By a local submonoid, of a regular semigroup $S$, we mean a subset of the form $eSe$, where $e$ is an idempotent of $S$. Many classes of regular semigroups can be defined in terms of properties of their local submonoids. For example, rectangular bands can be characterized as those regular semigroups all of whose local submonoids are trivial while completely simple semigroups are those whose local submonoids are groups. We say that a regular semigroup has a property $\mathcal{C}$ locally, or is a locally $\mathcal{C}$ semigroup, if each local submonoid of $S$ has property $\mathcal{C}$.

In a previous paper we showed that a regular semigroup is locally inverse if and only if it is an image, by a homomorphism which is one to one on local submonoids, of a regular Rees matrix semigroup over an inverse semigroup. In this paper we extend that result to various other classes of regular semigroups. In particular, we show the analog of this result for locally $E$-solid semigroups. (A regular semigroup is $E$-solid if the subsemigroup generated by its idempotents is a union of groups.) The class of locally $E$-solid regular semigroups is extremely extensive. It includes almost all classes of regular semigroups which have been studied from a structural point of view since inverse semigroups, orthodox semigroups, unions of groups semigroups and their localizations all belong to this class.

The first section of the paper contains preliminary results which we shall require later in the paper. Section 2 describes a general procedure for constructing Rees matrix covers for regular semigroups which is applied in subsequent sections to obtain Rees matrix covers for special classes of regular semigroups. These classes include locally $E$-solid regular semigroups, locally orthodox semigroups and locally $\mathcal{L}$-unipotent semigroups. The covering theorems on these classes of regular semigroups can be obtained directly through a painstaking analysis of special cases. We have, however, chosen to obtain them as applications of the covering theorem for locally inverse semigroups which was proved earlier [8] by making use of recent interesting results of Hall concerning the relationship between locally $E$-solid

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semigroups and locally inverse semigroups. Locally isomorphic images of regular Rees matrix semigroups over unions of groups are also characterized.

The final section is somewhat of a diversion. It shows that the results obtained earlier in the paper belong to semigroup theory proper. More precisely, we show that the multiplicative semigroup of a ring is locally $E$-solid if and only if the idempotents are central. This generalizes earlier results on rings whose multiplicative semigroup is completely 0-simple or a union of groups.

1. Preliminaries

Let $S$ and $T$ be regular semigroups. Then a homomorphism $\theta: S \to T$ is a local isomorphism if it is one-to-one on each subsemigroup $eSe$, $e' = e \in S$, of $S$; that is, if it is one-to-one on each local submonoid of $S$.

**Lemma 1.1** [8]. If $\theta: S \to T$ is a local isomorphism, then $\theta$ is one-to-one on each subsemigroup $xSy$, $x, y \in S$, of $S$.

**Proposition 1.2.** Let $S$ be a regular semigroup. Then $S$ has a greatest locally isomorphic image $T$. $T$ is the quotient of $S$ modulo the congruence $\rho$ where

$$ab \quad \text{if and only if} \quad xay = xby \quad \text{for all} \quad x, y \in S.$$

**Proof:** Suppose that $\theta$ is a local isomorphism of $S$ into a regular semigroup $U$. Then $a\theta = b\theta$ implies $(xay)\theta = (xby)\theta$ for all $x, y \in S$. Hence, by Lemma 1.1, $xay = xby$. Thus $\theta \circ \theta^{-1} \subseteq \rho$.

Conversely, $\rho$ is clearly a congruence on $S$. Suppose that $a, b \in eSe$ for some idempotent $e$. Then $(a, b) \in \rho$ implies $a = eae = ebe = b$. Hence the canonical homomorphism $\rho: S \to S/\rho$ is a local isomorphism.

**Corollary 1.3.** Let $S$ be a regular semigroup. Then every local isomorphism with domain $S$ is an isomorphism if and only if $xay = xby$, for all $x, y \in S$, implies $a = b$.

In the sequel we shall be determining conditions on a semigroup $S$ which ensure that $S/\rho$ belongs to some prescribed class of regular semigroups.

A regular semigroup $S$ is said to be $E$-solid if, whenever $e, f, g$ are idempotents of $S$ such that $e\not\rho f \not\rho g$, there is an idempotent $h$ such that $e\not\rho h \not\rho g$. Hall [3] and Fitz-Gerald (unpublished) have shown that $S$ is $E$-solid if and only if $IG(S)$, the subsemigroup generated by the idempotents of $S$, is a union of groups. It follows from this that the class of $E$-solid regular semigroups is closed under homomorphic images.
**Proposition 1.4.** Let $S$ be a regular semigroup. Then the following are equivalent:

(i) there is a local isomorphism of $S$ onto an $E$-solid semigroup;
(ii) $S/\rho$ is $E$-solid;
(iii) $S$ is $E$-solid.

**Proof.** By the remarks above, (i) implies (ii) while (iii) implies (i) so we need only show that (ii) implies (iii).

Suppose that $e, f, g$ are idempotents of $S$ and $e \not\Rightarrow f \not\Rightarrow g$. Then, by Lallement's lemma, there is an idempotent $h$ such that $e \not\Rightarrow h \not\Rightarrow gp$. Let $k = egh$. Then $(k, h) \in \rho$ since $e \not\Rightarrow hp \not\Rightarrow gp$. It is easy to see that $(k, k^2) \in \rho$ implies that $k$ is idempotent, from Lemma 1.1. Thus, since $(k, h) \in \rho$ implies $(ke, he) \in \rho$, and $e \not\Rightarrow hp$, we have $(ke, e) \in \rho$. Hence, since $e, ke \in eSe$ we find, since $\rho$ is a local isomorphism, that $e = ke$. It follows that $e \not\Rightarrow k$. Similarly $g \not\Rightarrow k$ so that $S$ is $E$-solid.

