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An Algorithm to Initialize the Search of Solutions of Polynomial Systems

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Abstract—One of the main problems dealing with iterative methods for solving polynomial systems is the initialization of the iteration. This paper provides an algorithm to initialize the search of solutions of polynomial systems.

Keywords—Polynomial systems, Zeros, Algorithm.

1. INTRODUCTION

Nonlinear systems, and in particular polynomial systems, arise, either directly or as a part of computing tasks, in many important mathematical areas, such as finite element methods, optimization, with or without constraints or nonlinear least square problems [1,2]. On the other hand they also appear in a large number of fields of science such as physics, chemistry, biology, geophysics, engineering, and industry. See [3]. In all these contexts most of the practical methods for solving them are iterative. In [4] the reader can see other no iterative methods for solving polynomial systems. Given an initial approximation, x_0, a sequence of iterates x_k, $k = 1, 2, \ldots$ is generated in such a way that, hopefully, the approximation to some solution is progressively improved. The convergence is not guaranteed in the general case and no global procedures are provided in order to find such a convenient approximation, x_0. In [5] and [6] the reader can find the motivation and theoretical bases, and in [7–9] complete and recent surveys of such algorithms can be consulted.

It is in the search for the above-mentioned approximations, x_0, where this paper might contribute to improving such algorithms, by giving a general method, still in its early steps, that lets us locate zeros inside *p-cubes* in $\mathcal{R}^p$, small enough to guarantee the convergence.

Throughout this paper we consider polynomial systems of equations, written in the form

$$F(x_1, \ldots, x_p) = (f_1(x_1, \ldots, x_p), \ldots, f_p(x_1, \ldots, x_p)) = (0, \ldots, 0). \tag{1}$$

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Given $x = (x_1, \ldots, x_p) \in \mathcal{R}^p$, then we set the following notation:

$$\begin{aligned} x_{-i} &= (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_p) \in \mathcal{R}^{p-1}, \\ x_{-i-j} &= (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_p) \in \mathcal{R}^{p-2}, \qquad \text{with } i < j, \\ \mathcal{R}_j^{p-1} &= \{x_{-j};\, x \in \mathcal{R}^p\}, \\ \mathcal{R}_{ij}^{p-2} &= \{x_{-i-j};\, x \in \mathcal{R}^p\}, \\ &\pi_j : \mathcal{R}^p \to \mathcal{R}_j^{p-1}; \qquad \pi_j(x) = x_{-j}, \\ &\pi_{ij} : \mathcal{R}^p \to \mathcal{R}_{ij}^{p-2}; \qquad \pi_{ij}(x) = x_{-i-j}. \end{aligned} \tag{2}$$

This article is organized as follows: in Section 2 a matrix model is introduced to establish a suitable order to solve the unknowns from the equations of the system. This order will be crucial in the following. Section 3 treats the necessary conditions for the existence of zeros in rectangles of $\mathcal{R}^p$. Section 4 deals with sufficient conditions, and the main result, Theorem 3, is introduced. In Section 5, we build a lower and upper bound of a kind of functions, defined in the next pages, that we will only need for practical calculations. Finally, Section 6 provides a provisional structure of the algorithm. An example is included to illustrate the main ideas.

Throughout this paper the empty set will be denoted by $\emptyset$.

2. A MATRIX MODEL OF THE PROBLEM

Let us start this section with an example of polynomial systems, given by

$$\begin{aligned} f_1(x, y, z) &= 6y^2 + 20y + 2x + 44z - 170 = 0, \\ f_2(x, y, z) &= 3y^3 - 43y - 7x - 6z + 100 = 0, \\ f_3(x, y, z) &= z^3 - 79z + 6x^2 - 10y + 4 = 0. \end{aligned} \tag{3}$$

The algorithm begins by setting up a suitable order to solve one different unknown from each equation of the system, in such a manner that the solved unknown from the first equation, say x, it also appears in the second one; the solved unknown from the second equation, for instance, y, different to the unknown x, also appears in the third one; and the solved unknown in the third equation, z, different to the unknowns x and y, appears in the first one again, closing a loop. In this section we show that a such choice can be done in the general case. To carry out this task a matrix model is developed.

DEFINITION 1.

1. *Let M_p be the set of matrices in $\mathcal{R}^{p \times p}$, with $p \geq 2$, defined by*

$$M_p = \{\mathcal{A};\ \mathcal{A} = \{a_{ij}\}_{1 \leq i,j \leq p};\ a_{ij} = 1 \text{ or } a_{ij} = 0\}. \tag{4}$$

2. *In M_p the relation "$\sqsubset$" is defined as*

$$(\mathcal{A} \sqsubset \mathcal{A}') \iff \left(\text{if } a_{ij} = 1,\ \text{then } a'_{ij} = 1\right). \tag{5}$$

3. *A matrix, $\mathcal{A}$, is said to be an α-matrix, if $\mathcal{A} \in M_p$ and all subsets of $k < p$ columns need, at least, $k + 1$ rows to cover all its ones.*
4. *Let M be an α-matrix, then M is said to be a minimum α-matrix, from now on ($M\alpha$)-matrix, if $\exists\, B \sqsubset M$ so that B is an α-matrix, then $B = M$.*

The following proposition will be used below.

PROPOSITION 1. CANONICAL FORM OF THE $(M\alpha)$-MATRIX $\mathcal{M}$. *If $\mathcal{M} \in M_p$ is an $(M\alpha)$-matrix, then by exchanging between themselves rows or columns (if required), it is always possible to obtain a matrix in the form*

$$\mathcal{M}_c = \begin{pmatrix} 1 & * & * & \cdots & * & 1 \\ 1 & * & * & \cdots & * & 0 \\ 0 & 1 & * & \cdots & * & * \\ 0 & 0 & 1 & \cdots & * & * \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \tag{6}$$

where the elements denoted by "$*$" are either zero or one. The matrix M_c is said to be the canonical form of the matrix $\mathcal{M}$.

Proposition 1 can be easily proven from the following lemmas.

LEMMA 1. *If $\mathcal{A} \in M_p$ is an α-matrix, then all its rows and columns have, at least, two entries equal to 1.*

LEMMA 2. *$\mathcal{A} \in M_p$ is an α-matrix if, and only if, there is no submatrix in $\mathcal{R}^{i\times k}$ of zeros, with $i + k \geq p$.*

LEMMA 3. *Let $\mathcal{M} \in M_p$, $p > 2$, be an α-matrix, then it is an $(M\alpha)$-matrix if, and only if, for each element, $m_{ij} = 1$, there is, at least, a submatrix in $\mathcal{R}^{r\times s}$, C, with $r + s = p$, with $m_{ij} = 1$, and all its remaining entries being equal to zero.*

LEMMA 4. *If $\mathcal{M} \in M_p$, $p > 2$, is an $(M\alpha)$-matrix then there are no submatrices in $\mathcal{R}^{2\times 2}$, so that all their entries are equal to 1.*

LEMMA 5. *Let $\mathcal{M} \in M_p$ be an $(M\alpha)$-matrix. Consider that there is a submatrix, $D \in R^{(k+1)\times k}$, $2 \leq k \leq p - 1$, with the following structure:*

$$\mathcal{D} = \begin{pmatrix} 1 & * & * & \cdots & * \\ 1 & * & * & \cdots & * \\ * & 1 & * & \cdots & * \\ * & * & 1 & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ * & * & * & * & 1 \end{pmatrix}, \tag{7}$$

where the elements denoted by "$*$" are either 0 or 1. Then, by exchanging either columns or rows between themselves, if needed, it is always possible to obtain another submatrix, D_1, with the same structure as $\mathcal{D}$, so that all the elements of D_1, m_{ij}, with $i > j + 1$ are equal to zero.

