The Dunkl–Williams equality in pre-Hilbert $C^*$-modules

Josip Pečarić $^a$, Rajna Rajić $^b$,*

$^a$ Faculty of Textile Technology, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia

$^b$ Faculty of Mining, Geology and Petroleum Engineering, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia

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Abstract

In this paper we consider the case of equality in some generalizations of the Dunkl–Williams inequality for elements of a pre-Hilbert $C^*$-module.

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1. Introduction

One of the most fundamental inequality is the triangle inequality, that is,

$$\|x + y\| \leq \|x\| + \|y\|$$

(1)

for any two elements $x$ and $y$ in a normed linear space. Over the years, this inequality has attracted the attention of a number of authors, and many interesting refinements and reverse inequalities of (1) have been obtained (see for instance [15,5,11]). Recently, Kato et al. [7] sharpened the triangle inequality by showing that

$$\left\| \sum_{j=1}^n x_j \right\| + \left( n - \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right) \min_{j \in \{1,\ldots,n\}} \|x_j\| \leq \sum_{j=1}^n \|x_j\|$$

(2)

* Corresponding author.

E-mail addresses: pecaric@element.hr (J. Pečarić), rajna.rajic@zg.t-com.hr (R. Rajić).

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for all nonzero elements $x_1, \ldots, x_n$ of a normed linear space. They also presented its reverse inequality, that is,

\[
\left\| \sum_{j=1}^{n} x_j \right\| + \left( n - \left\| \sum_{j=1}^{n} \frac{x_j}{\| x_j \|} \right\| \right) \max_{j \in \{1, \ldots, n\}} \| x_j \| \geq \sum_{j=1}^{n} \| x_j \|. \tag{3}
\]

In [13] we improved their results by showing that

\[
\left\| \sum_{j=1}^{n} \frac{x_j}{\| x_j \|} \right\| \leq \min_{i \in \{1, \ldots, n\}} \left\{ \frac{1}{\| x_i \|} \left( \left\| \sum_{j=1}^{n} x_j \right\| + \sum_{j=1}^{n} \| x_j \| - \| x_i \| \right) \right\} \tag{4}
\]

and

\[
\left\| \sum_{j=1}^{n} \frac{x_j}{\| x_j \|} \right\| \geq \max_{i \in \{1, \ldots, n\}} \left\{ \frac{1}{\| x_i \|} \left( \left\| \sum_{j=1}^{n} x_j \right\| - \sum_{j=1}^{n} \| x_j \| - \| x_i \| \right) \right\} \tag{5}
\]

holds for an arbitrary number of finitely many nonzero elements $x_1, \ldots, x_n$ of a normed linear space.

In the case of two elements (4) yields the inequality established by Maligranda in [10], which seems to be the sharpest refinement of the well-known Dunkl–Williams inequality (see [6]). When $n = 2$ the inequality (5) is precisely the inequality recently obtained by Mercer in [12].

The problem when the equality in (1) holds for Banach space operators was also studied by many authors; for example, see [1,9] and the references therein. In [3] Barraa and Boumazgour solved this problem for bounded linear operators acting on a complex Hilbert space. They characterized the triangle equality for Hilbert space operators in terms of the numerical range. Their result was generalized in [2] for elements of a pre-Hilbert $C^*$-module.

In this paper we further the study of this subject. Our aim is to consider the case of equality in each of the inequalities (4) and (5) for elements of a pre-Hilbert $C^*$-module.

2. Preliminaries

Pre-Hilbert $C^*$-modules generalize inner-product spaces by allowing the inner product to take values in a more general $C^*$-algebra than the field of complex numbers. The formal definition is as follows.