The proof of the next proposition is similar to that of Proposition 1.4, but easier.

**Proposition 1.5.** Let $S$ be a regular semigroup. Then the following are equivalent:

(i) there is a local isomorphism of $S$ onto an orthodox semigroup;
(ii) $S/\rho$ is orthodox;
(iii) $S$ is orthodox.

A regular semigroup is said to be $\mathcal{L}$-unipotent if each $\mathcal{L}$-class contains a unique idempotent. Such semigroups are necessarily orthodox so that locally $\mathcal{L}$-unipotent semigroups are locally orthodox. Blyth and Gomes [1] have shown that a regular semigroup $S$ is locally $\mathcal{L}$-unipotent if and only if the natural partial order $\leq$ introduced by Nambooripad [10],

\[ a \leq b \quad \text{if and only if} \quad a = aa'b \text{ and } a \in bS, \text{ for some } a' \in V(a), \]

is compatible with multiplication on the right.

**Proposition 1.6.** Let $S$ be a regular semigroup. Then the following are equivalent

(i) there is a local isomorphism from $S$ onto an $\mathcal{L}$-unipotent semigroup;
(ii) $S/\rho$ is $\mathcal{L}$-unipotent;
(iii) the idempotents of $S$ satisfy the identity $uef = ufef$. 
Proof. Clearly (ii) implies (i). Let \( \theta \) be a local isomorphism of \( S \) onto an \( \mathcal{L} \)-unipotent semigroup \( T \). Then, since \( T \) is orthodox, it follows from Proposition 1.5 that \( S \) is orthodox. Let \( u, e, f \) be idempotents in \( S \). Then \((ef)\theta \) and \((uf)\theta \) are \( \mathcal{L} \)-equivalent idempotents of the \( \mathcal{L} \)-unipotent semigroup \( T \). Thus \((ef)\theta = (uf)\theta \) so that \((uef)\theta = (uf)\theta \). Since \( uef \) and \( ufe \) are in \( uSf \) and \( \theta \) is a local isomorphism, it follows that \( uef = ufe \). Hence (i) implies (iii).

Suppose (iii) and let \( e, f \) be idempotents of \( S \) such that \( ep\mathcal{P}fp \). Then \((e, ef) \in \rho \) so that \( uev = uefv = ufev \) for all idempotents \( u, v \). But also \((f, fe) \in \rho \) which implies \( ufev = u, fe, fv = uf, fv = uf v \). Hence \( uev = uf v \) for all idempotents \( u, v \). But this implies \( xey = xfy \) for all \( x, y \in S \) so that \((e, f) \in \rho \). Hence \( S/\rho \) is \( \mathcal{L} \)-unipotent.

Hall [5] proves that various properties of regular semigroups can be extended from local subsemigroups to principal ideals. Among these we shall make use of the following.

**Lemma 1.7** (Hall [5]). Let \( e \) be an idempotent of a regular semigroup.

(i) If \( eSe \) is orthodox, then the idempotents of \( eS \) form a band;

(ii) if each \( \mathcal{L} \)-class of \( eSe \) contains at most one idempotent, then the same is true for \( eS \).

**Proposition 1.8.** Let \( S \) be a regular semigroup. Then \( S \) is locally orthodox if and only if, for all idempotents \( e, e, f, f, S(e, f) e, S(e, f) \).

**Proof.** Suppose that \( S \) is locally orthodox, and let \( h \in S(e, f) \). Then

(i) \[ f_1(f_1h) = f_1h = f_1(he) = (f_1h)e \] since \( f_1 \leq f \) and \( h \in S(e, f) \).

(ii) \[ e(f, h)f_1 = e, f f_1, h f_1 \] since \( f_1 \leq f \)

\[ = ehf_1f_1, h f_1 \] since \( h \in S(e, f) \)

\[ = ehf_1, h f_1 \] since \( f_1 \leq f \)

\[ = ehf_1 = ehf_1 = ef_1 \]

since \( h, f_1 \) idempotent in \( fS \) imply \( h f_1 \) is idempotent, by Lemma 1.7(i). Further, for the same reason \( f_1h \) is idempotent. Hence \( f_1h S(e, f_1) \). Dually, since \( e_1 \leq e \), we get \( f_1he_1 = (f_1h)e_1 S(e_1, f_1) \).

Conversely, suppose \( f, g \) are idempotents in \( eSe \), where \( e \) is idempotent. Then \( fg = feg \in fS(e, e)g \subseteq S(g, f) \) so that \( fg \) is idempotent.
2. REGULAR REES MATRIX COVERS

Let $S$ be a regular semigroup and let $I, A$ be a nonempty sets. Let $P$ be a $A \times I$ matrix over $S$. Then the set $\mathcal{M}(S; I, A; P)$ is a semigroup, the $I \times A$ Rees matrix semigroup over $S$ with sandwich matrix $P$. In general, it is not regular, however, it can be shown [7] that the set of regular elements forms a subsemigroup of $\mathcal{M}(S; I, A; P)$ which we denote by $\mathcal{R}\mathcal{M}(S; I, A; P)$;

$$(i, x, \lambda) \in \mathcal{R}\mathcal{M}(S; I, A; P) \quad \text{if and only if } V(x) \cap P_{j \lambda} S_{\mu i} \neq \emptyset$$

for some $j \in I, \mu \in A$, where, as usual, $V(x)$ denotes the set of inverses of $x \in S$. We call $\mathcal{R}\mathcal{M}(S; I, A; P)$ a regular Rees matrix semigroup over $S$.

