in [39,40], the entanglement entropy can be given by the area A_{Σ} of a minimal surface Σ anchored on $\partial\Sigma$, to wit

$$S = \frac{A_{\Sigma}(t)}{4\pi G}, \qquad (20)$$

where G is the Newton's constant. Based on the definition of area and (3), (20) can be rewritten as

$$S = \frac{\pi}{2} \int_0^{\theta_0} r \sin\theta \sqrt{\frac{(r')^2}{f(r)} + r^2}, \qquad (21)$$

in which $r' = dr/d\theta$ and θ_0 is the boundary of the entangling region in θ direction. Making use of the Euler–Lagrange equation, one can get the equation of motion of $r(\theta)$

$$0 = r'(\theta)^2 [\sin\theta r(\theta)^2 f'(r) - 2\cos\theta r'(\theta)]$$
$$\quad - 2r(\theta) f(r)[r(\theta)(\sin\theta r''(\theta) + \cos\theta r'(\theta)) - 3\sin\theta r'(\theta)^2]$$
$$\quad + 4\sin(\theta) r(\theta)^3 f(r)^2. \qquad (22)$$

It seems to be impossible to get the analytical solution of $r(\theta)$, so we will solve it numerically with the boundary conditions

$$r'(0) = 0, r(0) = r_0. \qquad (23)$$

Note that the entanglement entropy is divergent at the boundary, so it should be regularized by subtracting off the entanglement entropy in pure AdS with the same entangling surface and boundary values. We label the regularized entanglement entropy as δS. For the numerical computation, we choose $\theta_0 = 0.16$ and set the UV cutoff in the dual field theory to be $r(0.159)$. To compare with the phase transition of thermal entropy, we will study the relation between the entanglement entropy and Hawking temperature, which is regarded as the temperature of the dual field theory. The numeric results for different ω and a are shown in Fig. 3. As the same as that of the thermal entropy, in (a) and (b) in Fig. 3, the curves from top to down correspond to the isocharges for the case $Q = 0.6/6$, $1/6$, $1.4/6$ individually. In (c) and (d), these curves correspond to $Q = \frac{0.6}{6\sqrt{1-a}}$, $\frac{1}{6\sqrt{1-a}}$, $\frac{1.4}{6\sqrt{1-a}}$. For a fixed ω and a, we find the phase structure of entanglement entropy is the same as that of the thermal entropy. Namely the phase transition is similar to that of the Van der Waals Phase transition. In other words, the phase structure depends on the charge of the black hole. For the small charge, the small stable black hole will transfer to the large stable hole as the temperature is higher than the critical temperature, and this transition is first order. As the charge grows to the critical charge, the small hole and the large hole merge so that the unstable hole shrinks into an inflection point, where the transition for the small hole to the large hole is second order. For a large enough charge, a large stable hole forms and the entanglement entropy grows monotonously as the temperature rises. From Fig. 3, we can also observe how ω and a affect the phase structure of entanglement entropy. From the top curves in (b) and (d), we know that as ω decreases,

Table 2
Check of the equal area law in the $T - \delta S$ plane for different ω and a with $\theta_0 = 0.16$.

	$w = -2/3$	$w = -1$
$a = 0.5/2$	$T_\star = 0.2448$	$T_\star = 0.2465$
$a = 1.1/2$	$T_\star = 0.1971$	$T_\star = 0.19103$
$a = 0.5/2$	$\delta S_1 = 0.00021 \mid \delta S_3 = 0.00123$	$\delta S_1 = 0.000110 \mid \delta S_3 = 0.002128$
$a = 1.1/2$	$\delta S_1 = 0.0001453 \mid \delta S_3 = 0.0012605$	$\delta S_1 = 0.000731 \mid \delta S_3 = 0.003087$
$a = 0.5/2$	$A_1 = 0.000240 \mid A_3 = 0.000249$	$A_1 = 0.000479 \mid A_3 = 0.000477$
$a = 1.1/2$	$A_1 = 0.000221 \mid A_3 = 0.000220$	$A_1 = 0.000436 \mid A_3 = 0.000447$

Table 3
Check of the equal area law in the $T - \delta S$ plane for different θ_0 with $w = -2/3, a = 0.5/2$.

$\theta = 0.12$	0.0670316	0.0676132	0.000143	0.000142
$\theta = 0.2$	0.121277	0.123993	0.000669	0.000665
$\theta = 0.24$	0.152796	0.157537	0.001168	0.001160

the unstable stage is longer, and from the top curves in (c) and (d), we know that as a decreases, the unstable stage is shorter. These effects are the same as that of the black hole entropy in Fig. 1.

