Non-Abelian global vortices

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Abstract

We study topologically stable non-Abelian global vortices in the $U(N)$ linear sigma model. The profile functions of the solutions are numerically obtained. We investigate the behavior of vortices in two limits in which masses of traceless or trace parts of massive bosons are much larger than the others. In the limit that the traceless parts are much heavier, we find a somewhat bizarre vortex solution which can be identified with Abelian vortex with a non-integer $U(1)$ winding number $1/\sqrt{N}$ which is *irrational* in general.
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1. Introduction

Superfluid vortices appear in various condensed matter systems such as helium superfluid. They are global vortices in relativistic field theories [1]. When a global $U(1)$ symmetry is spontaneously broken, for instance, by the order parameter $\langle \phi \rangle = v$ with ϕ a complex scalar field in the Goldstone model, a $U(1)$ Nambu–Goldstone boson appears. Then in general there appears a global string asymptotically $\phi \sim v e^{i\theta}$ winding around the vacuum manifold (or the order parameter space) $U(1)$. The energy of global strings is logarithmically divergent in the infinitely large space, unlike local vortices with gauged $U(1)$. So less attention have been paid to global vortices as a candidate of cosmic strings compared with local counter part. When the net number

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of strings is zero, for instance, a pair of string and anti-string or string loops, their energy is finite and can be considered as cosmic strings [2,3]. Axion strings are such objects.

Natural non-Abelian extension of $U(1)$ vortices is $U(N)$ vortices in the $U(N)$ linear sigma model for which the order parameter is extended to an N by N complex matrix $\langle \Phi \rangle = v\mathbf{1}_N$.[1] Such non-Abelian vortices are expected to form during the chiral phase transition in QCD, in which case the field Φ of a 3 by 3 matrix ($N = 3$) is a condensate of quark–anti-quark $\langle \Phi \rangle \sim \langle \bar{q}q \rangle$. At the chiral phase transition [4], the chiral symmetry $SU(N)_L \times SU(N)_R$ is spontaneously broken down to its diagonal symmetry $SU(N)_V$. According to this breaking, massless Nambu–Goldstone bosons appear as pions (or more generally mesons). At the same time, the axial symmetry $U(1)_A$ is also spontaneously broken and the η' meson appears. However, $U(1)_A$ is explicitly broken by the axial anomaly at zero temperature, giving a mass to the η' meson. It has been argued that the axial anomaly might disappear and $U(1)_A$ is approximately recovered at high temperature [5]. Although there is still an ambiguity if it occurs below the temperature of the chiral phase transition, let us consider such a situation. Then the breaking pattern is $U(1)_A \times SU(N)_L \times SU(N)_R \to SU(N)_V$ apart from the discrete symmetry, and the vacuum manifold is

$$\frac{SU(N)_A \times U(1)_A}{\mathbb{Z}_N} \simeq U(N)_A. \tag{1.1}$$

The first homotopy group $\pi_1[U(N)_A] \simeq \mathbb{Z}$ is non-trivial and so there exist topologically stable vortex-strings in this breaking. The simplest vortex appearing is a $U(1)$ vortex-string called the η' string asymptotically given by $\Phi \sim ve^{i\theta}\mathbf{1}_N$, which winds around $U(1)_A$ once (2π) [6,7].[2] However, this is not the minimum vortex-string because there exists a smaller loop in the vacuum manifold (1.1), the minimum string is a non-Abelian string asymptotically given by $\Phi \sim v\,\mathrm{diag}(e^{i\theta}, 1, \ldots, 1) = ve^{i\frac{\theta}{N}}\,\mathrm{diag}(e^{i\frac{N-1}{N}\theta}, e^{-i\frac{\theta}{N}}, \ldots, e^{-i\frac{\theta}{N}})$, which winds $U(1)_A$ as well as the $SU(N)_A$ [9]. The important is that this string winds $2\pi/N$ of $U(1)_A$ and therefore its tension is $1/N$ of that of the $U(1)$ η'-string. Such vortices with fractional $U(1)$ winding number often appear in various condensed matter systems such as Bose–Einstein condensates and certain types of superconductors, and are called "fractional vortices".[3]

At low temperature the axial anomaly induces the periodic potential in $U(1)_A$. The $U(1)_A$ is broken to $\mathbb{Z}_N$ and there appear N disconnected vacua, in each of which the η' meson gets mass. A $U(1)$ η'-string is accompanied with N domain walls and the total configuration becomes N domain wall junction with a string at the junction line [10]. Balachandran et al. have discussed a possible role of such an object in the early universe [10].

The presence of a non-Abelian vortex breaks the $SU(N)_V$ symmetry of vacua to its subgroup $SU(N-1)_V \times U(1)_V$ and consequently there appear further Nambu–Goldstone modes

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[1] Other non-Abelian global vortices appear in the B-phase of ^{3}He in which symmetry is broken as $SO(3)_S \times SO(3)_L \times U(1) \to SO(3)_{S+L}$. In this case the corresponding vacuum manifold is $U(1) \times SO(3)$ and the first homotopy group is $\pi_1[U(1) \times SO(3)] \simeq \mathbb{Z} \oplus \mathbb{Z}_2$.

[2] Brandenberger et al. have discussed non-topological strings, called the pion strings, using massless Nambu–Goldstone (pions) $SU(2)_A$ [6–8]. They are topologically unstable because of $\pi_1[SU(N)] = 0$. In this sense non-Abelian strings below are made of both η' mesons and pions.

[3] For instance in the polar phase of a spin 1 spinor Bose–Einstein condensate, $U(1)_\Phi \times SO(3)_S$ is spontaneously broken down to $U(1)_{\Phi+S} \rtimes (\mathbb{Z}_2)_{\Phi+S}$ with $\rtimes$ denoting a semi-direct product. Then the vacuum manifold is $M \simeq [U(1)_\Phi \times SO(3)_S]/[U(1)_{\Phi+S} \rtimes (\mathbb{Z}_2)_{\Phi+S}] \simeq [U(1) \times S^2]/\mathbb{Z}_2$ [15] and the first homotopy group $\pi_1(M) \simeq \mathbb{Z}$ supports half quantized vortices [16]. Similarly to this, 1/3 quantized vortices exist in the cyclic phase of a spin 2 spinor Bose–Einstein condensate [17].

$\mathbb{C}P^{N-1} \simeq SU(N)/[SU(N-1) \times U(1)]$ which are orientations in the internal space [11]. This idea was brought from the local $U(N)$ vortices [12] for which $U(1)_A$ and $SU(N)_L$ are gauged. However, there exists a crucial difference between global and local $U(N)$ vortices. The Nambu–Goldstone modes $\mathbb{C}P^{N-1}$ of local $U(N)$ vortices are localized around the vortex and become the moduli (or collective coordinates) of the vortex [12] while those of global $U(N)$ vortices are not localized but spread to infinity (or the boundary of a finite space). Having this in mind, an inter-string force between two parallel global $U(N)$ vortex-strings with different orientations in the internal space has been calculated recently [13,14]. The force depends on the relative orientation: it reaches the maximum when two strings wind around the same component of the vacuum expectation value [for instance, diag($e^{i\theta}, 1, \ldots, 1$) and diag($e^{i\theta}, 1, \ldots, 1$)], and it vanishes when the two strings wind around different components of the vacuum expectation value [for instance, diag($e^{i\theta}, 1, 1, \ldots, 1$) and diag($1, e^{i\theta}, 1, \ldots, 1$)]. This result implies that a $U(1)$ η'-string is marginally decomposed into N pieces of non-Abelian strings: $e^{i\theta}\mathbf{1}_N \to$ diag($e^{i\theta}, 1, 1, \ldots, 1$) + diag($1, e^{i\theta}, 1, \ldots, 1$) + $\cdots$. Such decomposition necessary occurs at finite temperature where the free energy is minimized instead of the energy. Therefore at low temperature with the axial anomaly, a domain wall junction [10] is unstable, because a $U(1)$ string is pulled by each of N domain walls and is decomposed into N non-Abelian strings to each of which one domain wall is attached. In the end each piece is pulled to infinity in each of N directions.[4]

In this paper we study the purely string solution without domain walls in the linear sigma model without the axial anomaly. Numerical solutions themselves were previously obtained in [11]. Here we study profile functions of solutions in much more detail with more accuracy. By using the relaxation method, we numerically determine the shooting parameters of the solutions up to fifth order. We investigate the dependence of the profiles of the vortex to the parameters in the linear sigma model. We also study the two limits in which the masses of traceless or trace parts of the massive bosons in the linear sigma model are much larger than the others. In these limits, the model reduces some non-linear sigma models. In the limit that the trace parts are much heavier, the equations for the profiles of the $U(N)$ vortex become sine-Gordon-like. The solution remains to be regular. On the other hand, in the limit that the traceless parts are much heavier, we find somewhat surprising solution; the $U(N)$ vortex solution reduces to a singular $U(1)$ vortex with the $U(1)$ winding number $1/\sqrt{N}$ which is *irrational* specifically for $N = 2, 3, 5, 6, \ldots$. In general, profiles of $U(1)$ vortices with a non-integer $U(1)$ winding number ($\leqslant 1$) are of course singular. Interesting is that such an "*irrational vortex*" naturally appears in a particular limit of a regular non-Abelian vortex solution. As far as we know such a vortex has not been reported yet in the literature.

This paper is organized as follows. In Section 2 we review the $U(1)$ global vortex solution in the Goldstone model. We give numerical solutions and study their asymptotic behaviors. In Section 3 we study the non-Abelian vortex solution in the $U(N)$ linear sigma model. We derive profile functions of the minimum $U(N)$ vortex with the $U(1)$ winding number $1/N$ for $N = 2, 3, \ldots, 10$ and determine the shooting parameters up to fifth order. We also discuss various limits by sending some of masses to infinity. We find that in a particular limit the non-Abelian $U(N)$ vortex reduces to an Abelian vortex with the irrational $U(1)$ winding number $1/\sqrt{N}$. Section 4 is devoted to conclusion and discussion. Behaviors of vortex solutions in the large N limit is discussed in Appendix A.

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[4] Stable domain wall junctions or networks exist in gauged $U(N)$ linear sigma models with appropriate masses [18].

