On pedigree polytopes and Hamiltonian cycles

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Abstract

In this paper we define a combinatorial object called a pedigree, and study the corresponding polytope, called the pedigree polytope. Pedigrees are in one-to-one correspondence with the Hamiltonian cycles on \( K_n \). Interestingly, the pedigree polytope seems to differ from the standard tour polytope, \( Q_n \), with respect to the complexity of testing whether two given vertices of the polytope are nonadjacent. A polynomial time algorithm is given for nonadjacency testing in the pedigree polytope, whereas the corresponding problem is known to be \( \mathcal{NP} \)-complete for \( Q_n \). We also discuss some properties of the pedigree polytope and illustrate with examples.

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1. Introduction

The Traveling Salesman Problem (TSP) is about finding a minimum cost tour that starts from the home city and visits every city once and returns back to the home city. When the cost of traveling from city \( i \) to city \( j \) is the same as that of traveling from \( j \) to \( i \), we call the problem the Symmetric Traveling Salesman Problem (STSP). TSP is one of the typical \( \mathcal{NP} \)-hard combinatorial optimization problems, and has been extensively studied. The book The Traveling Salesman Problem—A Guided Tour of Combinatorial Optimization [16] is devoted to the history and various approaches for solving TSP and its extensions. Jünger et al. [15] survey TSP literature, with emphasis on computational achievements based on polyhedral combinatorics and facet defining inequalities. A more recent work dealing with TSP and its variants is the book edited by Gutin and Punnen [13].

In 1954, Dantzig et al. [9] formulated the Asymmetric Traveling Salesman Problem (ATSP) as a 0–1 integer program on a graph. The analogous formulation for the STSP, when the integrality constraints are relaxed, results in the Subtour Elimination Polytope, \( \text{SEP}_n \), where \( n \) denotes the number of cities.

Arthanari [2] posed the STSP as a multistage decision problem and gave a 0–1 programming formulation of the same, involving variables with three subscripts. We refer to this formulation as the multistage insertion (MI) and relaxing the integer restrictions we get the MI-relaxation. In the following section, we briefly outline the above formulation, after...
1.1. The multistage-insertion (MI) formulation

In local search and heuristic approaches to solve STSP, insertion is a commonly used strategy [10,16].
Consider the complete graph \( K_n \) on the vertex set, \( V = \{1, 2, \ldots, n\} \), \( n \geq 3 \). We call a Hamiltonian cycle in \( K_n \) an \( n \)-tour (see Section 2 for notations). Start with the unique 3-tour in \( K_3 \), namely \( (1, 2, 3, 1) \). We choose one of the edges in the 3-tour, \( \{(1, 2), (1, 3), (2, 3)\} \), to insert 4, to obtain a 4-tour. If our previous decisions yield a \((k - 1)\)-tour we select an edge available in the \((k - 1)\)-tour, say \( e_k = (i_k, j_k) \) for inserting \( k \), to obtain a \( k \)-tour. We proceed like this until we find a \( n \)-tour. We here have a multistage decision problem. We give a 0–1 integer formulation of the multistage insertion process below:

Define

\[
x_{ijk} = \begin{cases} 
1 & \text{if in stage } (k - 3) \text{ the decision is to insert } k \text{ between } i \text{ and } j, \\
1 \leq i < j \leq k - 1, \\
0 & \text{otherwise.} 
\end{cases}
\]

Let \( X = (x_{124}, \ldots, x_{n-2,n-1,n}) \in \{0, 1\}^{5n} \), where \( \tau_n = \sum_{k=4}^{n} (1 - (k - 2)/2 \right) \). Let \( \xi_{ijk} = c_{ik} + c_{jk} - c_{ij} \). Here \( \xi_{ijk} \)
gives the incremental cost of inserting \( k \) between \((i, j)\), where \( c_{ij} \) is the cost of visiting city \( j \) from city \( i \) and \( c_{ij} = c_{ji} \). Let \( x_k = (x_{12k}, \ldots, x_{k-2,k-1,k}) \), so we have, \( X = (x_4, \ldots, x_n) \). An integer programming formulation of the above multistage insertion problem given in [3], is presented as Problem 1.

**Problem 1.**

\[
\text{minimize } \sum_{k=4}^{n} \sum_{1 \leq i < j \leq k-1} \xi_{ijk} x_{ijk}
\]

subject to

\[
\sum_{1 \leq i < j \leq k-1} x_{ijk} = 1, \quad 4 \leq k \leq n, \quad (1)
\]

\[
\sum_{k=4}^{n} x_{ijk} \leq 1, \quad 1 \leq i < j \leq 3, \quad (2)
\]

\[
- \sum_{r=1}^{i-1} x_{rij} - \sum_{j=1}^{i-1} x_{isj} + \sum_{k=j+1}^{n} x_{ijk} \leq 0, \quad 4 \leq j \leq n - 1, \quad 1 \leq i < j, \quad (3)
\]

\[
x_{ijk} = 0 \text{ or } 1, \quad 1 \leq i < j \leq k - 1, \quad 4 \leq k \leq n. \quad (4)
\]

**Remark 1.**

- The cost of the insertion decisions is reflected by the objective function, which gives the total incremental cost of the insertions made.
- (1) with the 0–1 restriction on \( X \) ensures that each \( k \) is inserted in exactly one edge.
- (2) ensures that each of the edges \((1, 2), (1, 3)\) and \((2, 3)\) is used for insertion by at most one \( k \).
- (3) ensures that each of the other edges \((1, 4)\) to \((n - 2, n - 1)\) is used for insertion only when they are available.

Relaxing the integer constraints (4) with just nonnegativity constraints (as constraints \( x_{ijk} \leq 1 \) are implied by Eq. (1)) and adding the following constraints:

\[
- \sum_{r=1}^{i-1} x_{rin} - \sum_{s=i+1}^{n-1} x_{isn} \leq 0, \quad i = 1, \ldots, n - 1, \quad (5)
\]
we obtain the \( \text{MI} \)-relaxation of the \( \text{STSP} \). The number of constraints in the \( \text{MI} \)-formulation is \( (n - 3) + n(n - 1)/2 \) and there are \( O(n^3) \) variables. There is one inequality for each edge in \( E_n \), so we have one slack variable corresponding to each edge. Notice that constraints (5) are redundant and are added only because they define the slack variables corresponding to the edges \((i, n), 1 \leq i \leq n - 1\).

Some of the interesting properties of the \( \text{MI} \)-formulation/relaxation stated and proved in [3] are given below:

1. There is a one-to-one correspondence between the integer feasible solutions, \((X)'\)s\) to this formulation and the \( n \)-tours.
2. The slack variables, corresponding to an integer feasible solution, \(X\), give the edge-tour incidence vector of the corresponding \( n \)-tour.
3. The set of feasible slack variable vectors of the \( \text{MI} \)-relaxation, \( \mathcal{U}_n \subseteq \text{SEP}_n \).

See [3] for proofs of these and other properties of \( \text{MI} \)-formulation.

Recently, Carr has given a formulation of \( \text{STSP} \), which he calls cycle shrink. Carr [8] shows that the cycle shrink relaxation is a compact description of the \( \text{SEP} \). However, the cycle shrink formulation has more variables and constraints compared to \( \text{MI} \)-formulation. There is a natural transformation that puts the feasible solutions of \( \text{MI} \)-relaxation in one-to-one correspondence with that of the cycle shrink relaxation. It is shown in [4] that the two relaxations can be obtained as projections of a polytope embedded in a higher dimension.

### 1.2. Outline of the paper

In this paper, we call the sequence of edges selected during the multistage insertion process, \((e_4, \ldots, e_n)\), a pedigree.

The convex hull of these pedigrees yields a new polytope, called the pedigree polytope. The purpose of studying these polytopes, is to gain new insights into the \( \text{STSP} \). In particular, we study the pedigree polytope as being contained in another polytope, arising out of the \( \text{MI} \)-relaxation. Flow problems are defined for a given solution for the \( \text{MI} \)-relaxation, \(X\), for the purpose of obtaining necessary and sufficient conditions for \(X\) to be in the pedigree polytope.

Given two pedigrees, the problem of determining whether they correspond to nonadjacent vertices of the pedigree polytope is called the nonadjacency testing problem. It is well known that such a problem with respect to tours in the \( \text{STSP} \) polytope is \( \text{NP} \)-complete [22]. We show that nonadjacency testing of pedigrees can be done in polynomial time.

In Section 2 of this paper we introduce the preliminaries and notation used from graph theory, and the definition of the pedigree polytope. In Section 3, we show that the pedigree polytope is contained in a polytope arising out of the multistage insertion process. Section 4 presents a certain characterization of the pedigree polytope using a sequence of flow feasibility problems defined over bipartite graphs. The nonadjacency testing of pedigrees is discussed in Section 5. Section 6 discusses the consequences of the pedigree approach and future research directions.

### 2. Preliminaries and notations

This section gives a short review of the definitions and concepts from graph theory and flows in networks used in the paper.

Let \( R \) denote the set of reals. Similarly, \( Q, Z, N \) denote the rationals, integers and natural numbers, respectively, and \( B \) stands for the binary set of \([0, 1]\). Let \( R_+ \) denote the set of nonnegative reals. Similarly, the subscript \( \pm \) is understood with rationals. Let \( R^d \) denote the set of \( d \)-tuples of reals. Similarly, the superscript \( \pm \) is understood with rationals, etc. Let \( R^{m \times n} \) denote the set of \( m \times n \) real matrices.