A **pre-Hilbert $C^*$-module** $X$ over a $C^*$-algebra $\mathcal{A}$ (or a **pre-Hilbert $\mathcal{A}$-module**) is a (right) $\mathcal{A}$-module together with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle : X \times X \to \mathcal{A}$ satisfying the conditions:

(i) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$ for $x, y, z \in X, \alpha, \beta \in \mathbb{C},$
(ii) $\langle x, ya \rangle = \langle x, y \rangle a$ for $x, y \in X, a \in \mathcal{A},$
(iii) $\langle x, y \rangle^* = \langle y, x \rangle$ for $x, y \in X,$
(iv) $\langle x, x \rangle \geq 0$ for $x \in X,$
(v) $\langle x, x \rangle = 0$ if and only if $x = 0.$

It is straightforward that a $C^*$-algebra valued inner product is conjugate-linear in the first variable. We can define a norm on $X$ by $\| x \| = \| \langle x, x \rangle \|^{\frac{1}{2}}.$
For a pre-Hilbert \( \mathcal{A} \)-module \( X \) the Cauchy–Schwarz inequality holds, that is,
\[
\| (x, y) \| \leq \| x \| \| y \|
\]
for all \( x, y \in X \).

A pre-Hilbert \( \mathcal{A} \)-module which is complete with respect to its norm is called a Hilbert \( C^\ast \)-module over \( \mathcal{A} \), or a Hilbert \( \mathcal{A} \)-module.

Clearly, every inner-product space is a pre-Hilbert \( C \)-module (and every Hilbert space is a Hilbert \( C \)-module). Also, every \( C^\ast \)-algebra is a Hilbert \( C^\ast \)-module over itself with the inner product \( \langle a, b \rangle = a^\ast b \). The Banach space \( B(H_1, H_2) \) of all bounded linear operators between Hilbert spaces \( H_1 \) and \( H_2 \) is a Hilbert \( B(H_1) \)-module, where the inner product is defined as \( \langle T, S \rangle = T^\ast S \), and \( T^\ast \) denotes the adjoint operator of \( T \). General references for the theory of \( C^\ast \)-algebras are [4,14] or [16]. The basic theory of Hilbert \( C^\ast \)-modules can be found in [8] or [17].

To describe the case of equality in (4) and (5) in the case of a pre-Hilbert \( C^\ast \)-module, we use the following results obtained in [2] which characterize the triangle equality for elements of a pre-Hilbert \( C^\ast \)-module.

**Theorem 1** [2, Theorem 2.1]. Let \( \mathcal{A} \) be a \( C^\ast \)-algebra, \( X \) a pre-Hilbert \( \mathcal{A} \)-module and \( x, y \in X \). Then the equality \( \| x + y \| = \| x \| + \| y \| \) holds if and only if there exists a state \( \varphi \) on \( \mathcal{A} \) such that \( \varphi(\langle x, y \rangle) = \| x \| \| y \| \).

It was also noticed in [2] (see concluding remarks (b)) that the above characterization of the triangle equality can be generalized for an arbitrary number of finitely many elements of a pre-Hilbert \( C^\ast \)-module as follows:

**Theorem 2** [2]. Let \( \mathcal{A} \) be a \( C^\ast \)-algebra, \( X \) a pre-Hilbert \( \mathcal{A} \)-module, \( n \geq 2 \) a positive integer and \( x_1, \ldots, x_n \) nonzero elements of \( X \). Then the equality \( \| x_1 + \cdots + x_n \| = \| x_1 \| + \cdots + \| x_n \| \) holds if and only if there is a state \( \varphi \) on \( \mathcal{A} \) such that \( \varphi(\langle x_i, x_n \rangle) = \| x_i \| \| x_n \| \) for \( i = 1, \ldots, n - 1 \).

### 3. The results

**Theorem 3.1.** Let \( X \) be a pre-Hilbert \( \mathcal{A} \)-module and \( x_1, \ldots, x_n \) nonzero elements of \( X \) such that \( \| x_1 \| = \cdots = \| x_n \| \) does not hold and \( \sum_{j=1}^{n} x_j \neq 0 \). Then there is \( i \in \{1, \ldots, n\} \) such that the following two statements are mutually equivalent:

(i) \[
\left\| x_j \right\|_{\| x_j \|^{-1}} = \frac{1}{\| x_i \| \left( \sum_{j=1}^{n} x_j \right) + \sum_{j=1}^{n} \| x_j \| - \| x_i \|} \left( \sum_{j=1}^{n} x_j \right).
\]

(ii) There exists a state \( \varphi \) on \( \mathcal{A} \) such that
\[
\text{sgn}(\| x_i \| - \| x_k \|) \sum_{j=1}^{n} \varphi(\langle x_j, x_k \rangle) = \left\| \sum_{j=1}^{n} x_j \right\| \| x_k \|
\]
for all \( k \in \{1, \ldots, n\} \) for which \( \| x_k \| \neq \| x_i \| \).