Adopting the same strategy as that of the thermal entropy, we will check Maxwell's equal area law for the first order phase transition. In the $\delta S - T$ plane, we rewrite the equal area law as

$$A_1 \equiv \int_{\delta S_1}^{\delta S_3} T(\delta S) d\delta S = T_\star (\delta S_3 - \delta S_1) \equiv A_3, \qquad (24)$$

in which $T(\delta S)$ is an Interpolating Function obtained from the numeric result, and δS_1, δS_3 are the smallest and largest roots of the equation $T(\delta S) = T_\star$. Surely $T_\star$ is the first order phase transition temperature for a fixed ω and a, which can be read off from Fig. 2. For different ω and a, the results of δS_1, δS_3 and A_1, A_3 are listed in Table 2.

It is obvious that A_1 equals nearly A_3 for a fixed ω and a. Until now, our discussion is restricted in the case $\theta_0 = 0.16$. Next we will check whether the equal area law is valid for some other values of θ_0, which is shown in Table 3. Obviously, A_1 and A_3 are equal in our numeric accuracy, which implies that the equal area law holds regardless of the size of the entangling region. With Table 2 and Table 3, we can conclude that the equal area law is always valid for the first order phase transition of holographic entanglement entropy. Note that [3] once stressed that the value of θ_0 should be not too large in order to avoid the entanglement entropy to be contaminated by the surface that wraps the horizon. Our choice of the values of θ_0 satisfies this condition obviously for there is not saltation for our results in Table 3.

In order to study the critical exponent of the second order phase transition in the $\delta S - T$ plane, we define an analogous specific heat capacity

$$C_Q = T_{bh} \frac{\partial \delta S}{\partial T_{bh}} \mid_Q . \qquad (25)$$

Provided a similar relation as that in (19) is satisfied, then with (25) we can get the critical exponent of second order phase transition of entanglement entropy. Here we are interested in the logarithm of the quantities $T_{bh} - T_c$, $\delta S - \delta S_c$. For different ω and a, the relation between $\log |T_{bh} - T_c|$ and $\log |\delta S - \delta S_c|$ are plotted in Fig. 4, in which T_c is the critical temperature which have been defined in (12) as well as (15), and S_c is the corresponding critical entropy obtained numerically by the equation $T(\delta S) = T_c$. The analytical result of these straight lines can be fitted as

$$\log |T_{bh} - T_c| = \begin{cases} 19.9074 + 3.00361 \log |\delta S - \delta S_c|, \\ \quad \text{for } \omega = -2/3, a = 0.5/2 \\ 21.5846 + 3.08801 \log |\delta S - \delta S_c|, \\ \quad \text{for } \omega = -2/3, a = 1.1/2 \\ 20.1355 + 3.02826 \log |\delta S - \delta S_c|, \\ \quad \text{for } \omega = -1, a = 0.5/2 \\ 19.8945 + 3.02606 \log |\delta S - \delta S_c|, \\ \quad \text{for } \omega = -1, a = 1.1/2. \end{cases} \qquad (26)$$

We find for all the ω and a, the slope is always about 3, which is consistent with that of the thermal entropy. That is, the entanglement entropy has the same second order phase transition behavior as that of the thermal entropy.

3.2. Van der Waals Phase transition of two point correlation function

In previous section, we have shown that the entanglement entropy has the same phase structure as that of the thermal entropy. In this section, we intend to explore whether the two point correlation function has the similar behavior as that of the entanglement entropy. According to the AdS/CFT correspondence, the equal time two point correlation function under the saddle-point approximation can be holographically approximated as [41]

$$\langle \mathcal{O}(t_0, x_i) \mathcal{O}(t_0, x_j) \rangle \approx e^{-\Delta L}, \qquad (27)$$

if the conformal dimension Δ of scalar operator $\mathcal{O}$ is large enough, where L is the length of the bulk geodesic between the points (t_0, x_i) and (t_0, x_j) on the AdS boundary. Taking into account the spacetime symmetry of the quintessence Reissner–Nordström–AdS black hole, we can simply let $x_i = \theta$ with the boundary θ_0. We will also employ θ to parameterize the trajectory and in this case the proper length is given by

$$L = \int_0^{\theta_0} \mathcal{L}(r(\theta), \theta) d\theta, \quad \mathcal{L} = \sqrt{\frac{(r')^2}{f(r)} + r^2}, \qquad (28)$$

in which $r' = dr/d\theta$. Imagining θ as time, and treating $\mathcal{L}$ as the Lagrangian, one can get the equation of motion for $r(\theta)$ by making use of the Euler–Lagrange equation, that is

$$0 = r'(\theta)^2 f'(r) - 2f(r)r''(\theta) + 2r(\theta)f(r(\theta))^2. \qquad (29)$$

