MATHEMATICS

ON A CLASS OF REFLECTIVE SUBCATEGORIES

BY

PH. DWINGER

(Communicated by Prof. H. D. KLOOSTERMAN at the meeting of September 30, 1967)

1. We recall the definition of a reflective subcategory [2], [3]. Suppose $\mathscr{B}$ is a subcategory (not necessarily full) of a category $\mathscr{A}$. $\mathscr{B}$ is a *reflective subcategory* of $\mathscr{A}$ if the inclusion functor $\mathscr{B} \to \mathscr{A}$ has a left adjoint. This left adjoint is also called the *reflector*. Equivalently, for every object A of $\mathscr{A}$, there exists a morphism $g : A \to B$, where B is an object of $\mathscr{B}$, and such that the following condition is satisfied. Whenever $f : A \to C$ is a morphism, where C is an object of $\mathscr{B}$, there exists a unique morphism $h : B \to C$, $h \in \mathscr{B}$ such that $h \circ g = f$.

Reflective subcategories exist in abundance (cf. for example [1] and [2]). In many of these cases, the inclusion functor $\mathscr{B} \to \mathscr{A}$ is a monofunctor, i.e. it preserves monomorphisms. In addition, it may happen that the natural transformation η between the identity functor on $\mathscr{A}$, and the reflector R has the property that for every object A of $\mathscr{A}$, $\eta(A)$ is a monomorphism or an epimorphism, or both. The present paper is devoted to the situation that the inclusion functor is a *monofunctor*, and that $\eta(A)$ is a monomorphism (but not necessarily an epimorphism) for every object A of $\mathscr{A}$. If $\eta(A)$ is a monomorphism for every object A of $\mathscr{A}$, then η is also called a *pointwise monomorphism*. Our purpose is to find additional conditions in order that R be an epifunctor, thus that R preserves epimorphisms, and conditions in order that R be a monofunctor, thus that R preserves monomorphisms. Our two main results are stated in the theorems 1 and 2 (see below).

We will see that under rather weak conditions R is already an epifunctor. However we will need more conditions for R to be a monofunctor. The following condition is the crucial one in this case. For every object B of $\mathscr{B}$ there exists a monomorphism $h : B \to C$, where $h \in \mathscr{B}$ and where C is injective as an object of $\mathscr{A}$ (cf. [3], p. 131).

The work done in this paper was inspired by a situation that exists in Boolean algebras, where we have a reflective subcategory $\mathscr{B}$ of a category $\mathscr{A}$ such that the inclusion functor $\mathscr{B} \to \mathscr{A}$ is a monofunctor and such that the natural transformation η is a pointwise monomorphism (but η is not a pointwise epimorphism!). It turns out that here the reflector R is both an epifunctor and a monofunctor. We will describe this case briefly. It is well-known [5] that for every Boolean algebra A there exists a free α-extension A_α (α a fixed, infinite cardinal number). That is, there exists an α-complete Boolean algebra A_α in which A can be imbedded, and such that every homomorphism from A into an α-complete Boolean algebra B can be extended uniquely to an α-complete homomorphism from A_α to B. If we denote the category of Boolean algebras and homomorphisms by $\mathscr{A}$ and the subcategory of α-complete Boolean algebras and α-complete homomorphisms by $\mathscr{B}$, then this simply means that $\mathscr{B}$ is a reflective subcategory of $\mathscr{A}$ (notice that $\mathscr{B}$ is not a full subcategory of $\mathscr{A}$). It is not difficult to show that the following relationship exists between onto- (one-one) homomorphisms and epi-(mono)morphisms in the categories $\mathscr{A}$ and $\mathscr{B}$. A homomorphism in $\mathscr{A}$ is onto (one-one), if and only if it is an epi-(mono)morphism in $\mathscr{A}$. A homomorphism in $\mathscr{B}$ is one-one, if and only if it is a monomorphism in $\mathscr{B}$. Finally, an onto homomorphism in $\mathscr{B}$ is necessarily an epimorphism in $\mathscr{B}$. It easily follows that the inclusion functor $\mathscr{B} \to \mathscr{A}$ is a monofunctor. Denoting again the natural transformation between the identity functor on $\mathscr{A}$ and the reflector R by η, it also follows that η is a pointwise monomorphism. (It is not too hard to show that η is not a pointwise epimorphism.) (One can also show that R is faithful.) The proof that R preserves onto homomorphisms and thus, that R is an epifunctor is easy and standard. However, it is much more difficult to prove that R preserves one-one homomorphisms [5] and thus, that R is a monofunctor. In order to show this last property of R, one essentially needs the following two well-known properties of Boolean algebras [4]. First, every Boolean algebra A has a normal completion. Therefore, A can be imbedded in a complete Boolean algebra such that the imbedding is complete and hence, α-complete. Second, the complete Boolean algebras are precisely the injective objects in the category $\mathscr{A}$. (Notice that the condition stated in the previous paragraph is therefore satisfied.) Using these two properties, one can then prove that R preserves one-one homomorphisms and thus, that R is a monofunctor.