DEFINITION 2. *Given system (1), then the matrix $\mathcal{A} = \{a_{ij}\}_{1\leq i,j\leq p} \in M_p$ defined as $a_{ij} = 1$, if the unknown x_i belongs to the equation f_j, and $a_{ij} = 0$ if it does not, is said to be the unknown distribution matrix of system (1), from now on, (UD)-matrix.*

REMARK 1. Notice that all systems of p equations and p unknowns either are in the form $f_1(x_1, 0, \dots, 0) = 0, \dots, f_p(0, \dots, 0, x_p) = 0$ or its (UD)-*matrix* is an α-matrix or its (UD)-*matrix* can be decomposed into several submatrices to be α-*matrices*.

THEOREM 1. *Suppose that the (UD)-matrix of system (1) is an α-matrix, then it is always possible to build a sequence with all the equations and unknowns of (1), $f_{k_1}, f_{k_2}, \dots, f_{k_p}$ and $x_{k_1}, x_{k_2}, \dots, x_{k_p}$ (the subscripts are a permutation of $\{1, 2, \dots, p\}$), in such a way that*

1. *the unknown x_{k_1} appears in the equations f_{k_p} and f_{k_1};*
2. *each unknown x_{k_j}, $2 \leq j \leq p$, appears in the equations $f_{k_{j-1}}$ and f_{k_m}, with $m > j - 1$.* (8)

With the aim of simplifying the expression of the subscripts, hereafter sequences (8) will be denoted, without loss of generality, as $f_1, f_2, \dots, f_p$ and $x_1, x_2, \dots, x_p$, respectively, in such a way that x_1 is in f_p and f_1, x_2 is in f_1 and f_2, x_3 is in f_2 and f_3, ..., x_p is in f_{p-1} and f_p.

PROOF. As (1) is an α-system, then we follow the steps.

1. Compute the (UD)-matrix of (1), $\mathcal{A}$.
2. From $\mathcal{A}$ compute a $(M\alpha)$-matrix: $\mathcal{M} \sqsubset \mathcal{A}$.
3. From $\mathcal{M}$, compute its canonical form: $\mathcal{M}_c$.
4. The new order of the rows of $\mathcal{M}_c$ defines the sequence of unknowns: $x_{k_1}, \dots, x_{k_p}$, and the new order of the columns of $\mathcal{M}_c$ defines the sequence of equations: $f_{k_1}, \dots, f_{k_p}$, that satisfy the required hypothesis due to the structure of $\mathcal{M}_c$.

EXAMPLE 1. Coming back to system (3), the (UD)-matrix is as follows.

	f_1	f_2	f_3
z	1	1	1
x	1	1	1
y	1	1	1

(9)

From (9) we get an $(M\alpha)$-matrix

	f_1	f_2	f_3
z	1	0	1
x	1	1	0
y	0	1	1

(10)

that is its own M_c matrix. This leads us to the sequence of equations and unknowns

$$f_1,\ f_2,\ f_3\ z,\ x,\ y \begin{cases} z \text{ appears in } f_3 \text{ and } f_1, \\ x \text{ appears in } f_1 \text{ and } f_2, \\ y \text{ appears in } f_2 \text{ and } f_3, \\ \text{and the loop is closed.} \end{cases} \tag{11}$$

3. NECESSARY CONDITIONS FOR THE EXISTENCE OF ZEROS

For the sake of clarity in the presentation we recall some results of [10].

FIRST. Let $P(x)$ be the polynomial function

$$y = P(x) = a_0 + a_1 x + \cdots + a_m x^m, \tag{12}$$

where $a_0, a_1, \dots, a_m$ are real numbers with $a_1, a_m \neq 0$, then the series

$$f_P(y) = \frac{a_0 - y}{-a_1} \sum_{n=0}^{\infty} \sum_{q_1 + \cdots + q_{m-1} = n} d(q_1, \dots, q_{m-1}) \left(\frac{(a_0 - y) a_2}{(-a_1)^2} \right)^{q_1} \cdots \left(\frac{(a_0 - y)^{m-1} a_m}{(-a_1)^m} \right)^{q_{m-1}}, \tag{13}$$

with

$$d(q_1, \dots, q_{m-1}) = \frac{(2q_1 + \cdots + m q_{m-1})!}{q_1! \cdots q_{m-1}! (1 + q_1 + \cdots + (m-1) q_{m-1})!}, \tag{14}$$

is the inverse function of (12) in the region

$$D_{f_P} = \left\{ y \in \mathcal{R};\ \frac{|a_0 - y||a_2|}{|a_1|^2} + \cdots + \frac{|a_0 - y|^{m-1}|a_m|}{|a_1|^m} \leq \frac{(m-1)^{m-1}}{m^m} \right\}. \tag{15}$$

SECOND. If $0 \in D_{f_P}$, then $f_P(0)$ is the root of $P(x)$ closest to the origin.

THIRD.

$$\begin{aligned} &d(q_1,\dots,q_{m-1})q1!\cdots q_{m-1}! \le d(0,\dots,0,n)n!; \\ &\qquad d(0,\dots,0,n) \le \frac{1}{n}\left(\frac{m^m}{(m-1)^{m-1}}\right)^n, \qquad \forall n \ge 1. \end{aligned} \tag{16}$$

Before starting polynomial systems, we provide a new result of polynomial equations that will be needed in the following.

PROPOSITION 2. *The function f_P defined by (13) satisfies the inequality*

$$|f_P(y)| \le \left|\frac{a_0 - y}{-a_1}\right| \frac{m}{m-1}, \qquad \forall y \in D_{f_P}. \tag{17}$$

PROOF. From (16) we arrive at

$$\begin{aligned} |f_P(y)| &\le \left|\frac{a_0-y}{-a_1}\right| \sum_{n=0}^{\infty} \sum_{q_1+\cdots+q_{m-1}=n} d(q_1,\dots,q_{m-1}) \left|\frac{(a_0-y)a_2}{(-a_1)^2}\right|^{q_1} \\ &\quad \cdots \left|\frac{(a_0-y)^{m-1}a_m}{(-a_1)^m}\right|^{q_{m-1}} \\ &\le \left|\frac{a_0-y}{-a_1}\right| \sum_{n=0}^{\infty} d(0,\dots,n) \left(\left|\frac{(a_0-y)a_2}{(-a_1)^2}\right| + \cdots + \left|\frac{(a_0-y)^{m-1}a_m}{(-a_1)^m}\right|\right)^n \\ &\le \left|\frac{a_0-y}{-a_1}\right| \sum_{n=0}^{\infty} d(0,\dots,n) \left(\frac{(m-1)^{m-1}}{m^m}\right)^n. \end{aligned} \tag{18}$$

Taking into account that 1 is the closest root to the origin of $x^m - mx + (m-1) = 0$, then

$$\frac{m-1}{m} \sum_{n=0}^{\infty} d(0,\dots,n) \left(\frac{(m-1)^{m-1}}{m^m}\right)^n = 1, \tag{19}$$

and the result follows.