2. Global $U(1)$ vortices

2.1. The Goldstone model and vortex solutions

Let us begin with giving a review on a global vortex-string solution in the Goldstone model with a complex scalar field $\phi(x)$

$$\mathcal{L} = |\partial_\mu \phi|^2 - \lambda \big(|\phi|^2 - v^2\big)^2. \tag{2.1}$$

We choose $\lambda > 0$ and $v^2 > 0$ for stable vacua with broken $U(1)$ symmetry. The scalar potential is like a wine bottle, so we have an S^1 vacuum space with radius $|\phi| = v$. When we choose a vacuum $\phi = v$ and consider small fluctuations as $\phi = v + \frac{\varphi_1 + i\varphi_2}{\sqrt{2}}$ ($\varphi_{1,2} \in \mathbb{R}$), the Lagrangian in terms of the small fluctuations up to quadratic terms is of the form

$$\mathcal{L}^{(2)} = \frac{1}{2}(\partial_\mu \varphi_1)^2 + \frac{1}{2}(\partial_\mu \varphi_2)^2 - 2\lambda v^2 \varphi_1^2. \tag{2.2}$$

This shows that φ_1 is massive and φ_2 is a massless Nambu–Goldstone mode:

$$m_1^2 \equiv 4\lambda v^2, \qquad m_2^2 = 0. \tag{2.3}$$

The equation of motion of ϕ reads

$$\partial_\mu \partial^\mu \phi + 2\lambda \phi \big(|\phi|^2 - v^2\big) = 0. \tag{2.4}$$

A global vortex-string extending linearly to the x_3-axis is obtained by solving (2.4) with an axisymmetric vortex ansatz in the cylindrical coordinates $x_1 + ix_2 = re^{i\theta}$, given by

$$\phi(r, \theta) = ve^{ik\theta} f(r), \quad k \in \mathbb{Z}, \tag{2.5}$$

with the boundary conditions

$$\lim_{r \to \infty} f(r) = 1, \qquad \lim_{r \to 0} f(r) = 0. \tag{2.6}$$

Plugging the ansatz (2.5) into Eq. (2.4), we get a second order differential equation

$$f'' + \frac{f'}{r} - \frac{k^2 f}{r^2} - \frac{m_1^2}{2} f\big(f^2 - 1\big) = 0. \tag{2.7}$$

Numerical solutions for $k = 1, 2, \ldots, 10$ with the boundary conditions (2.6) are plotted in the left panel of Fig. 1.

The energy of the vortex solution can be expressed as

$$E = 2\pi \int_0^\infty dr\, r\mathcal{E} = 2\pi v^2 \int_0^\infty dr\, r \left[f'^2 + \frac{k^2 f^2}{r^2} + \frac{m_1^2}{4}\big(f^2 - 1\big)^2 \right], \tag{2.8}$$

where the first two terms come from the derivative of the field ϕ and the last term is from the scalar potential. Since $f \to 1$ as $r \to \infty$, the kinetic energy logarithmically diverges. So the energy of the vortex-string consists of a finite part and a logarithmically divergent part as

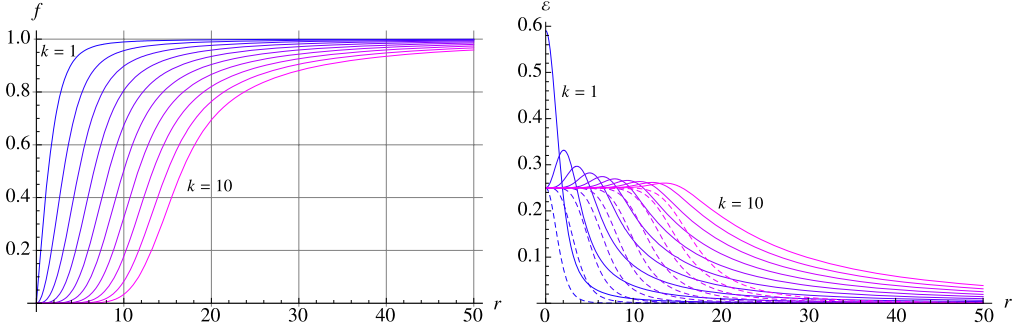$$E(k) = E_{\text{der}}(k) + E_{\text{div}}(k) + E_{\text{pot}}(k), \tag{2.9}$$

Fig. 1. Left panel shows the profile functions and the corresponding energy densities $\mathcal{E}(k)$ (solid lines) and $\mathcal{E}_{\text{pot}}(k)$ (broken lines) are plotted in the right panel for $k = 1, 2, \ldots, 10$ and $m_1 = 1$.

where we have defined

$$E_{\text{der}}(k) \equiv 2\pi v^2 \int_0^\infty dr\, r f'^2, \tag{2.10}$$

$$E_{\text{pot}}(k) = 2\pi \int_0^\infty dr\, r \mathcal{E}_{\text{pot}} \equiv \frac{\pi v^2 m_1^2}{2} \int_0^\infty dr\, r \left(f^2 - 1\right)^2, \tag{2.11}$$

$$E_{\text{div}}(k) \equiv 2\pi v^2 k^2 \int_0^\infty dr\, \frac{f^2}{r} = \text{const} + 2\pi v^2 k^2 \lim_{L \to \infty} \log \frac{L}{r_0}. \tag{2.12}$$

Here we have introduced an IR cut-off L which is the size of the system and $r_0 \simeq m_1^{-1}$ is a typical size of the $U(1)$ global vortex such as $|f - 1| \ll 1$ as $r \gg r_0$. We can analytically calculate E_{pot} as follows. Let us first introduce a dimensionless coordinate $\rho \equiv m_1 r$. The equation becomes independent of the coupling constants

$$F\big[f(\rho); k\big] \equiv f'' + \frac{f'}{\rho} - \frac{k^2 f}{\rho^2} - \frac{1}{2} f\left(f^2 - 1\right) = 0. \tag{2.13}$$

Then we see that E_{pot} is independent of the scalar mass m_1:

$$E_{\text{pot}}(k) = \frac{\pi v^2 m_1^2}{2} \int_0^\infty dr\, r \left(f^2 - 1\right)^2 = 2\pi v^2 \int_0^\infty d\rho\, \rho\, \frac{(f^2 - 1)^2}{4}. \tag{2.14}$$

To calculate this, let us make a trick

$$0 = 2\rho^2 f' F[f; k] = \frac{d}{d\rho}\left[\rho^2 f'^2 - k^2 f^2 - \frac{\rho^2}{4}\left(f^2 - 1\right)^2\right] + \frac{\rho}{2}\left(f^2 - 1\right)^2. \tag{2.15}$$

By using this equation, we can bring E_{pot} in the following form

$$E_{\text{pot}}(k) = \pi v^2 \left[\rho^2\left(-f'^2 + \frac{1}{4}\left(f^2 - 1\right)^2\right) + k^2 f^2\right]_0^\infty = \pi v^2 k^2. \tag{2.16}$$

This formula is called the Derrick–Pohozaev identity in the literature [19]. To derive the right-most equality, we have used the asymptotic behaviors $f \propto \rho^k$ at $\rho \ll 1$ and $f = 1 - a_2/\rho^2 + \mathcal{O}(\rho^{-4})$ for $\rho \gg 1$, which will be obtained in the next subsection. We plot the total and potential energy densities $\mathcal{E}$ and $\mathcal{E}_{\text{pot}}$, respectively of our numerical solutions in the right panel of Fig. 1. We also have numerically checked $E_{\text{pot}}/(\pi v^2 k^2) = 0.999 + \mathcal{O}(10^{-4})$. We can show that E_{der} is also finite.

Note that higher winding solutions ($k \geqslant 2$), especially co-axial vortices as we assumed above, are not stable. Since the energy of the co-axial vortices is almost proportional to k^2, distant vortices are energetically preferred ($k^2 > k$ for $k \geqslant 2$). Therefore, the above static solutions with $k > 1$ are artifacts of our co-axial ansatz and boundary conditions.

2.2. Asymptotic behaviors at $r \to 0$ and $r \to \infty$

Let us investigate asymptotic behavior at $m_1 r = \rho \ll 1$ where the profile function is very small $|f| \ll 1$. We expand the profile function as

$$f(\rho) = \sum_{n=1}^{\infty} a_n \rho^n. \tag{2.17}$$

By substituting this in Eq. (2.13), we find that the first non-zero coefficient is that of ρ^k

$$a_k = \lim_{\rho \to 0} \frac{f(\rho)}{\rho^k}, \tag{2.18}$$

which is sometimes called the shooting parameter and which may be determined by making use of numerical solutions. Such parameters are important since we can uniquely determine solutions with the parameters. In the minimal winding ($k = 1$) vortex, we got $a_1 = 0.4123772$ and

$$f_{k=1}(\rho) = a_1 \rho - \frac{a_1}{16} \rho^3 + \frac{a_1 + 16a_1^3}{768} \rho^5 - \frac{a_1 + 160a_1^3}{73728} \rho^7 + \mathcal{O}(\rho^9). \tag{2.19}$$

In principle, one can infinitely increase accuracy of the approximation with the unique parameter a_1. Although we assumed $\rho \ll 1$ in the beginning, we can reach at $\rho_0 \gtrsim 1$ with a good accuracy by increasing the order of expansion.

Let us next discuss asymptotics at $r \to \infty$. We expand solution in the following way

$$f(\rho) = 1 - \sum_{i=1}^{\infty} \frac{b_i}{\rho^i}, \quad \text{for } \rho \gg 1, \tag{2.20}$$

b_i being constant. Unlike the expansion parameters $\{a_i\}$ in Eq. (2.17), the expansion parameters $\{b_i\}$ can be precisely determined. To this end, we insert (2.20) into $F[f; k]$ given in Eq. (2.13) and determine b_i by comparing terms order by order. Then we get $b_{\text{odd}} = 0$ and

$$b_2 = k^2, \qquad b_4 = \frac{1}{2} k^2 (8 + k^2), \qquad b_6 = \frac{1}{2} k^2 (128 + 32k^2 + k^4), \quad \dots. \tag{2.21}$$

We can go on up to order which we desire.

3. Global $U(N)$ vortices

Let us study non-Abelian global vortices. Basic analysis on them have been done in [11]. Here we are going to push forward the analysis of [11] on vortex solutions by investigating the equations of motion in much detail.