We will now state the two main results that we will prove.

THEOREM 1. Suppose $\mathscr{B}$ is a reflective subcategory of the category $\mathscr{A}$ such that the following conditions are satisfied:

(i) $\mathscr{A}$ is balanced and has images

(ii) $\mathscr{B}$ has epimorphic images

(iii) the inclusion functor $\mathscr{B} \to \mathscr{A}$ is a monofunctor

(iv) the natural transformation η between the identity functor on $\mathscr{A}$ and the reflector R is a pointwise monomorphism.

Then, R is an epifunctor.

THEOREM 2. Suppose $\mathscr{B}$ is a reflective subcategory of the category $\mathscr{A}$ such that the following conditions are satisfied:

(i) $\mathscr{A}$ is locally small and balanced, and $\mathscr{A}$ and $\mathscr{B}$ have intersections, pullbacks and epimorphic images

(ii) the inclusion functor $\mathscr{B} \to \mathscr{A}$ is a monofunctor which preserves intersections, pullbacks and images

(iii) for every object B of $\mathscr{B}$, there exists a monomorphism $h: B \to C$, $h \in \mathscr{B}$, such that C is injective as an object of $\mathscr{A}$.

(iv) the natural transformation η between the identity functor on $\mathscr{A}$ and the reflector R is a pointwise monomorphism.

Then, R is a monofunctor.

Remark. It follows from theorem 1, that in theorem 2 one can also conclude that R is an epifunctor.

2. In general we will use the notations and definitions of [3] and we refer the reader to [3] for the meaning of those symbols and concepts which are not explained in this paper. For the sake of convenience, and also because in some cases our terminology slightly differs from [3], we will introduce some terminology and recall some definitions. We will always identify objects with the corresponding identity morphisms, and categories with the corresponding identity functors. Thus, if A is an object of $\mathscr{A}$, then A also denotes the identity morphism of A, and $\mathscr{A}$ also denotes the identity functor on $\mathscr{A}$. The class of objects of a category $\mathscr{A}$ will be denoted by Obj $\mathscr{A}$. A *subobject* of an object A of a category $\mathscr{A}$ is a monomorphism $h: B \to A$. An *image* of a morphism $h: A \to B$ of a category $\mathscr{A}$ is a subobject $v: C \to B$ such that the following condition is satisfied. There exists a morphism $u: A \to C$ such that $v \circ u = h$ and whenever $v': C' \to B$ is a monomorphism and $u': A \to C'$ a morphism satisfying $v' \circ u' = h$, then there exists a unique morphism $f: C \to C'$ such that $v' \circ f = v$. We also write $v = \operatorname{Im} h$. It is easy to see that f is monic and that $f \circ u = u'$ and that u is unique. If u is epic then v is an *epimorphic image* of h. We observe that if $\mathscr{A}$ is locally small and has intersections, then $\mathscr{A}$ has images.