In this paper, we shall be interested in obtaining various types of regular semigroups as locally isomorphic images of regular Rees matrix semigroups over simpler semigroups. In this section we provide a general procedure for obtaining such covers; we will apply the construction to special classes of semigroups in later sections.

Let $S$ be any regular semigroup with set of idempotents $E$ and let $Q$ be any $E \times E$ matrix over $S$ such that

(i) $q_{f, e} \in fSe$ for each $e, f \in E$,

(ii) $q_{e, e} = e$ for each $e \in E$.

Then $W = \{(e, x, f) : x \in eSf\}$ is easily seen to be a regular subsemigroup of $\mathcal{M}(S; E, E; Q)$. Indeed, since $q_{e, e} = e$ for each $e \in E$, $(e, x, f) \in W$ has an inverse $(f, x', e) \in W$ for each $x' \in V(x) \cap fSe \neq \emptyset$. Hence $W \subseteq \mathcal{R}\mathcal{M}(S; E, E, Q)$.

**Theorem 2.1.** Let $\psi: W \to T$ be a local isomorphism of $W$, as above, onto a regular semigroup $T$. Then $S$ is a locally isomorphic image of a regular Rees matrix semigroup over $T$.

Before giving the details of the proof we pause to indicate how this result might be applied to obtain matrix covers of a special type for a regular semigroup $S$.

**Corollary 2.2.** Let $\mathcal{C}$ be a class of regular semigroups which is closed under local isomorphic images and let $S$ be a regular semigroup with set of idempotents $E$. Suppose that there is an $E \times E$ matrix over $S$ such that (i) and (ii) hold and $W/p \in \mathcal{C}$. Then $S$ is a locally isomorphic image of a regular Rees matrix semigroup over a member of $\mathcal{C}$.

**Proof of Theorem 2.1.** Let $P$ be the $E \times E$ matrix over $T$ with

$$p_{e, f} = (e, ef, f)\psi$$
and form $\mathcal{M} = \mathcal{M}(T; E; P)$. For each $[e, (u, x, v)\psi, f] \in \mathcal{M}$ set

$$[e, (u, x, v)\psi, f] \theta = q_{ef} x q_{uf}. $$

Then $(u, x, v)\psi = (g, h, h)\psi$ implies, by Proposition 1.2, since $\psi$ is a local isomorphism, $(e, e, e)(u, x, v)(f, f, f) = (e, e, e)(g, y, h)(f, f, f)$; that is $(e, q_{ef} x q_{uf} f, f) = (e, q_{ef} y q_{uf} f, f)$. Hence $\theta$ is well defined.

Now let $[e_1, (u_1, x_1, v_1)\psi, f_1]$ and $[e_2, (u_2, x_2, v_2)\psi, f_2] \in \mathcal{M}$. Then

$$[e_1, (u_1, x_1, v_1)\psi, f_1][e_2, (u_2, x_2, v_2)\psi, f_2] \theta = [e_1, (u_1, x_1, v_1)\psi, f_1][e_2, (u_2, x_2, v_2)\psi, f_2] \theta $$

$$= q_{e_1 u_1} x_1 q_{v_1 f_1} q_{e_2 u_2} x_2 q_{v_2 f_2} $$

$$= [e_1, (u_1, x_1, v_1)\psi, f_1] \theta[e_2, (u_2, x_2, v_2)\psi, f_2] \theta $$

so that $\theta$ is a homomorphism of $\mathcal{M}$ into $S$. In particular, $\theta$ when restricted to $\mathcal{R} \mathcal{M} = \mathcal{R} \mathcal{M}(T; E, E; P)$ gives a homomorphism of $\mathcal{R} \mathcal{M}$ into $S$. To complete the proof, we show that $\theta$ is a local isomorphism of $\mathcal{R} \mathcal{M}$ onto $S$.

**Lemma 2.3.** $\theta$ is a homomorphism of $\mathcal{R} \mathcal{M}$ onto $S$.

**Proof.** Let $x \in S$. Then $x \in eSf$ for some $e, f \in E$. Then $x$ has an inverse $x' \in fSe$ and

$$[e, (e, x, f)\psi, f][f, (f, x', e)\psi, e] = [e, (e, x, f)\psi, f] $$

so that $[e, (e, x, f)\psi, f] \in \mathcal{R} \mathcal{M}$ and $[e, (e, x, f)\psi, f] \theta = x$ since $q_{ef} = e, q_{ff} = f$. Hence $\theta$ is onto $\mathcal{R} \mathcal{M}$.

**Lemma 2.4.** (i) $\mathcal{R} \mathcal{M} = \{[e, (u, x, v)\psi, f] : V(x) \cap q_{vf} S q_{uv} \neq \emptyset\}$;

(ii) $[e, (u, x, v)\psi, f]$ is idempotent if and only if $x = x q_{uf} q_{uv} x$.