Let \( n \) be an integer, \( n \geq 3 \). Let \( V_n \) be a set of vertices. Assuming, without loss of generality, that the vertices are numbered in some fixed order, we write \( V_n = \{1, \ldots, n\} \). Let \( E_n = \{(i, j) | i, j \in V_n, i < j\} \) be the set of edges. The cardinality of \( E_n \) is denoted by \( p_n = n(n - 1)/2 \). Let \( K_n = (V_n, E_n) \) denote the complete graph of \( n \) vertices.

We denote the elements of \( E_n \) by \( e \), where \( e = (i, j) \). We also use the notation \( ij \) for \((i, j)\). Notice that, unlike the usual practice, an edge is assumed to be written with \( i < j \).

**Definition 1 (Edge label).** Let the elements of \( E_n \) be labeled as follows: \((i, j) \in E_n\), has the label, \( l_{ij} = p_{j-1} + i \).
This means edges \((1, 2), (1, 3), (2, 3) \in E_3\) are labeled 1, 2, and 3, respectively. Once the elements in \(E_{n-1}\) are labeled then the elements of \(E_n \setminus E_{n-1}\) are labeled in increasing order of the first coordinate, namely \(i\).

For a subset \(F \subset E_n\) we write the characteristic vector of \(F\) by \(x_F \in \mathbb{R}^{p_n}\) where

\[
x_F(e) = \begin{cases} 
1 & \text{if } e \in F, \\
0 & \text{otherwise}.
\end{cases}
\]

We assume that the edges in \(E_n\) are ordered in increasing order of the edge labels.

For a subset \(S \subset V_n\) we write

\[
E(S) = \{(i, j) \mid i, j \in E, i, j \in S\}.
\]

Given \(u \in \mathbb{R}^{p_n}\), \(F \subset E_n\), we define,

\[
u(F) = \sum_{e \in F} u(e).
\]

For any subset \(S\) of vertices of \(V_n\), let \(\delta(S)\) denote the set of edges in \(E_n\) with one end in \(S\) and the other in \(S^c = V_n \setminus S\). For \(S = \{i\}\), we write \(\delta(i) = \delta(i)\).

A subset \(H\) of \(E_n\) is called a Hamiltonian cycle in \(K_n\) if it is the edge set of a simple cycle in \(K_n\), of length \(n\). We also call such a Hamiltonian cycle a \(n\)-tour in \(K_n\). At times we represent \(H\) by the vector \((1_{i_2 \ldots i_n} 1)\), where \((i_2 \ldots i_n)\) is a permutation of \((2 \ldots n)\), corresponding to \(H\).

For details on graph related terms see any standard text on graph theory such as [7].

2.1. Forbidden arcs transportation problem

Consider a balanced transportation problem, in which, some arcs called the forbidden arcs are not available for transportation. We call the problem of finding whether a feasible flow exists in such an incomplete bipartite network, a forbidden arcs transportation (FAT) problem [20].

The celebrated work by Ford and Fulkerson, Flows in Networks [11], is a classic on this subject. For recent developments in bipartite network flow problems see [1].

We prove the following lemma on such a flow feasibility problem arising with respect to nonempty partitions of a finite set.

**Lemma 2.1.** Suppose \(\mathcal{D} \neq \emptyset\) is a finite set and \(g: \mathcal{D} \to \mathbb{Q}_+\) is a nonnegative rational function, such that, \(g(\emptyset) = 0\), and \(g(\emptyset) = 1\). Let \(\mathcal{D}^1 = \{D_1^1, x = 1, \ldots, n\}_1\), \(\mathcal{D}^2 = \{D_2^2, \beta = 1, \ldots, n\}_2\) be two nonempty partitions of \(\mathcal{D}\). (That is, \(\bigcup_{x=1}^n D_1^x = \mathcal{D}\) and \(D_1^x \cap D_1^y = \emptyset, x \neq y\). Similarly \(\mathcal{D}^2\) is understood.) Consider the FAT problem defined as follows:

Let the origins correspond to \(D_1^x\), with availability \(a_x = g(D_1^x), x = 1, \ldots, n\) and the destinations correspond to \(D_2^x\), with requirement \(b_\beta = g(D_2^\beta), \beta = 1, \ldots, n\). Let the set of arcs be given by

\[
\mathcal{A} = \{(x, \beta) \mid D_1^x \cap D_2^\beta \neq \emptyset\}.
\]

Then \(f_{x\beta} = g(D_1^x \cap D_2^\beta) \geq 0\) is a feasible solution for the FAT problem considered.

**Proof.** Since \(\mathcal{D}^1\) and \(\mathcal{D}^2\) are partitions of \(\mathcal{D}\), we have \(\sum_{x} a_x = \sum_{\beta} b_\beta = g(\mathcal{D}) = 1\). \(f_{x\beta} \geq 0\) can be easily seen. Now \(a_x = g(D_2^\beta) = \sum_{\beta} g(D_1^x \cap D_2^\beta) = \sum_{\beta} f_{x\beta}, \forall x\). Similarly, \(\sum_{x} f_{x\beta} = b_\beta, \forall \beta\). Hence the feasibility of \(f\). \(\square\)

This lemma is used in the proof of Theorem 4.2, given in the Appendix. Several other FAT problems are defined and studied in the later sections.

2.2. Definition of the pedigree polytope

In this subsection we present an alternative polyhedral representation of the STSP, using the definition of pedigrees. Since the cardinality of \(E_k\) is denoted by \(p_k\) we have, \(\tau_n = \sum_{k=4}^n p_k - 1\).
Let $Q_n$ denote the standard STSP polytope, given by

$$Q_n = \text{conv}(\{X_H : X_H \text{ is the characteristic vector of } H \in \mathcal{H}_n\})$$

where $\mathcal{H}_n$ denotes the set of all Hamiltonian cycles (or n-tours) in $K_n$.

**Problem 2.1 (STSP optimization).** Given $K_n$, $c \in \mathbb{Z}_+^{2n}$, find $H^* \in \mathcal{H}_n$ such that,

$$c(H^*) \leq c(H) \quad \forall H \in \mathcal{H}_n.$$ 

Traditionally, in polyhedral combinatorics, $Q_n$ is studied while solving STSP (see [16]). However, we deviate from this and consider an alternative polytope for this purpose. The required notations and concepts follow.

Given $H \in \mathcal{H}_{k-1}$, the operation insertion is defined as follows: let $e = (i, j) \in H$. Inserting $k$ in $e$ is equivalent to replacing $e$ in $H$ by $\{(i, k), (j, k)\}$ obtaining a $k$-tour. When we denote $H$ as a subset of $E_{k-1}$, then inserting $k$ in $e$ gives us a $H' \in \mathcal{H}_k$ such that,

$$H' = (H \cup \{(i, k), (j, k)\}) \setminus \{e\}.$$ 

We write $H \rightarrow e, k \leftarrow H'$.

Given $H \in \mathcal{H}_k$, the operation shrinking is defined as follows: let $\delta(k) \cap H = \{(i, k), (j, k)\}$. Shrinking $H$ is equivalent to replacing $\{(i, k), (j, k)\}$ in $H$ by $\{(i, j)\}$ obtaining a $(k-1)$-tour. When we denote $H$ as a subset of $E_k$ then shrinking $H$ gives us a $H' \in \mathcal{H}_{k-1}$, such that,

$$H' = (H \setminus \{(i, k), (j, k)\}) \cup \{(i, j)\}.$$ 

We write $H \leftarrow e, k \rightarrow H'$, and read this as $H$ shrinks to $H'$.

Notice that shrinking is the inverse operation of insertion. However, in shrinking only vertex $k$ is chosen for shrinking, but for insertion $e$ needs to be specified, as well.

**Definition 2 (Pedigree).** The vector $W = (e_4, \ldots, e_n) \in E_3 \times \cdots \times E_{n-1}$ is called a pedigree if and only if there exists a $H \in \mathcal{H}_n$ such that $H$ is obtained from the 3-tour by the sequence of insertions, viz.,

$$3\text{-tour } e_4, 4 \rightarrow H^4 \ldots H^{n-1} e_n, n \rightarrow H.$$ 

The pedigree $W$ is referred to as the pedigree of $H$. Pedigree is a compact way of writing $H$. The pedigree of $H$ can be obtained by shrinking $H$ sequentially to the 3-tour and noting the edge created at each stage. We then write the edges obtained in the reverse order of their occurrence.

Let the set of all pedigrees, corresponding to $H \in \mathcal{H}_n$ be denoted by $\mathcal{P}_n$. For any $4 \leq k \leq n$, given an edge $e \in E_{k-1}$, with edge label $l$, we can associate a 0–1 vector, $x(e) \in B^{p_{k-1}}$, such that, $x(e)$ has a 1 in the $l$th coordinate, and zeros elsewhere. That is, $x(e)$ is the indicator of $e$.

Similarly, we can associate a $X = (x_4, \ldots, x_n) \in B^{7n}$, the characteristic vector of the pedigree $W$, where $(W)_k = e_k$, the $(k-3)$rd component of $W$, $4 \leq k \leq n$ and $x_k$ is the indicator of $e_k$.