Next, we apply the above results to the case of polynomial system.

DEFINITION 3. *With the notation (8) let us assume that system (1) satisfies the conditions*

1. *The (UD)-matrix of (1), $\mathcal{A}$, is an α-matrix.*
2. *They verify the following properties in Ω:*

$$\frac{\partial f_p}{\partial x_1}, \frac{\partial f_1}{\partial x_1} \ne 0; \quad \frac{\partial f_1}{\partial x_2}, \frac{\partial f_2}{\partial x_2} \ne 0; \quad \dots; \quad \frac{\partial f_{p-1}}{\partial x_p}, \frac{\partial f_p}{\partial x_p} \ne 0. \tag{20}$$

3. *The determinants*

$$\begin{vmatrix} \dfrac{\partial f_p}{\partial x_p} & \dfrac{\partial f_1}{\partial x_p} \\ \dfrac{\partial f_p}{\partial x_1} & \dfrac{\partial f_1}{\partial x_1} \end{vmatrix}, \quad \begin{vmatrix} \dfrac{\partial f_{j-1}}{\partial x_{j-1}} & \dfrac{\partial f_j}{\partial x_{j-1}} \\ \dfrac{\partial f_{j-1}}{\partial x_j} & \dfrac{\partial f_j}{\partial x_j} \end{vmatrix}, \qquad 2 \le j \le p, \tag{21}$$

are nonzero in Ω. Then system (1) is said to be an α-system in Ω.

EXAMPLE 2. Consider polynomial system (3) and sequences (11) then

$$\begin{aligned} &\frac{\partial f_3}{\partial z} = 3z^2 - 79; \qquad \frac{\partial f_1}{\partial z} = 44; \qquad \frac{\partial f_1}{\partial x} = 2; \\ &\frac{\partial f_2}{\partial x} = -7; \qquad \frac{\partial f_2}{\partial y} = 9y^2 - 43; \qquad \frac{\partial f_3}{\partial y} = -10; \end{aligned} \tag{22}$$

$$\begin{vmatrix} \frac{\partial f_3}{\partial y} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_3}{\partial z} & \frac{\partial f_1}{\partial z} \end{vmatrix} = -440 - (20+12y)\left(-79+3z^2\right); \tag{23}$$

$$\begin{vmatrix} \frac{\partial f_1}{\partial z} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial x} \end{vmatrix} = -296; \tag{24}$$

$$\begin{vmatrix} \frac{\partial f_2}{\partial x} & \frac{\partial f_3}{\partial x} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_3}{\partial y} \end{vmatrix} = 20 + 12x(20+12y); \tag{25}$$

and, finally, define Ω_1 as the open subset of $\mathcal{R}^3$, so that (22)–(25) are different from zero, then we can say that system (3) is an α-system with sequence (11) in Ω_1.

Let (1) be an α-system in Ω, $\mathcal{A}$, its (UD)-matrix, $\mathcal{M}$, an $(M\alpha)$-matrix so that $\mathcal{M} \sqsubset \mathcal{A}$, $\mathcal{M}_c$, the canonical form of $\mathcal{M}$, and finally the set of subscripts $S = \{(i,j);\ \text{entries of } \mathcal{M}_c \text{ equal to } 1\}$. Consider $r \in \mathcal{R}^p \cap \Omega$, then for each $(i,j) \in S$, we can make the change of variables in the equation f_j

$$x = y + r \qquad \text{and} \qquad g_j(y) = f_j(y+r). \tag{26}$$

Arranging the terms of $f_j(x)$ according to the powers of x_i, one gets

$$\begin{aligned} f_j(x) &= f_j(y+r) = g_j(y) \\ &= s_{m,j}(y_{-i}+r_{-i})(y_i+r_i)^m + s_{m-1,j}(y_{-i}+r_{-i})(y_i+r_i)^{m-1} + \cdots \\ &\quad + s_{1,j}(y_{-i}+r_{-i})(y_1+r_1) + s_{0,j}(y_{-i}+r_{-i}) = 0. \end{aligned} \tag{27}$$

Therefore,

$$g_j(y) = t_{m,j}(y_{-i})y_i^m + t_{m-1,j}(y_{-i})y_i^{m-1} + \cdots + t_{1,j}(y_{-i})y_i + t_{0,j}(y_{-i}) = 0. \tag{28}$$

Then, for the equation f_j, we define the auxiliary functions K_j and $Y_{1,j}, Y_{2,j}, \dots, Y_{m-1,j}$ as

$$\begin{aligned} &K_j(y_{-i}) = \frac{t_{0,j}(y_{-i})}{-t_{1,j}(y_{-i})}, \quad Y_{1,j}(y_{-i}) = \frac{t_{0,j}(y_{-i})t_{2,j}(y_{-i})}{(-t_{1,j}(y_{-i}))^2}, \quad Y_{2,j}(y_{-i}) = \frac{t_{0,j}^2(y_{-i})t_{3,j}(y_{-i})}{(-t_{1,j}(y_{-i}))^3}, \\ &\qquad \vdots \\ &Y_{m-2,j}(y_{-i}) = \frac{t_{0,j}^{m-2}(y_{-i})t_{m-1,j}(y_{-i})}{(-t_{1,j}(y_{-i}))^{m-1}}, \quad Y_{m-1,j}(y_{-i}) = \frac{t_{0,j}^{m-1}(y_{-i})t_{m,j}(y_{-i})}{(-t_{1,j}(y_{-i}))^m}. \end{aligned} \tag{29}$$

PROPOSITION 3. *Let (1) be an α-system in Ω and $r = (r_1, \dots, r_p) \in \mathcal{R}^p \cap \Omega$. For all $(i,j) \in S$, the functions*

$$\begin{aligned} x_i = \varphi_j^i(x_{-i}) &= r_i \\ +K_j(x_{-i}-r_{-i}) &\sum_{n=0}^{\infty} \sum_{q_1+\cdots+q_{m-1}=n} d(q_1,\dots,q_{m-1}) Y_{1,j}^{q_1}(x_{-i}-r_{-i}), \dots, Y_{m-1,j}^{q_{m-1}}(x_{-i}-r_{-i}), \end{aligned} \tag{30}$$

are well defined in

$$D_{ij} = \left\{ x_{-i} \in \mathcal{R}_{-j}^p;\ |Y_{1,j}(x_{-i}-r_{-i})| + \cdots + |Y_{m-1,j}(x_{-i}-r_{-i})| \le \frac{(m-1)^{m-1}}{m^m} \right\} \cap \Omega, \tag{31}$$

and they satisfy $f_j(x_1, \dots, x_{i-1}, \varphi_j^i(x_{-i}), x_{i+1}, \dots, x_p) = 0$.