**Proof.** (i) Suppose that $[e, (u, x, v)\psi, f] \in \mathcal{R} \mathcal{M}$. Then it has an inverse $[a, (b, y, c)\psi, d]$ so that

$$[e, (u, x, v)\psi, f] = [e, (u, x, v)\psi, f][a, (b, y, c)\psi, d][e, (u, x, v)\psi, f]. $$

Multiplying and comparing middle components, we get

$$(u, x)\psi = (u, x q_{uf} q_{ab} y q_{cd} q_{uv} x, v)\psi. $$

Hence, since $\psi$ is a local isomorphism, $x = x q_{uf} q_{ab} y q_{cd} q_{uv} x$ so that $x$ has an inverse in $q_{uf} y q_{uv}$. Conversely, suppose that $x$ has an inverse $q_{uf} y q_{uv}$; without loss of
generality, we may assume \( fy = y = ye \). Then straightforward calculation shows that

\[
[e, (u, x, v)\psi, f] [f, (f, y, e)\psi, e] [e, (u, x, v)\psi, f] = [e, (u, x, v)\psi, f]
\]

so that \([e, (u, x, v)\psi, f] \in \mathcal{R}. \mathcal{M}\).

(ii) \([e, (u, x, v)\psi, f] \) is idempotent if and only if \((u, x, v)\psi = (u, xq_v q_{eu} x, v)\psi\). Since \(\psi\) is a local isomorphism, this occurs if and only if \(x = xq_v q_{eu} x\).

**Lemma 2.5.** \(\theta\) is a local isomorphism.

**Proof.** Let \(e_i = [e_i, (u_i, x_i, v_i)]\psi, f_i\}, i = 1, 2\) be idempotents and suppose that \(a = [a, (b, y, c)\psi, d] \in \mathcal{R}. \mathcal{M}\). Then

\[
e_i e_2 = [e_1, (u_1, x_1 q_{v_1} f_1 q_{ab} q_{cd} q_{e_2 u_2} x_2, v_2)]\psi, f_2]
\]

which has the form

\[
[e_1, (u_1, x_1 q_{v_1} f_1, w q_{e_2 u_2} x_2, v_2)\psi, f_2],
\]

where \(w \in f_1 S e_2\). Thus

\[
e_i \mathcal{R}. \mathcal{M} e_2 \subseteq \{[e_1, (u_1, x_1 q_{v_1} f_1, w q_{e_2 u_2} x_2, v_2)\psi, f_2] : w \in f_1 S e_2\}.
\]

Suppose that \(w_1, w_2 \in f_1 S e_2\) and that

\[
[e_1, (u_1, x_1 q_{v_1} f_1, w_1 q_{e_2 u_2} x_2, v_2)\psi, f_2] \theta = [e_1, (u_1, x_1 q_{v_1} f_1, w_2 q_{e_2 u_2} x_2, v_2)\psi, f_2] \theta.
\]

Then

\[
q_{e_1 u_1} x_1 q_{v_1 f_1} w_1 q_{e_2 u_2} x_2 q_{v_2 f_2} = q_{e_1 u_1} x_1 q_{v_1 f_1} w_2 q_{e_2 u_2} x_2 q_{v_2 f_2}.
\]

Since \(e_1\) and \(e_2\) are idempotents, Lemma 2.4 shows that \(x_1 = x_1 q_{v_1 f_1} q_{e_2 u_2} x_1\). Hence, when the equation above is premultiplied by \(x_1 q_{v_1 f_1}\) and post multiplied by \(q_{e_2 u_2} x_2\), we get

\[
x_1 q_{v_1 f_1} w_1 q_{e_2 u_2} x_2 = x_1 q_{v_1 f_1} w_2 q_{e_2 u_2} x_2.
\]

Hence \(\theta\) is one-to-one on \(e_1 \mathcal{R}. \mathcal{M} e_2\).

The results in the following lemma, whose proof is straightforward, if some what tedious, are useful for proving converses of a number of the theorems in later sections.

**Lemma 2.6.** Let \(S\) be a regular semigroup, \(I, A\) nonempty sets and let \(P\) be a \(A \times I\) matrix over \(S\). Then
(i) each local submonoid of \( R = \mathcal{R}(S; I, A; P) \) is isomorphic to a local submonoid of \( S \);
(ii) the partially ordered set of principal ideals of \( R \) is isomorphic to the partially ordered set of principal ideals of \( SPS = \bigcup \{ Sp_{\lambda} S : i \in I, \lambda \in A \} \). (Note that \( R = \mathcal{R}(SPS; I, A; P) \).

**Corollary 2.7.** Let \( \mathcal{C} \) be a class of regular semigroups which is closed under local submonoids. Then

(i) every regular Rees matrix semigroup over a member of \( \mathcal{C} \) is locally in \( \mathcal{C} \);
(ii) every locally isomorphic image of a regular Rees matrix semigroup over a member of \( \mathcal{C} \) is locally in \( \mathcal{C} \).

Corollary 2.7 provides a sort of converse for Theorem 2.1.

### 3. Locally E-Solid Semigroups

Hall [5] has proved the following theorem which characterizes locally E-solid and locally orthodox semigroups in terms of locally inverse semigroups.

**Theorem 3.1.** Let \( S \) be a locally E-solid semigroup. Then there is a homomorphism \( \theta \) of \( S \) onto a locally inverse semigroup \( T \) such that \( e\theta^{-1} \) is completely simple for each idempotent \( e \) of \( S \).

If \( S \) is locally orthodox then there exists a homomorphism \( \theta \) of \( S \) onto a locally inverse semigroup \( T \) such that \( e\theta^{-1} \) is a rectangular band for each idempotent \( e \) of \( S \).

