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Some consequences of the generalised uncertainty principle: statistical mechanical, cosmological, and varying speed of light

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Abstract

We study the dynamical consequences of Maggiore's unique generalised uncertainty principle (GUP). We find that it leads naturally, and generically, to novel consequences. In the high temperature limit, there is a drastic reduction in the degrees of freedom, of the type found, for example, in strings far above the Hagedorn temperature. In view of this, the present GUP may perhaps be taken as the new version of the Heisenberg uncertainty principle, conjectured by Atick and Witten to be responsible for such reduction. Also, the present GUP leads naturally to varying speed of light and modified dispersion relations. They are likely to have novel implications for cosmology and black hole physics, a few of which we discuss qualitatively.

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1. Based on gedanken experiments in string theory [1] and black holes [2], the Heisenberg uncertainty principle is found to be modified to

$$\Delta x \geqslant \frac{\hbar}{\Delta p} + \text{const} \cdot \frac{\lambda^2 \Delta p}{\hbar}, \tag{1}$$

where Δx and Δp denote position and momentum uncertainties, and λ is a length parameter, given by string length and Planck length in the above contexts. Under certain assumptions, Maggiore has derived in [3,4] unique generalised commutation relations (GCRs) which lead to a generalised uncertainty principle (GUP) which, in turn, leads to (1) in a suitable limit. Under a different set of assumptions it is possible to obtain more general commutation relations, e.g., as in [5], which also lead to (1). However, they are not unique. Hence, in the following, we will consider Maggiore's generalisation only, although our analysis is applicable to other cases also.

The GCRs are kinematical. The dynamics is determined by specifying a Hamiltonian H. To illustrate explicitly the dynamical consequences of the GUP, we choose two candidates for H and study the statistical mechanics and the particle dynamics of free particle systems obeying the GUP.[1] We find that the GUP leads naturally to many novel consequences.

In the high temperature limit, we find that there is a drastic reduction in the degrees of freedom (d.o.f)

[1] In the case of the GCRs given by [3,4], some aspects of the particle dynamics have been studied in [3]. In the case of the GCRs given by [5], some aspects of the particle dynamics, statistical mechanics, and black hole physics have been studied in [6], [7], and [8], respectively.

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and the corresponding free energy is analogous to that found in certain topological field theories and in strings far above the Hagedorn temperature [9,10]. In this context, Atick and Witten had indeed conjectured in [9] that a new version of the Heisenberg uncertainty principle may be responsible for such a reduction in the d.o.f. Here, we see that such a reduction emerges naturally as a consequence of the GUP. In view of this, the present GUP may perhaps be taken as the conjectured new version of the Heisenberg uncertainty principle.

Another consequence of the GUP is the natural emergence of the varying speed of light (VSL) and the modified dispersion relations which, in turn, have non trivial implications for cosmology and black hole physics. [2] For one of the H's considered here, the VSL and the free energy together is likely to solve the horizon problem in cosmology, as in [18]. The corresponding VSL is likely to have novel implications for black hole physics also.

The plan of the Letter is as follows. In Section 2, we present the details of Maggiore's GCR and GUP, and the two candidate Hamiltonians. In Section 3, we study the statistical mechanics and discuss the consequences. In Section 4, we study the particle dynamics and discuss the consequences. In Section 5, we present a brief summary and close by mentioning a few issues for further study.

2. The uncertainty principle (1) can be thought of as arising from a generalisation of the commutation relations between the position operators X_i and the momentum operators P_j in d-dimensional space where $i, j = 1, 2, \ldots, d$. Seeking the most general deformed Heisenberg algebra in $d = 3$, Maggiore has derived in [3,4] the generalised commutation relations (GCRs) between X_i and P_j, determined uniquely under the following assumptions: (i) The spatial rotation group and, hence, the commutators $[J_i, J_j]$, $[J_i, X_j]$, and $[J_i, P_j]$ are undeformed. (ii) The translation group and, hence, the commutators $[P_i, P_j]$ are undeformed. (iii) The commutators $[X_i, X_j]$ and $[X_i, P_j]$ depend on a deformation parameter λ, with dimension of length, and reduce to the undeformed ones in the limit $\lambda \to 0$. The GCRs that follow uniquely from these assumptions are given by