Let $P_n = \{X \in B^{7n} : X \text{ is the characteristic vector of } W, \text{ the pedigree of } H \in \mathcal{H}_n\}$. Thus, there is a one-to-one correspondence between $H \in \mathcal{H}_n$ and $X \in P_n$. We can also write equivalently, $P_n = \{X \in B^{7n} : X \text{ is the characteristic vector of the pedigree } W \in \mathcal{P}_n\}$.

Consider the convex hull of $P_n$. We call this the pedigree polytope, denoted by $\text{conv}(P_n)$.

Our study is devoted to the discovery of the properties of the pedigree polytope. A pedigree $W = (e_4, \ldots, e_n)$ is such that $(e_4, \ldots, e_k)$ is a pedigree for a $k$-tour, for $4 \leq k \leq n$. We state this as a fact below:

**Fact.** An interesting property of $X = (x_4, \ldots, x_n) \in P_n$ is that, for any $k, 4 \leq k \leq n$, $X$ restricted to the first $k - 3$ stage(s), written as

$$X/k = (x_4, \ldots, x_k)$$

is in $P_k$. 
Similarly, $X/k - 1$ and $X/k + 1$ are interpreted as restrictions of $X$. We use this notation for any $X \in R^\infty$ as well, like in Definition 6.

**Example 2.1.** Consider the pedigree, $W$ given by

$$W = (e_4 = (1, 2), e_5 = (2, 3), e_6 = (1, 3), e_7 = (2, 5)).$$

Starting with the 3-tour and inserting 4 in $e_4 = (1, 2)$, we obtain the 4-tour, $\{(1, 4), (2, 4), (2, 3), (1, 3)\}$. Now $e_5 = (2, 3)$ is available in the 4-tour. We insert 5 in $e_5$ and obtain a 5-tour and so on. Finally, inserting 7 in $e_7$ we get the 7-tour, given by $H = (1, 4, 2, 7, 5, 3, 6, 1)$. The characteristic vector, $X = (x_4, \ldots, x_7)$, corresponding to $W$ is given by $x_4 = (100), x_5 = (001, 000), x_6 = (010, 000, 0000), x_7 = (000, 000, 0100, 00000)$.

**Definition 3.** Given $e = (i, j) \in E_n$, we call

$$G(e) = \begin{cases} \delta(i) \cap E_{j-1} & \text{if } j \geq 4, \\ E_3 \setminus \{e\} & \text{otherwise,} \end{cases}$$

the set of *generators* of the edge $e$.

Since an edge $e = (i, j), j > 3$ is generated by inserting $j$ in any $e'$ in the set $G(e)$, the name *generator* is used to denote any such edge.

**Example 2.2.** Consider $n = 5, e = (3, 5)$. Here $j \geq 4$, so, we have $G(e) = \delta(i) \cap E_{j-1}$. Since $i = 3, \delta(3) = \{(1, 3), (2, 3), (3, 4), (3, 5)\}$, and $E_{j-1} = E_4 = \{(1, 2), (1, 3), (2, 3), (1, 4), (2, 4), (3, 4)\}$. Therefore, $G(e) = \{(1, 3), (2, 3), (3, 4)\}$.

Consider $e = (1, 3)$. Since $j \leq 3$, we have $G(e) = E_3 \setminus \{e\} = \{(1, 2), (2, 3)\}$.

The following lemma gives the consistency conditions for a pedigree. This ensures that for inserting node $k$ in the $(k - 3)$rd stage, the edge used must be available.

**Lemma 2.2.** Given $n$, consider $W = (e_4, \ldots, e_n)$, where $e_k = (i_k, j_k)$ for $1 \leq i_k < j_k \leq k - 1, 4 \leq k \leq n$. $W$ corresponds to a pedigree in $P_n$ if and only if

1. $e_k, 4 \leq k \leq n$, are all distinct,
2. $e_k \in E_{k-1}, 4 \leq k \leq n$, and
3. for every $k, 5 \leq k \leq n$, there exists a $e' \in G(e_k)$ such that, $e_q = e'$, where $q = \max\{4, j_k\}$.

**Proof.** Follows from the definition of a pedigree. Suppose $W$ corresponds to a pedigree in $P_n$, then by definition, assertion (1) is necessarily true, as any edge $e_k$ used for inserting $k$ is not available in the subsequent tours obtained. Assertion (2) is true, as the set of edges available in $H_k$ for inserting $k + 1$ is a subset of $E_k, 4 \leq k < n$. Suppose assertion (3) is not true for some $k$ then $e_k = (i_k, j_k)$ and $e_q \notin G(e_k)$. This means $H^{(k)}$ does not contain $e_k$ and so subsequently, $H^{k-1}$ also does not contain $e_k$. Contradiction. This proves one way.

Consider $W = (e_4, \ldots, e_n)$. Let $n = 4$. Notice that $W = (e_4)$ corresponds to a pedigree in $P_4$ as $e_4 \in E_3$. Let $l, 4 < l \leq n$ be the smallest $l$ such that $W = (e_4, \ldots, e_l)$ does not correspond to a pedigree in $P_l$. This means $W' = (e_4, \ldots, e_{l-1})$ corresponds to a pedigree in $P_{l-1}$. So, we have a ($l - 1$)-tour, $H(W')$, obtained by the sequence of insertions, as per $W'$.

By assumption $e_l \notin H(W')$. But $W$ satisfies (3) and so a generator of $e_j$ appears as $e_q$. This implies $e_l$ is available in the $q$-tour corresponding to the pedigree $(e_4, \ldots, e_q)$. Also, for every $s, q < s \leq l - 1$, we have $e_l$ distinct from $e_s$, and so $e_l$ is available in the $s$-tour corresponding to the pedigree $(e_4, \ldots, e_s)$. So $e_l$ is in $H(W')$. Hence $l$ cannot be the smallest such index. Contradiction.

Hence the lemma. □

This lemma allows us to define a pedigree without explicitly considering the corresponding Hamiltonian cycle.
Definition 4 (Extension of a pedigree). Let \( y(e) \) be the indicator of \( e \in E_k \). Given a pedigree, \( W = (e_4, \ldots, e_k) \) (with the characteristic vector, \( X \in P_k \)) and an edge \( e \in E_k \), we call \( (W, e) = (e_4, \ldots, e_k, e) \) an extension of \( W \) in case \((X, y(e)) \in P_{k+1}\).

Using Lemma 2.2, observe that given \( W \) a pedigree in \( P_k \) and an edge \( e = (i, j) \in E_k \), \((W, e)\) is a pedigree in \( P_{k+1} \) if and only if (1) \( e_l \neq e \), \( 4 \leq l \leq k \) and (2) there exists a \( q = \max(4, j) \) such that \( e_q \) is a generator of \( e = (i, j) \).

3. A polytope that contains \( \text{conv}(P_n) \)

Let \( X = (x_4, \ldots, x_n) \in P_n \) correspond to the pedigree \( W = (e_4, \ldots, e_n) \). We state the following results without proof:

**Lemma 3.1.** \( X \in P_n \) implies \( X \geq 0 \) and
\[
x_k(E_{k-1}) = 1, \quad k \in V_n \setminus V_3.
\]

**Lemma 3.2.** \( X \in P_n \) implies
\[
\sum_{k=4}^{n} x_k(e) \leq 1, \quad e \in E_3.
\]

**Lemma 3.3.** \( X \in P_n \) implies
\[
-x_j(\delta(i) \cap E_{j-1}) + \sum_{k=j+1}^{n} x_k(e) \leq 0, \quad e = (i, j) \in E_{n-1} \setminus E_3.
\]

Notice that the equations and inequalities considered in the above lemmas are same as that of the MI-relaxation discussed in Section 1.1, except that we have a three subscripted notation and some redundant constraints in the MI-formulation.

Definition 5 (\( P_{MI}(n) \)-polytope). Consider \( X \in R^* \) satisfying the nonnegativity restrictions, \( X \geq 0 \) and Eq. (6) and the inequalities (7) and (8).

The set of all such \( X \) is a polytope, as we have defined it using linear equalities and inequalities. We call this polytope, \( P_{MI}(n) \).

As every pedigree satisfies the equalities and inequalities that define \( P_{MI}(n) \), we can now conclude that \( \text{conv}(P_n) \subset P_{MI}(n) \). In addition we have,

**Theorem 3.1.** \( X \in P_n \) implies \( X \) is an extreme point of \( P_{MI}(n) \).

**Proof.** We have shown by Lemmas 3.1–3.3, that \( X \in P_{MI}(n) \). Now, consider the first \((n-3)\) rows of the submatrix formed by the columns corresponding to positive components of \( X \), that is, \( x_k(e), k \in V_n \setminus V_3 \). Since this is an identity matrix of size \( n-3 \), the columns corresponding to the positive components of \( X \) are linearly independent. Now these \( n-3 \) columns with the identity columns \( (p_{n-1} \text{ in all}) \) corresponding to the slack variables of the inequality constraints, form a basis for the \( P_{MI}(n) \)—in standard form. Hence, \( X \in P_n \) corresponds to an extreme point \( P_{MI}(n) \).