3.1. The $U(N)$ linear sigma model and vortex solutions

The model that we consider here is a natural extension of the Goldstone model given in Eq. (2.1). It is the $SU(N)_{\text{L}} \times SU(N)_{\text{R}} \times U(1)_{\text{A}}$ linear sigma model for an $N \times N$ complex matrix of scalar fields $\Phi(x)$, given by

$$\mathcal{L} = \text{Tr}\big[\partial_\mu \Phi^\dagger \partial^\mu \Phi - \lambda_2 (\Phi^\dagger \Phi)^2 + \mu^2 \Phi^\dagger \Phi\big] - \lambda_1 \big(\text{Tr}[\Phi^\dagger \Phi]\big)^2 - \frac{\mu^4 N}{4(N\lambda_1 + \lambda_2)}, \tag{3.1}$$

where the last constant term is introduced for the vacuum energy to vanish. For a stability of vacua, we consider the parameter region $\mu^2 > 0$, $\lambda_2 > 0$ and $N\lambda_1 + \lambda_2 > 0$. The chiral symmetry $SU(N)_{\text{L}} \times SU(N)_{\text{R}}$ and the axial symmetry $U(1)_{\text{A}}$ act on Φ as

$$\Phi \to e^{i\theta} g_L \Phi g_R, \quad \big(e^{i\theta}, g_L, g_R\big) \in U(1)_{\text{A}} \times SU(N)_{\text{L}} \times SU(N)_{\text{R}}. \tag{3.2}$$

However, unlike the usual case in the absence of $U(1)_{\text{A}}$ broken by the axial anomaly, the structure of discrete symmetries becomes somewhat complicated in the presence of $U(1)_{\text{A}}$. Here we explain it in detail. First, the group action G on Φ is not Eq. (3.2) itself but is given by

$$G = \frac{U(1)_{\text{A}} \times SU(N)_{\text{L}} \times SU(N)_{\text{R}}}{\mathbb{Z}_N \times \mathbb{Z}_N}, \tag{3.3}$$

where the following $\mathbb{Z}_N \times \mathbb{Z}_N$ action does not act on Φ and therefore is removed from G:

$$\big(\omega^{-k-l}, \omega^k \mathbf{1}_N, \omega^l \mathbf{1}_N\big) \in U(1)_{\text{A}} \times SU(N)_{\text{L}} \times SU(N)_{\text{R}},$$
$$\omega \equiv e^{2\pi i/N} \quad (k, l = 0, 1, 2, \ldots, N-1). \tag{3.4}$$

For later use let us redefine these discrete groups as $\mathbb{Z}_N \times \mathbb{Z}_N \simeq (\mathbb{Z}_N)_{\text{V}} \times (\mathbb{Z}_N)_{\text{A}}$ with

$$(\mathbb{Z}_N)_{\text{V}}: \quad \big(1, \omega^k \mathbf{1}_N, \omega^{-k} \mathbf{1}_N\big) \in U(1)_{\text{A}} \times SU(N)_{\text{L}} \times SU(N)_{\text{R}}, \tag{3.5}$$

$$(\mathbb{Z}_N)_{\text{A}}: \quad \big(\omega^{-2k}, \omega^k \mathbf{1}_N, \omega^k \mathbf{1}_N\big) \in U(1)_{\text{A}} \times SU(N)_{\text{L}} \times SU(N)_{\text{R}}. \tag{3.6}$$

By using the symmetry G, any vacuum can be transformed into the form

$$\langle \Phi \rangle = v \mathbf{1}_N, \quad v^2 = \frac{\mu^2}{2(N\lambda_1 + \lambda_2)}. \tag{3.7}$$

We can consider this vacuum without loss of generality. All other degenerate vacua are obtained from this by the G transformations. The symmetry G is spontaneously broken down to the isotropy group

$$H = \frac{SU(N)_{\text{V}} \times (\mathbb{Z}_N)_{\text{A}}}{\mathbb{Z}_N \times \mathbb{Z}_N} = \frac{SU(N)_{\text{V}}}{(\mathbb{Z}_N)_{\text{V}}}, \tag{3.8}$$

where $\mathbb{Z}_N \times \mathbb{Z}_N$ is the one of Eq. (3.4) which can be rewritten by Eqs. (3.5) and (3.6), and $SU(N)_{\text{V}}$ is given by

$$SU(N)_{\text{V}}: \quad (1, g, g^\dagger) \in U(1)_{\text{A}} \times SU(N)_{\text{L}} \times SU(N)_{\text{R}}. \tag{3.9}$$

When the vacuum $\langle \Phi \rangle$ is transformed by G, the isotropy group H is also transformed by a similarity transformation, but all of them are isomorphic to the original isotropy group (3.8). Therefore the vacuum manifold (the order parameter space) can be written as a coset space:

$$\frac{G}{H} = \frac{(U(1)_A \times SU(N)_L \times SU(N)_R)/(\mathbb{Z}_N \times \mathbb{Z}_N)}{SU(N)_V/(\mathbb{Z}_N)_V}$$
$$\simeq \frac{U(1)_A \times SU(N)_A}{(\mathbb{Z}_N)_A} \simeq U(N)_A, \tag{3.10}$$

which is eventually a group manifold (because H is a normal subgroup of G). Since the first homotopy group of the vacuum manifold

$$\pi_1[U(N)_A] \simeq \mathbb{Z} \tag{3.11}$$

is non-trivial, it admits topological vortex-string solutions.

In order to find the mass spectrum, let us perturb Φ with small fluctuations as

$$\Phi(x) = v\mathbf{1}_N + \phi(x)\mathbf{1}_N + \chi_a(x)T^a \quad (a = 1, 2, \ldots, N^2 - 1), \tag{3.12}$$

with the generators T^a of $SU(N)$ ($\text{Tr}[T^a T^b] = \delta^{ab}$). Then it is turned out that imaginary parts of ϕ and χ_a are massless Nambu–Goldstone modes parametrizing the vacuum manifold $U(N)_A$ in Eq. (3.10), while their real parts are massive:

$$m_\phi^2 = 2\mu^2, \qquad m_\chi^2 = \frac{2\lambda_2}{N\lambda_1 + \lambda_2}\mu^2 = 4\lambda_2 v^2. \tag{3.13}$$

The original coupling constants are expressed by these masses as

$$\mu^2 = \frac{m_\phi^2}{2}, \qquad \lambda_1 = \frac{m_\phi^2 - m_\chi^2}{4Nv^2}, \qquad \lambda_2 = \frac{m_\chi^2}{4v^2}. \tag{3.14}$$

The scalar potential can be rewritten by the dimensionful parameters v, m_ϕ and m_χ as

$$V = \frac{m_\phi^2}{4Nv^2}\big(\text{Tr}[\Phi^\dagger\Phi - v^2\mathbf{1}_N]\big)^2 + \frac{m_\chi^2}{4v^2}\text{Tr}[\langle\Phi^\dagger\Phi\rangle^2], \tag{3.15}$$

where we have introduced a notation $\langle X \rangle \equiv X - \frac{\text{Tr}[X]}{N}\mathbf{1}_N$ for $N \times N$ matrix X (note $\langle\mathbf{1}_N\rangle = 0$). Let us introduce a ratio of the two masses[5]

$$\tau \equiv \frac{m_\chi}{m_\phi} = \sqrt{\frac{\lambda_2}{N\lambda_1 + \lambda_2}}. \tag{3.16}$$

We will see that τ determines all properties of non-Abelian global vortices.

Let us construct vortex-string solutions in this model. Firstly, one may consider the following simple boundary condition

$$\lim_{r \to \infty} \Phi(r, \theta) = ve^{i\theta}\mathbf{1}_N. \tag{3.17}$$

A natural ansatz with this boundary condition is

$$\Phi(r, \theta) = ve^{i\theta}f(r)\mathbf{1}_N, \tag{3.18}$$

with boundary conditions $f(0) = 0$ and $f(\infty) = 1$. The equation of motion for the profile function $f(r)$ is identical to that of the global $U(1)$ vortex with $k = 1$

$$f'' + \frac{f'}{r} - \frac{f}{r^2} - \frac{m_\phi^2}{2}f(f^2 - 1) = 0. \tag{3.19}$$

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[5] Instead of this, $\kappa = \lambda_1/\lambda_2$ was used to parametrize solutions in [11].

So the solution is just embedding of the global $U(1)$-vortex of Eq. (2.13). It is important to observe that neither N nor m_χ appears. Furthermore, only the overall $U(1)$ phase winds once when we go around the vortex solution. This Abelian vortex solution is called the η' string [7]. However, as we will see, this solution is not minimal [11] and is broken into N minimal solutions [13].

A minimal winding vortex configuration can be obtained by taking the following boundary condition

$$\lim_{r \to \infty} \Phi(r, \theta) = v \operatorname{diag}(e^{i\theta}, 1, \ldots, 1) = v e^{i\frac{\theta}{N}} \operatorname{diag}(e^{i\frac{N-1}{N}\theta}, e^{-i\frac{\theta}{N}}, \ldots, e^{-i\frac{\theta}{N}}). \tag{3.20}$$

The key point here is that the overall $U(1)$-phase winds only $2\pi/N$ around the vortex and the rests $2\pi(N-1)/N$ or $-2\pi/N$ are inside the non-Abelian group $SU(N)_A$. As a consequence, the tension of the vortex is $1/N$ of the above Abelian solution (3.18). Because the vacuum space is $U(N)_A$, the above vortex is called a non-Abelian global vortex or more specifically a global $U(N)$ vortex. Corresponding vortex ansatz is of the form

$$\Phi(r, \theta) = v \operatorname{diag}(e^{i\theta} f(r), g(r), \ldots, g(r)), \tag{3.21}$$

with the boundary conditions

$$\lim_{r \to \infty} f = \lim_{r \to \infty} g = 1, \qquad \lim_{r \to 0} f = 0, \qquad \lim_{r \to 0} g' = 0. \tag{3.22}$$

Since f is the profile function of a winding scalar field, it must go to zero at the origin. On the other hand, g does not have to vanish.

For a fixed $\theta = \theta_0$, the field Φ behaves at the boundary as $\Phi(\theta = \theta_0, r \to \infty) = v \operatorname{diag}(e^{i\theta_0}, 1, \ldots, 1)$, where the isotropy group is not the same with the one in Eq. (3.8). Instead, it is obtained from H in Eq. (3.8) by a G transformation $(e^{i\frac{\theta_0}{N}}, g_0, g_0) \in U(1)_A \times SU(N)_L \times SU(N)_R$ with $g_0 = \operatorname{diag}(e^{i\frac{N-1}{2N}\theta_0}, e^{-i\frac{\theta_0}{2N}}, \ldots, e^{-i\frac{\theta_0}{2N}})$, and therefore it is isomorphic to $H = SU(N)_V/(\mathbb{Z}_N)_V$ in Eq. (3.8). Around the vortex where $g(r)$ differs from $f(r)$ (especially at the center $\Phi(0) = v \operatorname{diag}(0, g(0), \ldots, g(0))$), the isotropy group $H_{\theta=\theta_0} = SU(N)_{V,\theta_0}/(\mathbb{Z}_N)_{V,\theta_0}$ is further broken to its subgroup $[SU(N-1)_V \times U(1)_V]/(\mathbb{Z}_N)_{V,\theta_0}$. This gives rises to further Nambu–Goldstone modes [11]

$$\mathbb{C}P^{N-1} \simeq \frac{SU(N)_V}{SU(N-1)_V \times U(1)_V}. \tag{3.23}$$

In the case of the local $U(N)$ vortices for which $U(1)$ and $SU(N)_L$ are gauged [12], the corresponding modes are called the orientational zero modes of a vortex. However, these are non-normalizable for a global $U(N)$ vortex (in the infinite space), unlike the local $U(N)$ vortex, because the isotropy groups H_θ depends on $\theta = \theta_0$ and differ from each other, and consequently the wave functions of the Nambu–Goldstone modes $\mathbb{C}P^{N-1}$ spread to infinity.