We will mostly be dealing with a situation, where we have a category $\mathscr{A}$ and a subcategory $\mathscr{B}$ of $\mathscr{A}$. In such a case, if we talk about morphisms (monomorphisms, epimorphisms) objects, etc., without further specification, then we will always assume that they belong to $\mathscr{A}$. Also, if h is a morphism in $\mathscr{B}$ and it is stated that h is monic (epic) in $\mathscr{B}$ then this does not necessarily imply that h is monic (epic) in $\mathscr{A}$. However in most cases we will assume that the inclusion functor $\mathscr{B} \to \mathscr{A}$ is a monofunctor and then, a monomorphism in $\mathscr{B}$ is also a monomorphism in $\mathscr{A}$. Again, if B is an object in $\mathscr{B}$ and if we talk about a subobject $h: A \to B$, then this will mean that h is a subobject of B in $\mathscr{A}$. Thus h is a monomorphism in $\mathscr{A}$ and does not necessarily belong to $\mathscr{B}$, unless stated otherwise. On the other hand if it is stated that h is a subobject of B in $\mathscr{B}$, then h is a monomorphism in $\mathscr{B}$ but then, h is not necessarily a subobject of B in $\mathscr{A}$, unless again the inclusion functor $\mathscr{B} \to \mathscr{A}$ is a monofunctor. Similarly, if $h: A \to B$ is a morphism in $\mathscr{B}$ and we talk about $\operatorname{Im} h$, then this is the image of h in $\mathscr{A}$ and not necessarily the image of h in $\mathscr{B}$. On the other hand if the image of h exists in $\mathscr{B}$, then this need not necessarily be the image of h in $\mathscr{A}$, unless the inclusion functor preserves images. A similar argument applies to pullbacks, intersections, etc. Finally, if we say that an object B of $\mathscr{B}$ is injective, then we mean that B is injective as an object of $\mathscr{A}$.

In the remaining part of this section we will prove two lemmas which we will need in the sequel.

Lemma 1. Suppose $\mathscr{A}$ is a category with finite intersections. Let $h: A \to B$ be a morphism in $\mathscr{A}$ and suppose h has an epimorphic image $v: C \to B$. Let $f: B \to B'$ be a subobject of B'. Then $f \circ v = \operatorname{Im} f \circ h$.

Proof. There exists an epimorphism $u: A \to C$ such that $v \circ u = h$. Thus $f \circ v \circ u = f \circ h$. Suppose $q: B'' \to B'$ is a subobject of B' and $p: A \to B''$ is a morphism such that $q \circ p = f \circ h$. We must show that there exists a unique morphism $x: C \to B''$ such that $q \circ x = f \circ v$. Let r be the intersection of f and q. Thus there exists a monomorphism $r_1: D \to B$ and a monomorphism $r_2: D \to B''$ such that $f \circ r_1 = q \circ r_2 = r$. Since $q \circ p = f \circ h$, there exists a unique morphism $s: A \to D$ such that $q \circ p = f \circ h = r \circ s$ and $r_2 \circ s = p$, $r_1 \circ s = h$. r_1 is monic (and $r_1 \circ s = h$), thus there exists a unique monomorphism $t: C \to D$ such that $r_1 \circ t = v$ and $t \circ u = s$. Let $x = r_2 \circ t$. We have $q \circ x \circ u = q \circ r_2 \circ t \circ u = q \circ r_2 \circ s = q \circ p = f \circ h = f \circ v \circ u$. But u is epic, thus $q \circ x = f \circ v$. It remains to show that x is unique. Suppose $x': C \to B''$ such that $q \circ x' = f \circ v$. But $v \circ u = h = r_1 \circ s = r_1 \circ t \circ u$. But u is epic, hence $v = r_1 \circ t$ and thus $q \circ x' = f \circ r_1 \circ t$. It follows that there exists a unique morphism $t': C \to D$ such that $r_2 \circ t' = x'$ and $r_1 \circ t = r_1 \circ t'$. But r_1 is monic, thus $t = t'$ and it follows that $x = x'$.

Lemma 2. Suppose $\mathscr{A}$ is a balanced category with images. Let $f: A \to B$, $g: B \to D$, $h: A \to C$ and $k: C \to D$ be morphisms such that f is epic, g is monic and $g \circ f = k \circ h$. Suppose $v = \operatorname{Im} k$ exists, $v: E \to D$. Then there exists a unique monomorphism $s: B \to E$ such that $v \circ s = g$.

Proof. There exists a morphism $u: C \to E$ such that $v \circ u = k$. f is epic and g is monic and moreover $\mathscr{A}$ is balanced. It follows (Prop. 10.2, p. 12, [3]) that $g = \operatorname{Im} g \circ f$. Again, v is monic and $v \circ u \circ h = g \circ f$. Hence there exists a unique monomorphism $s: B \to E$ such that $v \circ s = g$ (and $s \circ f = u \circ h$).