$$[X_i, X_j] = -i\epsilon\hbar^2\lambda^2\epsilon_{ijk}J_k, \quad J_k = -i\epsilon_{klm}P_l\frac{\partial}{\partial P_m}, \tag{2}$$

$$[X_i, P_j] = i\hbar\delta_{ij}f, \quad f = \sqrt{1 + \frac{\epsilon\lambda^2}{\hbar^2}\left(P^2 + m^2c^2\right)}, \tag{3}$$

where $\epsilon = \pm 1$, $P^2 = \sum_i P_i^2$, and λ is a length parameter. The GCR (3) then leads to the generalised uncertainty principle (GUP)

$$\Delta x_i \Delta p_j \geqslant \frac{\hbar}{2}\delta_{ij}\langle f \rangle. \tag{4}$$

In the limit $\lambda^2(p^2 + m^2c^2) \ll \hbar^2$ and $\lambda\Delta p \lesssim \hbar$, where p^2 is the eigenvalue of P^2, Eq. (4) reduces to (1). See [3,4] for details.

In the following, we consider d-dimensional space. [3] We set $\hbar = c = 1$ unless indicated otherwise. The case $\epsilon = -1$ implies a bound $\lambda^2(p^2 + m^2) < 1$, whose physical significance is not clear. Hence, we will consider the case $\epsilon = 1$ only. Also, we consider non rotating systems only and, therefore, set $J_k = 0$ in (2). The only nontrivial GCR is then given by (3) with

$$f = \sqrt{1 + \lambda^2(P^2 + m^2)}, \quad P^2 = \sum_{i=1}^{d} P_i^2. \tag{5}$$

We study the consequences of the GCR (3), equivalently of the GUP (4), with f given by (5). It is important to note that the commutation relations are kinematical only. The dynamics is determined by the Hamiltonian H which, therefore, needs to be specified. In the following, we assume that H depends on P only through the rotationally invariant combination $\sqrt{P^2 + m^2}$. Furthermore, for the sake of simplicity, we consider here only the free particle case, for which H is independent of X.

To illustrate the nontrivial consequences of the GUP (4), we consider two choices for H. They are given by

$$H' = 1 \quad \longleftrightarrow \quad H = \sqrt{P^2 + m^2}, \tag{6}$$

[2] VSL theories were first postulated in [11] and studied further in [12,13]. Their implications have been studied in [12–15]. The implications of modified dispersion relations for cosmology have been studied in [16,17].

[3] With the right-hand side of (2) written explicitly in terms of P_i and $\partial/\partial P_j$, Eqs. (2)–(4) are valid in d-dimensional space also, as can be easily verified.

$$f H' = 1 \quad \longleftrightarrow \quad \operatorname{Sinh} \lambda H = \lambda \sqrt{P^2 + m^2}, \tag{7}$$

where H' is the derivative of H with respect to $\sqrt{P^2 + m^2}$ and f is given by (5). The Hamiltonian H in (6) is, perhaps, the simplest and a natural choice; H in (7) is obtained in [3] from the first Casimir operator and, hence, is also a natural choice from group theoretic point of view.

3. Consider the statistical mechanics of a system of free particles confined in a d-dimensional volume V, which obey the GUP (4) with f given by (5). The calculations in the microcanonical or canonical ensemble approach are complicated. But they are simple in the grand canonical ensemble approach which we, therefore, use.

In the standard case where $\lambda = 0$, the one-particle phase space measure is given by h^{-d} where h is the Planck's constant. Physically, this is because the phase space is divided into cells of volume h^d as a consequence of the Heisenberg uncertainty principle. Various phase space integrals will be of the form $\int d^d x\, d^d p\, h^{-d} (*)$, where x and p denote the eigenvalues of X and P. In the present case, the particles are assumed to obey the GUP (4). Consequently, the phase space must be divided into cells of volume $h^d f^d$. Then, the one-particle phase space measure is given by $h^{-d} f^{-d}$, and various phase space integrals are of the form $\int d^d x\, d^d p\, h^{-d} f^{-d} (*)$.