Checking whether \( X \in P_{MI}(n) \) can be done by checking whether \( X \) is a feasible solution to the MI-relaxation. And it can be done in polynomial time.

4. Characterization theorems

In this section, given a \( X \in P_{MI}(n) \) we wish to know whether \( X \) is indeed in \( \text{conv}(P_n) \). Some of these results presented here find use in Section 5. Let \(|P_k|\) denote the cardinality of \( P_k \). Assume that the pedigrees in \( P_k \) are numbered (say, according to the lexicographical ordering of the edge labels of the edges appearing in a pedigree).
Definition 6. Given $X = (x_4, \ldots, x_n) \in P_{M1}(n)$ we denote by $X/k = (x_4, \ldots, x_k)$, the restriction of $X$, for $4 \leq k \leq n$.

Given $X \in P_{M1}(n)$ and $X/k \in \text{conv}(P_k)$, consider $\lambda \in R_{+}[P_k]$ that can be used as a weight to express $X/k$ as a convex combination of $X' \in P_k$. Let $I(\lambda)$ denote the index set of positive coordinates of $\lambda$. Let $A_k(X)$ denote the set of all possible weight vectors, for a given $X$ and $k$, that is,

$$A_k(X) = \left\{ \lambda \in R_{+}[P_k] \left| \sum_{r \in I(\lambda), X' \in P_k} \lambda_r X' = X/k, \sum_{r \in I} \lambda_r = 1 \right. \right\}.$$

Definition 7. Consider a $X \in P_{M1}(n)$ such that $X/k \in \text{conv}(P_k)$. We denote the $k$-tour corresponding to a pedigree $X^2$ by $H^2$. Given a weight vector $\lambda \in A_k(X)$, we define a FAT problem with the following data:

- $O$ -- Origin: $\alpha, \alpha \in I(\lambda)$
- $a$ -- Supply: $a_2 = \lambda_2$
- $D$ -- Destinations: $\beta, \beta \in E_k, x_{k+1}(e_{\beta}) > 0$
- $b$ -- Demand: $b_\beta = x_{k+1}(e_{\beta})$
- $\wedge$ -- Arcs: $\{(\alpha, \beta) \in O \times D | e_{\beta} \in H^2\}$.

We designate this problem as $\text{FAT}_k(\lambda)$. Notice that arcs $(\alpha, \beta)$ not satisfying $e_{\beta} \in H^2$ are the forbidden arcs. We also say $\text{FAT}_k$ is feasible if problem $\text{FAT}_k(\lambda)$ is feasible for some $\lambda \in A_k(X)$.

Equivalently, the arcs in $\wedge$ can be interpreted as follows: If $W^2$ is the pedigree corresponding to $X^2 \in P_k$ for an $a \in I(\lambda)$ then the arcs $(\alpha, \beta) \in \wedge$ are such that the $(W^2, e_{\beta})$ is an extension of $W^2$. (Recall Definition 4.)

Example 4.1. Consider $X = (0 \frac{1}{2} \frac{1}{3} \frac{1}{4}, 0 \frac{1}{6} 0 \frac{1}{6} \frac{1}{4} \frac{1}{5})$. We wish to check whether $X$ is in $\text{conv}(P_5)$. It is easy to check that $X$ indeed satisfies the constraints of $P_{M1}(5)$. Also $X/4 = (0 \frac{1}{2} \frac{1}{2})$ is obviously in $\text{conv}(P_4)$. And $A_4(X) = \{(0 \frac{1}{3}, \frac{1}{3})\}$. Assume that the pedigrees in $P_4$ are numbered such that, $X_1 = (1 0 0), X_2 = (0 1 0)$ and $X_3 = (0 0 1)$ and the edges in $E_4$ are numbered according to their edge labels. Then $I(\lambda) = \{2, 3\}$. Here $k = 4$ and the FAT$_4$ is given by a problem with origins, $O = \{2, 3\}$ with supply $a_2 = \frac{1}{4}, a_3 = \frac{2}{3}$ and destinations, $D = \{2, 4, 5, 6\}$ with demand $b_2 = b_4 = \frac{1}{5}, b_3 = b_6 = \frac{1}{7}$. Corresponding to origin 2 we have the pedigree $W^2 = \{(1, 3)\}$. And the edge corresponding to destination 2 is $e_2 = (1, 3)$. As $(W^2, e_2)$ is not an extension of $W^2$, we do not have an arc from origin 2 to destination 2. Similarly, $(W^3, e_4), (W^2, e_5)$ are not extensions of $W^3$ and $W^2$, respectively, so we do not have arcs from origin 3 to destination 4 and origin 2 to destination 5. We have the set of arcs given by

$$\wedge = \{(2, 4), (2, 6), (3, 1), (3, 5) \text{ and } (3, 6)\}.$$

Notice that $f$ given by $f_{24} = f_{26} = f_{32} = f_{36} = \frac{1}{6}, f_{35} = \frac{1}{3}$ is feasible to $\text{FAT}_4(\lambda)$. (See Fig. 1). This $f$, in fact, gives a weight vector to express $X$ as a convex combination of the vectors in $P_5$, which are the extensions corresponding to arcs with positive flow. This role of $f$ is in general true and we state this as Theorem 4.1.

It is easy to check that $f$ is the unique feasible flow in this example, so no other weight vector exists to certify $X$ in $\text{conv}(P_5)$. Thus, we have expressed $X$ as a convex combination of the incidence vectors of the pedigrees $W^7 = ((1, 3)(1, 4)), W^8 = ((1, 3)(3, 4)), W^{10} = ((2, 3)(1, 3)), W^{12} = ((2, 3)(3, 4))$ (each of them receive a weight of $\frac{1}{6}$) and $W^{11} = ((2, 3)(2, 4))$ (which receives a weight of $\frac{1}{7}$).

Theorem 4.1. Let $k \in V_{n-1} \backslash V_3$. Suppose $\lambda \in A_k(X)$ is such that $\text{FAT}_k(\lambda)$ is feasible. Consider any feasible flow $f$ for the problem. Let $W^{2\beta}$ denote the extension $(W^2, e_{\beta})$, a pedigree in $P_{k+1}$, corresponding to the arc $(\alpha, \beta)$. Let $\wedge_f$ be the set of such pedigrees, $W^{2\beta}$ with positive flow $f_{2\beta}$. Then $f$ provides a weight vector to express $X/k + 1$ as a convex combination of pedigrees in $\wedge_f$.

Next, we observe that $\text{conv}(P_n)$ can be characterized using a sequence of flow feasibility problems as stated in the following theorems:

Theorem 4.2. If $X \in \text{conv}(P_n)$ then $\text{FAT}_k$ is feasible $\forall k \in V_{n-1} \backslash V_3$. 
Theorem 4.3. Let $k \in V_{n-1}/V_3$. If $\lambda \in A_k(X)$ is such that FAT$_k(\lambda)$ is feasible, then $X/(k+1) \in \text{conv}(P_{k+1})$.

The proofs of these theorems are given in the Appendix. In general, we do not have to explicitly give the set $A_k(X)$. The set is used in the proofs. Thus, for a given $X \in P_{M}(n)$ the condition

$$\forall k \in V_{n-1}/V_3, \exists \lambda \in A_k(X) \text{ such that FAT}_k(\lambda) \text{ is feasible}$$

is both necessary and sufficient for $X$ to be in $\text{conv}(P_n)$.

In Theorem 4.3, we have a procedure to check whether a given $X \in P_{M}(n)$, is in the pedigree polytope, $\text{conv}(P_n)$. Since feasibility of a FAT$_k(\lambda)$ problem for a weight vector $\lambda$ implies $X/(k+1)$ is in $\text{conv}(P_{k+1})$, we can sequentially solve FAT$_k(\lambda_k)$ for each $k = 4, \ldots, n-1$ and if FAT$_k(\lambda_k)$ is feasible we set $k = k + 1$ and while $k < n$ we repeat; at any stage if the problem is infeasible we stop. So if we have reached $k = n$ we have a proof that $X \in \text{conv}(P_n)$. However, if for a $\lambda \in A_k(X)$ the problem is infeasible we cannot conclude that $X \notin \text{conv}(P_n)$. Example 4.2 illustrates this.

Example 4.2. Consider $X$ given by

$$x_4 = (1/2, 1/2, 0),$$
$$x_5 = (0, 0, 1/2, 1/2, 0, 0),$$
$$x_6 = (0, 0, 0, 1/2, 1/2, 0, 0, 0, 0).$$

FAT$_4(\lambda)$ for the unique $\lambda = x_4$ is feasible; $f$ given by $f((1, 2), (2, 3)) = f((1, 3), (1, 4)) = 1/2$ with flow along other arcs zero is a feasible flow for FAT$_4(\lambda)$.

The problem FAT$_5(\lambda)$ corresponding to the $\lambda$ given by $f$ is infeasible as the maximum flow in the corresponding network is only $1/2$.

We are not able to conclude whether $X \in \text{conv}(P_6)$. But we can check that $X = 1/2(X^1 + X^2)$, where $X^1$ is given by $x^1_4((1, 2)) = x^1_5((1, 4)) = x^1_6((2, 4)) = 1$ and $X^2$ is given by $x^2_4((1, 3)) = x^2_5((2, 3)) = x^2_6((1, 4)) = 1$.