**Proof.** Let us denote \( J = \{j \in \{1, \ldots, n\} : \| x_j \| \neq \| x_i \|\} \). Let us also put \( x'_j = \text{sgn}(\| x_i \| - \| x_j \|)x_j \), \( j \in J \). Since
\[
\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} = -\left| \frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} \right| \operatorname{sgn}(\|x_i\| - \|x_j\|),
\]

it follows that
\[
\sum_{j=1}^{n} \left( \frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} \right) x_j = -\sum_{j=1}^{n} \left| \frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} \right| x_j'.
\]

Passing the proof of the inequality (4) (see [13, Theorem 2.1]) we conclude that (i) holds precisely when
\[
\left\| \sum_{j=1}^{n} \frac{x_j}{\|x_i\|} - \sum_{j=1}^{n} \left( \frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} \right) x_j \right\| = \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_i\|} \right\| + \left\| \sum_{j=1}^{n} \frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} \|x_j\|. \tag{6}
\]

By using (6) we see that (7) is equivalent to
\[
\left\| \sum_{j=1}^{n} \frac{x_j}{\|x_i\|} + \sum_{j \in J} \left( \frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} \right) x_j' \right\| = \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_i\|} \right\| + \left\| \sum_{j \in J} \frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} \|x_j'\|. \tag{7}
\]

Let us now put \( y = \sum_{j=1}^{n} \frac{x_j}{\|x_i\|} \) and \( z_j = \left| \frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} \right| x_j', \ j \in J \). Then (8) can be written as
\[
\left\| y + \sum_{j \in J} z_j \right\| = \|y\| + \sum_{j \in J} \|z_j\|. \tag{9}
\]

Note that \( y \neq 0 \) and \( z_j \neq 0 \) for all \( j \in J \). By Theorem 2, (9) holds if and only if there is a state \( \varphi \) on \( \mathcal{A} \) satisfying \( \varphi((y, z_k)) = \|y\| \|z_k\|, \ k \in J \). Thus, for \( k \in J \) we have
\[
\varphi\left( \left( \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \left| \frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} \right| x_j' \right) \right) = \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_i\|} \right\| \frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} \|x_j'\|,
\]

that is,
\[
\sum_{j=1}^{n} \varphi((x_j, x_j')) = \left\| \sum_{j=1}^{n} x_j \right\| \|x_j'\|,
\]

Hence,
\[
\operatorname{sgn}(\|x_i\| - \|x_k\|) \sum_{j=1}^{n} \varphi((x_j, x_k)) = \left\| \sum_{j=1}^{n} x_j \right\| \|x_k\|, \ k \in J,
\]

which proves the theorem. \( \square \)

**Theorem 3.2.** Let \( X \) be a pre-Hilbert \( \mathcal{A} \)-module and \( x_1, \ldots, x_n \) nonzero elements of \( X \) such that \( \|x_1\| = \cdots = \|x_n\| \) does not hold and \( \sum_{j=1}^{n} x_j = 0 \). Then there is \( i \in \{1, \ldots, n\} \) such that the following two statements are mutually equivalent:

(i) \( \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| = \frac{1}{\|x_i\|} \sum_{j=1}^{n} \left| \|x_j\| - \|x_i\| \right| \). 

(ii) There exist \( k \in \{1, \ldots, n\} \) such that \( \|x_k\| \neq \|x_i\| \) and a state \( \varphi \) on \( \mathcal{A} \) such that
Thus, for all \( j \) inequality (4) and Theorem 3.1.