Let us investigate concrete profile functions of vortex solutions. The Hamiltonian density and the energy densities in terms of the profile functions $f(r)$ and $g(r)$ is given by

$$\mathcal{H} = 2\pi v^2 r \mathcal{E}, \tag{3.24}$$

$$\mathcal{E} = f'^2 + \frac{f^2}{r^2} + (N-1)g'^2 + \mathcal{V}, \tag{3.25}$$

$$\mathcal{V} = \frac{m_\phi^2}{4N}(f^2 + (N-1)g^2 - N)^2 + \frac{(N-1)m_\chi^2}{4N}(f^2 - g^2)^2. \tag{3.26}$$
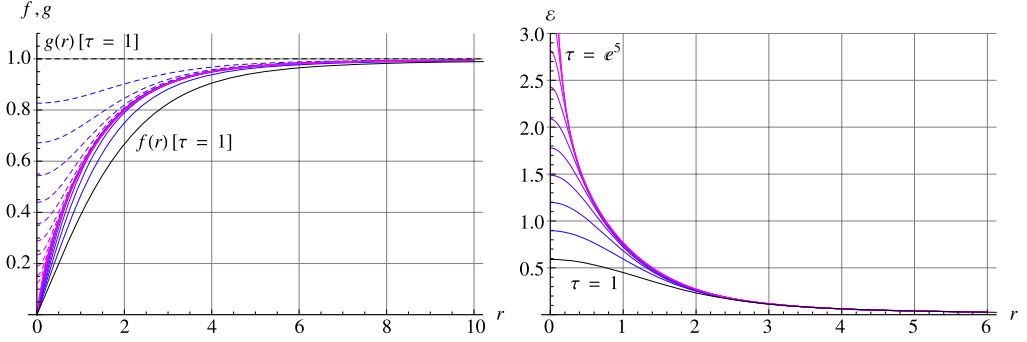
Fig. 2. Configurations (solid lines for $f(r)$ and broken lines for $g(r)$) in the left panel and energy densities in the right panel for $\log \tau = 0, 1/2, 1, 3/2, \ldots, 9/2, 5$ for $N = 2$.
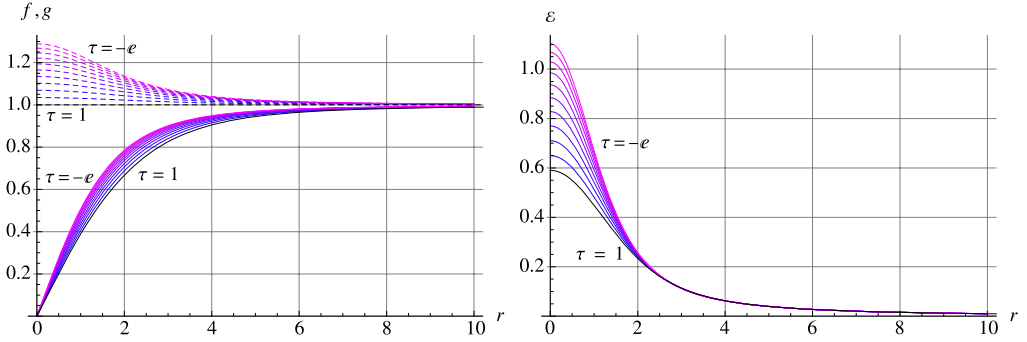

Fig. 3. Configurations (solid lines for $f(r)$ and broken lines for $g(r)$) in the left panel and energy densities in the right panel for $-\log \tau = 0, 1/10, 1/5, \ldots, 9/10, 1$ with $N = 2$.

By minimizing the Hamiltonian $\mathcal{H}$, we get the equations of motion for f and g

$$f'' + \frac{f'}{r} - \frac{f}{r^2} - \frac{m_\phi^2}{2N} f\left(f^2 + (N-1)g^2 - N\right) - \frac{(N-1)m_\chi^2}{2N} f\left(f^2 - g^2\right) = 0, \qquad (3.27)$$

$$g'' + \frac{g'}{r} - \frac{m_\phi^2}{2N} g\left(f^2 + (N-1)g^2 - N\right) + \frac{m_\chi^2}{2N} g\left(f^2 - g^2\right) = 0. \qquad (3.28)$$

The third term in the left-hand side of Eq. (3.27) is typical for global vortices. This leads to logarithmic energy divergence. With respect to a dimensionless coordinate $\rho = m_\phi r$, the above equations can be written in the following forms

$$\mathcal{F}_N[f, g; \tau] \equiv f'' + \frac{f'}{\rho} - \frac{f}{\rho^2} - \frac{f(f^2 + (N-1)g^2 - N)}{2N} - \frac{(N-1)\tau^2 f(f^2 - g^2)}{2N} = 0,$$
$$\qquad (3.29)$$

$$\mathcal{G}_N[f, g; \tau] \equiv g'' + \frac{g'}{\rho} - \frac{g}{2N}\left(f^2 + (N-1)g^2 - N\right) + \frac{\tau^2 g}{2N}\left(f^2 - g^2\right) = 0. \qquad (3.30)$$

It is clear in this form that solutions depend on τ only. We should solve these ordinary differential equations with boundary conditions (3.22) with replacing r by ρ.

Let us see the non-Abelian vortex in a special case $\tau = 1$. In this case, $\mathcal{G}_N[f, g; 1] = 0$ can be solved by $g = 1$ while the other equation $\mathcal{F}_N[f, g = 1; \tau = 1] = 0$ is the same as (2.13).
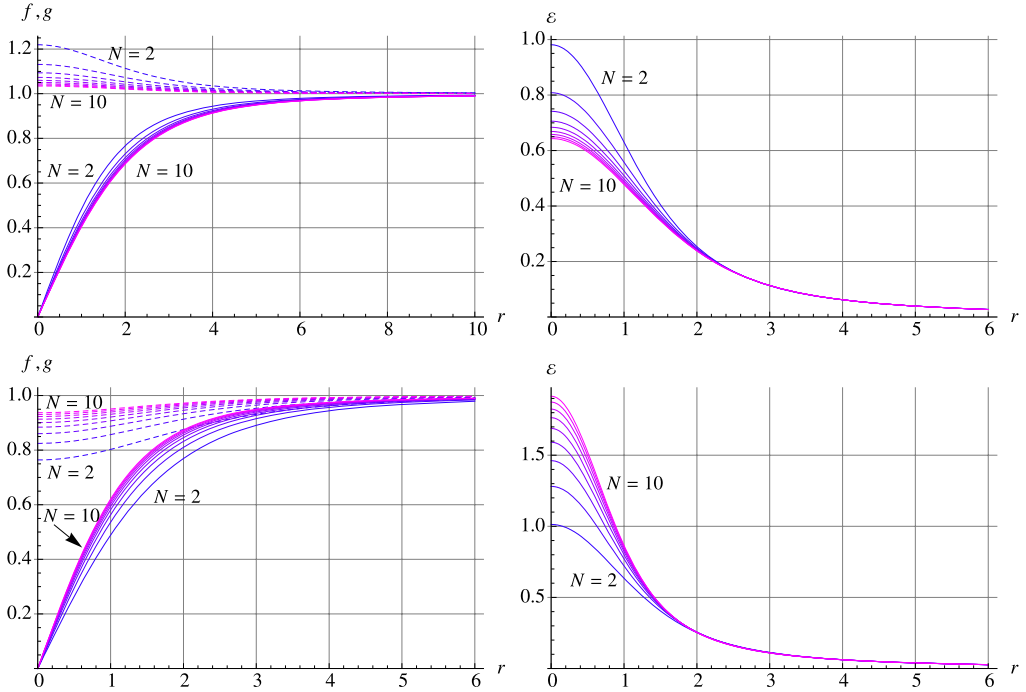
Fig. 4. Configurations (solid lines for $f(r)$ and broken lines for $g(r)$) in the left two panels and energy densities in the right two panels, with $N = 2, 3, \ldots, 10$. The upper (lower) two figures are for $\tau = 1/2$ ($\tau = 2$).

Therefore in the case of $\tau = 1$ the Abelian vortex is embedded into Φ as a non-Abelian vortex solution. Qualitative behaviors of vortex profiles change at $\tau = 1$. As we take τ to larger than 1, the value $g(0)$ gradually goes down toward zero and the energy density becomes sharp (but remains regular for finite τ), see Fig. 2. On the other hand, $g(0)$ grows and the vortex remains regular and finite when we take τ smaller than 1, see Fig. 3. As we will see below, $g(0)$ is an important value.

In order to see N dependence of the non-Abelian global vortex, we plot two figures in Fig. 4 by changing N form 2 to 10 with τ being fixed to 1/2 and 2.

3.2. Asymptotic behaviors

Let us investigate asymptotics of the non-Abelian global vortex. This has been studied in [11] and here we want to extend it with a better accuracy and determine some fundamental parameters (shooting parameters) accompanied by the differential equations (3.29) and (3.30).

We start with study on asymptotics at $r \gg \max\{m_\phi^{-1}, m_\chi^{-1}\}$. As usual, we expand the fields f, g in the following

$$f = 1 - \sum_{i=1}^{\infty} a_i r^{-i}, \qquad g = 1 - \sum_{i=1}^{\infty} b_i r^{-i}. \tag{3.31}$$

By plugging these into Eqs. (3.29) and (3.30) and equating them with 0 order by order, we can analytically determine the coefficients. It is immediately turned out that a_{2i+1} and b_{2i+1} vanish.