The function f and, in the free particle case, the Hamiltonian H depend on P only. x-integration then simply gives a volume factor V. Writing p in terms of energy E, which is the eigenvalue of H, the phase space integrals can be written as

$$\int \frac{d^d x\, d^d p}{h^d f^d} (*) \equiv \int dE\, g(E) (*). \tag{8}$$

The measure $g(E)$ is the analog of the one-particle density of states. It can be easily calculated and is given by

$$g(E) = \frac{\mathcal{C} p^{d-2}}{f^d E'} \sqrt{p^2 + m^2},$$
$$\text{where} \quad \mathcal{C} \equiv \frac{\Omega_{d-1} V}{h^d}, \tag{9}$$

Ω_{d-1} is the area of a unit $(d-1)$-dimensional sphere, and E' is the derivative of E with respect to $\sqrt{p^2 + m^2}$. In Eq. (9), p is to be expressed in terms of E.

Consider the grand canonical ensemble. It can be easily verified that the definitions of, and the relations between, various thermodynamical quantities all remain unchanged, with $g(E)$ given as in (9) [19]. Thus, we have

$$-\beta F = \beta \mathcal{P} V = \ln \mathcal{Z}$$
$$= \frac{1}{a} \int_0^\infty dE\, g(E) \ln\bigl(1 + a e^{-\beta(E-\mu)}\bigr), \tag{10}$$

where we have used the standard notation: $\beta = T^{-1}$ is the inverse temperature, F is the free energy, $\mathcal{P}$ is the pressure, $\mathcal{Z}$ is the grand canonical partition function, and μ is the chemical potential. Also, $a = -1$, 0, or $+1$ depending on whether the particles obey, respectively, Bose–Einstein, Maxwell–Boltzmann, or Fermi–Dirac statistics. When $a = 0$, $\ln \mathcal{Z}$ is to be evaluated in the limit $a \to 0$. Various thermodynamical quantities can then be calculated using (10): for example, the internal energy $U = -\partial \ln \mathcal{Z} / \partial \beta$, the particle number $N = \partial \ln \mathcal{Z} / (\beta \partial \mu)$, the entropy $S = \beta(U + \mathcal{P} V - \mu N)$, etc.

It is clear from the above formulae, or from physical arguments, that the effect of λ will be considerable only when the temperature/energy is of $\mathcal{O}(\lambda^{-1})$ or higher. Therefore, the limit of interest here is the high temperature limit $\beta \ll \lambda$. Also, we expect that λ is extremely small, in particular, $\lambda m \ll 1$. (For example, $\lambda \simeq$ string/Planck length.) Furthermore, in the high temperature limit, the statistics is irrelevant. Therefore, for the sake of simplicity, we set $m = 0$ and $a = 0$ in the following. Then $\mu = 0$ since $m = 0$. One then obtains U, S, N, etc. in terms of V and T using (10) [19].

To proceed further, p and f in equation (9) are to be expressed in terms of E, for which an explicit form of H is required. We consider H given in (6) and (7), with $m = 0$. A simple algebra then shows that $g(E)$ is given by

$$H' = 1 \quad \longleftrightarrow \quad g(E) = \frac{\mathcal{C} E^{d-1}}{(1 + \lambda^2 E^2)^{d/2}}, \tag{11}$$

$$f H' = 1 \quad \longleftrightarrow \quad g(E) = \frac{\mathcal{C}}{\lambda^{d-1}} (\tanh \lambda E)^{d-1}. \tag{12}$$

The partition function $\mathcal{Z}$ and other quantities can now be evaluated in closed form in terms of special functions, as described in the Appendix A.