However if we have chosen the alternative $f^*$, feasible solution for FAT$_4(\lambda)$, given by $f^*((1, 2), (1, 4)) = f^*((1, 3), (2, 3)) = 1$ with flow along other arcs zero, we have the problem FAT$_5(\lambda^*)$ corresponding to $f^*$. And this problem is feasible and so we conclude $X \in \text{conv}(P_6)$.

Current research is directed towards devising methods to find a suitable $\lambda$ for which the FAT$_k(\lambda)$ problem is feasible or to show that for no $\lambda \in A_k(X)$ the problem is feasible.

5. Nonadjacency in the pedigree polytopes

Papadimitriou [22] has shown that the nonadjacency testing on the Traveling Salesman Polytope is $NP$-complete. The asymmetric version of the problem was considered by Heller in 1955 [14]. Murty [18] gave a purported necessary

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{FAT$_4$ Problem for Example 4.1.}
\end{figure}
and sufficient condition for two tours to be nonadjacent in the convex hull of tours. Theorem 2 in [18], states that $T^{[1]}$ and $T^{[2]}$ are nonadjacent if and only if there exists a tour $T^{[3]}$ different from both $T^{[i]}$, $i = 1, 2$ such that

$$T^{[3]} \subset \bigcup_{i=1,2} T^{[i]} \quad \text{and} \quad \bigcap_{i=1,2} T^{[i]} \subset T^{[3]}.$$ 

Rao [23] gives a counter example to show that this condition, though a necessary one, is not sufficient.

Next, we state the corresponding problem for pedigrees.

**Problem 5.1 (Nonadjacency of pedigrees).** Instance:
- number of cities: $n$.
- Pedigrees: $X^{[1]}, X^{[2]} \in P_n$.
- Question: Are $X^{[1]}, X^{[2]}$ nonadjacent vertices of $\text{conv}(P_n)$?

Next, we give a formal definition of nonadjacency in pedigree polytopes. Given $X^{[1]}, X^{[2]} \in P_n$, let $\overline{x} = \frac{1}{2}(X^{[1]} + X^{[2]})$. (In this section $\overline{x}$ has only this meaning, unless otherwise specified.) Let $\lambda^0 \in \Lambda_{n-1}(\overline{x})$ correspond to the convex combination, $\overline{x}/n - 1 = \frac{1}{2}(X^{[1]}/n - 1 + X^{[2]}/n - 1)$. However, we define $\lambda^0(X^{[1]}/n - 1) = 1$, if $X^{[1]}/n - 1 = X^{[2]}/n - 1$.

**Theorem 5.1.** Given vertices $X^{[i]} \in P_n$, $i = 1, 2$, they are nonadjacent in $\text{conv}(P_n)$ if and only if there exists a $S \subset P_n$ and $\mu \in \Lambda_n(\overline{x})$ such that

- $S \cap \{X^{[i]}, i = 1, 2\} \neq \{X^{[i]}, i = 1, 2\}$,
- $\sum_{Y \in S} \mu(Y)Y = \overline{x}$, $\sum_{Y \in S} \mu(Y) = 1$, $\mu(Y) > 0$, $Y \in S$.

Such a $S$ is called a witness for nonadjacency of the given pedigrees, or witness for short.

**Proof.** This result can be derived from the definition of adjacency of vertices of a polytope, that is, the line segment joining the two vertices is an edge (one-dimensional face) of the polytope.

5.1. **FAT problems and adjacency in $\text{conv}(P_n)$**

Given $X^{[1]}, X^{[2]} \in P_n$, let the corresponding pedigrees be $W^{[i]}, i = 1, 2$. Let the $2 \times (n - 3)$ array $L = (e_{ij})$ denote the edges in $W^{[1]}$, $W^{[2]}$ as rows, respectively. That is, $x^{[i]}_j(e_{ij}) = 1$, $i = 1, 2$, and $4 \leq j \leq n$. We also informally say, $e_{ij}$ is in $X^{[i]}$, if the corresponding edge is the $i$th element of $L$.

**Definition 8.** Given $X/n - 1 \in \text{conv}(P_{n-1})$, consider any $\lambda \in \Lambda_{n-1}(X)$, and the FAT$_{n-1}(\lambda)$ problem. Then $\lambda$ is called inadmissible, rigid or flexible depending on FAT$_{n-1}(\lambda)$ has no solution, unique solution or infinitely many solutions, respectively. We also say that a $\lambda$ is admissible if it is either rigid or flexible.

**Lemma 5.1.** For any admissible $\lambda \in \Lambda_{n-1}(X)$, if problem FAT$_{n-1}(\lambda)$ has a single source or a single sink then $\lambda$ is rigid.

**Proof.** This is so because if there is a single source/sink then the requirement/availability at any sink/source is the only feasible flow along the arc connecting the source/sink and the sink/source. In other words, the flow is unique and hence $\lambda$ is rigid.

So, if $x^{[1]}/n - 1 = x^{[2]}/n - 1$ we have a single source in FAT$_{n-1}(\lambda)$ and so by Lemma 5.1, $\lambda^0$ is rigid. Similarly, if $x^{[i]}_n(e) = 1$, $i = 1, 2$, for some $e \in E_{n-1}$ then FAT$_{n-1}(\lambda)$ has a single sink and any admissible $\lambda$ is rigid. We have

**Corollary 5.1.** If $\lambda \in \Lambda_{n-1}(\overline{x})$ is flexible then FAT$_{n-1}(\lambda)$ has at least two sources and exactly two sinks.

**Theorem 5.2.** Given $X^{[1]}, X^{[2]} \in P_n$, $X^{[1]} \neq X^{[2]}$, are adjacent in $\text{conv}(P_n) \iff \lambda^0$ is the only admissible $\lambda \in \Lambda_{n-1}(\overline{x})$ and it is rigid.
Theorem. Let \( f(g, h) \) denote the flow along the arc \((g, h)\) in \( \text{FAT}_n^{-1}(\lambda) \) corresponding to a feasible flow \( f \). Consider 
\[
\lambda^0 \in \mathcal{A}_{n-1}(\overline{X}).
\]
Notice that the solution with \( f(X[i]/n - 1, e_{1n}) = \frac{1}{2}, i = 1, 2, \) and \( f(g, h) = 0 \), for all other arcs is 
feasible for \( \text{FAT}_n^{-1}(\lambda^0) \). This implies \( \lambda^0 \) is admissible. Suppose \( \lambda \) is the unique admissible \( \lambda \in \mathcal{A}_{n-1}(\overline{X}) \) and \( \lambda^0 \) is 
rigid, we shall show that \( X[1], X[2] \) are adjacent in \( \text{conv}(P_n) \).

\( \lambda^0 \) is rigid implies that the flow given above is the only solution to \( \text{FAT}_n^{-1}(\lambda^0) \). This together with the fact \( \lambda^0 \) is the 
unique admissible \( \lambda \in \mathcal{A}_{n-1}(\overline{X}) \) implies that the extensions of \( X[i]/n - 1, i = 1, 2, \) as per \( f \) are the only pedigrees in \( P_n \)
which receive positive weights while representing \( \overline{X} \). But the extensions are nothing but \( X[1] \) and \( X[2] \). This completes the 
proof one way.

Let \( X[1], X[2] \) be adjacent in \( \text{conv}(P_n) \), and if possible let either (a) the unique admissible \( \lambda \in \mathcal{A}_{n-1}(\overline{X}) \) be flexible or (b) the set of admissible \( \lambda \in \mathcal{A}_{n-1}(\overline{X}) \) be not a singleton set.

Case a: This implies \( \lambda^0 \), which is the unique admissible \( \lambda \in \mathcal{A}_{n-1}(\overline{X}) \), is flexible. So, from an application of 
Corollary 5.1, \( X[1]/n - 1 \neq X[2]/n - 1 \) and similarly \( e_{1n} \neq e_{2n} \). Notice that we have exactly two sources and 
two sinks in this problem. Flexibility of \( \lambda^0 \) means that there exists another solution \( f' \neq f \) for \( \text{FAT}_n^{-1}(\lambda^0) \). Since 
\( f(X[i]/n - 1, e_{1n}) = \frac{1}{2}, i = 1, 2, \) and \( f(g, h) = 0 \) otherwise, \( f' \), which differs from \( f \) must have \( f'(X[i]/n - 1, e_{1n}) \neq \frac{1}{2} \) 
for some \( i = 1, 2 \), say, \( i = 1 \).

Now, \( f'(X[i]/n - 1, e_{1n}) = 0 \), is necessarily \( < \frac{1}{2} \) as the availability at \( X[i]/n - 1 \) is only \( \lambda^0(X[i]/n - 1) = \frac{1}{2} \). So the 
sink \( e_{1n} \) must get its remaining requirement \( (\frac{1}{2} - \theta) \) form the other source, namely, \( X[2]/n - 1 \). This further implies 
that the sink \( e_{2n} \) receives only \( \theta \) from \( X[2]/n - 1 \) and so it must receive its remaining requirement \( (\frac{1}{2} - \theta) \) from 
the other source \( X[1]/n - 1 \). Thus, an alternative flow \( f' \) is possible for any \( 0 \leq \theta < \frac{1}{2} \) with 
\[
\begin{align*}
f'(X[i]/n - 1, e_{1n}) &= f'(X[2]/n - 1, e_{2n}) = 0, \\
f'(X[i]/n - 1, e_{2n}) &= f'(X[2]/n - 1, e_{1n}) = 1/2 - \theta.
\end{align*}
\]
Thus, corresponding to \( \theta = 0 \), we have found a new set of pedigrees in \( P_n \), that is a witness for \( X[1], X[2] \) being 
nonadjacent in \( \text{conv}(P_n) \). Contradiction. So (a) is not possible.