3. We start this section with introducing a new notion which will be useful. Suppose $\mathscr{B}$ is a subcategory of the category $\mathscr{A}$ and suppose that the inclusion functor is a monofunctor. Let B be an object of $\mathscr{B}$ and let $h: A \to B$ be a subobject of B. A subobject $h^*: A^* \to B$ of B in $\mathscr{B}$ is said to be *$\mathscr{B}$-generated* by h, if the following condition is satisfied. There exists a morphism $f: A \to A^*$ such that $h^* \circ f = h$ and whenever $h^{*\prime}: C \to B$ is a subobject of B in $\mathscr{B}$ and $f': A \to C$ is a morphism such that $h^{*\prime} \circ f' = h$, then there exists a unique morphism $p: A^* \to C$, $p \in \mathscr{B}$ such that $h^{*\prime} \circ p = h^*$. It is not difficult to show that f, f' and p are monomorphisms. Moreover, f is unique and $p \circ f = f'$.

The notion of "$\mathscr{B}$-generated" will play an essential rôle in most of the following lemmas. We will need these lemmas in the next section for the proofs of our main results.

Lemma 3. Suppose $\mathscr{B}$ is a subcategory of the category $\mathscr{A}$ such that the inclusion functor is a monofunctor. If $h: A \to B$, $B \in \mathrm{Obj}\ \mathscr{B}$ and if $h^*: A^* \to B$ is $\mathscr{B}$-generated by A then A^* is $\mathscr{B}$-generated by f, where f is the unique morphism determined by $h^* \circ f = h$.

Proof. Suppose $h': C \to A^*$ is a subobject of A^* in $\mathscr{B}$, and $f': A \to C$ is a morphism (necessarily monic) such that $h' \circ f' = f$. We must show that there exists a unique morphism $p: A^* \to C$ such that $h' \circ p = A^*$. We have $h^* \circ h' \circ f' = h$ and $h^* \circ h'$ is a monomorphism in $\mathscr{B}$. Thus there exists a unique morphism (necessarily monic) $p: A^* \to C$, $p \in \mathscr{B}$ such that $h^* \circ h' \circ p = h^*$. But h^* is monic, hence $h' \circ p = A^*$. Now suppose $p': A^* \to C$, $p' \in \mathscr{B}$ also satisfies $h' \circ p = A^*$. Then $h^* \circ h' \circ p' = h^*$. But by uniqueness of p, it follows $p = p'$.

Lemma 4. Suppose $\mathscr{B}$ is a subcategory of $\mathscr{A}$ such that the following conditions are satisfied:

(i) $\mathscr{A}$ is locally small and $\mathscr{A}$ and $\mathscr{B}$ have both intersections

(ii) the inclusion functor $\mathscr{B} \to \mathscr{A}$ is a monofunctor which preserves intersections.

If $h: A \to B$ is a subobject of B, $B \in \mathrm{Obj}\ \mathscr{B}$, then there exists subobject of B in $\mathscr{B}$ which is $\mathscr{B}$-generated by h.

Proof. Consider a representative class (cf. [3], p. 7) $\{k_i: A_i \to B, i \in I\}$ of subobjects of B in $\mathscr{B}$ which is representative for the property that there exists for every $i \in I$ a monomorphism $h_i: A \to A_i$ such that $k_i \circ h_i = h$ for every $i \in I$. Notice that such a class exists because of (i). Also observe that because of (ii), the k_i are also subobjects of B in $\mathscr{B}$. Let $h^*: A^* \to B$ be the intersection in $\mathscr{B}$ (and thus also in $\mathscr{A}$ by (ii)) of the set $\{k_i: \mathrm{i} \in I\}$. h^* is a monomorphism in $\mathscr{B}$ (and thus in $\mathscr{A}$) and there exists for every $i \in I$ a unique monomorphism $u_i: A^* \to A_i$ in $\mathscr{B}$ such that $k_i \circ u_i = h^*$ for every $i \in I$. Since h^* is also the intersection in $\mathscr{A}$, and since $k_i \circ h_i = h$ for every $i \in I$, there exists a unique morphism $f: A \to A^*$, such that $h^* f = h$ (and $u_i \circ f = h_i$ for every $i \in I$). Observe that f is monic since h is monic. We claim that h^* is $\mathscr{B}$-generated by h.