PROOF. For each $(i,j) \in S$, functions (29) are well defined, since from (20) and (27) one gets

$$0 \neq \left.\frac{\partial f_j}{\partial x_i}\right|_{x=r} = \frac{\partial g_j}{\partial x_i}\frac{\partial x_i}{\partial y_i} = \left.\frac{\partial g_j}{\partial y_i}\right|_{y=0} = t_{1,j}(y_{-i}).$$

Observe that (28) can be considered as a polynomial in the variable y_i. So, using (13) one gets

$$y_i = K_j(y_{-i}) \sum_{n=0}^{\infty} \sum_{q_1+\cdots+q_{m-1}=n} d(q_1, \ldots, q_{m-1}) Y_{1,j}^{q_1}(y_{-i}), \ldots, Y_{m-1,j}^{q_{m-1}}(y_{-i}), \tag{32}$$

that is convergent in the region D_{ij} (undoing the change of variables (26)) in agreement with (15).

DEFINITION 4. *Let (1) be an α-system in Ω, $r \in \mathcal{R}^p \cap \Omega$ and let G_r be the set of functions given by*

$$G_r = \left\{\varphi_j^i(x_{-i});\ \text{with } f_j(x_1, \ldots, x_{i-1}, \varphi_j^i(x_{-i}), x_{i+1}, \ldots, x_p) = 0;\ (i,j) \in S\right\}, \tag{33}$$

with φ_j^i defined in (30). Then G_r is said to be a complete set of explicit functions of system (1), around the point r, from now on (CSEF)$_r$ of system (1).

THEOREM 2. *Let (1) be an α-system in Ω. Let $a \in \Omega$ be a root of (1). Then there exists a closed p-cube, $K = I_1 \times \cdots \times I_p \subset \Omega$, $a \in K$, where all the functions of the set G_a are well defined.*

PROOF. If $x = a$, then

$$f_j(a_1, a_2, \ldots, a_p) = 0 = g_j(0, \ldots, 0) = t_{0,j}(0, \ldots, 0), \qquad 1 \le j \le p,$$

and (31) holds. By continuity there is a neighbourhood of a, $\mathcal{U}^{ij}$, where (31) is verified. The result follows, by considering the intersection $V = \cap\, \mathcal{U}^{ij}$, with $(i,j) \in S$, since we can take $K \subset V$ and then $\varphi_j^i \in G_a$ is well defined in $I_1 \times \cdots \times I_{i-1} \times I_{i+1} \times I_p$, $1 \le i \le p$, respectively.

EXAMPLE 3. Consider again (3); in agreement with this theorem the algorithm must search regions, K, in Ω_1, where the functions φ_j^i are convergent. To accomplish this aim one begins by looking for a point $r \in \Omega_1$, in such a way that the series function $\varphi_3^1 \in G_r$, corresponding to the first row of the matrix M_c, is convergent at the point r_{-1}. In this example we can take $r = 0$, since

(1) $0 \in \Omega_1$,
(2) by using (32), it is obtained the series function

$$z = \varphi_3^1(x,y) = \frac{4 + 6x^2 - 10y}{79} \sum_{n=0}^{\infty} d(0,n) Z^n(x,y) \tag{34}$$

where, according to (14),

$$d(0,n) = \frac{1}{2n+1}\binom{3n}{n} \qquad \text{and} \qquad Z(x,y) = \frac{\left(4 + 6x^2 - 10y\right)^2}{493039},$$

that is uniformly convergent in the region

$$D_{13} = \left\{(x,y);\ -270.2 \le 4 + 6x^2 - 10y \le 270.2\right\}, \tag{35}$$

and that, obviously, contains the point $(x,y) = (0,0)$.

Once this is done, the remaining functions of the set G_0 are considered, starting with φ_1^1, also corresponding to the first row of the matrix M_c.

$$z = \varphi_1^1(x,y) = \frac{85 - x - 10y - 3y^2}{22}, \tag{36}$$

defined in

$$D_{11} = \left\{(x,y), (x,y) \in \mathcal{R}^2\right\}. \tag{37}$$

On the other hand, taking into account (17),

$$|z| = |\varphi_1^3|(x,y) \le \left|\frac{4+6x^2-10y}{79}\right| \frac{3}{2} \le 5.13, \qquad \text{with } (x,y) \in D_{13}. \tag{38}$$

Therefore the ranges of the functions φ_3^1 and φ_1^1, defined in D_{13} and D_{11}, are contained in the intervals,

$$I_{13} = \{-5.13 \le z \le 5.13\}, \qquad I_{11} = \mathcal{R}, \tag{39}$$

respectively. Compute now

$$R_1 = D_{11} \cap D_{13}, \qquad I_1 = I_{11} \cap I_{13}. \tag{40}$$

Then we conclude that inside the region

$$A_1 = R_1 \times I_1 \neq \emptyset \tag{41}$$

the graphs of the functions φ_3^1 and φ_1^1 are well defined. We continue with the functions $\varphi_1^2 \in G_0$ and $\varphi_2^2 \in G_0$ corresponding to the second row of the matrix M_c,

$$\begin{aligned} x &= \varphi_1^2(y,z) = 85 - 10y - 3y^2 - 22z, \\ x &= \varphi_2^2(y,z) = \frac{100 - 43y + 3y^3 - 6z}{7}. \end{aligned} \tag{42}$$

In an equivalent way the regions

$$\begin{aligned} &D_{21} = D_{22} = \left\{(y,z);\ (y,z) \in \mathcal{R}^2\right\}, \\ &I_{21} = I_{22} = \mathcal{R}, \\ &R_2 = D_{21} \cap D_{22}, \qquad I_2 = I_{21} \cap I_{22}, \qquad \text{and} \\ &A_2 = (R_1 \times I_1) \cap (R_2 \times I_2) \neq \emptyset \end{aligned} \tag{43}$$

are computed. Then we deduce that the graphs of the functions φ_3^1, φ_1^1, φ_1^2, and φ_2^2 are well defined inside A_2. Finally the functions $\varphi_2^3 \in G_0$ and $\varphi_3^3 \in G_0$, corresponding to the third row of the matrix M_c, are considered

$$\begin{aligned} y &= \varphi_2^3(x,z) = \frac{100 - 7x - 6z}{43} \sum_{n=0}^{\infty} d(0,n) Y^n(x,z), \\ y &= \varphi_3^3(x,z) = \frac{4 + 6x^2 - 79z + z^3}{10}, \end{aligned} \tag{44}$$

where

$$Y(y,z) = \frac{3(100 - 7x - 6z)^2}{1953125}.$$

Then,

$$\begin{aligned} D_{32} &= \{(x,z);\ -62.66 \le 100 - 7x - 6z \le 62.66\}, \\ D_{33} &= \left\{(x,z);\ (x,z) \in \mathcal{R}^2\right\}. \end{aligned} \tag{45}$$

As

$$|y| = \left|\varphi_3^2(x,z)\right| \le \left|\frac{100 - 7x - 6z}{43}\right| \frac{3}{2} \le 2.18, \qquad \text{with } (x,z) \in D_{32}, \tag{46}$$

then

$$I_{32} = \{-2.18 \le y \le 2.18\}; \qquad I_{33} = \mathcal{R}. \tag{47}$$

And as a consequence all the graphs of the functions of G_0 are well defined inside the region

$$A_3 = (R_1 \times I_1) \cap (R_2 \times I_2) \cap (R_3 \times I_3) \neq \emptyset. \tag{48}$$

Finally the set $K \subset A_3 \cap \Omega_1$, given by

$$K = \{(x,y,z);\ 3.93 \le x \le 6.43;\ 0.82 \le y \le 1.82;\ 4.5 \le z \le 1.87\} \tag{49}$$

is computed. Note that K is a compact rectangle in Ω_1, where all functions of G_0 are well defined. So K verifies the necessary conditions of Theorem 2, and it is a candidate region to contain a root.