Case b: This implies, there exists an admissible \( \lambda \in \mathcal{A}_{n-1}(\overline{X}), \lambda \neq \lambda^0 \), with a feasible flow \( f' \) for \( \text{FAT}_n^{-1}(\lambda) \).

So there exists a pedigree \( Y \in P_{n-1} \), such that \( \lambda(Y) > 0 \) and \( Y \neq X[1]/n - 1 \) (say, without loss generality). Let 
\( f'((Y, e)) > 0 \) for some \( e = e_{1n}, i = 1, 2 \). So corresponding to the admissible \( \lambda \), and the feasible flow, \( f' \), we have a \( S \subset P_n \) 
such that \( S \neq \{X[1], X[2]\} \) and \( \overline{X} \) can be written as a convex combination of pedigrees in \( S \). If \( S \cap \{X[1], X[2]\} \neq \{X[1], X[2]\} \) are through and \( X[1], X[2] \) are not adjacent in \( \text{conv}(P_n) \). Otherwise, let \( S' = S - \{X[1], X[2]\} \). Let 
\( \mu \in \mathcal{A}_{n}(\overline{X}) \) be the weight vector corresponding to \( S \). Let \( \min (\mu(X[1]), \mu(X[2])) = \epsilon \). Thus 
\[
\overline{X} = 1/(1 - 2\epsilon) \left\{ \sum_{Y \in S'} \mu(Y)Y + \sum_{i=1,2} (\mu(X[i]) - \epsilon)X[i] \right\}.
\]
In other words, we have found a witness, which is a subset of \( S \), that includes at most one of the two given pedigrees, \( X[1] \) and \( X[2] \). Contradiction. So (b) is not possible. Hence the theorem. \( \square \)

Lemma 5.2. Given \( X[1], X[2] \in P_n \), suppose for some \( k, 4 \leq k < n \), and some \( e \in E_k \), \( x_k^{1}[i] = 1, i = 1, 2 \), then every 
\( \lambda \in \mathcal{A}_k(\overline{X}) \) is rigid.

Proof. Since \( X[1], X[2] \in P_n \), \( X[1]/k + 1, X[2]/k + 1 \in P_{k+1} \). Therefore, \( x_k^{1}[i] = 1, i = 1, 2 \) for some \( e \in E_k \), 
implies that \( e \neq e_{1l}, i = 1, 2 \), for any \( l, 4 \leq l \leq k \). Also, there exists a \( l \), such that, \( e_{1l} \) is a generator of \( e \), for \( i = 1, 2 \) 
(\( e_{1l}, e_{2l} \) may or may not be distinct).

Claim. For any \( \lambda \in \mathcal{A}_k(\overline{X}) \), every \( Y \in P_k \), with \( \lambda(Y) > 0 \), must agree with the zeros of \( \overline{X} \). That is, \( \forall l, 4 \leq l \leq k, \)
\[
y_{l} = x_{k}^{1}[i] \text{ for some } i = 1, 2.
\]

Proof. Suppose, for some \( l \) if \( y_{l} \neq x_{k}^{1}[i] \) for both \( i = 1, 2 \) then \( y_{l}(\bar{e}) = 1 \) for some \( \bar{e}, e_{1l} \neq \bar{e} \neq e_{2l} \). But \( x_{k}^{1}(\bar{e}) = 0 \). So, 
no such \( Y \) appears in any convex combination representing \( X/k \). Hence the claim. \( \square \)
Consider any \( \lambda \in A_k(X) \). Every \( Y \in P_k \), with \( \lambda(Y) > 0 \) obeys Eq. (9). So, every \( Y \) has a generator of \( e \). Thus, we have the feasible flow \( f \) in \( FAT_k(\lambda) \) given by \( f(Y, e) = \lambda(Y) \). In other words \( \lambda \) is admissible. And the uniqueness of the flow implies \( \lambda \) is rigid as well. Hence the lemma. \( \square \)

As a consequence of Theorem 5.2, we have the following fact about inheritance of the adjacency property.

**Theorem 5.3.** Given \( \bar{X} \in X^{[1]} \), \( X^{[2]} \in P_n \), suppose \( X^{[1]}/k, X^{[2]}/k \) are adjacent/nonadjacent in \( \text{conv}(P_k) \), for some \( k, 4 \leq k < n \), and \( X^{[1]}_k(e) = 1, i = 1, 2 \) for some \( e \in E_k \), then \( X^{[1]}/k + 1, X^{[2]}/k + 1 \) are adjacent/nonadjacent in \( \text{conv}(P_{k+1}) \), accordingly.

**Proof.** We have from Lemma 5.2, any \( \lambda \in A_k(\bar{X}) \) is rigid. Now \( X^{[1]}/k, X^{[2]}/k \) are adjacent in \( \text{conv}(P_k) \) implies we have a unique \( \lambda \). And hence \( A_{k+1}(\bar{X}) \) is a singleton set. So \( X^{[1]}/k + 1, X^{[2]}/k + 1 \) are adjacent. And \( X^{[1]}/k, X^{[2]}/k \) are nonadjacent in \( \text{conv}(P_k) \) implies we have more than one \( \lambda \in A_k(\bar{X}) \). And from Lemma 5.2 all these \( \lambda \)'s are rigid. Hence \( A_{k+1}(\bar{X}) \) has more than one element. Hence the inheritance of adjacency property is established. \( \square \)

**Remark 2.**

1. Theorem 5.3 can be repeatedly applied if \( X_q^{[1]}(e_q) = 1, i = 1, 2 \) for some \( e_q \in E_{q-1} \), for all \( k + 1 \leq q \leq s \), and we can conclude that \( X^{[1]}/s, X^{[2]}/s \) are adjacent/nonadjacent in \( \text{conv}(P_s) \) depending on \( X^{[1]}/k, X^{[2]}/k \) are adjacent/nonadjacent in \( \text{conv}(P_k) \).
2. Notice that nothing certain can be said about inheritance of adjacency property, the moment we encounter a \( q \) with \( X_q^{[1]} \neq X_q^{[2]} \). Examples 5.2 and 5.3 illustrate this point.
3. Due to inheritance, it is sufficient to concentrate on components \( q \) such that \( X^{[1]} \) and \( X^{[2]} \) disagree. This leads to the definition of the set of discords.

**Definition 9.** Given \( X^{[1]} \), \( X^{[2]} \in P_n \), we call \( D = \{ q | X_q^{[1]} \neq X_q^{[2]}, 4 \leq q \leq n \} \) the set of discordant components or discords. This means, in terms of \( L, e_{1q} \neq e_{2q} \), \( q \in D \).

**Lemma 5.3.** Given \( X^{[1]} \), \( X^{[2]} \in P_n \), consider the set \( D. \) If \( |D| = 1 \) then \( X^{[1]} \) and \( X^{[2]} \) are adjacent in \( \text{conv}(P_n) \).

**Proof.** Let \( q \in D. \) We have \( X^{[1]}/q - 1 = X^{[2]}/q - 1 \) as \( q \) is the first component of discord. Let \( \lambda^0 \) correspond to the degenerate convex combination \( \bar{X} = X^{[1]}/q - 1 \). Consider \( FAT_{q-1}(\lambda^0) \). From Lemma 5.1, \( \lambda^0 \) is rigid. So by Theorem 5.2 \( X^{[1]}/q, X^{[2]}/q \) are adjacent in \( \text{conv}(P_q) \). If \( q = n \) we are through, otherwise from item 1 of Remark 2, \( X^{[1]}, X^{[2]} \) are adjacent in \( \text{conv}(P_n) \). Hence the lemma. \( \square \)

This lemma provides an easy to check sufficient condition for adjacency in the pedigree polytopes. So we have a nontrivial problem of determining nonadjacency, only when \( |D| > 1 \).

**Lemma 5.4.** Given \( X^{[1]} \), \( X^{[2]} \in P_n \), consider the set \( D = \{ q_1 < \cdots < q_r \} \). Let \( \lambda^0 \) correspond to the convex combination \( X = \frac{1}{r}(X^{[1]}/q_r - 1 + X^{[2]}/q_r - 1) \). If \( r > 1 \) and \( \lambda^0 \) is flexible then \( X^{[1]} \) and \( X^{[2]} \) are nonadjacent in \( \text{conv}(P_n) \).