We will also use Eq. (23) to solve this equation. We choose $\theta_0 = 0.16$ and set the UV cutoff in the dual field theory to be $r(0.159)$. We label the regularized two point correlation function as δL. For simplicity in this section, we fix $\omega = -2/3$. The relations between δL and T_{bh} for different a ares shown in Fig. 5. In each figure, the curves from top to down correspond to the isocharges for $Q = 0.6/6$, $1/6$, $1.4/6$. Comparing Fig. 5 with (a) and (b) in Fig. 3, we find they are the same nearly. That is, the phase structure of the two point correlation function is also similar to that of the Van der Waals-like phase transition. To further confirm this conclusion,
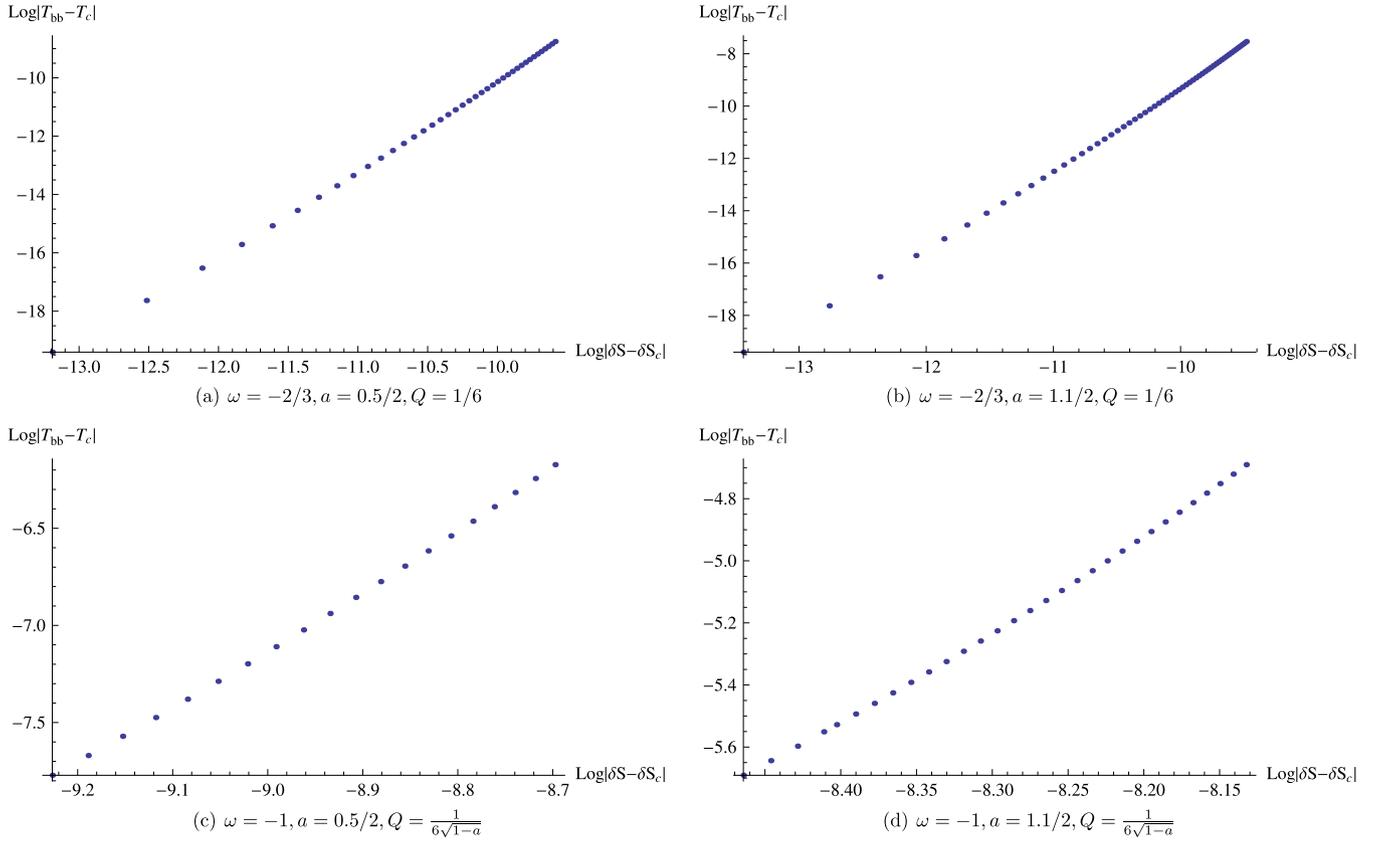
Fig. 4. Relations between $\log | T_{bh} - T_c |$ and $\log | \delta S - \delta S_c |$ for different ω and a.
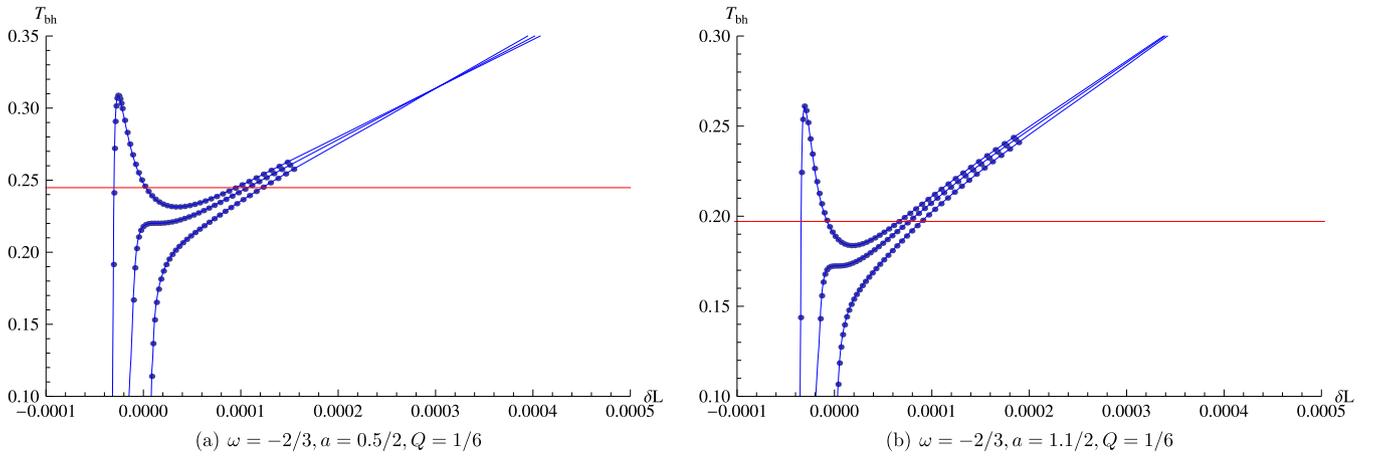