4. SUFFICIENT CONDITIONS. THE MAIN RESULT

DEFINITION 5. *Let (1) be an α-system in Ω. Assume that there is a closed p-cube, $K = I_1 \times \cdots \times I_p \subset \Omega$ and $r \in \mathcal{R}^p$ so that all the functions of G_r are well defined. Then the following functions can be defined:*

$$\begin{aligned} G^1(x_{-1}) &= \left(\varphi_p^1 - \varphi_1^1\right)(x_2, \dots, x_{p-1}, x_p), \\ G^i(x_{-i}) &= \left(\varphi_i^i - \varphi_{i-1}^i\right)(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_p), \qquad 2 \le i \le p, \end{aligned} \tag{50}$$

in $I_2 \times \cdots \times I_p$, $I_1 \times \cdots \times I_{i-1}$, $I_{i+1} \times \cdots \times I_P$, $2 \le i \le p$, respectively. On the other hand, if we take a fixed $x^0 \in K \cap \mathcal{R}^{p-2}_{1,p}$ for G^1 and $x^0 \in K \cap \mathcal{R}^{p-2}_{i-1,i}$ for G^i, $2 \le i \le p$, then the real functions can be introduced

$$\begin{aligned} G^1_{x^0}(x_p) &= \left(\varphi_p^1 - \varphi_1^1\right)\left(x_2^0, \dots, x_{p-1}^0, x_p\right), \\ G^i_{x^0}(x_{i-1}) &= \left(\varphi_i^i - \varphi_{i-1}^i\right)\left(x_1^0, \dots, x_{i-2}^0, x_{i-1}, x_{i+1}^0, \dots, x_p^0\right), \qquad 2 \le i \le p, \end{aligned} \tag{51}$$

where $x_p \in I_p$, $x_{i-1} \in I_{i-1}$, with $2 \le i \le p$, respectively.

REMARK 2. From (20) and (21), we can deduce that

$$\frac{\partial G^1(x_{-1})}{\partial x_p} \ne 0; \quad \frac{\partial G^i(x_{-i})}{\partial x_{i-1}} \ne 0, \qquad \text{with } 2 \le i \le p, \tag{52}$$

since

$$\begin{aligned} &\begin{vmatrix} \dfrac{\partial f_p}{\partial x_p} & \dfrac{\partial f_1}{\partial x_p} \\ \dfrac{\partial f_p}{\partial x_1} & \dfrac{\partial f_1}{\partial x_1} \end{vmatrix} = \frac{\partial f_p}{\partial x_p}\frac{\partial f_1}{\partial x_1} - \frac{\partial f_1}{\partial x_p}\frac{\partial f_p}{\partial x_1} \ne 0 \\ &\qquad \Rightarrow -\frac{\frac{\partial f_p}{\partial x_p}}{\frac{\partial f_p}{\partial x_1}} + \frac{\frac{\partial f_1}{\partial x_p}}{\frac{\partial f_1}{\partial x_1}} = \frac{\partial \varphi_p^1}{\partial x_p} - \frac{\partial \varphi_1^1}{\partial x_p} \ne 0 \Rightarrow \frac{dG^1_{x^0}(x_p)}{dx_p} \ne 0. \end{aligned} \tag{53}$$

In the same way

$$\begin{aligned} &\begin{vmatrix} \dfrac{\partial f_{i-1}}{\partial x_{i-1}} & \dfrac{\partial f_i}{\partial x_{i-1}} \\ \dfrac{\partial f_{i-1}}{\partial x_i} & \dfrac{\partial f_i}{\partial x_i} \end{vmatrix} = \frac{\partial f_{i-1}}{\partial x_{i-1}}\frac{\partial f_i}{\partial x_i} - \frac{\partial f_i}{\partial x_{i-1}}\frac{\partial f_{i-1}}{\partial x_i} \ne 0 \\ &\qquad \Rightarrow -\frac{\frac{\partial f_{i-1}}{\partial x_{i-1}}}{\frac{\partial f_{i-1}}{\partial x_i}} + \frac{\frac{\partial f_i}{\partial x_{i-1}}}{\frac{\partial f_i}{\partial x_i}} = \frac{\partial \varphi_{i-1}^i}{\partial x_{i-1}} - \frac{\partial \varphi_i^i}{\partial x_{i-1}} \ne 0 \\ &\qquad \Rightarrow \frac{dG^i_{x^0}(x_p)}{dx_{i-1}} \ne 0, \qquad 2 \le i \le p. \end{aligned} \tag{54}$$

Suppose that sets $\mathcal{V}_1$, $\mathcal{V}_i$, $2 \le i \le p$, given by

$$\begin{aligned} \mathcal{V}_1 &= \left\{(z_1, \dots, z_p) \in \mathcal{R}^p \cap \Omega;\ z_1 = \varphi_1^1(z_2, \dots, z_p) = \varphi_p^1(z_2, \dots, z_p);\ G^1(z_2, \dots, z_p) = 0\right\}, \\ \mathcal{V}_i &= \left\{(z_1, \dots, z_p) \in \mathcal{R}^p \cap \Omega;\ z_i = \varphi_{i-1}^i(z_{-i}) = \varphi_i^i(z_{-i});\ G^i(z_{-i}) = 0\right\} \end{aligned} \tag{55}$$

are not empty. Therefore, by applying the implicit function theorem to the functions G^i, there are suitable open sets, where the functions

$$\begin{aligned} z_p &= X_p(z_2, \dots, z_{p-1}), \quad \text{with } G^1(z_2, \dots, X_p(z_2, \dots, z_{p-1})) = 0, \\ z_{i-1} &= X_{i-1}(z_{-(i-1)-i}), \quad \text{with } G^i(z_1, \dots, X_{i-1}(z_{-(i-1)-i}), \dots, z_p) = 0, \ \text{with } 2 \le i \le p, \end{aligned} \tag{56}$$

are well defined. And finally, the following diffeomorphisms between open sets of $\mathcal{R}^p$ can be built:

$$\begin{aligned}&\Gamma_1 : W^1 \to T^1,\\ &\Gamma_1(z) = \left(z_1 - \varphi_1^1(z_2,\dots,X_p(z_{-1-p})), z_2,\dots,z_p - X_p(z_{-1-p})\right),\\ &\Gamma_1 : W^1 \cap \mathcal{V}_1 \to T^1 \cap \mathcal{R}^{p-2}_{1,p}.\end{aligned} \tag{57}$$