The leading order terms are of the form

$$f = 1 - \frac{1}{Nr^2}\left(\frac{1}{m_\phi^2} + \frac{N-1}{m_\chi^2}\right), \qquad g = 1 - \frac{1}{Nr^2}\left(\frac{1}{m_\phi^2} - \frac{1}{m_\chi^2}\right). \tag{3.32}$$

Note that f is always less than 1 while $g \gtrless 1$ for $1 \gtrless \tau$. As observed in [11], the behavior of $g(r)$ depends on the coupling constants λ_1, λ_2. With respect to the physical masses m_χ and m_ϕ, we now understand that the behavior depends on the ratio τ of the two masses. The higher order terms are determined as

$$a_4 = \frac{(-1+N)(-1+9N)m_\phi^4 + 2(-1+N)m_\phi^2 m_\chi^2 + (1+8N)m_\chi^4}{2N^2 m_\phi^4 m_\chi^4}, \tag{3.33}$$

$$b_4 = \frac{(1-8N)m_\phi^4 - 2m_\phi^2 m_\chi^2 + (1+8N)m_\chi^4}{2N^2 m_\phi^4 m_\chi^4}, \tag{3.34}$$

$$a_6 = \frac{1}{2N^3 m_\phi^6 m_\chi^6}\left((N-1)\left(1 + N(-58+161N)\right)m_\phi^6 + \left(3 - (86-83N)N\right)m_\phi^4 m_\chi^2\right.$$
$$\left. - \left(3 + (5-8N)N\right)m_\phi^2 m_\chi^4 + \left(1 + 32N(1+4N)\right)m_\chi^6\right), \tag{3.35}$$

$$b_6 = \frac{m_\chi^2 - m_\phi^2}{2N^3 m_\phi^6 m_\chi^6}\left(\left(1 - 56N + 152N^2\right)m_\phi^4 + 2\left(-1 + 12N + 64N^2\right)m_\phi^2 m_\chi^2\right.$$
$$\left. + \left(1 + 32N + 128N^2\right)m_\chi^4\right), \tag{3.36}$$

$$\vdots$$

Let us next consider asymptotics at $r \ll \min\{m_\phi^{-1}, m_\chi^{-1}\}$. We expand the fields by

$$f = \sum_{i=0}^{\infty} c_i r^i, \qquad g = \sum_{i=0}^{\infty} d_i r^i. \tag{3.37}$$

By plugging this into Eqs. (3.29) and (3.30), one can determine $c_{2m} = d_{2m+1} = 0$. The leading order of the approximation gives us

$$f = c_1 r + \mathcal{O}(r^3), \qquad g = d_0 + \mathcal{O}(r^2). \tag{3.38}$$

The two parameters $c_1 = f'(0)$ and $d_0 = g(0)$ cannot be obtained analytically, but we can do it numerically. Note that they depend on τ and N, see Table 1.

Higher order terms are determined by c_1 and d_0:

$$d_2 = -\frac{N(1-d_0^2)m_\phi^2 + (m_\phi^2 - m_\chi^2)d_0^2}{8N} d_0, \tag{3.39}$$

$$c_3 = \frac{-Nm_\phi^2 + (N-1)(m_\phi^2 - m_\chi^2)d_0^2}{16N}c_1, \tag{3.40}$$

$$d_4 = \frac{d_0}{256N^2}\left[8Nc_1^2(m_\phi^2 - m_\chi^2)\right.$$
$$\left. + \left(N(3d_0^2-1)m_\phi^2 - 3d_0^2(m_\phi^2 - m_\chi^2)\right)\left(N(d_0^2-1)m_\phi^2 - d_0^2(m_\phi^2 - m_\chi^2)\right)\right], \tag{3.41}$$

Table 1
Numerical data for the shooting parameters. The parameter d_0 at $\log \tau \to -\infty$ ($\tau \to 0$) has analytic values, given in Eq. (3.52).

$\log \tau$	$N = 2$		$N = 3$		$N = 4$		$N = 5$	
	c_1	d_0	c_1	d_0	c_1	d_0	c_1	d_0
$+\infty$	–	0	–	0	–	0	–	0
3	0.89928	0.28858	1.35826	0.36708	1.77660	0.42829	2.15432	0.47785
5/2	0.82145	0.35595	1.17489	0.44008	1.47803	0.5032	1.73919	0.55277
2	0.74680	0.43959	1.00794	0.52694	1.21660	0.58945	1.38688	0.63682
3/2	0.67273	0.54356	0.85220	0.62924	0.98383	0.68702	1.08462	0.72889
1	0.59541	0.67207	0.70243	0.74686	0.77302	0.79353	0.82316	0.82555
1/2	0.50986	0.82689	0.55511	0.87497	0.58130	0.90209	0.59840	0.91953
0	0.41238	1	0.41238	1	0.41238	1	0.41238	1
$-1/2$	0.50651	1.16517	0.46970	1.10162	0.45361	1.07342	0.44457	1.05748
-1	0.57045	1.28860	0.50278	1.16697	0.47578	1.11765	0.46121	1.09086
$-3/2$	0.60330	1.35937	0.51809	1.20061	0.48563	1.13948	0.46845	1.10697
-2	0.61733	1.39235	0.52430	1.21534	0.48955	1.14881	0.47131	1.11377
$-5/2$	0.62280	1.40589	0.52667	1.22120	0.49104	1.15249	0.47239	1.11644
-3	0.62486	1.41111	0.52755	1.22343	0.49160	1.15388	0.47279	1.11744
$-\infty$	0.62607	$\sqrt{2}$	0.52807	$\sqrt{3/2}$	0.49192	$\sqrt{4/3}$	0.47302	$\sqrt{5/4}$

$$
\begin{aligned}
c_5 = \frac{c_1}{768N^2}\Big[&\big(N^2 - 6(N-1)Nd_0^2 + 5(N-1)^2 d_0^4\big)m_\phi^4 \\
&- 2(N-1)d_0^2\big(-3N + (-5+3N)d_0^2\big)m_\phi^2 m_\chi^2 + \big(5 + (N-6+)N\big)d_0^4 m_\chi^4 \\
&+ 16Nc_1^2\big(m_\phi^2 + (N-1)m_\chi^2\big)\Big].
\end{aligned}
\tag{3.42}
$$

3.3. Heavy particle limits

In this subsection, we are going to study how the vortices behave in two regions (i) $m_\phi \gg m_\chi$ and (ii) $m_\chi \gg m_\phi$. Before doing it, remember that the linear sigma model (3.1) becomes the chiral Lagrangian in the limit $m_\chi, m_\phi \to \infty$. This is because, after the heavy fields are integrated out, only massless NG modes ($\Phi \in U(N)$) survive in a low energy theory. Our purpose of this subsection is to clarify effects of lightest massive fields to the chiral Lagrangian. For later convenience, let us rewrite the equations of motion (3.27) and (3.28) as follows

$$
\frac{f''}{f} + \frac{f'}{rf} - \frac{1}{r^2} + (N-1)\left(\frac{g''}{g} + \frac{g'}{rg}\right) - \frac{m_\phi^2}{2}\big(f^2 + (N-1)g^2 - N\big) = 0,
\tag{3.43}
$$

$$
\frac{f''}{f} + \frac{f'}{rf} - \frac{1}{r^2} - \frac{g''}{g} - \frac{g'}{rg} - \frac{m_\chi^2}{2}\big(f^2 - g^2\big) = 0.
\tag{3.44}
$$

A point is that m_ϕ appears only in the first equation while m_χ is in the second equation.

Similar limits (with some particles being infinitely heavy) have been studied recently for a *local* $U(N)$ vortex, for which $U(1)$ and $SU(N)_L$ are gauged, at the critical coupling (with a certain relation between gauge and scalar couplings) [20].

3.3.1. $m_\phi \to \infty$ limit ($\tau \to 0$)

Let us first consider the mass of the trace part of Φ is much greater than that of traceless part,

$$
m_\phi \gg m_\chi.
\tag{3.45}
$$

In other words, we consider a limit $\tau \to 0$ by sending $m_\phi \to \infty$ with m_χ being fixed.[6] Then we integrated out the heavy modes with mass m_ϕ. The first term in the scalar potential in Eq. (3.15) becomes very high and sharp, so that the scalar fields are squeezed into the following manifold

$$\text{Tr}\big[\Phi^\dagger \Phi - v^2 \mathbf{1}_N\big] = 0. \tag{3.46}$$

To solve this condition, let us expand Φ by

$$\Phi = \vec{\varphi} \cdot \vec{T}, \quad \vec{T} = \big(\mathbf{1}_N/\sqrt{N}, T^1, T^2, \ldots, T^{N^2-1}\big), \tag{3.47}$$

with N^2 complex vector $\vec{\varphi} = (\varphi_0, \varphi_1, \ldots, \varphi_{N^2-1})$. Then the condition becomes

$$|\vec{\varphi}|^2 = Nv^2. \tag{3.48}$$

Therefore the linear sigma model (3.1) reduces to S^{2N^2-1} non-linear sigma model. The scalar potential takes the form

$$V_0 = \frac{m_\chi^2}{4v^2} \text{Tr}\big[\langle \Phi^\dagger \Phi \rangle^2\big] = \frac{m_\chi^2}{4v^2} \text{Tr}\big[(\Phi^\dagger \Phi - v^2 \mathbf{1}_N)^2\big]. \tag{3.49}$$

The vacuum remains as $U(N)_A$. By using the chiral symmetry, we can choose

$$\varphi_0 = \sqrt{N}v, \ \varphi_{a \geqslant 1} = 0 \iff \Phi = v\mathbf{1}_N. \tag{3.50}$$

Let us next consider a vortex-string solution in this model by taking the same diagonal ansatz (3.21) as before. However, f and g are no longer independent because of the condition (3.48). They should satisfy

$$f(r)^2 + (N-1)g(r)^2 = N. \tag{3.51}$$

With this at hand, we are aware of an interesting phenomenon that the shooting parameter $g(0) = d_0$ approaches to an analytic value as

$$\lim_{\tau \to 0} d_0 = \lim_{\tau \to 0} g(0) = \sqrt{\frac{N}{N-1}}. \tag{3.52}$$

Here we have used $f(0) = 0$. Our numerical result matches with this result, see Table 1. The same thing can be found from the view point of equations of motion (3.43). By dividing its both hands by m_ϕ^2 and taking the limit $m_\phi \to \infty$, we get Eq. (3.51) again.

To simplify the other equation (3.44), we may rewrite the fields by

$$f = \sqrt{N} \cos \Theta, \qquad g = \sqrt{\frac{N}{N-1}} \sin \Theta. \tag{3.53}$$

The reduced model is like the sine-Gordon model: the potential is

$$\tau = 0: \quad V = m_\chi^2 \frac{(N \cos^2 \Theta - 1)^2}{4(N-1)}. \tag{3.54}$$

The corresponding equation of motion is of the form

$$\tau = 0: \quad \Theta'' + \frac{\Theta'}{r} + \frac{1}{4} \sin 2\Theta \left(\frac{2}{r^2} + m_\chi^2 \frac{N(N \cos^2 \Theta - 1)}{N-1}\right) = 0. \tag{3.55}$$

---

[6] The same limit can be realized by taking $m_\chi \to 0$ with m_ϕ being fixed. However, the massless limit is tricky because the vortex infinitely spreads out and dilutes.
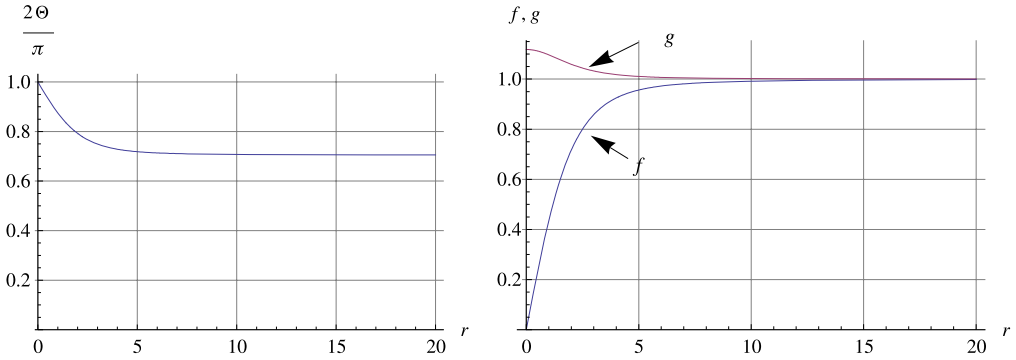
Fig. 5. Profiles of Θ and corresponding f, g for $N = 2$ case ($m_\chi = 1$).