Fig. 5. Relations between the two point correlation function and temperature for different a with a fixed $\omega = -2/3$. The red solid lines correspond to the locations of first order phase transition.

we will also pay attention to the equal area law for the first order phase transition and critical exponent for the second order phase transition. In the $\delta L - T$ plane, we define the equal area law as

$$A_1 \equiv \int_{\delta L_1}^{\delta L_3} T(\delta L) d\delta L = T_\star(\delta L_3 - \delta L_1) \equiv A_3, \qquad (30)$$

in which $T(\delta L)$ is an Interpolating Function obtained from the numeric result, and δL_1, δL_3 are the smallest and largest roots of the equation $T(\delta L) = T_\star$.

For different a, the results of δL_1, δL_3 and A_1, A_3 are listed in Table 4. From this table, we can see that A_1 equals A_3 in our numeric accuracy.

Similarly, we will check whether the equal area law is valid for some other values of θ_0, which is shown in Table 5. It is obvious that A_1 and A_3 are equal for different θ within reasonable error. From Table 4 and Table 5, we can conclude that the equal area law is also valid in the $\delta L - T$ plane. In other words, similar to the holographic entanglement entropy, the two point correlation function also exhibits the first order phase transition as that of the thermal entropy.

(a) $\omega = -2/3, a = 0.5/2, Q = 1/6$

(b) $\omega = -2/3, a = 1.1/2, Q = 1/6$
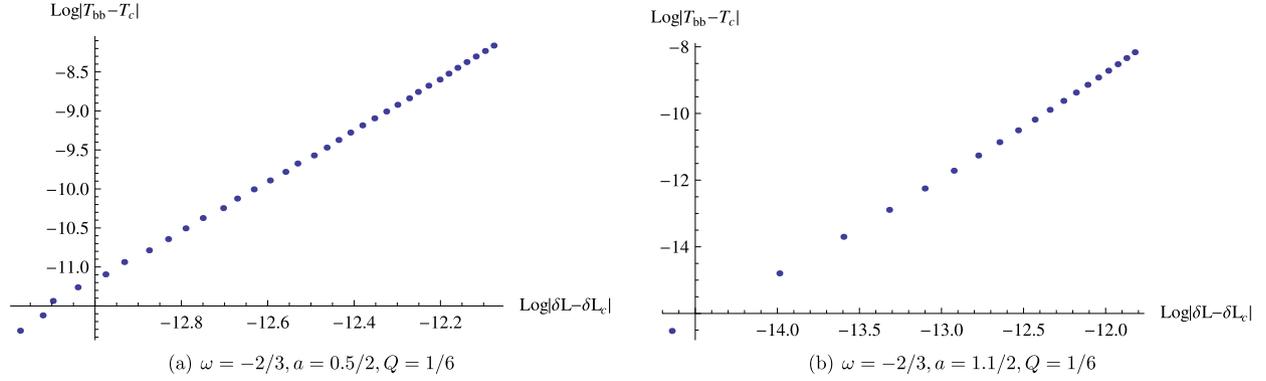
Fig. 6. Relations between $\log |T_{bh} - T_c|$ and $\log |\delta L - \delta L_c|$ for different a.

Table 4
Check of the equal area law in the $T - \delta L$ plane for different a with $\theta_0 = 0.16$.

	δL_1	δL_3	A_1	A_3
$a = 0.5/2, T_\star = 0.2448$	0.000002483098	0.000095059	0.0000220	0.0000227
$a = 1.1/2, T_\star = 0.1971$	−0.000007475310	0.000066343	0.0000140	0.0000146

Table 5
Check of the equal area law in the $T - \delta L$ plane for different θ_0 with $w = -2/3, a = 0.5/2$.

	δL_1	δL_3	A_1	A_3
$\theta = 0.12$	0.00956964	0.0096248	0.0000136	0.0000135
$\theta = 0.2$	0.0159275	0.016183	0.0000629	0.0000626
$\theta = 0.24$	0.0192298	0.0196722	0.0001090	0.0001083

By defining an analogous heat capacity

$$C = T_{bh} \frac{\partial \delta L}{\partial T_{bh}}, \qquad (31)$$

we can also study the critical exponent of the heat capacity for the second order phase transition in the $\delta L - T$ plane. Similar to that of the entanglement entropy, we are interested in the logarithm of the quantities $T_{bh} - T_c$, $\delta L - \delta L_c$, in which T_c is the critical temperature defined in (15), and L_c is obtained numerically by the equation $T(\delta L) = T_c$. The relations between $\log |T_{bh} - T_c|$ and $\log |\delta L - \delta L_c|$ are plotted in Fig. 6. The analytical results of these curves can be fitted as

$$\log |T_{bh} - T_c| = \begin{cases} 27.2601 + 3.00747 \log |\delta L - \delta L_c|, \\ \quad \text{for } \omega = -2/3, a = 0.5/2 \\ 31.0971 + 3.15294 \log |\delta L - \delta L_c|, \\ \quad \text{for } \omega = -2/3, a = 1.1/2. \end{cases} \qquad (32)$$

It is obvious that the slope is also about 3, which is consistent with that of the thermal entropy. That is, the two point correlation function also has the same second order phase transition as that of the thermal entropy.

4. Concluding remarks

We have investigated detailedly the Van der Waals-like phase transition in the quintessence Reissner–Nordström–AdS black hole in a fixed charge ensemble. We first investigated the phase structure of the thermal entropy in the $T - S$ plane and found that the phase structure depends on the charge of the black hole. For the small charge, there is always an unstable hole interpolating between the small stable black hole and large stable black hole. The transition for the small hole to the large hole is first order and

the equal area law holds. As the charge of the hole increases to the critical value, the unstable hole merges into an inflection point where the phase transition is second order. While the charge is larger than the critical charge, the hole is stable always. We also discussed the effect of ω as well as a on the phase structure and found ω promote the hole to reach the stable phase while a delays. Interestingly, we found that the nonlocal observables such as entanglement entropy and two point correlation function also exhibit the similar phase structure in the $T - \delta S$ plane and $T - \delta L$ plane respectively. The influence for ω and a on the phase structure is the same as that of the thermal entropy. To confirm this observation, we further showed that the equal area law holds and the critical exponent of the heat capacity is consistent with that of the mean field theory for both the entanglement entropy and two point correlation function. These results imply that the entanglement entropy and the two point correlation function are indeed a good probe to the phase transition. In addition, our results also reveal a fact that the quintessence dark energy in the bulk will not affect the definition of field theory on the boundary of an AdS space time though its dual field theoretical interpretation is still unclear.

Acknowledgements

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