**Proof.** Let \( J = \{ j \in \{ 1, \ldots, n \} : \| x_j \| \neq \| x_i \| \} \), \( x'_j = \text{sgn}(\| x_i \| - \| x_j \|) x_j \) and \( z_j := \frac{1}{\| x_j \|} - \frac{1}{\| x'_j \|} \), \( j \in J \), be as in the proof of Theorem 3.1. Passing the proof of Theorem 3.1, one can easily see that (i) holds if and only if

\[
\| \sum_{j \in J} z_j \| = \sum_{j \in J} \| z_j \|. 
\]

Using Theorem 2 we deduce that (10) is equivalent to the fact that there exist \( k \in J \) and a state \( \varphi \) on \( \mathcal{A} \) satisfying

\[
\varphi(\langle z_j, z_k \rangle) = \| z_j \| \| z_k \|, \quad j \in J \setminus \{ k \}.
\]

Thus, for all \( j \in J \setminus \{ k \} \) we get

\[
\varphi \left( -\frac{1}{\| x_j \|} - \frac{1}{\| x'_j \|} \right) = \frac{1}{\| x_i \|} - \frac{1}{\| x_k \|} \| | x'_j \| \| x'_k \|,
\]

that is,

\[
\text{sgn}(\| x_i \| - \| x_j \|) \text{sgn}(\| x_i \| - \| x_k \|) \varphi(\langle x_j, x_k \rangle) = \| x_j \| \| x_k \|.
\]

This proves the theorem. \( \square \)

**Corollary 3.3.** Let \( X \) be a pre-Hilbert \( \mathcal{A} \)-module and \( x_1, \ldots, x_n \) nonzero elements of \( X \) such that \( \sum_{j=1}^{n} x_j \neq 0 \). Then the following two statements are mutually equivalent:

(i) \( \left\| \sum_{j=1}^{n} \frac{x_j}{\| x_j \|} \right\| = \min_{i \in \{1, \ldots, n\}} \left\{ \| x_i \| \sum_{j=1}^{n} \left| \frac{x_j}{\| x_j \|} \right| \right\} \).

(ii) \( \| x_1 \| = \cdots = \| x_n \| \) or there exist \( i \in \{1, \ldots, n\} \) and a state \( \varphi \) on \( \mathcal{A} \) satisfying

\[
\text{sgn}(\| x_i \| - \| x_k \|) \sum_{j=1}^{n} \varphi(\langle x_j, x_k \rangle) = \sum_{j=1}^{n} x_j \| x_k \|
\]

for all \( k \in \{1, \ldots, n\} \) such that \( \| x_k \| \neq \| x_i \| \).

**Proof.** If \( \| x_1 \| = \cdots = \| x_n \| \) we are done. If this is not the case, our corollary follows from the inequality (4) and Theorem 3.1. \( \square \)

**Corollary 3.4.** Let \( X \) be a pre-Hilbert \( \mathcal{A} \)-module and \( x_1, \ldots, x_n \) nonzero elements of \( X \) such that \( \sum_{j=1}^{n} x_j = 0 \). Then the following two statements are mutually equivalent:

(i) \( \left\| \sum_{j=1}^{n} \frac{x_j}{\| x_j \|} \right\| = \min_{i \in \{1, \ldots, n\}} \left\{ \| x_i \| \sum_{j=1}^{n} \left| \frac{x_j}{\| x_j \|} \right| \right\} \).

(ii) \( \| x_1 \| = \cdots = \| x_n \| \) or there exist \( i, k \in \{1, \ldots, n\} \) satisfying \( \| x_i \| \neq \| x_k \| \) and a state \( \varphi \) on \( \mathcal{A} \) such that

\[
\text{sgn}(\| x_i \| - \| x_j \|) \text{sgn}(\| x_i \| - \| x_k \|) \varphi(\langle x_j, x_k \rangle) = \| x_j \| \| x_k \|
\]

for all \( j \in \{1, \ldots, n\} \setminus \{ k \} \) such that \( \| x_j \| \neq \| x_i \| \).
Proof. If $\|x_1\| = \cdots = \|x_n\|$ we are done. So, suppose that this is not the case. Then our corollary follows immediately from the inequality (4) and Theorem 3.2. □

In what follows we consider the case of equality in (5) for elements of a pre-Hilbert $C^*$-module over a $C^*$-algebra $\mathcal{A}$.

Theorem 3.5. Let $X$ be a pre-Hilbert $\mathcal{A}$-module and $x_1, \ldots, x_n$ nonzero elements of $X$ such that $\|x_1\| = \cdots = \|x_n\|$ does not hold and $\sum_{j=1}^n \frac{x_j}{\|x_j\|} \neq 0$. Then there is $i \in \{1, \ldots, n\}$ such that the following two statements are mutually equivalent:

(i) $\left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| = \frac{1}{\|x_i\|} \left( \left\| \sum_{j=1}^n x_j \right\| - \sum_{j=1}^n (\|x_j\| - \|x_i\|) \right)$.