We expect to obtain the standard results in the limit $\beta \gg \lambda$ and to obtain the non trivial features, if any, in the limit $\beta \ll \lambda$.[4] A simple calculation in the limit $\beta \gg \lambda$ shows that, to the leading order in λ/β, the thermodynamical quantities are independent of λ, and are indeed the standard ones for a gas of massless free particles in d-dimensional space obeying the Heisenberg uncertainty principle. They are given by [19]

$$-\beta F = \frac{\mathcal{C}(d-1)!}{\beta^d},$$
$$U = \frac{\mathcal{C}d!}{\beta^{d+1}}, \qquad S = \frac{\mathcal{C}(d+1)!}{d\beta^d}. \tag{13}$$

Note that the free energy has the behaviour $|\beta F| \sim T^d$.

Consider the limit $\beta \ll \lambda$. The results now depend on whether $g(E)$ is given by Eq. (11) or (12). In the case where $g(E)$ is given by (11), the thermodynamical quantities are given, to the leading order in β/λ, by

$$-\beta F = \text{const} + \frac{\mathcal{C}}{\lambda^d}\ln(\lambda T), \qquad U = \frac{\mathcal{C}}{\beta\lambda^d},$$
$$S = \text{const} + \frac{\mathcal{C}}{\lambda^d}\big(1 + \ln(\lambda T)\big). \tag{14}$$

Note that in the high temperature limit $\beta \ll \lambda$, the free energy has the behaviour $|\beta F| \sim \text{const} + \ln T$ for any value of d. This indicates a drastic reduction in the degrees of freedom (d.o.f). Such reduction may be possible, in the context of Heisenberg uncertainty principle, if a continuum field theory is replaced by a lattice theory, with a finite number of Bose oscillators at each site [9]; or, in certain topological theories [10] with general covariance restored at short distances [9]. Here, we see that such a behaviour emerges naturally as a consequence of the GUP (4), for systems whose Hamiltonian H is given by (6).

The thermodynamical relations (14), which, upto polarisation factors, are valid for photons also, may have interesting cosmological consequences. In a recent paper [18], the authors analyse the thermodynamical behaviour of photons in the framework of non commutative geometry, postulating a model dependent varying speed of light (VSL). Amazingly, although their set up bears no discernible relation to the present one, the equation of state

$$U = \frac{\mathcal{C}}{\beta\lambda^d}, \qquad \frac{\mathcal{P}}{U} = \text{const} + \ln(\lambda T), \tag{15}$$

in the present case, obtained from (10) and (14), is essentially the same as that obtained in [18] (Eqs. (38) and (39) of [18]). It is shown in [18] that such an equation of state, with an additional ingredient to be mentioned in the next section, solves the horizon problem.

Consider the limit $\beta \ll \lambda$, now in the case where $g(E)$ is given by (12). The thermodynamical quantities are given, to the leading order in β/λ, by

$$-\beta F = \text{const} + \frac{\mathcal{C}}{\lambda^{d-1}\beta},$$
$$U = \frac{\mathcal{C}}{\lambda^{d-1}\beta^2},$$
$$S = \text{const} + \frac{2\mathcal{C}}{\lambda^{d-1}\beta}. \tag{16}$$

Note that in the high temperature limit $\beta \ll \lambda$, the free energy has the behaviour $|\beta F| \sim \text{const} + T$ for any value of d. This indicates a drastic reduction in the d.o.f. Precisely such a free energy, and hence such a reduction, has been found in [9] in the case of the strings at temperatures far above the Hagedorn temperature. In this context, Atick and Witten had indeed conjectured in [9] that a new version of the Heisenberg uncertainty principle may be responsible for such a reduction in the d.o.f. Here, we see that such a drastic reduction in the d.o.f emerges naturally as a consequence of the GUP (4), for systems whose Hamiltonian H is given by (7). In view of this, the GUP (4) may perhaps be taken as the conjectured new version of the Heisenberg uncertainty principle.