Hall points out that the second part of Theorem 3.1 can be deduced from [5, Theorem 1] and the results of Meakin and Nambooripad [9]. Among other results, [5, Theorem 1] shows that if \( S \) is locally orthodox the idempotents of \( eS \) and \( Se \) are bands for each idempotent \( e \). In [9], Meakin and Nambooripad considered regular semigroups with the latter property. They showed that such semigroups are coextensions of locally inverse semigroups by rectangular bands and provided a spined product decomposition for these semigroups analogous to Hall's decomposition theorem [14] for orthodox semigroups. Theorem 3.2 gives an alternative structure theorem for locally orthodox semigroups, as well as a structure theorem for locally E-solid semigroups.

**Theorem 3.2.** Let \( S \) be a regular semigroup.
(A) $S$ is locally $E$-solid if and only if it is a locally isomorphic image of a regular Rees matrix semigroup over an $E$-solid semigroup.

(B) $S$ is locally orthodox if and only if it is a locally isomorphic image of a regular Rees matrix semigroup over an orthodox semigroup.

Proof. Suppose that $S$ is a locally $E$-solid semigroup and fix an idempotent $e \in S$. For each idempotent $f \in S$, let $f^* \in S(e, f)$ and, for idempotents $f, g$ set

$$q_{f,g} = \begin{cases} f & \text{if } f = g \\ f^*g & \text{otherwise} \end{cases}$$

and form the semigroup $W = \{(f, x, g) \in E \times S \times E : x \in fSg\}$, using the matrix $Q = [q_{f,g}]$ for multiplication, as in Section 2. We aim to show that $W$ is $E$-solid and that, if $S$ is locally orthodox, then $W$ is orthodox. The construction in Section 2 then proves the theorem. That $W$ is $E$-solid, or orthodox, can be proved by lengthy case by case analyses like those used in the proof of [8, Theorem 2.11]. However, Theorem 3.1 allows us to take advantage of the work done in the proof of [8, Theorem 2.1] to obtain a more conceptual and cleaner verification. To do this we introduce another semigroup.

Let $\theta$ be a homomorphism of $S$ onto a locally inverse semigroup $T$, as in Theorem 3.1, and let

$$U = \{(f, t, g) \in E \times T \times E : t \in (fSg)\theta\}$$

be the semigroup obtained by using the sandwich matrix $N$, with $n_{f,g} = (q_{f,g})\theta$, to define multiplication. Then, since

$$n_{f,g} = \begin{cases} f & \text{if } f = g \\ f^*g & \text{if } f \neq g \end{cases}$$

wherin, e.g., $\bar{f} = f\theta$, the arguments involved in [8, Section 2] show that $U$ is an orthodox semigroup; indeed a locally inverse orthodox semigroup.

Further, the mapping $\phi: W \to U$ defined by $(f, x, g)\phi = (f, x\theta, g)$ is easily seen to be a homomorphism.

Lemma 3.3. $W$ is $E$-solid.

Proof. Suppose first that $(f, x, g)\phi$ belongs to a subgroup of $U$. Then $(f, x, g)\phi \mathcal{H}(f, xn_{x,f}, x, g)\phi$ which implies $x\theta \mathcal{H}(xq_{x,f}x)\theta$. Let $x' \in V(x)$, then $(x'x') \theta \mathcal{H}(xq_{x,f}x'x')\theta$ so that $xq_{x,f}x'$ and $xx'$ belong to a completely simple subsemigroup $(xx')\theta^{-1}$ of $S$. It follows that $xx'xq_{x,f}x'$ so that $xq_{x,f}x$. But this implies $(f, x, g)\mathcal{H}(f, xq_{x,f}x, g) = (f, x, g)^2$; that is, $(f, x, g)$ belongs to a subgroup of $W$. 


Now suppose that \( u, v, w \) are idempotents of \( W \) and that \( u \mathrel{\mathcal{L}} v \mathrel{\mathcal{R}} w \). Then \( u \mathrel{\mathcal{R}} u w \mathrel{\mathcal{L}} w \) and, since \( U = W \phi \) is orthodox, \( u w \phi \) is idempotent. Hence, by the first paragraph, \( u w \) is in a subgroup of \( W \). That is, there is an idempotent \( y \) such that \( u \mathrel{\mathcal{R}} y \mathrel{\mathcal{L}} w \). Hence \( W \) is \( E \)-solid.

The proof that \( W \) is orthodox, when \( S \) is locally orthodox, follows a similar pattern.

**Lemma 3.4.** If \( S \) is locally orthodox, then \( W \) is orthodox.

**Proof.** Suppose that \( (f, x, g) \phi \) is an idempotent of \( U \). Then \( x \theta = (x q_{\mathcal{R}} f x) \theta \) so that for \( x' \in V(x) \), \( (x x') \theta = (x q_{\mathcal{R}} f x x') \theta \) so that \( x x' \) and \( x q_{\mathcal{R}} f x x' \) belong to the rectangular band \( x x' \theta \theta^{-1} \) of \( S \). Hence \( x x' = x x'. x q_{\mathcal{R}} f x x' \) so that \( x = x q_{\mathcal{R}} f x \). Thus \( (f, x, g) \) is idempotent.

If \( u, w \) are idempotents of \( W \) then, since \( W \) is orthodox, \( (u, w) \phi \) is idempotent and, hence, so is \( u w \).