Indeed, there exists a morphism $f : A \to A^*$ such that $h^* \circ f = h$. It remains to show the following. Suppose $h^{*\prime} : C \to B$ is a subobject of B in $\mathscr{B}$ such that there exists a morphism (necessarily monic) $f' : A \to C$ such that $h^{*\prime} \circ f' = h$. Then there exists a unique morphism $p : A^* \to C$, $p \in \mathscr{B}$ such that $h^{*\prime} \circ p = h^*$. Now, we have $h^{*\prime} \circ f' = h$. Hence we may assume without loss of generality, that $h^{*\prime}$ is a member of the class $\{k_i : i \in I\}$. Hence there exists a unique (recall that the u_i are unique!) morphism $p : A^* \to C$, $p \in \mathscr{B}$ (necessarily monic) such that $h^{*\prime} \circ p = h^*$. This completes the proof of the lemma.

Lemma 5. Suppose $\mathscr{B}$ is a subcategory of the category $\mathscr{A}$ such that the inclusion functor is a monofunctor. Let $h : A \to B$, $B \in \operatorname{Obj} \mathscr{B}$ be a subobject of B and suppose that B is $\mathscr{B}$-generated by h. Let $g : A \to C$ and $k : C \to B$, $C \in \operatorname{Obj} \mathscr{B}$, $k \in \mathscr{B}$ be monomorphisms such that $k \circ g = h$. Then k is an isomorphism.

Proof. B is $\mathscr{B}$-generated by h. Hence, there exists a monomorphism $p : B \to C$, $p \in B$, such that $k \circ p = B$ and $p \circ h = g$. It easily follows that $p \circ k = C$. Indeed, $k \circ p \circ k = k$. But k is monic, hence $p \circ k = C$. It follows that k is an isomorphism.

Lemma 6. Suppose $\mathscr{B}$ is a subcategory of the category $\mathscr{A}$ such that the following conditions are satisfied:

(i) $\mathscr{A}$ is balanced and has images

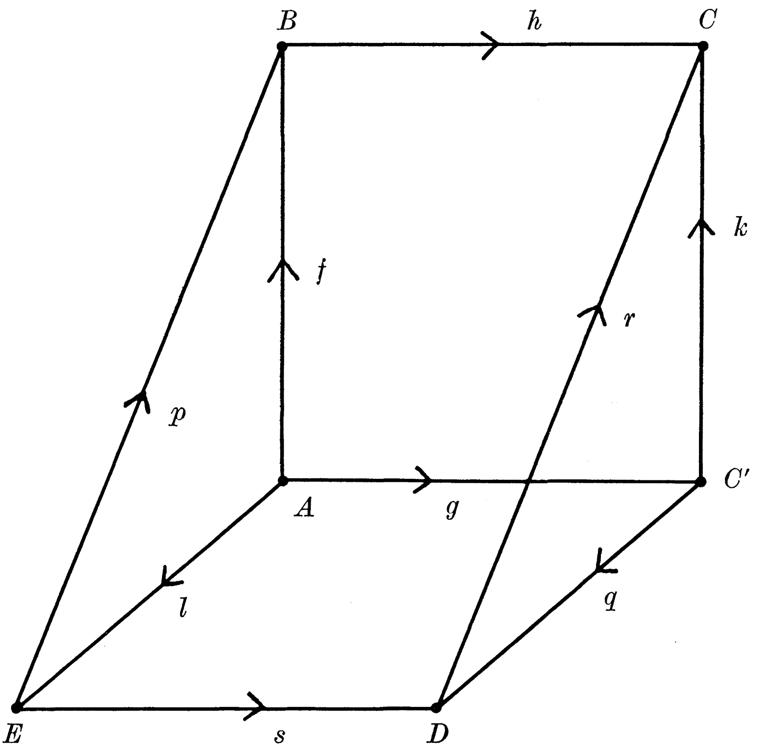
(ii) $\mathscr{B}$ has pullbacks

(iii) the inclusion functor $\mathscr{B} \to \mathscr{A}$ is a monofunctor and preserves pullbacks

Suppose $f : A \to B$, $B \in \operatorname{Obj} \mathscr{B}$ is a subobject of B such that B is $\mathscr{B}$-generated by f, and suppose $h : B \to C$ is a morphism in $\mathscr{B}$ such that h is epic (in $\mathscr{A}$!). Then C is $\mathscr{B}$-generated by $\operatorname{Im} h \circ f$.