(ii) There exists a state $\varphi$ on $\mathcal{A}$ such that $\text{sgn}(\|x_k\| - \|x_i\|) \sum_{j=1}^n \varphi \left( \left( \frac{x_j}{\|x_j\|}, x_k \right) \right) = \sum_{j=1}^n \frac{x_j}{\|x_j\|} \|x_k\|$ for all $k \in \{1, \ldots, n\}$ such that $\|x_k\| \neq \|x_i\|$.

Proof. Passing the proof of the inequality (5) (see [13, Theorem 2.1]), one can see that (i) holds if and only if the following two conditions are satisfied:

\[
\left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| = \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| - \left\| \sum_{j=1}^n \frac{1}{\|x_j\|} - \frac{1}{\|x_j\|} \right\| x_j \right\|.
\]

and

\[
\left\| \sum_{j=1}^n \left( \frac{1}{\|x_j\|} - \frac{1}{\|x_j\|} \right) x_j \right\| = \sum_{j=1}^n \left| \frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} \right| \|x_j\|. \quad (12)
\]

Let us denote $J = \{ j \in \{1, \ldots, n\} : \|x_j\| \neq \|x_i\| \}$. Let us put $x'_j = \text{sgn}(\|x_j\| - \|x_i\|) x_j$, $j \in J$. Since

\[
\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} = \left| \frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} \right| \text{sgn}(\|x_j\| - \|x_i\|),
\]

we have

\[
\left( \frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} \right) x_j = \left| \frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} \right| x'_j, \quad j \in J. \quad (13)
\]

Define $y = \sum_{j=1}^n \frac{x_j}{\|x_j\|}$, $z_j = \left( \frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} \right) x_j$, $j = 1, \ldots, n$, and $z = \sum_{j=1}^n z_j$. Then (11) and (12) have the following forms:

\[
\|y - z\| = \|y\| - \|z\| \quad (14)
\]

and

\[
\left\| \sum_{j \in J} z_j \right\| = \sum_{j \in J} \|z_j\|, \quad (15)
\]

respectively. Let us now show that (14) and (15) together are equivalent to the fact that there is a state $\varphi$ on $\mathcal{A}$ satisfying
\[ \phi((y - z, z)) = \|y - z\| \sum_{j \in J} \|z_j\|. \] (16)

Indeed, by Theorem 1 the equality (14) holds if and only if there is a state \( \phi \) on \( \mathcal{A} \) such that
\[ \phi((y - z, z)) = \|y - z\| \|z\|. \]

Thus, it is evident that ((14) and (15)) \( \Rightarrow \) (16).

Let us now assume that there is a state \( \phi \) on \( A \) satisfying (16). Then we get
\[ \|y - z\| \sum_{j \in J} \|z_j\| = \phi((y - z, z)) \leq \|y - z\| \|z\|, \]
from which it follows that \( \sum_{j \in J} \|z_j\| \leq \|z\| \), since \( \|y - z\| / \|z\| = 0 \). So, we have
\[ \sum_{j \in J} \|z_j\| \leq \|z\|, \]
that is, \( \| \sum_{j \in J} z_j \| = \| \sum_{j \in J} \|z_j\| \), which is the equality (15). Now, (15) and (16) imply \( \phi((y - z, z)) = \|y - z\| \|z\| \), which is by Theorem 1 equivalent to (14). Hence, (16) \( \Rightarrow \) ((14) and (15)).