In the same manner, for $2 \le i \le p$,

$$\begin{aligned}&\Gamma_i : W^i \to T^i,\\ &\Gamma_i(z) = \left(z_1,\dots,z_{i-1} - X_{i-1}(z_{-(i-1)-i}), z_i - \varphi_i^i(z_1,\dots,X_{i-1}(z_{-(i-1)-i}),\dots,z_p),\dots,z_p\right),\\ &\Gamma_i : W^i \cap \mathcal{V}_i \to T^i \cap \mathcal{R}^{p-2}_{i-1,i}.\end{aligned} \tag{58}$$

THEOREM 3. *Let (1) be an α-system in Ω. Then it has a zero, a, in Ω if, and only if, there exists a closed p-cube, $K = I_1 \times \dots \times I_p \subset \Omega$, so that each function $G^i_{x^0}$, $1 \le i \le p$, introduced in (50), is well defined and it has a zero in I_i. Besides $a \in K$.*

PROOF. Let $a = (a_1,\dots,a_p)$ be a zero of (1) in Ω, then there is a compact p-cube $K = I_1 \times \dots \times I_p \subset \Omega$, with $a \in K$, that satisfies the hypothesis of Theorem 2. So the functions G^i exist. As a is a root, then $a \in \mathcal{V}_1 \cap \dots \cap \mathcal{V}_p$ and K can be taken so that $K \subset W^1 \cap \dots \cap W^p$. Hence, in agreement with (57) and (58), it follows that

$$\Gamma_1(K \cap \mathcal{V}_1) = I_2 \times \dots \times I_{p-1}, \tag{59}$$

and for $2 \le i \le p$,

$$\Gamma_i(K \cap \mathcal{V}_i) = I_1 \times \dots \times I_{i-2} \times I_{i+1} \times \dots \times I_p. \tag{60}$$

From (59)

$$\begin{aligned}&\left(x_2^0,\dots,x_{p-1}^0\right) \in I_2 \times \dots \times I_{p-1} \to \Gamma_1^{-1}\left(x_2^0,\dots,x_{p-1}^0\right) = \left(y_1, x_2^0,\dots,x_{p-1}^0, y_p\right) \in K \cap \mathcal{V}_1;\ \text{with}\\ &\quad y_p = X_p\left(x^0_{-1,-p}\right) \in I_p;\\ &\quad y_1 = \varphi_1^1\left(x_2^0,\dots,x_{p-1}^0, X_p\left(x^0_{-1,-p}\right)\right) = \varphi_p^1\left(x_2^0,\dots,X_p\left(x^0_{-1,-p}\right)\right) \in I_1.\end{aligned} \tag{61}$$

Therefore $G^1_{x^0}(X_p(x^0_{-1,-p})) = 0$. On the other side (52) implies that $(X_p(x^0_{-1,-p}))$ is the unique zero. In a similar way, from (60), the remaining cases can be proven.

Conversely, assume now that there is a closed p-cube, K, to verify the required hypothesis. First of all, we are going to consider the projections $\pi_{p,1}$ and $\pi_{i-1,i}$, introduced in (2). Having done this, $\forall\,(x_1,\dots,x_p) \in I_1 \times \dots \times I_p$, we arrive at

$$\begin{aligned}\pi_{p,1}(x_1,\dots,x_p) &= (0,x_2,\dots,x_{p-1},0),\\ (\Gamma_1)^{-1}(0,x_2,\dots,x_{p-1},0) &= (z_1,x_2,\dots,x_{p-1},y_p),\\ G^1(x_2,\dots,x_{p-1},y_p) &= 0, && \text{with } y_p \in I_p,\\ z_1 &= \varphi_1^1(x_2,\dots,x_{p-1},y_p), && \text{with } z_1 \in I_1.\end{aligned} \tag{62}$$

From (62), we take $(z_1,x_2,\dots,x_{p-1},y_p)$ and define the following functions:

$$\begin{aligned}\pi_{1,2}(z_1,x_2,\dots,x_{p-1},y_p) &= (0,0,x_3,\dots,x_{p-1},y_p),\\ (\Gamma_2)^{-1}(0,0,x_3,\dots,x_{p-1},y_p) &= (y_1,z_2,x_3\dots,x_{p-1},y_p),\\ G^2(y_1,x_3,\dots,x_{p-1},y_p) &= 0, && \text{with } y_1 \in I_1,\\ z_2 &= \varphi_2^2(y_1,x_3,\dots,x_{p-1},y_p), && \text{with } z_2 \in I_2.\end{aligned} \tag{63}$$

By going on this process, we take $(y_1, \dots, y_{i-1}, z_i, x_{i+1}, \dots, x_{p-1}, y_p)$ from the i^{th} previous step and define the following functions:

$$\begin{aligned} \pi_{i(i+1)}(y_1, \dots, y_{i-1}, z_i, x_{i+1}, \dots, x_{p-1}, y_p) &= (y_1, \dots, y_{i-1}, 0, 0, x_{i+2}, \dots, x_{p-1}, y_p), \\ (\Gamma_{i+1})^{-1}(y_1, \dots, y_{i-1}, 0, 0, x_{i+2}, \dots, x_{p-1}, y_p) &= (y_1, \dots, y_i, z_{i+1}, x_{i+2}, \dots, x_{p-1}, y_p), \\ G^{i+1}(y_1, \dots, y_{i-1}, y_i, x_{i+2}, \dots, x_{p-1}, y_p) &= 0, \qquad \text{with } y_i \in I_i, \\ z_{i+1} = \varphi_{i+1}^{i+1}(y_1, \dots, y_{i-1}, y_i, x_{i+2}, \dots, x_{p-1}, y_p), &\qquad \text{with } z_{i+1} \in I_{i+1}. \end{aligned} \tag{64}$$

And the last step will be

$$\begin{aligned} \pi_{p-1,p}(y_1, \dots, z_{p-1}, y_p) &= (y_1, \dots, y_{p-2}, 0, 0), \\ (\Gamma_p)^{-1}(y_1, \dots, y_{p-2}, 0, 0) &= (y_1, \dots, y_{p-2}, y_{p-1}, z_p), \\ G^p(y_1, \dots, y_{p-1}) &= 0, \qquad \text{with } y_{p-1} \in I_{p-1}, \\ z_p &= \varphi_p^p(y_1, \dots, y_{p-1}), \qquad \text{with } z_p \in I_p. \end{aligned} \tag{65}$$

Equations (62)–(65) allow us to define the continuous function

$$F = (\Gamma_p)^{-1} \circ \cdots \circ (\Gamma_2)^{-1} \circ \pi_{1,2} \circ (\Gamma_1)^{-1} \circ \pi_{p,1} \tag{66}$$

to verify $F(I_1 \times \cdots \times I_p) \subset I_1 \times \cdots \times I_p$, therefore, from the point fixed theorem of Brawer, F has a fixed point $(p_1, \dots, p_p)$, that is a root of system (1) and it is in K. The result is proved.