**Proof.** From an application of Theorem 5.2 we have \( X^{[1]}/q_r, X^{[2]}/q_r \) nonadjacent in \( \text{conv}(P_{q_r}) \). And from Remark 2, \( X^{[1]}, X^{[2]} \) are nonadjacent in \( \text{conv}(P_n) \). Hence the lemma. \( \square \)

This lemma provides an easy to check sufficient condition for nonadjacency in the pedigree polytopes. So we have a nontrivial problem of determining nonadjacency, only when \( |D| > 1 \) and \( FAT_{q_r-1}(\lambda^0) \) has a unique solution. This is equivalent to checking (recall definition of extension of a pedigree) whether

\[
(X^{[i]}/q_r - 1, X^{[i]}/q_r) \text{ is a pedigree in } P_{q_r}, \quad i = 1, 2,
\]

where \( \bar{i} = 3 - i \). This means a generator of \( e_{q_r} \) appears in \( W^{[\bar{i}]}/q_r - 1 \) and \( e_{q_r} \) itself does not appear in \( W^{[\bar{i}]}/q_r - 1 \).
Stated differently, failure to meet the conditions (10) can happen for two reasons: Either

(i) a generator of $e_{iq}$ is not available in $W^{[i]}/q_r - 1$. Or

(ii) $e_{iq}$ itself appears in $W^{[i]}/q_r - 1$, as it is used to insert some $l \leq q_r - 1$.

In case (i), in every $Y \in P_{qr}$ which is eligible to be in any convex representation of $X$ with a positive weight, we have $y_{qr}(e_{iq}) = 1$ along with $y_{l}(e_{il}) = 1$ for some $l < q_r$ corresponding to a generator of $e_{iq}$ available in $W^{[i]}/q_r - 1$. In case (ii), suppose $e_{il} = e_{iq}$, for some $l < q_r$, in every $Y \in P_{q_r}$ which is eligible to be in any convex representation of $X$ with a positive weight, we have $y_{qr}(e_{iq}) = 1$ along with $y_{l}(e_{il}) = 1$. Otherwise $y_{l}(e_{il}) = y_{l}(e_{iq})$ has to be 1. In that case, $Y$ cannot be a pedigree, as we have $y_{qr}(e_{iq})$ also equal to 1.

Thus, checking conditions (10) can be done using the procedure $\text{find flexible}$ (Fig. 2). This involves at most $4n$ comparisons. But can be done more efficiently using Lemma 2.2. We illustrate this procedure with Example 5.1.

**Example 5.1.** Consider the pedigrees in $\mathcal{P}_6$, corresponding to

$$L = \left( \begin{array}{ccc}
(1, 2) & (2, 4) & (2, 5) \\
(1, 3) & (1, 2) & (2, 3) \\
\end{array} \right).$$

So $D = \{4, 5, 6\}$ is the set of discords, with $q_r = 6$. In Step 1 of the procedure $\text{find flexible}$, we have $i = 1$, so $\bar{t} = 3 - i = 2$. In Step 3, we check whether $e_{16} = (2, 5) = e_{25}$, for some $s \leq 5$. Since no such $s$ exists we go to Step 5 and check whether a generator $e'$ of $e_{16}$, is available in $W^{[2]}/5$. Since $e_{25} = (1, 2)$ is a generator of $(2, 5)$, the answer is yes, and we go to Step 7. As $i = 1$ we continue and go to Step 1. Now $i = 2$. And so $\bar{t} = 1$. In Step 3, we check whether $e_{26} = (2, 3) = e_{15}$, for some $s \leq 5$. Since no such $s$ exists we go to Step 5 and check whether a generator $e'$ of $e_{26}$, is available in $W^{[1]}/5$. Since $e_{14} = (1, 2)$ is a generator of $(2, 3)$, the answer is yes and so in Step 7 we conclude that $\lambda^0$ is flexible and stop.

From Lemma 5.4, we conclude that the given pedigrees are nonadjacent.

**5.2. Graph $G_R$ and its implications**

In this section we define the graph of rigidity for a given pair of pedigrees, and show that the connectedness of the graph indicates that the pedigrees are adjacent. Otherwise by swapping the edges in the pedigrees, corresponding to a component of the graph, we can produce a pair of new pedigrees, that forms a witness for nonadjacency of the given pedigrees.

**Definition 10.** Let $\bar{t} = 3 - i$. Given $L$, giving the pedigrees $W^{[1]}$, $W^{[2]} \in \mathcal{P}_n$, a $q \in D$ and an $i \in \{1, 2\}$, we say that a generator of $e_{iq} = (u, v)$ is not available in $W^{[i]}$ in case $e_{is} \notin G(e_{iq})$, where $s = \max(4, v)$. Equivalently, $(W^{[i]}/q - 1, e_{iq})$ is not a pedigree.
Definition 11. Let \( \bar{t} = 3 - i \). Given the pedigrees \( W^{[i]}, i = 1, 2 \). Let \( D \) be the set of discords. We say \( q \in D \) is welded to \( s, s \in D, s < q \) if either

1. no generator of \( e_{iq} \) is available in the pedigree \( W^{[\bar{t}]} \), for some \( i = 1, 2 \) or
2. \( e_{is} = e_{iq} \) for some \( i = 1, 2 \).

Definition 12. Given a pair of pedigrees in \( \mathcal{P}_n \), we define the graph of rigidity denoted by \( G_R \). The vertex set of \( G_R \) is the corresponding set of discords \( D \), and the edge set is given by \( \{(s, q) | s, q \in D, s < q, \text{ and } q \text{ is welded to } s\} \).

Remark 3.

1. The graph \( G_R \) expresses the restriction imposed on the elements of \( D \) as far as producing a witness for nonadjacency of \( X^{[1]} \) and \( X^{[2]} \) in \( \text{conv}(P_n) \) is concerned. Any \( Y \in S \subset P_n \), a witness, has to agree with \( 0/1 \)'s of \( X \) and has to have exactly one edge from \( \{e_{iq}, i = 1, 2\}, q \in D \). And so we may visualize \( Y \) as the incidence vector of a pedigree obtained from \( X^{[1]} \) or \( X^{[2]} \) by swapping \( (e_{iq}, e_{2q}) \), for some \( q \in D \). (Definition 13 formalizes this idea.)
2. Next, we find conditions on \( G_R \) that will ensure nonadjacency of pedigrees. Notice that all \( q \) in a connected component of \( G_R \) are required to be swapped simultaneously, to ensure feasibility. Thus, if \( G_R \) is a connected graph then we have no witness for nonadjacency, and so we can declare \( X^{[1]} \) and \( X^{[2]} \) are adjacent in \( \text{conv}(P_n) \).

Definition 13. Given \( C \subset V_n \setminus V_3 \), let \( Y^{[i]} = \text{swap}(X^{[i]}, C) \in B^{5n} \) denote the characteristic vector, obtained from \( X^{[i]} \) by swapping \( q \in C \), where, by operation swap we mean:

\[
y_q^{[i]} = \begin{cases} 
x_q^{[i]} & \text{if } q \in C, \\
x_q^{[i]} & \text{otherwise.}
\end{cases}
\]

Lemma 5.5. Given \( X^{[i]} \in P_n, i = 1, 2 \) consider the graph \( G_R \). If \( C = \{l\} \), is a component of \( G_R \), then (i) \( Y^{[i]}/l \mid l \in P_l, i = 1, 2 \), and so (ii) \( Y^{[i]} \in P_n \).

Proof. Suppose \( Y^{[i]}/l \notin P_l \), for some \( i = 1, 2 \). That is \( Y^{[i]}/l = (X^{[i]}/l - 1, x_l^{[i]}) \notin P_l \). In other words, there exists a \( s < l \) such that \( l \) is welded to \( s \) and so \( (s, l) \) is an edge in \( G_R \). But \( C \) is a component of \( G_R \) implies that \( s \in C \). Contradiction. This proves part (i). If \( l = n \) this also proves assertion (ii) of the lemma.

Let \( l < n \). Suppose (ii) is false. Then there exists a \( q, l < q \leq n \), the smallest such, for which \( Y^{[i]}/q \notin P_q \), for some \( i = 1, 2 \).

Notice that,

1. \( e_{iq} \neq e_{i[l]} \), as otherwise \( q \) would be welded to \( l \).
2. \( e_{i[l]} \neq e_{is}, s < q \), as otherwise that would contradict the fact \( X^{[i]}/q \) is a pedigree in \( P_q \).

Let \( u < q \) be such that \( e_{iu} \) is the generator of \( e_{iq} \) in \( X^{[i]}/q \).

Case 1: \( u = l \). That is \( e_{i[l]} \) is the generator of \( e_{iq} \) in \( X^{[i]}/q \). Since \( q \) is not welded to \( l \), \( e_{i[l]} \) is also a generator of \( e_{iq} \).

And since \( y_l^{[i]} = x_l^{[i]} \), we have \( e_{i[l]} \) available in \( Y^{[i]}/l \).

Case 2: \( u \neq l \). Then \( e_{iu} \) is still available in \( Y^{[i]} \) as \( u \notin C \).

Thus, we have shown that a generator of \( e_{iq} \) exists in \( Y^{[i]}/q - 1 \) and \( e_{iq} \) is not in \( Y^{[i]}/q - 1 \). So, \( Y^{[i]}/q \in P_q \), \( i = 1, 2 \).

Contradiction. This completes the proof of (ii).

Hence the lemma. \( \square \)

Theorem 5.4. Given \( X^{[i]} \in P_n, i = 1, 2 \), if \( C \) is a component of \( G_R \), then \( Y^{[i]} \in P_n \).