Finally, it remains to show that (16) is equivalent to (ii). Let us denote \( \alpha_k = \frac{1}{\|x_k\|} - \frac{1}{\|z_k\|} \), \( k \in J \). Then, by (13), it is clear that (16) reads
\[ \phi (\langle y - z, \sum_{k \in J} \alpha_k x'_k \rangle) = \|y - z\| \sum_{k \in J} \|\alpha_k x'_k\|, \]
that is,
\[ \sum_{k \in J} \alpha_k \phi(\langle y - z, x'_k \rangle) = \sum_{k \in J} \alpha_k \|y - z\| \|x'_k\|. \] (17)

Note that (17) implies
\[ \sum_{k \in J} \alpha_k \|y - z\| \|x'_k\| = \sum_{k \in J} \alpha_k \Re \phi(\langle y - z, x'_k \rangle) \]
\[ \leq \sum_{k \in J} \alpha_k |\phi(\langle y - z, x'_k \rangle)| \]
\[ \leq \sum_{k \in J} \alpha_k \|y - z\| \]
\[ \leq \sum_{k \in J} \alpha_k \|y - z\| \|x'_k\|; \]

hence
\[ \sum_{k \in J} \alpha_k \Re \phi(\langle y - z, x'_k \rangle) = \sum_{k \in J} \alpha_k \phi(\langle y - z, x'_k \rangle) = \sum_{k \in J} \alpha_k \|y - z\| \|x'_k\|. \] (18)

Since
\[ \alpha_k \Re \phi(\langle y - z, x'_k \rangle) \leq \alpha_k |\phi(\langle y - z, x'_k \rangle)| \leq \alpha_k \|y - z\| \|x'_k\| \leq \alpha_k \|y - z\| \|x'_k\|, \]
the equality (18) holds if and only if
\[ \alpha_k \Re \phi(\langle y - z, x'_k \rangle) = \alpha_k |\phi(\langle y - z, x'_k \rangle)| = \alpha_k \|y - z\| \|x'_k\| \]
for every $k \in J$, that is, if and only if
\[
\text{Re } \varphi((y - z, x'_k)) = |\varphi((y - z, x'_k))| = \|y - z\| \|x'_k\|, \quad k \in J,
\]
(19)
since $\alpha_k > 0$ for all $k \in J$. Clearly, (19) is equivalent to
\[
\varphi((y - z, x'_k)) = \|y - z\| \|x'_k\|, \quad k \in J.
\]
Thus, we have proved the equivalence of (17) and (20). It remains to note that (20) reads
\[
\varphi\left(\left(\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}, \text{sgn}(\|x_k\| - \|x_i\|)x_k\right)\right) = \left(\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right) \|x_k\|, \quad k \in J,
\]
which is (ii). □

**Theorem 3.6.** Let $X$ be a pre-Hilbert $\mathcal{A}$-module and $x_1, \ldots, x_n$ nonzero elements of $X$ such that $\|x_1\| = \cdots = \|x_n\|$ does not hold and $\sum_{j=1}^{n} \frac{x_j}{\|x_j\|} = 0$. Then there is $i \in \{1, \ldots, n\}$ such that the following two statements are mutually equivalent:

(i) $\left\| \sum_{j=1}^{n} x_j \right\| = \sum_{j=1}^{n} \|x_j\| - \|x_i\|.$

(ii) There exist $k \in \{1, \ldots, n\}$ such that $\|x_k\| \neq \|x_i\|$ and a state $\varphi$ on $\mathcal{A}$ such that
\[
\text{sgn}(\|x_i\| - \|x_j\|)\text{sgn}(\|x_k\| - \|x_j\|)\varphi(x_j, x_k) = \|x_j\| \|x_k\|
\]
for all $j \in \{1, \ldots, n\} \setminus \{k\}$ for which $\|x_j\| \neq \|x_i\|$.