EXAMPLE 4. Coming back to system (3), for all $(x^0, y^0, z^0) \in K$, K being the region introduced in (49), we introduce the functions

$$\begin{aligned} G^1_{y^0}(z) &= \varphi_1^1(y^0, z) - \varphi_3^1(y^0, z), \qquad z \in [1.87, 4.5], \\ G^2_{z^0}(x) &= \varphi_2^2(x, z^0) - \varphi_1^2(x, z^0), \qquad x \in [3.93, 6.43], \\ G^3_{x^0}(y) &= \varphi_3^3(x^0, y) - \varphi_2^3(x^0, y), \qquad y \in [0.82, 1.82]. \end{aligned} \tag{67}$$

As required in Theorem 3, we must show that $\forall (x^0, y^0, z^0) \in K$, $G^1_{y^0}(1.87) G^1_{y^0}(4.5) < 0$, $G^2_{z^0}(3.93) G^2_{z^0}(6.43) < 0$, and $G^3_{x^0}(0.82) G^1_{x^0}(1.82) < 0$. In other words, functions (67) have a zero in their intervals of definition. This issue will be treated in the following section.

5. LOWER AND UPPER BOUND FOR THE FUNCTIONS φ_j^i

PROPOSITION 4. *A lower bound for (14) is given by*

$$d(q_1, \dots, q_{m-1}) \geq \frac{n!}{q_1! \cdots q_{m-1}!} \frac{1}{n} \frac{1}{3.75} 3.75^n, \qquad \forall n \geq 1. \tag{68}$$

PROOF. First we are going to prove that

$$d(n, 0, \dots, 0) \geq \frac{1}{n} 3,75^{n-1}, \qquad n \geq 1. \tag{69}$$

Given the sequence

$$a_n = \frac{(4n+2)(n+1)}{(n+2)n}, \qquad \forall n \geq 1,$$

it is easy to show that it is increasing for $n \geq 4$. As $a_4 = 3.75$, then $a_n \geq 3.75$, $\forall n \geq 4$. For $n = 1, 2$, and 3 inequality (69) holds true. For $n = k$, $k > 3$ we have that

$$\frac{d(k+1, 0, \dots, 0)(k+1)}{d(k, 0, \dots, 0)k} = a_k \geq 3.75.$$

Hence,

$$d(k+1,0,\dots,0) \geq 3.75\, d(k,0,\dots,0)\frac{k}{k+1}.$$

By induction, hypothesis (69) is concluded.

$$\begin{aligned} d(q_1,\dots,q_{m-1})q_1!\cdots q_{m-1}! &= \frac{(2q_1+3q_2+\cdots+mq_{m-1})!}{(q_1+2q_2+\cdots+(m-1)q_{m-1}+1)!} \\ &= \frac{2q_1+3q_2+\cdots+mq_{m-1}}{q_1+2q_2+\cdots+(m-1)q_{m-1}+1} \\ &\quad\cdot\frac{2q_1+3q_2+\cdots+mq_{m-1}-1}{q_1+2q_2+\cdots+(m-1)q_{m-1}} \\ &\quad\cdots\frac{2q_1+2q_2+\cdots+2q_{m-1}+1}{q_1+q_2+\cdots+q_{m-1}+2}\frac{(2n)!}{(n+1)!}. \end{aligned} \tag{70}$$

Note that the last term of product (70) is equal to $d(n,0,\dots,0)n!$. As

$$\frac{2n+K}{n+1+K} \geq 1,$$

for all integer number $K \geq 1$, then all the terms of product (70) are greater than or equal to 1, so the proof is completed.

THEOREM 4. *Assume that each function $\varphi_j^i \in G_r$ is well defined, G_r defined in (33). Consider*

$$\begin{aligned} V_j(y_{-i}) &= Y_{1,j}(y_{-i})+\cdots+Y_{m-1,j}(y_{-i}), \\ U_j(y_{-i}) &= |Y_{1,j}(y_{-i})|+\cdots+|Y_{m-1,j}(y_{-i})|, \end{aligned} \tag{71}$$

where $Y_{1,j},\dots,Y_{m-1,j}$ are the functions introduced in (29), and

$$\begin{aligned} H_{u,j}^{+}(y_{-i}) &= 1-\frac{1}{2T}\log\left(1-TV_j(y_{-i})\right)-\frac{1}{2T}\log\left(1-TU_j(y_{-i})\right), \\ H_{l,j}^{+}(y_{-i}) &= 1-\frac{1}{7.5}\log\left(1-3.75V_j(y_{-i})\right)-\frac{1}{7.5}\log\left(1-3.75U_j(y_{-i})\right), \\ H_{l,j}^{-}(y_{-i}) &= \frac{1}{7.5}\log\left(\frac{1-3.75V_j(y_{-i})}{1-3.75U_j(y_{-i})}\right), \\ H_{u,j}^{-}(y_{-i}) &= \frac{1}{2T}\log\left(\frac{1-TV_j(y_{-i})}{1-TU_j(y_{-i})}\right), \end{aligned} \tag{72}$$

where $T = m^m/(m-1)^{m-1}$, then the functions

$$U_j^i(y_{-i}) = \begin{cases} K_j(y_{-i})H_{u,j}^{+}(y_{-i})-K_j(y_{-i})H_{l,j}^{-}(y_{-i}), & \text{if } K_j(y_{-i})\geq 0, \\ K_j(y_{-i})H_{l,j}^{+}(y_{-i})-K_j(y_{-i})H_{u,j}^{-}(y_{-i}), & \text{if } K_j(y_{-i})<0, \end{cases} \tag{73}$$

and

$$L_j^i(y_{-i}) = \begin{cases} K_j(y_{-i})H_{l,j}^{+}(y_{-i})-K_j(y_{-i})H_{u,j}^{-}(y_{-i}), & \text{if } K_j(y_{-i})\geq 0, \\ K_j(y_{-i})H_{u,j}^{+}(y_{-i})-K_j(y_{-i})H_{l,j}^{-}(y_{-i}), & \text{if } K_j(y_{-i})<0, \end{cases} \tag{74}$$

with $K_j(y_{-i})$ introduced in (29), and defined in region (31), they satisfy the inequalities

$$L_j^i(y_{-i}) \leq \varphi_j^i(y_{-i}) \leq U_j^i(y_{-i}). \tag{75}$$

PROOF. Assume, with loss of generality, that $Y_{1,j}(y_{-i}),\dots,Y_{p,j}(y_{-i})$ are negative and the remaining $Y_{p+1,j}(y_{-i}),\dots,Y_{m-1,j}(y_{-i})$, positive. Then, $\varphi_j^i(y_{-i})$ can be written as

$$\varphi_j^i(y_{-i}) = K_j(y_{-i})\left(H_j^{+}(y_{-i})-H_j^{-}(y_{-i})\right), \tag{76}$$

where

$$H_j^+(y_{-i}) = 1 + \sum_{n=1}^{\infty} \sum_{2t+v=n} \sum_{\substack{q_1+\cdots+q_p=2t \\ q_{p+1}+\cdots+q_{m-1}=v}} d(q_1,\ldots,q_{m-1}) |Y_{1,j}(y_{-i})|^{q_1} \cdots |Y_{m-1,j}(y_{-i})|^{q_{m-1}} \tag{77}$$

and

$$H_j^-(Y(y_{-i})) = \sum_{n=1}^{\infty} \sum_{2t+1+v=n} \sum_{\substack{q_1+\cdots+q_p=2t+1 \\ q_{p+1}+\cdots+q_{m-1}=v}} d(q_1,\ldots,q_{m-1}) |Y_{1,j}(y_{-i})|^{q_1} \cdots |Y_{m-1,j}(y_{-i})|^{q_{m-1}}. \tag{78}$$