Proof. Let $k = \operatorname{Im} h \circ f$, $k : C' \to C$. Thus there exists a morphism $g : A \to C'$ such that $k \circ g = h \circ f$. Now, suppose $r : D \to C$ is a subobject of C in $\mathscr{B}$ and suppose $q : C' \to D$ is a morphism (necessarily monic) such that $r \circ q = k$ (observe that r is monic by (iii)). We must show that that there is a unique morphism (necessarily monic) $r' : C \to D$, $r' \in \mathscr{B}$, such that $r \circ r' = C$. We will in fact show that r is an isomorphism. Let the commutative diagram $\langle p, s, r, h \rangle$ be the pullback of h and r in $\mathscr{B}$. By (i) this is also the pullback in $\mathscr{A}$. r is monic and it easily follows that p is monic.

Moreover $h \circ f = r \circ q \circ g$. Hence, there exists a unique morphism $l : A \to E$ such that $p \circ l = f$ and $s \circ l = q \circ g$. Since p is monic, it follows that l is monic. B is $\mathscr{B}$-generated by f. Hence by lemma 5, p is an isomorphism. We claim that r is epic. Indeed, suppose $u_1 \circ r = u_2 \circ r$, then $u_1 \circ h \circ p = u_1 \circ r \circ s = u_2 \circ r \circ s = u_2 \circ h \circ p$. But h is epic and p is an isomorphism, thus $u_1 = u_2$ and thus r is epic. But r is also monic, thus r is an isomorphism.



Thus if $r' = r^{-1}$, then $r \circ r' = C$. The uniqueness of r' follows easily.

Lemma 7. Suppose $\mathscr{B}$ is a subcategory of the category $\mathscr{A}$ such that

(i) the inclusion functor $\mathscr{B} \to \mathscr{A}$ is a monofunctor.

Suppose $h : A \to B$ is a subobject of B, $B \in \text{Obj}\, \mathscr{B}$ such that the following holds:

(ii) whenever $g : A \to C$, $C \in \text{Obj}\, \mathscr{B}$ is a morphism, then there exists a unique morphism $p : B \to C$, $p \in \mathscr{B}$ such that $p \circ h = g$.

Then B is $\mathscr{B}$-generated by h.

Proof. Suppose $k : C \to B$ is a subobject of B in $\mathscr{B}$, and $g : A \to C$ is a morphism such that $k \circ g = h$. By (ii) there exists a unique morphism $p : B \to C$, $p \in \mathscr{B}$, such that $p \circ h = g$. Now $k \circ p \circ h = k \circ g = h$. Also $B \circ h = h$. Hence, by uniqueness $k \circ p = B$. Now suppose $p' : B \to C$. $p' \in \mathscr{B}$ such that $k \circ p' = B = k \circ p$. But k is monic, hence $p = p'$. This shows that B is $\mathscr{B}$-generated by h.

Lemma 8. Suppose $\mathscr{B}$ is a subcategory of the category $\mathscr{A}$ such that the following conditions are satisfied:

(i) $\mathscr{B}$ has pullbacks and epimorphic images

(ii) the inclusion functor $\mathscr{B} \to \mathscr{A}$ is a monofunctor which preserves pullbacks

Suppose $h : A \to A^*$ is a subobject of A^*, $A \in \text{Obj}\, \mathscr{B}$ such that A^* is $\mathscr{B}$-generated by h. Then, whenever $f_1 \circ h = f_2 \circ h$, $f_1, f_2 \in \mathscr{B}$, it follows that $f_1 = f_2$.