**Proof.** Let us denote $J = \{j \in \{1, \ldots, n\} : \|x_j\| \neq \|x_i\|\}$. Let us put $x'_j = \text{sgn}(\|x_j\| - \|x_i\|)x_j$, $j \in J$. Denote $y = \sum_{j=1}^{n} \frac{x_j}{\|x_j\|}$, $z_j = \left(\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|}\right)x_j$, $j = 1, \ldots, n$, and $z = \sum_{j=1}^{n} z_j$. As $y - z = 0$ by assumption, passing the proof of Theorem 3.5 we conclude that (i) is equivalent to (15), that is,
\[
\left\| \sum_{j \in J} z_j \right\| = \sum_{j \in J} \|z_j\|.
\]
Now, Theorem 2 implies that (15) holds if and only if there exist $k \in J$ and a state $\varphi$ on $\mathcal{A}$ such that
\[
\varphi(z_j, z_k) = \|z_j\| \|z_k\|, \quad j \in J \setminus \{k\}.
\]
Therefore, for $j \in J \setminus \{k\}$ we have
\[
\varphi\left(\left(\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|}, \frac{1}{\|x_i\|} - \frac{1}{\|x_k\|}\right) x'_j, \frac{1}{\|x_i\|} - \frac{1}{\|x_k\|} \|x_j\| \|x_k\|, \frac{1}{\|x_i\|} - \frac{1}{\|x_k\|}\right),
\]
that is,
\[
\text{sgn}(\|x_j\| - \|x_i\|)\text{sgn}(\|x_k\| - \|x_j\|)\varphi(x_j, x_k) = \|x_j\| \|x_k\|.
\]
This completes the proof. □

**Corollary 3.7.** Let $X$ be a pre-Hilbert $\mathcal{A}$-module and $x_1, \ldots, x_n$ nonzero elements of $X$ such that $\sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \neq 0$. Then the following two statements are mutually equivalent:

(i) $\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\| = \max_{i \in \{1, \ldots, n\}} \left\{ \frac{1}{\|x_i\|} \left(\left\| \sum_{j=1}^{n} x_j \right\| - \sum_{j=1}^{n} \|x_j\| - \|x_i\|\right)\right\}.$
(ii) \( \|x_1\| = \cdots = \|x_n\| \) or there exist \( i \in \{1, \ldots, n\} \) and a state \( \varphi \) on \( \mathcal{A} \) satisfying

\[
\text{sgn}(\|x_k\| - \|x_i\|) \sum_{j=1}^{n} \varphi \left( \left( \frac{x_j}{\|x_j\|}, x_k \right) \right) = \left| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right| \|x_k\|
\]

for all \( k \in \{1, \ldots, n\} \) such that \( \|x_k\| \neq \|x_i\| \).

**Proof.** If \( \|x_1\| = \cdots = \|x_n\| \) we are done. If this is not the case, our corollary follows from the inequality (5) and Theorem 3.5. □

**Corollary 3.8.** Let \( X \) be a pre-Hilbert \( \mathcal{A} \)-module and \( x_1, \ldots, x_n \) nonzero elements of \( X \) such that

\[
\sum_{j=1}^{n} \frac{x_j}{\|x_j\|} = 0.
\]

Then the following two statements are mutually equivalent:

(i) \( 0 = \max_{i \in \{1, \ldots, n\}} \left\{ \frac{1}{\|x_i\|} \left( \|\sum_{j=1}^{n} x_j\| - \sum_{j=1}^{n} \|x_j\| - \|x_i\| \right) \right\} \).
(ii) \( \|x_1\| = \cdots = \|x_n\| \) or there exist \( i, k \in \{1, \ldots, n\} \) satisfying \( \|x_i\| \neq \|x_k\| \) and a state \( \varphi \) on \( \mathcal{A} \) such that

\[
\text{sgn}(\|x_i\| - \|x_j\|)\text{sgn}(\|x_i\| - \|x_k\|) \varphi(\langle x_j, x_k \rangle) = \|x_j\| \|x_k\|
\]

for all \( j \in \{1, \ldots, n\} \setminus \{k\} \) such that \( \|x_j\| \neq \|x_i\| \).

**Proof.** If \( \|x_1\| = \cdots = \|x_n\| \) we are done. So, suppose that this is not the case. Then our corollary follows immediately from the inequality (5) and Theorem 3.6. □

**Remark 3.9.** It is known that every inner-product space is strictly convex. Characterizations of the case of equality in (4) and (5) for elements of a strictly convex normed linear space are given in [13, Corollary 2.7] and [13, Corollary 2.9], respectively. It was said before that every inner-product space can be regarded as a pre-Hilbert \( C^* \)-module over the \( C^* \)-algebra of all complex numbers. Thus, our results are also valid for elements of an arbitrary inner-product space. As the only state on the field of complex numbers is the identity operator, we can reformulate our results in this special case and get the characterizations of the case of equality in (4) and (5) obtained in [13, Corollaries 2.7 and 2.9]. The details are left to the reader.

**References**