Lemmas 3.3 and 3.4, coupled with Theorem 2.1, provide the information necessary for the proof of Theorem 3.2. In the next section we shall prove an analog of Theorem 3.2 for locally \( \mathcal{L} \)-unipotent semigroups. Unfortunately, the argument employed in this section is not sufficiently precise to prove that \( W \phi / \rho \) is \( \mathcal{L} \)-unipotent; \( W \) itself need not be. It will be necessary to analyze the structure of \( W \) in a more detailed manner.

### 4. Locally \( \mathcal{L} \)-Unipotent Semigroups

In this section we shall show that a regular semigroup is locally \( \mathcal{L} \)-unipotent if and only if it is a locally isomorphic image of a regular Rees matrix semigroup over a locally \( \mathcal{L} \)-unipotent semigroup. The proof follows the procedure outlined in Section 2, and used in Section 3, except that we show that \( W \phi / \rho \) instead of \( W \) itself, is \( \mathcal{L} \)-unipotent. Since locally \( \mathcal{L} \)-unipotent regular semigroups are locally orthodox, Lemma 3.4 shows that \( W \), and thus \( W \phi / \rho \), is orthodox. To show that \( W \phi / \rho \) is locally \( \mathcal{L} \)-unipotent, it suffices, by Proposition 1.6, to show that the idempotents of \( W \) satisfy the identity

\[
ug_\mathcal{L} = u g_\mathcal{L}.
\]

To prove this we need some more information on the form of the idempotents of \( W \).

**Lemma 4.1.** Suppose that \( S \) is a locally orthodox semigroup and fix an idempotent \( e \) in \( S \). For idempotents \( f, g \in S \) set

\[
u_{f, g} = \begin{cases} f & \text{if } f = g \\
fg & \text{otherwise,} \end{cases}
\]
where $f^* \in S(e, f)$ and let $W = \{(f, x, g) \in E \times S \times E : x \in fSg\}$. Then every idempotent of $W$ is of the form

$$(f, f_1, f)(g, g_1, g) = (f, f_1, q_{fg}, g_1, g),$$

where $f_1 = f_1 \in fSf$, $g_1 = g_1 \in gSg$.

**Proof.** First, let $\theta : S \to T$ be a homomorphism of $S$ onto a locally inverse semigroup $T$ such that $f^*\theta^*$ is a rectangular band, for each idempotent $f \in S$; cf. Theorem 3.1. Then, by [8, Lemma 2.2]

$$n_{\theta f}n_{\theta g}n_{\theta f} = n_{\theta f} \quad \text{where} \quad n_{\theta f} = q_{\theta f}\theta.$$

It follows, as in Lemma 3.4, that $q_{\theta f}q_{\theta g}q_{\theta f} = q_{\theta f}$.

Now suppose that $(f, x, g)$ is idempotent, then

$$x = xq_{\theta f}x = xq_{\theta f} \cdot q_{\theta g} \cdot q_{\theta f}x = f_1q_{\theta g}g_1,$$

where $f_1 \in fSf$, $g_1 \in gSg$ are idempotents. Thus

$$(f, x, g) = (f, f_1, f)(g, g_1, g).$$

The relevance of Lemma 4.1 to our investigations stems from the following lemma.

**Lemma 4.2.** Let $B$ be a band with generators $A = \{a_i : i \in I\}$ and suppose that $a_ia_ia_j = a_ia_ja_j$ for all $i, j, k \in I$. Then $B$ satisfies the identity

$$u_{fg} = u_{fg}$$

for all $u, f, g \in B$.

**Proof.** Let $f = b_1 \cdots b_n$ where each $b_i \in A$. We first use induction on the length $n$ of $f$ to show that $a_ka_jfaj = a_ka_jfaj$ for all $a_k, a_j \in A$. If $n = 1$, this is immediate, by assumption, so assume $n > 1$. Then

$$a_ka_jfaj = a_ka_j(b_1 \cdots b_{n-2}) b_{n-1}b_na_j$$

$$= a_ka_j(b_1 \cdots b_{n-2}) b_{n-1}a_jb_na_j \quad \text{since} \quad b_{n-1}b_na_j = b_{n-1}a_jb_n a_j$$

$$= a_ka_j(b_1 \cdots b_{n-1}) a_jb_na_j$$

$$= a_k(b_1 \cdots b_{n-1}) a_jb_na_j \quad \text{by induction hypothesis}$$

$$= a_k(b_1 \cdots b_{n-2}) b_{n-1}a_jb_na_j$$

$$= a_k(b_1 \cdots b_{n-2}) b_{n-1}a_jb_na_j \quad \text{since} \quad b_{n-1}a_jb_na_j = b_{n-1}b_na_j$$

$$= a_kfaj.$$
Next we use induction on the length of $g = c_1 \cdots c_m$, $c_i \in A$, to show that $a_k f g = a_k g f g$ for each $a_k \in A$.

Finally, let $u = d_1 \cdots d_r$, where each $d_i \in A$. Then $d_r f g = d_r g f g$ immediately implies $u f g = u g f g$.

It follows from Lemmas 4.1 and 4.2 that, to show $W/\rho$ is $L$-unipotent, we need only verify that

$$(u, u_1, u)(f, f_1, f)(g, g_1, g) = (u, u_1, u)(g, g_1, g)(f, f_1, f)(g, g_1, g)$$

for idempotents $u_1 \leq u, f_1 \leq f, g_1 \leq g$ in $S$. This is the content of Lemma 4.3.