Proof. Let $k : C \to A^*$, $k \in \mathscr{B}$ be such that $\langle k, f_1, f_2 \rangle$ is the pullback of f_1 and f_2 in $\mathscr{B}$. Thus $f_1 \circ k = f_2 \circ k$. Also $f_1 \circ h = f_2 \circ h$. By (ii) we have that $\langle k, f_1, f_2 \rangle$ is also the pullback in $\mathscr{A}$. Hence, there exists a unique morphism $l : A \to C$ such that $k \circ l = h$. By (i) $\mathscr{B}$ has epimorphic images. Let $p_1 = \text{Im}\, k$ in $\mathscr{B}$, $p_1 : D \to A^*$. Thus there exists a morphism $p_2 : C \to D$, $p_2 \in \mathscr{B}$, such that p_2 is epic in $\mathscr{B}$ and such that $p_1 \circ p_2 = k$. Observe that $p_1 \circ p_2 \circ l = h$. But h is monic, hence $p_2 \circ l$ is monic. Also observe that p_1 is monic in $\mathscr{B}$. It follows from lemma 5 that p_1 is an isomorphism. Now $f_1 \circ p_1 \circ p_2 = f_1 \circ k = f_2 \circ k = f_2 \circ p_1 \circ p_2$. But p_2 is epic in $\mathscr{B}$ and $f_1 \circ p_1$ and $f_2 \circ p_1 \in \mathscr{B}$. Hence $f_1 \circ p_1 = f_2 \circ p_1$. But p_1 is an isomorphism. Hence $f_1 = f_2$.

4. The machinery developed in the previous sections will now enable us to prove the theorems 1 and 2 which were stated in section 1.

Proof of theorem 1. Suppose $f : A \to B$ is an epimorphism. We must show that $R(f)$ is epic in $\mathscr{B}$. It follows from (iv) that $\eta(B)$ is monic. It follows from (i) that $\eta(B) = \text{Im}\, \eta(B) \circ f$ (cf. Prop. 10.2, p. 12, [3]). Let $p = \text{Im}\, R(f)$ in $\mathscr{B}$. There exists a morphism $v : R(A) \to C$, $v \in \mathscr{B}$, such that $p \circ v = R(f)$ and by (ii) v is epic in $\mathscr{B}$. We have $p \circ v \circ \eta(A) = R(f) \circ \eta(A) = \eta(B) \circ f$. p is monic in $\mathscr{B}$, thus by (iii) p is monic in $\mathscr{A}$. Hence, there exists a unique morphism $u : B \to C$ such that $p \circ u = \eta(B)$. R is a reflector and $\eta(B)$ is monic, thus it follows from lemma 7 that $R(B)$ is $\mathscr{B}$-generated by $\eta(B)$. u is monic and p is monic in $\mathscr{B}$, hence by lemma 5, p is an isomorphism. Again, since $p \circ v = R(f)$ and since v is epic in $\mathscr{B}$, it follows that $R(f)$ is epic in $\mathscr{B}$. This completes the proof of theorem 1.

Remarks.

1. Observe that we did not require in theorem 1 that $\mathscr{A}$ and $\mathscr{B}$ have the same images.
2. The conditions that $\mathscr{A}$ and $\mathscr{B}$ have images can be replaced by the stronger condition that $\mathscr{A}$ and $\mathscr{B}$ have inverse limits (of intersections).
3. The only lemmas that we used in the proof, are lemmas 5 and 7.

Proof of theorem 2. Let $f : A \to B$ be a monomorphism. We must show that $R(f)$ is monic in $\mathscr{B}$. By (iv) $\eta(B)$ is monic, thus by (i), (ii) and lemma 4, there exists a subobject of $R(B)$ in $\mathscr{B}$ which is generated by $\eta(B) \circ f$. Thus there exists a subobject $k : A^* \to R(B)$ of $R(B)$ in $\mathscr{B}$ and a monomorphism $j : A \to A^*$ such that

(1) $$k \circ j = \eta(B) \circ f$$

and such that k is $\mathscr{B}$-generated by $\eta(B) \circ f$. Notice that by (ii) k is also monic in $\mathscr{A}$. We will show that $A^* = R(A)$ and that $j = \eta(A)$. It then follows from the properties of a reflection that $k = R(f)$ and hence that $R(f)$ is monic. Suppose $h : A \to C$, $C \in \text{Obj}\,\mathscr{B}$. We must show that there exists a unique morphism $x : A^* \to C$, $x \in \mathscr{B}$ such that

(2) $$x \circ j = h.$$

By (iii) there exists a monomorphism $l : C \to C'$, $l \in \mathscr{B}$, C' injective. Hence there exists a morphism $h_1 : B \to C'$ such that

(3) $$h_1 \circ f = l \circ h.$$

Since R is a reflector, there exists a unique morphism $h_2 : R(B) \to C'$, $h_2 \in \mathscr{B}$, such that