The physical origin of the reduction in the d.o.f is easy to understand. When the system obeys the GUP (4), the volume of the phase space cells is $h^d f^d$. It grows at high temperatures/energies and, since the function f is given by (5), $\sim p^d$. Consequently, the number of available cells is enormously reduced, compared with the standard case. This, essentially, is

[4] In both of these limits, it is easier to evaluate the integral in (10) directly, using Eqs. (11) and (12) and appropriate Taylor expansions, than to work with the special functions.

the origin of the reduction in the d.o.f seen above. The precise amount of reduction depends on the choice of the Hamiltonian H. For the H given by $f^\alpha H' = 1$, where $\alpha = 0, 1,$ [5] it can be seen from Eq. (9) that the reduction is such that the resulting d.o.f are equal to that of an α-dimensional system obeying the Heisenberg uncertainty principle and with an effective volume (effective number of sites in the $\alpha = 0$ case) $V_\alpha \sim V\lambda^{\alpha-d}$. This is precisely the result seen explicitly in Eq. (14) for $\alpha = 0$ and in Eq. (16) for $\alpha = 1$.

4. Consider the dynamics of particles which obey the GUP (4), equivalently the GCR (3), with f given by (5). We first note in passing that the time energy uncertainty relation remains unchanged. The derivation proceeds in the standard way, e.g., as in [22], with the result that

$$\Delta t_Q \Delta E \geqslant \frac{\hbar}{2}, \tag{17}$$

where $\Delta t_Q \simeq \Delta Q (dQ/dt)^{-1}$ is the time uncertainty, time being measured by measuring the variation of an observable Q, which has no explicit time dependence and, hence, obeys $dQ/dt = (i/\hbar)[H, Q]$, where H is the Hamiltonian.

We now define the velocity operator V_i by (see [3] also)

$$V_i \equiv \frac{dX_i}{dt} = \frac{i}{\hbar}[H, X_i]. \tag{18}$$

In the case of a non rotating system for which $[X_i, X_j] = 0$, or a free particle system for which H is independent of X_i, one obtains using (3) that

$$V_i = \frac{f H' P_i}{\sqrt{P^2 + m^2}}, \tag{19}$$

where H' is the derivative of H with respect to $\sqrt{P^2 + m^2}$. Denoting the eigenvalue of V_i by v_i, the speed v of a particle with mass m is given by

$$v = \left(\sum_{i=1}^{d} v_i^2 \right)^{1/2} = \frac{p f E'}{\sqrt{p^2 + m^2}}. \tag{20}$$

[5] For $\alpha > 1$, H becomes bounded, i.e., its eigenvalue $E \to$ const as $p \to \infty$, a behaviour whose significance is not clear.

In quantum mechanics, with $\hbar = 1$, the energy eigenfunction will have a time dependence $e^{-i\omega t}$ where $\omega = E$, the eigenvalue of H. Its group velocity $d\omega/dk$ can then be identified naturally with the velocity v in (20), where now $E = \omega$. This leads to a modified dispersion relation, given by

$$\frac{d\omega}{dk} = \frac{p f E'}{\sqrt{p^2 + m^2}}, \tag{21}$$

where one sets $E = \omega$ on the right hand side. Note that Eq. (21) can be thought of as defining the wave number k in the position space. Indeed, in $d = 1$, k is the eigenvalue of the operator K defined by $f(dK/dP) = 1$, so that $[X, K] = i$. Eq. (21) then follows since

$$\frac{dH}{dK} = \frac{PH'}{\sqrt{P^2 + m^2}}, \qquad \frac{dP}{dK} = \frac{PfH'}{\sqrt{P^2 + m^2}}.$$

Furthermore, the speed of light, denoted by C, can be identified naturally with the speed of a particle with mass $m = 0$. Eq. (20) then gives

$$C = f E' \quad \text{and} \quad v \leqslant C. \tag{22}$$

In Eqs. (20), (21), and (22) p and f are to be expressed in terms of E, for which an explicit form of H is required. For H given by (6) we have, after a simple integration,

$$v = \frac{\sqrt{(E^2 - m^2)(1 + \lambda^2 E^2)}}{E},$$
$$C = \sqrt{1 + \lambda^2 E^2}, \tag{23}$$

$$\lambda^2 \omega^2 = \left(1 + \lambda^2 m^2\right) \mathrm{Sinh}^2 \lambda k + \lambda^2 m^2. \tag{24}$$