We solve this with the boundary conditions

$$\lim_{r \to 0} \Theta = \frac{\pi}{2}, \qquad \lim_{r \to \infty} \Theta = \arccos\left(\frac{1}{\sqrt{N}}\right). \tag{3.56}$$

We numerically solved this and the solution is shown in Fig. 5.

3.3.2. $m_\chi \to \infty$ limit ($\tau \to \infty$)

Next let us investigate the other limit

$$m_\chi \gg m_\phi \quad (\tau \to \infty). \tag{3.57}$$

We send m_χ infinity with m_ϕ being kept finite. Then $N^2 - 1$ real scalar fields in Φ become infinitely heavy and we integrated them out from the theory. The scalar potential (3.15) gives us the following condition

$$\mathrm{Tr}\big[\langle \Phi^\dagger \Phi \rangle^2\big] = 0 \iff \langle \Phi^\dagger \Phi \rangle = 0 \iff \Phi^\dagger \Phi \propto \mathbf{1}_N. \tag{3.58}$$

We can solve this by

$$\Phi(x) = s(x)U(x) \quad \text{with } s \in \mathbb{C}, \ U \in SU(N)_A. \tag{3.59}$$

This decomposition is up to the $\mathbb{Z}_k$ identification $(s, U) \sim (\omega^k s, \omega^{-k} U)$ with $\omega = e^{2\pi i/N}$ ($k = 0, 1, 2, \ldots, N - 1$). Thus the theory in this limit is the $SU(N)$ chiral Lagrangian coupled with a complex scalar field s. The kinetic term is given by

$$\begin{aligned} K_{\tau \to \infty} &= N|\partial_\mu s|^2 + |s|^2 \,\mathrm{Tr}\big(\partial_\mu U \partial^\mu U^\dagger\big) \\ &= N(\partial_\mu \sigma)^2 + \sigma^2 \,\mathrm{Tr}\big(\partial_\mu \hat{U} \partial^\mu \hat{U}^\dagger\big) \end{aligned} \tag{3.60}$$

with $s \equiv \sigma e^{i\alpha}$ and $\hat{U} \equiv e^{i\alpha} U \in U(N) \simeq [SU(N) \times U(1)]/\mathbb{Z}_N$. From the second expression, we see that the metric of the target space is a cone over $U(N)$. The scalar potential becomes in this limit as

$$V_{\tau \to \infty} = \frac{m_\phi^2}{4v^2}\big(\mathrm{Tr}\big[\Phi^\dagger \Phi - v^2 \mathbf{1}_N\big]\big)^2 = \frac{N^2 m_\phi^2}{4v^2}\big(|s|^2 - v^2\big)^2. \tag{3.61}$$

Thus the vacuum manifold remains to be $U(N)_A$. The vacuum expectation value v of $|s|$ gives the pion decay constant $F_\pi^2 = 16|s|^2$ of the $U(N)$ chiral Lagrangian.
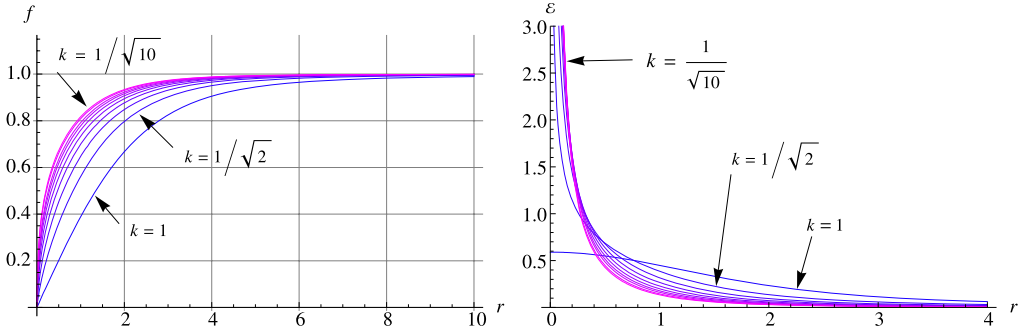
Fig. 6. The left panel shows the profile functions and the corresponding energy densities $\mathcal{E}$ in Eq. (2.8) are plotted in the right panel for $k = 1/\sqrt{N}$ with $N = 1, 2, \ldots, 10$. The Abelian solution of $N = 1$ is only regular but the others are singular.

We are ready to consider vortex-string solutions in the limit. With respect to the profile functions f, g, the condition (3.58) forces us the following condition

$$f(r) = g(r). \tag{3.62}$$

In terms of s and U, the non-Abelian vortex solution (3.21) can be expressed as

$$s = e^{i\frac{\theta}{N}} f(r), \qquad U = \text{diag}\big(e^{\frac{i(N-1)\theta}{N}}, e^{-i\frac{\theta}{N}}, \ldots, e^{-i\frac{\theta}{N}}\big). \tag{3.63}$$

Note that the overall $U(1)$ winding of the non-Abelian vortex is $1/N$ as before. The condition (3.62) explains the behavior of $g(0)$ which tends to go down toward zero as τ is sent to ∞, see Fig. 2 and Table 1. Because of this behavior, the scalar fields Φ vanish at the origin entirely so that the full chiral symmetry is recovered at the center of vortex. Furthermore with recalling the discussion around Eq. (3.23), we find that no second symmetry breaking occurs in the presence of the vortex solution (3.63), because the isotropy group is H_θ even for finite $r(\neq 0)$ which is isomorphic to $H = SU(N)_V/(\mathbb{Z}_N)_V$ in Eq. (3.8). Therefore, we find that the solution loses the internal orientations of $\mathbb{C}P^{N-1}$. Eq. (3.44) is automatically satisfied while Eq. (3.43) reduces to

$$\tau = \infty: \quad f'' + \frac{f'}{r} - \frac{1}{Nr^2} - \frac{m_\phi^2}{2} f\big(f^2 - 1\big) = 0. \tag{3.64}$$

Let us compare this with Eq. (2.7) for the Abelian vortex string. Interestingly, the non-Abelian global vortex in the $m_\chi \to \infty$ limit can be identified with the Abelian global vortex with a non-integer $U(1)$ winding number

$$k = \frac{1}{\sqrt{N}}, \tag{3.65}$$

which is smaller than unity, and can be an *irrational* numbers for N's which are not able to be expressed by squared integers. Of course, $U(1)$ vortices with a non-integer $U(1)$ winding number are generically singular because $\phi \sim v e^{i\theta/\sqrt{N}}$ (at $r \sim 0$) is not single valued. Such singular solutions for $F[f(\rho), \frac{1}{\sqrt{N}}] = 0$ in Eq. (2.13) with the boundary conditions (2.6) are shown in Fig. 6.

There is no smooth interpolation in the solutions between $m_\chi < \infty$ and $m_\chi = \infty$ in the following sense: Eqs. (3.43) and (3.44) with finite m_χ always provide $f \sim r$ to leading order in $r \sim 0$ (see Table 1), and keep the energy density finite, while in the limit where $m_\chi = \infty$ the

resultant Eq. (3.64) gives $f \sim r^{\frac{1}{\sqrt{N}}}$ for $r \sim 0$ and energy density gets divergent as $r^{2(\frac{1}{\sqrt{N}}-1)}$ at the center of vortex. Furthermore, the profile function g (in this limit $g = f$) does not satisfy the original boundary condition $g'(0) = 0$. Instead, it is replaced by $g(0) = 0$.

Although this singular solution is an artifact appearing only in the limit where we have discarded the heavy modes completely, it reasonably accounts for the fact that the energy profile of the non-Abelian vortex becomes very sharp and finally looks singular when $m_\chi \gg m_\phi$, see Fig. 2. Taking into account such heavy modes, the singularity is smeared.

4. Conclusion and discussion

In this article, we have investigated non-Abelian global vortices in $SU(N)_L \times SU(N)_R \times U(1)_A$ linear sigma model in detail. We push forward the analysis in [11] and determined important numerical parameters c_1 and d_0 which determines all the properties of the solutions. Furthermore, we have obtained expansion formulae for asymptotics at large distance. We have found that interesting two limits (i) $m_\phi \gg m_\chi$, (ii) $m_\chi \gg m_\phi$. The original linear sigma model reduces S^{2N^2-1} non-linear sigma model in the (i) limit and we have found a sort of non-Abelian global string solution there. In the second limit (ii), we have obtained the $SU(N)$ chiral Lagrangian coupled with a complex scalar field. We have also found a sort of Abelian global vortex solution. It is a singular solution and can be identified with Abelian global vortex with an irrational $U(1)$ winding number $k = 1/\sqrt{N}$.[7]

Here we give several discussions. The coupling between a global $U(1)$ string and the $U(1)$ Nambu–Goldstone boson can be constructed by using a duality between a boson ϕ and a two-form field $B_{\mu\nu}$ [21]. In the same way the coupling of a global $U(N)$ string and the $U(N)$ Nambu–Goldstone bosons will be possible by using non-Abelian two-form [22].

In the presence of the $U(1)_A$ axial anomaly a term $V_1(\Phi, \Phi^\dagger) = c(\det \Phi + \det \Phi^\dagger)$ is induced, which gives a sine-Gordon potential to the phase. Then a $U(N)$ vortex becomes a boundary of a domain wall [9,10]. In the presence of quark masses, a term $V_2(\Phi, \Phi^\dagger) = \text{Tr}[H(\Phi + \Phi^\dagger)]$ exists with H a mass matrix. It remains as a future problem to study detailed structure of vortex solutions in the presence of these terms because the authors in [9,10] assumed constant profiles.