**Lemma 4.3.** Let $S$ be a locally $L$-unipotent regular semigroup and let $u, f_1 \leq f, g_1 \leq g$ be idempotents in $S$. Then

$$q_{uf} f_1 q_{fg} g_1 = q_{ug} g_1 q_{uf} f_1 q_{fg} g_1.$$  (**I**)

**Proof:** If $f = g$, this equation reduces to $q_{uf} f_1 g_1 = q_{ug} g_1 f_1 g_1$. This is true since $f_1, g_1$ are idempotents in $fSf$, which is $L$-unipotent, so that $f_1 g_1 = g_1 f_1 g_1$. Hence we may assume $f \neq g$.

We distinguish three subcases.

(a) $u = f$. Then the right side of (**I**) is

$$u^* g_1 g^* f_1 f^* g_1 = u^* g_1^* f_1^* g_1$$

$$= u^* g_1 f_1^* g_1$$

$$= u^* g_1 f_1^* g_1,$$

where, by Proposition 1.8, $g_1^* - g_1 g^* \in S(e, g_1), f_1^* = f_1 f^* \subseteq S(e, f_1)$. Thus, the right side of (**I**) is

$$u^* e g_1^* e f_1^* e g_1^* g_1$$

since, for example, $u^* \in S(e, u)$ implies $u^* = u^* e$. But then $e g_1^*, e f_1^*$ are idempotents in $eSe$ which is $L$-unipotent so that $e g_1^*, e f_1^* e f_1^* = e f_1^* e g_1^*$. Hence the right side of (**I**) reduces to

$$u^* e f_1^* e g_1^* g_1 = u^* f_1^* g_1$$

$$= f_1^* f g_1$$

$$= f_1 f g_1 = f_1 g_1$$

which is the left side of (**I**) since $f_1, f^*$ are idempotents of $fS$ and these, by Lemma 1.7, form an $L$-unipotent band.
(b) \( u = g \). The right side of \( (**') \) is \( g^*_1 f_1^* f_1 g_1 = g^*_1 f_1 g_1 = g^*_1 f_1^* \) where \( g^*_1 = g_1^* f_1^* = f_1^* g^*_1 \) are in \( S(e, g_1), S(e, f_1) \), respectively. The left side is \( g^*_1 f_1^* g^*_1 g_1 \), so it suffices to show that \( g^*_1 f_1^* g^*_1 = g^*_1 f_1^* g^*_1 \).

Now \( g^*_1, g^*_1, f^*_1 \) are idempotents in \( S(e, g_1) \) and, by Lemma 1.7(i), these form a band \( B = E(S(e, g_1)) \). Hence \( g^*_1 f_1^* g^*_1 = g^*_1 f_1^* \) and \( g^*_1 f_1^* g^*_1 = g^*_1 f_1^* g^*_1 = g^*_1 f_1^* g^*_1 \). Next, \( g^*_1 f_1^* g^*_1 = g^*_1 f_1^* \) since \( f_1^*, g^*_1 \) is a band. Thus \( g^*_1 f_1^* g^*_1 = g^*_1 f_1^* g^*_1 \) and so, since these are idempotents in \( gS \), it follows from Proposition 1.7(ii) that \( g^*_1 f_1^* g^*_1 = g^*_1 f_1^* g^*_1 \).

(c) \( u, f, g \) are all different. In this case, the left side of \( (**') \) is \( u^* f_1^* g_1 = u^* f_1^* g_1 \), where \( f_1^* = f_1^* \in S(e, f_1) \), \( g_1^* \in S(e, g_1) \). The right side is \( u^* g_1^* f_1^* g_1 \), so it suffices to show \( u^* g_1^* f_1^* g_1 = u^* g_1^* f_1^* g_1 \). Now

\[
\begin{align*}
  u^* f_1^* g_1 &= u^* e f_1^* e g_1^* \\
  &= u^* e g_1^* f_1^* e g_1^* \\
  &= u^* g_1^* f_1^* g_1^* \
\end{align*}
\]

since, e.g., \( u^* = u^* e g_1^* f_1^* e g_1^* \) since the idempotents of \( eS \) form an \( L \)-unipotent band, by Lemma 1.7(ii).

Hence the required equation holds.

**Corollary 4.4.** Let \( u, u_1, f, f_1, g, g_1 \) be idempotents in \( S \). Then

\[
(u, u_1, u)(f, f_1, f)(g, g_1, g) = (u, u_1, u)(g, g_1, g)(f, f_1, f)(g, g_1, g)
\]

If we combine Lemmas 4.2 and Corollary 4.4, it follows that \( W/\rho \) is \( L \)-unipotent and hence, by Theorem 2.1, \( S \) is a locally isomorphic image of a regular Rees matrix semigroup over the \( L \)-unipotent semigroup \( W/\rho \). Thus we have the direct half of the following theorem.

**Theorem 4.5.** Let \( S \) be a regular semigroup. Then \( S \) is locally \( L \)-unipotent if and only if it is a locally isomorphic image of a regular Rees matrix semigroup over an \( L \)-unipotent semigroup.

The proof of the converse of Theorem 4.5 is straightforward.

The arguments involved in proving Lemma 4.1 are typical of those required to prove the results of Section 3, without recourse to [8].

5. Semigroups Which Locally Are Unions of Groups

In this section, we shall characterize those regular semigroups which are locally isomorphic images of regular Rees matrix semigroups over unions of groups. The local submonoids of such a semigroup are automatically unions of groups. However, the converse is not true.
THEOREM 5.1. Let $S$ be a regular semigroup. Then $S$ is a locally isomorphic image of a regular Rees matrix semigroup over a union of groups if and only if

(i) $S$ is locally a union of groups;

(ii) the principal ideals of $S$ form a semilattice under intersection.