For H given by (7) (see [3] also) we have, after a simple integration,

$$v = \frac{\sqrt{\mathrm{Sinh}^2 \lambda E - \lambda^2 m^2}}{\mathrm{Sinh}\, \lambda E}, \qquad C = 1, \tag{25}$$

$$\mathrm{Cosh}\, \lambda\omega = \sqrt{1 + \lambda^2 m^2}\, \mathrm{Cosh}\, \lambda k. \tag{26}$$

We now discuss the physical significance of these results. Consider first the modified dispersion relation (21). For $\lambda = 0$, it reduces to $\omega^2 = k^2 + m^2$. For $\lambda \neq 0$, the modification is generically non trivial. The exception is when the Hamiltonian H is given by (7), in which case the modification is only marginal. These can be seen explicitly by considering the high energy 'transplanckian' limit $\lambda\omega \gg 1$ of Eqs. (24) and (26).

Recent studies [16,17] have suggested that such 'transplanckian' modifications of dispersion relation may have observable consequences for the density perturbations that arise during inflation. In these studies, the modified dispersion relations need to be postulated. Here, however, we see that generically the GUP (4) leads naturally to modified dispersion relations. It is clearly of interest to study their consequences for the density fluctuations that arise during inflation.

Consider the speed of light C. It is clear from Eq. (22) that, generically, C is varying and is a non trivial function of energy E. The exception is when the Hamiltonian H is given by (7), in which case $C = 1$. Such 'varying speed of light' (VSL) theories have been extensively studied [11–13], and found to have non trivial implications for cosmology [14] and black hole physics [15]. In these theories, VSL needs to be postulated. Here, however, we see that generically the GUP (4) leads naturally to VSL. It is clearly of interest to study its consequences, which are likely to be non trivial.

For example, C given by (23) increases with energy E. This is precisely the ingredient, alluded to below Eq. (15), that is necessary, but postulated, in [18] to solve the horizon problem. Here, it arises naturally. Moreover, a preliminary analysis shows that the VSL, given by (23), and the photon distribution at energies $E \gg \lambda^{-1}$, derivable from (10) and (11), both have the right behaviour needed to solve the horizon problem, as in [18], but within the present framework. A detailed analysis, however, is beyond the scope of the present Letter.

Eq. (23) are likely to have novel implications for black hole physics also. The horizon of a black hole can naively be thought of as the place where the escape velocity $= 1$, in units where $c = 1$. Particles can then escape from, or from even inside, the horizon if their energy E is sufficiently high since their speed can then be > 1. This may, therefore, provide a mechanism for the transfer of information from inside the horizon to the outside, a process for which no mechanism is known at present [23].

Perhaps more correctly, the horizon is to be thought of as the place from where nothing can escape. Then the escape velocity at the horizon must be infinite since v and $C \to \infty$ as $E \to \infty$. Very likely, therefore, the horizon size must be infinitesimally small or, perhaps equivalently, no black holes can form. [6] These implications for the black hole physics of the GUP (4), with f given by (5) and the Hamiltonian H given by (6), are very interesting but the present arguments are, admittedly, qualitative. Unfortunately, in the absence of a Lorentz and/or general coordinate invariant formulation of the GUP, these issues cannot be addressed rigorously.

5. In summary, we have studied the physical consequences of the GUP (4) that follows from the GCR (3) which is determined uniquely by Maggiore under a set of assumptions. We studied the statistical mechanics and the particle dynamics of systems obeying the GUP (4) and found novel consequences arising in a natural way. For example, the GUP leads naturally to free energies of the form found in certain topological field theories and in strings far above the Hagedorn temperature. It also leads naturally to VSL and to modified dispersion relations. Among other things, these features are likely to solve the horizon problem in cosmology, and may provide novel insights into black hole physics also.

There are numerous issues that require further study. We close by mentioning a few of them.