In Section 3.3.2 we have simply sent the mass m_χ to infinity to obtain the $SU(N)$ chiral Lagrangian coupled with a complex scalar field s. However quantum mechanically we should integrate out the massive fields. This procedure generally induces higher derivative terms for remaining massless fields. The Skyrme term is such a term of the fourth order [23]. In our case with massless field s we will obtain the effective Lagrangian of the form

$$\mathcal{L}_{\text{eff.}} = N|\partial_\mu s|^2 + |s|^2 \text{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{|s|^4}{e^2} \text{Tr}([U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]^2) - V(|s|^2)$$

$$= N(\partial_\mu \sigma)^2 + \sigma^2 \text{Tr}(\partial_\mu \hat{U} \partial^\mu \hat{U}^\dagger) + \frac{\sigma^4}{e^2} \text{Tr}([\hat{U}^\dagger \partial_\mu \hat{U}, \hat{U}^\dagger \partial_\nu \hat{U}]^2) - V(\sigma^2) \qquad (4.1)$$

where e is a parameter determined by an explicit calculation and V is the potential in Eq. (3.61). Note that there is no fourth order term for s. Because of the relation $\Phi^\dagger \partial_\mu \Phi = s^* \partial_\mu s \mathbf{1}_N + |s|^2 U^\dagger \partial_\mu U$, the possible fourth order term $[\Phi^\dagger \partial_\mu \Phi, \Phi^\dagger \partial_\nu \Phi]^2$ reduces to the Skyrme-like term

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[7] We have also investigated the large N limit, and found that there exists a regular vortex string solution which has the same form with the usual $U(1)$ vortex Eq. (3.19) with winding number $k = 1$ but with replacement $m_\phi \to m_\chi$. See Appendix A for details.

in the Lagrangian (4.1). When s is fixed to the vacuum expectation value, the Lagrangian (4.1) reduces to the $U(N)$ Skyrme model and admits the usual Skyrmion solution for $U(x) = \exp\{i[F(r)\sigma \cdot \mathbf{r}/|\mathbf{r}|]\}$.[8] It is interesting to note that the Abelian vortex $s \sim ve^{i\theta}$ does not interact with the Skyrmion while a non-Abelian vortex does. It remains as a future problem to study interaction, scattering or absorption of baryons (Skyrmions) by non-Abelian strings.

In this paper we have studied $U(N)$ vortices. Local and semi-local vortices with different groups $[U(1) \times G]/\mathbb{Z}_{n_0}$, where G is arbitrary group with the center n_0 [25], have been studied recently. In this framework the $U(N)$ vortex corresponds to the case of $G = SU(N)$ with $n_0 = N$. Global version of these vortices are also possible, especially the case of $G = SO$ is related to vortices in the B-phase of ^{3}He superfluids.

Before closing this paper let us compare our global $U(N)$ vortices with other types of $U(N)$ vortices in the related models: (1) semi-superfluid $U(N)$ vortices in high density QCD and (2) local $U(N)$ vortices. In these models, the group structure is completely the same with the global case in this paper. However, the energetics/interactions of vortices and the (non-)normalizability of the zero modes are significantly different.

(1) In high density QCD it is expected that color superconductivity is realized. There, the color symmetry $SU(N)_{\rm C}$ and the flavor symmetry $SU(N)_{\rm F}$ (with $N = 3$) as well as the baryon $U(1)_{\rm B}$ symmetry are spontaneously broken down to the color-flavor locked symmetry $SU(N)_{\rm C+F}$ apart from the discrete symmetries. The corresponding vacuum manifold $[SU(N) \times U(1)]/\mathbb{Z}_N \simeq U(N)$ is the same with that of the global $U(N)$ vortices. In this case the $SU(N)$ subgroup of the vacuum manifold $U(N)$ is gauged and therefore only one massless Nambu–Goldstone boson for the $U(1)_{\rm B}$ exists. The $U(N)$ vortices here are called semi-superfluid vortices [26]. In the asymptotic behavior of the scalar field of a $U(N)$ vortex, $\Phi \sim v\, {\rm diag}(e^{i\theta}, 1, \ldots, 1) = ve^{i\frac{\theta}{N}}\, {\rm diag}(e^{i\frac{N-1}{N}\theta}, e^{-i\frac{\theta}{N}}, \ldots, e^{-i\frac{\theta}{N}})$, the latter non-Abelian part can be eliminated by a gauge transformation $U(r, \theta) = {\rm diag}(e^{-i\frac{N-1}{N}\theta F(r)}, e^{i\frac{\theta}{N}F(r)}, \ldots, e^{i\frac{\theta}{N}F(r)})$ with an arbitrary function $F(r)$ satisfying the boundary conditions $F(r = 0) = 0$ and $F(r \to \infty) = 1$.[9]

(2) Next, let us to compare the global $U(N)$ vortices studied in this paper with the local $U(N)$ vortices [12]. In this case too the symmetry breaking pattern is the same but a crucial difference is that the $U(1)$ symmetry is also gauged in addition to the color $SU(N)$, and therefore there remain no Nambu–Goldstone bosons. Namely the vacuum is the unique at the infinity even in the presence of a $U(N)$ vortex because the $U(1)$ symmetry is gauged. Nevertheless, we cannot gauge transform the asymptotic behavior of the scalar fields from $\Phi \sim ve^{i\frac{\theta}{N}}\mathbf{1}_N$ to $\mathbf{1}_N$ because we can define no regular gauge transformation well-defined in the entire space because of non-triviality of the first homotopy group: $\pi_1[U(1)] \neq 0$. Unlike the global $U(N)$ vortices or semi-superfluid $U(N)$ vortices, these local $U(N)$ vortices have finite energy because of the gauged $U(1)$. The $\mathbb{C}P^{N-1}$ zero modes are normalizable. At the critical (BPS) coupling with a

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[8] The Skyrme model admits a topologically unstable string solution [24] which may be related to the pion string [6].

[9] This transformation is well-defined because of the triviality of the first homotopy group: $\pi_1[SU(N)] = 0$. This transformation brings Φ to $\Phi \sim ve^{i\frac{\theta}{N}}\mathbf{1}_N$. This property makes the orientational zero modes of $\mathbb{C}P^{N-1}$ to be normalizable and to become the moduli (collective coordinates) of the vortex [27]. In other words, the isotropy groups H_θ at the infinities in the presence of a global $U(N)$ vortex were physically different and depend on θ (although they are isomorphic), whereas, in the $SU(N)$ gauged case, the isotropy group at the infinities in the presence of a semi-superfluid $U(N)$ vortex is physically equivalent for any θ. The $U(1)_{\rm B}$ is global and so the energy of a vortex remains logarithmically divergent. The asymptotic interaction between two semi-superfluid $U(N)$ vortices is also essentially the same with the one between $U(1)$ global vortices because of the above property [14,27]. Therefore it gives the universal repulsion between separated vortices.

particular relation between gauge and scalar couplings, there is no static force among multiple vortices, with allowing the multi-vortex moduli space [28,29] (see [30] for the moduli spaces of local $U(N)$ vortices on a cylinder and a torus). Static interactions exist between vortices at non-critical (non-BPS) couplings. The force between two $U(N)$ vortices was shown to depend on both $\mathbb{C}P^{N-1}$ orientations and positions [31].

If we gauge all the symmetry, the vortices are those in quiver gauge theories [32]. In this case, the diagonal gauge symmetry remains unbroken, and the vortices do not have orientations in the internal space. The final possibility which was not studied so far is the case that only $U(1)$ is gauged.

Acknowledgements

Appendix A. Large N limit

In this appendix we will derive an asymptotic form of the vortex solution in the large N limit. Before implementing the large N limit, we have to know how the parameters in the Lagrangian scale in N. From the observation of loop corrections (perturbation series in λ_1 and λ_2) to two-body meson scattering amplitude in a single channel, one can set $\lambda_1 \sim \mathcal{O}(N^{-2})$ and $\lambda_2 \sim \mathcal{O}(N^{-1})$ in order to have a tenable perturbative expansion [33]. $\mu^2 \sim \mathcal{O}(1)$ because of no flavor degeneracy in a single channel. As consequences, one finds $m_{\phi,\chi}^2 \sim \mathcal{O}(1)$ thus $\tau \sim \mathcal{O}(1)$.

Now we are ready to see what happens in large N limit. After taking $N \to \infty$ as keeping m_ϕ and m_χ finite, Eq. (3.28) can be solved by

$$g(r) = 1, \tag{A.1}$$

and the other equation (3.27) becomes

$$f'' + \frac{f'}{r} - \frac{f}{r^2} - \frac{m_\chi^2}{2} f(f^2 - 1) = 0. \tag{A.2}$$

This is the equation for $k = 1$ Abelian global vortex. Note that m_χ is shown up in the equation and solutions are independent of m_ϕ unlike the case of Abelian global vortex given in Eq. (3.19). The solutions themselves happen to be identical to those for $m_\phi = m_\chi$ ($\tau = 1$) with any finite N.

References

[1] R.L. Davis, E.P.S. Shellard, Global strings and superfluid vortices, Phys. Rev. Lett. 63 (1989) 2021.
[2] A. Vilenkin, A.E. Everett, Cosmic strings and domain walls in models with Goldstone and pseudo-Goldstone bosons, Phys. Rev. Lett. 48 (1982) 1867;
E.P.S. Shellard, Cosmic string interactions, Nucl. Phys. B 283 (1987) 624;
L. Perivolaropoulos, Instabilities and interactions of global topological defects, Nucl. Phys. B 375 (1992) 665.
[3] A. Vilenkin, E.P.S. Shellard, Cosmic Strings and Other Topological Defects, Cambridge Univ. Press, 1994;
M.B. Hindmarsh, T.W.B. Kibble, Cosmic strings, Rep. Prog. Phys. 58 (1995) 477, arXiv:hep-ph/9411342.
[4] R.D. Pisarski, F. Wilczek, Remarks on the chiral phase transition in chromodynamics, Phys. Rev. D 29 (1984) 338.