We recall that using the Taylor expansion we have

$$\log((1 - V_j(y_{-i}))(1 - U_j(y_{-i}))) = -2\sum_{n=1}^{\infty} \sum_{2t+v=n} \sum_{\substack{q_1+\cdots+q_p=2t \\ q_{p+1}+\cdots+q_{m-1}=v}} \frac{n!}{q_1!\cdots q_{m-1}!} \frac{1}{n} |Y_{1,j}(y_{-i})|^{q_1} \cdots |Y_{m-1,j}(y_{-i})|^{q_{m-1}}, \tag{79}$$

$$\log\left(\frac{1 - V_j(y_{-i})}{1 - U_j(y_{-i})}\right) = 2\sum_{n=1}^{\infty} \sum_{2t+1+v=n} \sum_{\substack{q_1+\cdots+q_p=2t+1 \\ q_{p+1}+\cdots+q_{m-1}=v}} \frac{n!}{q_1!\cdots q_{m-1}!} \frac{1}{n} |Y_{1,j}(y_{-i})|^{q_1} \cdots |Y_{m-1,j}(y_{-i})|^{q_{m-1}}. \tag{80}$$

Taking into account inequalities (16) and (68), from the Taylor series (79) and (80) it follows that

$$H_{l,j}^+(Y(y_{-i})) \le H_j^+(Y(y_{-i})) \le H_{u,j}^+(Y(y_{-i})) \tag{81}$$

and

$$H_{l,j}^-(Y(y_{-i})) \le H_j^-(Y(y_{-i})) \le H_{u,j}^-(Y(y_{-i})). \tag{82}$$

And the result is proved.

EXAMPLE 5. Taking again system (3), by computing functions (73) and (74), the change of sign of the functions $G_{y^0}^1(z)$, $G_{z^0}^2(x)$, and $G_{x^0}^3(y)$ can be calculated following the method, for instance, of consecutive subdivisions of intervals. Finally we get a root for (3) in the region

$$\left\{(x,y,z) \in \mathcal{R}^3; 6.05 \ge x \ge 5.8;\ 1.5 \ge y \ge 0.82;\ 3.15 \ge z \ge 2.76\right\}. \tag{83}$$

Then taking as initial value $x = 6.05$, $y = 1.5$, and $z = 3.15$, for instance, using Newton method, the root $x = 6$, $y = 1$, and $z = 3$ is obtained.

6. ALGORITHM STRUCTURE

This algorithm, whose structure is introduced, searches initial values for solving polynomial system (1) in a bounded and closed region of $\mathcal{R}^p$, Γ, so that $\Gamma \subset \Omega$, Ω being the region where (1) is α-system (see Definition 3).

STEP 1. Compute the (UD)-matrix of system (1), $\mathcal{A}_1$, according to Definition 2.

STEP 2. If $\mathcal{A}_1$ is an α-matrix, then go to the following step. In another case, in agreement with Remark 1, decompose the matrix $\mathcal{A}_1$ into several α-matrices, denoting by $\mathcal{A}$ each one of them.

STEP 3. Compute a $(M\alpha)$-matrix, $\mathcal{M} \sqsubset \mathcal{A}$, in accordance with Definition 1.

STEP 4. Compute the canonical form of $\mathcal{M}$, $\mathcal{M}_c$, in agreement with Proposition 1.

STEP 5. Introduce sequence S satisfying Theorem 1.

STEP 6. Taking into account the sequence S, search a point $r \in \mathcal{R}^p$ verifying inequalities (20) and (21), so that the function φ_1^1 (see Proposition 3) is convergent at the point r_{-1}.

STEP 7. Compute the functions φ_j^i of G_r, introduced in Definition 4.

STEP 8.

1. Compute the functions φ_1^1, $\varphi_p^1 \in G_r$, corresponding to the first row of M_c.
2. Find D_{11} and D_{1p}, the regions of convergence of the series functions φ_1^1 and φ_p^1, respectively, according to Proposition 3.
3. Find $I_{11} = [-M_1, M_1] \neq \emptyset$ and $I_{1p} = [-N_1, N_1]$, where, in agreement with (17), M_1 satisfies the inequalities $-M_1 \leq \varphi_1^1(x_{-1}) \leq M_1$, with $x_{-1} \in D_{11}$, and N_1, $-N_1 \leq \varphi_p^1(x_{-1}) \leq N_1$, with $x_{-1} \in D_{1p}$.
4. Find $R_1 = D_{11} \cap D_{1p}$ and $I_1 = I_{11} \cap I_{1p}$.
5. If $A_1 = R_1 \times I_1 = \emptyset$, then, from Theorem 2, there are no zeros in $(D_{11} \times I_{11}) \neq \emptyset$, and return to Step 6, taking other r outside $(D_{11} \times I_{11})$. If, on the contrary, $I_1 \times R_1 \neq \emptyset$, then go to the following step.

STEP 9.

1. Compute φ_1^2, $\varphi_j^2 \in G_r$, corresponding to the second row of M_c, and the regions $R_2 = D_{21} \cap D_{22}$, $I_2 = I_{21} \cap I_{22}$.
2. If $A_2 = (I_1 \times R_1) \cap (R_2 \times I_2) = \emptyset$, return to Step 6. Otherwise go to the following step.

STEP 10. Continue this routine until covering all the rows of the matrix M_c and all the functions of G_r, then there are no zeros in the bounded region, Γ, and the algorithm ends, or $A_p = (I_1 \times R_1) \cap \cdots \cap (R_p \times I_p) \neq \emptyset$ and go to the following step.

STEP 11. Compute $K = J_1 \times \cdots \times J_p \subset A_p$, so that (1) is an α-system in K.

STEP 12. Introduce the functions G^i, $1 \leq i \leq p$, in K (see Definition 5), and the functions F_l^i, F_u^i, so that $F_l^i \leq G^i \leq F_u^i$, $1 \leq i \leq p$; such functions are built from (73) and (74).

STEP 13. By using the functions F_l^i and F_u^i, compute the sign of the functions G^i in the established conditions in Theorem 3, then,

1. if all the functions G^i, $1 \leq i \leq p$, change their signs, there exists a zero of system (1), going to Step 6 for the search of other possible zeros;
2. if some function G^i preserves its sign, there is no zero and algorithm goes to Step 6,
3. otherwise the intervals J_i, $1 \leq i \leq p$, are divided into subintervals until Theorem 3 is satisfied.
4. The algorithm ends when all the bounded region Γ is covered, by regions of type A_j.

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