(i) Understanding the physical significance of the bound $\lambda^2(p^2 + m^2) < 1$ in the $\epsilon = -1$ case.
(ii) Finding the physical principle, if any, which selects a given Hamiltonian H, e.g., the one given by $f^\alpha H' = 1$ for a given value of α.
(iii) Finding a Lorentz and/or general coordinate invariant formulation of the GUP which is crucial, for example, for the study of black hole physics.
(iv) Exploring relations, if any, between GUP and string theory, topological field theory, and VSL theories. That such a relation may exist is, perhaps, indicated by Eqs. (16), (14), and (22).

[6] In this context, note that a certain class of VSL theories are shown in [13] to be equivalent to generalised Brans–Dicke theories. For a certain class of the later theories, it is argued in [24] that black holes are unlikely to form. For a discussion of black hole physics in VSL theories from another point of view, see [15].

Appendix A

The partition function in (10), with $m = a = 0$, can be obtained in a closed form in terms of special functions both for the cases where $H' = 1$, and where $fH' = 1$. We need to evaluate integrals of the form

$$I_{m,n} = \int_0^\infty dt\, t^n \left(\alpha^2 + t^2\right)^{-m/2} e^{-st} \quad \text{for } H' = 1, \tag{A.1}$$

$$J_{m,n} = \int_0^\infty dt\, t^n (\tanh t)^m e^{-st} \quad \text{for } fH' = 1, \tag{A.2}$$

where $t = \lambda E$, $s = \beta/\lambda$, and $\alpha^2 = 1$. It is easy to see that

$$I_{2k,n} = \frac{(-1)^{n+k-1}}{(k-1)!} \left(\frac{d}{ds}\right)^n \left(\frac{d}{d\alpha^2}\right)^{k-1} I_{2,0},$$

$$I_{2k+1,n} = \frac{(-1)^{n+k} 2^{k+1}}{(2k-1)!!} \left(\frac{d}{ds}\right)^n \left(\frac{d}{d\alpha^2}\right)^{k+1} I_{-1,0},$$

$$J_{m,n} = (-1)^n \left(\frac{d}{ds}\right)^n J_{m,0}.$$

Clearly, $J_{0,0} = 1/s$. Moreover,

$$I_{2,0} = \frac{1}{\alpha}\left(\mathrm{ci}(\alpha s)\sin(\alpha s) - \mathrm{si}(\alpha s)\cos(\alpha s)\right),$$

$$I_{-1,0} = \frac{\pi\alpha}{2s}\left(\mathbf{H}_1(\alpha s) - \mathbf{Y}_1(\alpha s)\right),$$

$$J_{1,0} = \beta(s/2) - \frac{1}{s},$$

where $I_{2,0}$ is given in Eq. (3.354.1) of [20]; $J_{1,0}$ in Eq. (3.541.7) of [20]; and $I_{-1,0}$ is obtained using Eq. (4.2.27) of [21] and the properties of the Laplace transform of the derivative of a function. Here, ci and si are the cosine and sine integrals, respectively (see Section (8.23) of [20]), $\mathbf{H}_1$ is the Struve's function, $\mathbf{Y}_1$ is the Bessel function of the second kind, and the function $\beta(x)$ is related to the derivatives of the gamma function (see Section 8.37 of [20]).

We now obtain a recursion relation for $J_{m,0}$. We have, for $m \neq 1$,

$$T_m(x) \equiv \int_0^x dx' \left(\tanh x'\right)^m = -\frac{(\tanh x)^{m-1}}{m-1} + T_{m-2}(x). \tag{A.3}$$

Thus, after a partial integration in $J_{m,0}$ and using $T_m(0) = 0$, one obtains

$$J_{m,0} = s \int_0^\infty dx\, T_m(x) e^{-sx}. \tag{A.4}$$

Eqs. (A.2), (A.3), and (A.4) now lead to the recursion relation

$$J_{m,0} = -\frac{s}{m-1} J_{m-1,0} + J_{m-2,0}, \tag{A.5}$$

using which $J_{m,0}$ can be obtained for $m \geqslant 2$. Using these formulae, the partition function and other quantities can be evaluated in closed form.

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