Proof. Let $T$ be a union of groups and let $R = \mathcal{M}(T; I, A; P)$ be a regular Rees matrix semigroup over $T$. Suppose further that $\theta$ is a local isomorphism of $R$ onto $S$. Then, by Lemma 2.6, $R$ is locally a union of groups and the partially ordered set of principal ideals of $R$ is isomorphic to that of $TPT$. Since the latter is a union of groups, its principal ideals form a semilattice under intersection. Hence the same is true of $R$.

Since $\theta$ is a local isomorphism, the local submonoids of $S$ are isomorphic to local submonoids of $R$ and so are unions of groups. Further, since $\theta$ is a local isomorphism, $a \theta = b \theta$, for $a, b \in R$, implies $a \mathcal{J} b$ so that $S$ and $R$ have isomorphic sets of principal ideals. Thus (i) and (ii) hold for $S$.

Conversely, suppose that (i) and (ii) hold and let $e, f$ be idempotents in $S$. Then there exists an idempotent $g$ such that $SeS \cap SfS = SgS$. Since $g \in SeS$, there exist $x \in S$, $x' \in V(x)$ such that $x'x = g$, $xx' \leq e$; likewise there exist $y \in S$, $y' \in V(y)$ such that $yy' = g$, $y'y \leq f$. Hence $xy \in eSf$ and $xyy'x' = xgx' = xx'$ so that $xy \mathcal{L} y g$; similarly $xy \mathcal{L} y g$. In particular $xy \mathcal{L} g$ so that $SeS \cap SfS = SgS = SxyS$.

For each pair of idempotents $e, f$ pick $q_{ef} \in eSf$ such that $SeS \cap SfS = Sq_{ef}S$ with $q_{ee} = e$. Let $W = \{(e, x, f) \in E \times S \times E : x \in eSf\}$ with multiplication induced by the matrix $Q$. We show that $W$ is a union of groups. Theorem 2.1 then completes the proof.

LEMMA 5.2. If $x \in eSf$, then $x \mathcal{L} q_{ef}x \mathcal{R} q_{ef}x$.

Proof. Let $x' \in V(x) \cap fSe$, $q_{ef} \in V(q_{ef}) \cap eSf$. Then $xx' \in SeS \cap SfS = Sq_{ef}S$, we have $xx' \leq x q_{ef} q_{ef}$ in $eSf$ which is a union of groups. Thus $xx' \mathcal{J} q_{ef} q_{ef} xx'$ in $eSe$ and so, since $q_{ef} q_{ef} xx' \in Sxx'$, we must have $xx' \mathcal{L} q_{ef} q_{ef} xx'$ so that $x \mathcal{L} q_{ef} x$.

That $x \mathcal{R} q_{ef} x$ follows dually.

COROLLARY 5.3. If $x \in eSf$, then $x \mathcal{L} q_{ef} x \mathcal{R} x q_{ef} x$.

Proof. We have, from Lemma 5.2,

$$Sx = Sq_{ef} x = S(q_{ef} x)^2 \subseteq Sx q_{ef} x \subseteq Sx$$

since $q_{ef} x \in fSf$ which is a union of groups. Thus $x \mathcal{L} q_{ef} x$. Similarly $x \mathcal{R} q_{ef} x$.

Now suppose that $(e, x, f) \in W$. Then $(e, x, f)^2 = (e, x q_{ef} x, f)$ and, since
flxqfQx, it is easy to see that $(e, xq_{x}, f) \mathcal{H}(e, x, f)$. Hence $(e, x, f) \mathcal{H}(e, x, f)^{2}$ which implies that $(e, x, f)$ belongs to a subgroup of $W$. That is, $W$ is a union of groups.

6. **Locally E-Solid Regular Semigroups**

Lallement [6] has shown that a regular ring is completely 0-simple as a multiplicative semigroup only if it is a division ring, while Chaptal [2] has shown that any ring whose multiplicative semigroup is a union of groups has central idempotents. More recently, Zeleznikow [11] proved that orthodox regular rings are also inverse. All these results are special cases of the following result.

**Proposition 6.1.** Let $R$ be a singular ring. Then its multiplicative semigroup is locally E-solid if and only if idempotents are central.

**Proof.** If the idempotents are central, then, certainly, $R$ is locally E-solid. To prove the converse, we shall make use of the following lemma due to Zeleznikow.

**Lemma 6.2** [11]. Let $R$ be a regular ring. Then the idempotents are central if and only if, for all idempotents $e, f \in R$, $ef = 0$ implies $fe = 0$.

Suppose now that $R$ is locally E-solid and that $ef = 0$ for some idempotents $e, f$. Then $g = e - fe$ is idempotent and $fg = 0 = gf$ so that $u = f + g$ is also idempotent. Further, $eu = e = ue$ and $fu = f = uf$ so that $e, f \in uRu$. Since $R$ is locally E-solid, $uSu$ is E-solid so the subsemigroup generated by its idempotents is a union of groups. Since $ef = 0$ we have $(fe)^{2} = 0$. Thus $fe$, being in the same subgroup as $(fe)^{2}$, must be 0. Hence $ef = 0$ implies $fe = 0$ and so, by Lemma 6.2, the idempotents of $R$ are central.

*Note added in proof.* Proposition 1.6 has been found independently by G. Gomes (Proc. Roy Soc. Edinburgh A 95 (1983), 59–71, Theorem 5.6).

**References**