(4) $$h_2 \circ (B) = h_1.$$

Now let

(5) $$h^* = h_2 \circ k, \ h^* \in \mathscr{B}.$$

Let $q = \text{Im}\,h^*$ in $\mathscr{B}$, $q : D \to C'$. q exists by (i). Thus there exists a unique morphism $p : A^* \to D$ such that

(6) $$q \circ p = h^*.$$

Observe that q is monic in $\mathscr{B}$ and thus monic in $\mathscr{A}$. In addition, p is epic in $\mathscr{B}$ by (i). But by virtue of (i), $\mathscr{A}$ also has epimorphic images and it follows from (ii) that q is also $\text{Im}\,h^*$ in $\mathscr{A}$. Furthermore, it follows from the uniqueness of p and again from (ii) that p is also epic in $\mathscr{A}$. Let $s : E \to C'$ be the intersection in $\mathscr{B}$ of l and q. This intersection exists by (i). Hence there exist unique morphisms $r : E \to C$, $t : E \to D$, $r, t \in \mathscr{B}$ such that

(7) $$l \circ r = q \circ t = s$$

r, s and t are monomorphisms in $\mathscr{B}$ and thus in $\mathscr{A}$. Notice that by (ii) s is also the intersection of l and q in $\mathscr{A}$ (with the same corresponding monomorphisms r and t). Let $u : A' \to D$ be the image of $p \circ j$, u exists by (i). Thus there exists a morphism $m : A \to A'$ such that

(8) $$u \circ m = p \circ j$$

u is monic and m is epic by (i). By lemma 3, A^* is $\mathscr{B}$-generated by j.

Now p is epic, hence it follows from (i), (ii) and from lemma 6 that D is $\mathscr{B}$-generated by u. Now $u = \mathrm{Im}\ p \circ j$, q is monic. Thus by (i) and by lemma 1, $q \circ u = \mathrm{Im}\ q \circ p \circ j$ and thus by (6)

$$q \circ u = \mathrm{Im}\ h^* \circ j. \tag{9}$$

Now it follows from (1), (3) and (4) that $l \circ h = h^* \circ j$. Recall that l is monic. Thus by (9) there exists a unique monomorphism $v : A' \to C$ such that

$$l \circ v \circ m = h^* \circ j \text{ and } v \circ m = h. \tag{10}$$

Again it follows from (6), (8) and (10) that $l \circ v \circ m = q \circ u \circ m$. But m is epic, hence

$$l \circ v = q \circ u. \tag{11}$$

s is the intersection of l and q in $\mathscr{B}$ and thus by (ii) also the intersection in $\mathscr{A}$. It follows from (11) that there exists a unique morphism $w : A' \to E$ such that

$$s \circ w = l \circ v = q \circ u,\ r \circ w = v,\ t \circ w = u \tag{12}$$

D is $\mathscr{B}$-generated by u, t is monic in $\mathscr{B}$. v is monic and by (12) we have $r \circ w = v$ and thus w is monic. It follows from (12) and from lemma 5 that t is an isomorphism. Let $x : A^* \to C$ be defined by $x = r \circ t^{-1} \circ p$. Notice that $x \in \mathscr{B}$. It easily follows from (8), (10) and (12) that $x \circ j = h$. Finally, we must show that x is unique. Suppose $x' : A^* \to C$, $x' \in \mathscr{B}$, such that $x' \circ j = h$. Recall that A^* is $\mathscr{B}$-generated by j. Now $x \circ j = x' \circ j\ (= h)$. It follows from lemma 8 that $x = x'$. This complete the proof of theorem 2.

Remark. It is not difficult to show that the example described in section 1 satisfies the conditions of theorem 2 and hence, of theorem 1.

University of Illinois, Chicago

References

1. Herrlich, Horst, On the concept of reflections in general topology, to appear in "Symp. on Extension Theory of Topological Structures and its Applications", Berlin 1967.
2. Freyd, Peter, An introduction to the theory of functors, Harper and Row, New York, Evanston and London (1964).
3. Mitchell, Barry, Theory of categories, Academic Press, New York and London (1965).
4. Sikorski, R., Boolean algebras, Second edition, Springer Verlag, Berlin-Göttingen-Heidelberg-New York (1964).
5. Yaqub, F. M., Free extensions of Boolean algebras, Pac. Journ. of Math., 13, no. 2, 761–771 (1963).