  [5] H. Itoyama, A.H. Mueller, The axial anomaly at finite temperature, Nucl. Phys. B 218 (1983) 349;
      E.V. Shuryak, Which chiral symmetry is restored in hot QCD? Comments Nucl. Part. Phys. 21 (1994) 235;
      T.D. Cohen, The high temperature phase of QCD and $U(1)_A$ symmetry, Phys. Rev. D 54 (1996) 1867;
      C.W. Bernard, et al., Which chiral symmetry is restored in high temperature QCD? Phys. Rev. Lett. 78 (1997) 598;
      S. Chandrasekharan, D. Chen, N.H. Christ, W.J. Lee, R. Mawhinney, P.M. Vranas, Anomalous chiral symmetry breaking above the QCD phase transition, Phys. Rev. Lett. 82 (1999) 2463;
      R.A. Janik, M.A. Nowak, G. Papp, I. Zahed, $U(1)$ problem at finite temperature, AIP Conf. Proc. 494 (1999) 408;
      K. Fukushima, K. Ohnishi, K. Ohta, Topological susceptibility at zero and finite temperature in the Nambu–Jona-Lasinio model, Phys. Rev. C 63 (2001) 045203;
      H. Nagahiro, M. Takizawa, S. Hirenzaki, η- and η'-mesic nuclei and $U(A)(1)$ anomaly at finite density, Phys. Rev. C 74 (2006) 045203;
      D. Horvatic, D. Klabucar, A.E. Radzhabov, η and η' mesons in the Dyson–Schwinger approach at finite temperature, Phys. Rev. D 76 (2007) 096009, arXiv:0708.1260 [hep-ph];
      J.W. Chen, K. Fukushima, H. Kohyama, K. Ohnishi, U. Raha, $U_A(1)$ anomaly in hot and dense QCD and the critical surface, arXiv:0901.2407 [hep-ph].
  [6] X. Zhang, T. Huang, R.H. Brandenberger, Pion and eta strings, Phys. Rev. D 58 (1998) 027702, arXiv:hep-ph/9711452.
  [7] R.H. Brandenberger, X.M. Zhang, Anomalous global strings and primordial magnetic fields, Phys. Rev. D 59 (1999) 081301, arXiv:hep-ph/9808306.
  [8] M. Nagasawa, R.H. Brandenberger, Stabilization of embedded defects by plasma effects, Phys. Lett. B 467 (1999) 205, arXiv:hep-ph/9904261;
      B. Carter, R.H. Brandenberger, A.C. Davis, Thermal stabilisation of superconducting sigma strings and their drum vortons, Phys. Rev. D 65 (2002) 103520, arXiv:hep-ph/0201155;
      H. Mao, Y. Li, M. Nagasawa, X.M. Zhang, T. Huang, Signal of the pion string at LHC Pb–Pb collisions, Phys. Rev. C 71 (2005) 014902, arXiv:hep-ph/0404132.
  [9] A.P. Balachandran, S. Digal, Non-Abelian topological strings and metastable states in linear sigma model, Phys. Rev. D 66 (2002) 034018, arXiv:hep-ph/0204262.
  [10] A.P. Balachandran, S. Digal, Topological string defect formation during the chiral phase transition, Int. J. Mod. Phys. A 17 (2002) 1149, arXiv:hep-ph/0108086.
  [11] M. Nitta, N. Shiiki, Non-Abelian global strings at chiral phase transition, Phys. Lett. B 658 (2008) 143, arXiv:0708.4091 [hep-ph].
  [12] A. Hanany, D. Tong, Vortices, instantons and branes, JHEP 0307 (2003) 037, arXiv:hep-th/0306150;
      R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi, A. Yung, Non-Abelian superconductors: Vortices and confinement in $N = 2$ SQCD, Nucl. Phys. B 673 (2003) 187, arXiv:hep-th/0307287.
  [13] E. Nakano, M. Nitta, T. Matsuura, Interactions of non-Abelian global strings, Phys. Lett. B 672 (2009) 61, arXiv:0708.4092 [hep-ph].
  [14] E. Nakano, M. Nitta, T. Matsuura, Non-Abelian strings in hot or dense QCD, Prog. Theor. Phys. Suppl. 174 (2008) 254, arXiv:0805.4539 [hep-ph].
  [15] F. Zhou, Spin correlation and discrete symmetry in spinor Bose–Einstein condensates, Phys. Rev. Lett. 87 (2001) 080401, arXiv:cond-mat/0102372;
      S. Mukerjee, C. Xu, J.E. Moore, Topological defects and the superfluid transition of the $s = 1$ spinor condensate in two dimensions, Phys. Rev. Lett. 97 (2006) 120406, arXiv:cond-mat/0605102.
  [16] U. Leonhardt, G.E. Volovik, How to create an Alice string (half-quantum vortex) in a vector Bose–Einstein condensate, Pis'ma Zh. Eksp. Teor. Fiz. 72 (2000) 66–70, JETP Lett. 72 (2000) 46–48, arXiv:cond-mat/0003428.
  [17] G.W. Semenoff, F. Zhou, Discrete symmetries and 1/3-quantum vortices in condensates of $F = 2$ cold atoms, Phys. Rev. Lett. 98 (2007) 100401, arXiv:cond-mat/0610162;
      M. Kobayashi, Y. Kawaguchi, M. Nitta, M. Ueda, Collision dynamics and gluon rung of non-Abelian vortices, arXiv:0810.5441 [cond-mat.other].
  [18] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, N. Sakai, Webs of walls, Phys. Rev. D 72 (2005) 085004, arXiv:hep-th/0506135;
      M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, N. Sakai, Non-Abelian webs of walls, Phys. Lett. B 632 (2006) 384, arXiv:hep-th/0508241;
      M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta, N. Sakai, D-brane configurations for domain walls and their webs, AIP Conf. Proc. 805 (2006) 354, arXiv:hep-th/0509127;
      M. Eto, T. Fujimori, T. Nagashima, M. Nitta, K. Ohashi, N. Sakai, Effective action of domain wall networks, Phys. Rev. D 75 (2007) 045010, arXiv:hep-th/0612003;

M. Eto, T. Fujimori, T. Nagashima, M. Nitta, K. Ohashi, N. Sakai, Dynamics of domain wall networks, Phys. Rev. D 76 (2007) 125025, arXiv:0707.3267 [hep-th].

[19] N.S. Manton, P. Sutcliffe, Topological Solitons, Cambridge Univ. Press, Cambridge, UK, 2004.

[20] M. Eto, T. Fujimori, T. Nagashima, M. Nitta, K. Ohashi, N. Sakai, Multiple layer structure of non-Abelian vortex, Phys. Lett. B 678 (2009) 254–258, arXiv:0903.1518 [hep-th].

[21] M. Kalb, P. Ramond, Classical direct interstring action, Phys. Rev. D 9 (1974) 2273;
F. Lund, T. Regge, Unified approach to strings and vortices with soliton solutions, Phys. Rev. D 14 (1976) 1524;
R.L. Davis, E.P.S. Shellard, Antisymmetric tensors and spontaneous symmetry breaking, Phys. Lett. B 214 (1988) 219.

[22] K. Seo, M. Okawa, A. Sugamoto, Dual transformation in non-Abelian gauge theories, Phys. Rev. D 19 (1979) 3744;
D.Z. Freedman, P.K. Townsend, Antisymmetric tensor gauge theories and nonlinear sigma models, Nucl. Phys. B 177 (1981) 282.

[23] T.H.R. Skyrme, A nonlinear field theory, Proc. R. Soc. London A 260 (1961) 127;
T.H.R. Skyrme, A unified field theory of mesons and baryons, Nucl. Phys. 31 (1962) 556.

[24] A. Jackson, String-like solutions in the Skyrme model, Nucl. Phys. A 493 (1989) 365;
A. Jackson, Decay of the string in the Skyrme model, Nucl. Phys. A 496 (1989) 667;
M. Nitta, N. Shiiki, Skyrme strings, Prog. Theor. Phys. 119 (2008) 829, arXiv:0706.0316 [hep-ph].

[25] L. Ferretti, S.B. Gudnason, K. Konishi, Non-Abelian vortices and monopoles in $SO(N)$ theories, Nucl. Phys. B 789 (2008) 84, arXiv:0706.3854 [hep-th];
M. Eto, T. Fujimori, S.B. Gudnason, K. Konishi, M. Nitta, K. Ohashi, W. Vinci, Constructing non-Abelian vortices with arbitrary gauge groups, Phys. Lett. B 669 (2008) 98, arXiv:0802.1020 [hep-th];
M. Eto, T. Fujimori, S.B. Gudnason, M. Nitta, K. Ohashi, SO and USp Káhler and hyper-Káhler quotients and lumps, Nucl. Phys. B, in press, arXiv:0809.2014 [hep-th].

[26] A.P. Balachandran, S. Digal, T. Matsuura, Semi-superfluid strings in high density QCD, Phys. Rev. D 73 (2006) 074009, arXiv:hep-ph/0509276.

[27] E. Nakano, M. Nitta, T. Matsuura, Non-Abelian strings in high density QCD: Zero modes and interactions, Phys. Rev. D 78 (2008) 045002, arXiv:0708.4096 [hep-ph].

[28] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, N. Sakai, Moduli space of non-Abelian vortices, Phys. Rev. Lett. 96 (2006) 161601, arXiv:hep-th/0511088;
M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, N. Sakai, Manifestly supersymmetric effective Lagrangians on BPS solitons, Phys. Rev. D 73 (2006) 125008, arXiv:hep-th/0602289;
M. Eto, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci, N. Yokoi, Non-Abelian vortices of higher winding numbers, Phys. Rev. D 74 (2006) 065021, arXiv:hep-th/0607070;
M. Eto, K. Hashimoto, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci, Universal reconnection of non-Abelian cosmic strings, Phys. Rev. Lett. 98 (2007) 091602, arXiv:hep-th/0609214;
M. Eto, et al., Non-Abelian duality from vortex moduli: A dual model of color-confinement, Nucl. Phys. B 780 (2007) 161, arXiv:hep-th/0611313;
M. Eto, et al., On the moduli space of semilocal strings and lumps, Phys. Rev. D 76 (2007) 105002, arXiv:0704.2218 [hep-th].

[29] Y. Isozumi, M. Nitta, K. Ohashi, N. Sakai, All exact solutions of a 1/4 Bogomol'nyi–Prasad–Sommerfield equation, Phys. Rev. D 71 (2005) 065018, arXiv:hep-th/0405129;
M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, N. Sakai, Instantons in the Higgs phase, Phys. Rev. D 72 (2005) 025011, arXiv:hep-th/0412048;
M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, 1/2, 1/4 and 1/8 BPS equations in SUSY Yang–Mills–Higgs systems: Field theoretical brane configurations, Nucl. Phys. B 752 (2006) 140, arXiv:hep-th/0506257;
M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, N. Sakai, Solitons in the Higgs phase: The moduli matrix approach, J. Phys. A 39 (2006) R315, arXiv:hep-th/0602170.

[30] M. Eto, T. Fujimori, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta, N. Sakai, Non-Abelian vortices on cylinder: Duality between vortices and walls, Phys. Rev. D 73 (2006) 085008, arXiv:hep-th/0601181;
M. Eto, T. Fujimori, M. Nitta, K. Ohashi, K. Ohta, N. Sakai, Statistical mechanics of vortices from D-branes and T-duality, Nucl. Phys. B 788 (2008) 120, arXiv:hep-th/0703197.

[31] R. Auzzi, M. Eto, W. Vinci, Type I non-Abelian superconductors in supersymmetric gauge theories, JHEP 0711 (2007) 090, arXiv:0709.1910 [hep-th];
R. Auzzi, M. Eto, W. Vinci, Static interactions of non-Abelian vortices, JHEP 0802 (2008) 100, arXiv:0711.0116 [hep-th].

[32] A.D. Popov, R.J. Szabo, Quiver gauge theory of non-Abelian vortices and noncommutative instantons in higher dimensions, J. Math. Phys. 47 (2006) 012306, arXiv:hep-th/0504025;
O. Lechtenfeld, A.D. Popov, R.J. Szabo, Quiver gauge theory and noncommutative vortices, Prog. Theor. Phys. Suppl. 171 (2007) 258, arXiv:0706.0979 [hep-th].

[33] A.V. Manohar, Large N QCD, arXiv:hep-ph/9802419.