Proof. We prove this theorem by induction on the cardinality of \( C \) and on the cardinality of \( D \), the vertex set of \( G_R \). Lemma 5.5 provides the basis for induction. Suppose the theorem is true for any component with cardinality up to \( r - 1 \), and the set of discards having up to \( s - 1 \) vertices.
Let \( C = \{l_1, <, l_2, \ldots, < l_r\} \subset D \), be a component of \( G_R \). Consider the graph of rigidity \( G'_R \) with vertex set \( D \setminus \{l_r\} \). Now \( C \setminus \{l_r\} \) in \( G'_R \) may or may not be a single component of \( G'_R \). Consider \( Y^{[i]}/l_r - 1, i = 1, 2 \) obtained by swapping \( C \setminus \{l_r\} \) in \( X^{[i]}/l_r - 1, i = 1, 2 \) by induction hypothesis. Since \( l_r \) is welded to some element(s) in \( C \setminus \{l_r\} \) in \( G_R \), \( Y^{[i]}/l_r - 1, s^{[i]}_{l_r} \in P_{l_r}, i = 1, 2 \). Which is equivalent to swapping \( C \) in \( X^{[i]} \), \( i = 1, 2 \). Therefore, \( Y^{[i]}/l_r \in P_{l_r} \). The proof of \( Y^{[i]} \in P_n, i = 1, 2 \) is similar to that of Lemma 5.5. Hence the theorem. □

5.3. Characterization of nonadjacency through the graph of rigidity

From the results obtained on the graph of rigidity, we are in a position to interpret nonadjacency of a given pair of pedigrees, using the graph \( G_R \). We have shown that for any component \( C \) of \( G_R \), swap \( (X^{[i]}, C) \) produces a pedigree in \( P_n \). However, if \( C = D \) then the swapping produces, trivially, the same pedigrees, as swap \( (X^{[i]}, D) \) is \( X^{[i]} \). So if \( C \neq D \) we get a pair of pedigrees from \( X^{[i]}, i = 1, 2 \), by swapping \( C \) and it is easy to check that, we have a witness for nonadjacency of the given pedigrees, that is \( X = \frac{1}{2}(Y^{[1]} + Y^{[2]}) \) and \( X^{[i]}, i = 1, 2 \), are all different. Thus, we have the following theorem characterizing nonadjacency in pedigree polytopes.

**Theorem 5.5.** Given \( X^{[i]} \in P_n, i = 1, 2 \), consider the graph of rigidity \( G_R \). The given pedigrees are nonadjacent in \( \text{conv}(P_n) \) if and only if \( G_R \) is not connected.

Given two pedigrees, the set of discords, \( D \), can be found in at most \( n - 3 \) comparisons. If \( |D| = 1 \) we stop as the pedigrees are adjacent. Otherwise, \( D = \{q_1 < \cdots < q_r\} \). Construction of \( G_R \) requires finding the edges in \( G_R \), which can be done starting with \( q_r \) and checking whether it is welded to any \( s < q_r, s \in D \). Something similar to the procedure find flexible is required. At most \( 4n \) comparisons are required, as noted earlier. Then we do the same with \( q_{r-1} \) and so on. So construction of \( G_R \) is of complexity \( O(n^2) \). However, it is well known that the components of an undirected graph can be found efficiently [1]. Thus, we have an algorithm that can check nonadjacency in the pedigree polytope, \( \text{conv}(P_n) \) in time polynomial in \( n \). The following examples illustrate the simple algorithm.

**Example 5.2.** Consider the pedigrees in \( \mathcal{P}_6 \), corresponding to

\[
L = \begin{pmatrix}
(1, 2) & (2, 3) & (2, 5) \\
(1, 2) & (2, 4) & (2, 3)
\end{pmatrix}.
\]

So \( D = \{5, 6\} \) is the set of discords. Here 6 is welded to 5 as \( e_{26} = (2, 3) = e_{15} \). So \( G_R = ((5, 6), (5, 6)) \).

Since \( G_R \) has a single component, from Theorem 5.5, the pedigrees are adjacent in \( \text{conv}(\mathcal{P}_6) \).

**Example 5.3.** Consider the pedigrees in \( \mathcal{P}_6 \), corresponding to

\[
L = \begin{pmatrix}
(1, 3) & (2, 3) & (3, 4) \\
(1, 2) & (1, 4) & (1, 3)
\end{pmatrix}.
\]

So \( D = \{4, 5, 6\} \). Here 6 is welded to 4 for more than one reason (\( e_{26} = (1, 3) = e_{14} \) and no generator of \( e_{16} = (3, 4) \) in the second pedigree). But 5 is not welded to any other element of \( D \). So \( G_R = ((4, 5, 6), (4, 6)) \).

Since \( G_R \) has two components, from Theorem 5.5, the pedigrees are nonadjacent. The new set of pedigrees \( \{(1, 3), (1, 4), (3, 4)\}, ((1, 2), (2, 3), (1, 3)) \}, obtained by swapping the component, \( C = \{5\} \) is a witness.

**Example 5.4.** Consider the pedigrees in \( \mathcal{P}_7 \), corresponding to

\[
L = \begin{pmatrix}
(1, 2) & (2, 4) & (2, 3) & (2, 6) \\
(1, 3) & (2, 3) & (2, 5) & (3, 4)
\end{pmatrix}.
\]

So \( D = \{4, 5, 6, 7\} \). Here 7 is welded to 4 as no generator of \( e_{27} = (3, 4) \) is in the first pedigree. But 6 is welded to 5 as \( e_{16} = (2, 3) = e_{25} \). Finally, 5 is welded to 4 as no generator of \( e_{15} = (2, 4) \) is in the second pedigree. So \( G_R = ((4, 5, 6, 7), (4, 5), (6, 7), (5, 6)) \).

\( G_R \) is connected can be easily seen. So from Theorem 5.5, the pedigrees are adjacent.
5.4. Adjacency in conv($P_n$) does not imply adjacency in $Q_n$

The examples provided in the previous section have something in common, namely, whenever the pedigrees are adjacent/nonadjacent in conv($P_n$) the corresponding $n$-tours are also adjacent/nonadjacent in $Q_n$. If this were in general true that would imply $NP = P$, since we have a one-to-one correspondence between pedigrees and tours. So we are interested in the question: Does adjacency/nonadjacency of a pair of pedigrees in conv($P_n$) imply adjacency/nonadjacency of the corresponding $n$-tours in $Q_n$? The answer is in the negative. We give a counter example to show that adjacency of pedigrees does not imply adjacency of the corresponding $n$-tours.

Example 5.5. Consider the pedigrees $W[1], W[2]$ in $P_{10}$, corresponding to $L = (1, 2) (1, 3) (2, 4) (2, 6) (3, 5) (1, 4) (5, 8)$. So $D = \{4, 5, 6, 7, 10\}$. Here 10 is welded to 7 as no generator of $e_{210} = (4, 7)$ is in the first pedigree. But 7 is welded to 6 as no generator of $e_{17} = (2, 6)$ is in the second pedigree; 6 is welded to 4, as no generator of $e_{26} = (3, 4)$ is in the first pedigree. And finally, 5 is welded to 4 as $e_{15} = (1, 3) = e_{24}$. So $G_R = (\{4, 5, 6, 7, 10\}, \{(4, 5), (4, 6), (6, 7), (7, 10)\})$. As $G_R$ is connected, from Theorem 5.5, the pedigrees $W[1], W[2]$ are adjacent in conv($P_n$).

Let the corresponding 10-tours be called Tour$_1$, Tour$_2$, respectively. Let the incident vector corresponding to Tour$_i$ be denoted by $T_i$. Let $T = \frac{1}{2}[T[1] + T[2]]$. We have

Tour$_1 = (1, 9, 4, 6, 7, 2, 3, 8, 10, 5, 1)$ and Tour$_2 = (1, 9, 4, 10, 7, 6, 3, 8, 5, 2, 1)$.

Now consider the tours Tour$_3$, Tour$_4$ given by

Tour$_3 = (1, 9, 4, 10, 8, 3, 6, 7, 2, 5, 1)$ and Tour$_4 = (1, 9, 4, 6, 7, 10, 5, 8, 3, 2, 1)$. 

Fig. 3. Support graph of $T$—Example 5.5.
Let $\mathcal{T}' = \frac{1}{2}[T^{[3]} + T^{[4]}]$. It can be verified that $\mathcal{T} = \mathcal{T}'$. In other words, we have shown that $T^{[1]}$, $T^{[2]}$ are nonadjacent in $Q_{10}$. Fig. 3 gives the support graph of $\mathcal{T}$.

6. Conclusions

The necessity of FAT is the main theme of this paper. The pedigree polytope is defined and its properties are highlighted. The motivation for studying this polytope comes from the MI formulation of the STSP problem given by the author. For a $X \in \mathbb{R}^{n}$ to be in $\text{conv}(P_{n})$, it is necessary that $X$ is feasible for the MI-relaxation. The necessity of FAT feasibility for all $k \in \mathbb{N}$, for a vector $X$ to be in $\text{conv}(P_{n})$, gives a sequential approach, for testing whether $X$ is indeed in $\text{conv}(P_{n})$. A polynomial algorithm for nonadjacency testing in the pedigree polytope is presented. Implication of the nonadjacency in the pedigree polytope on the nonadjacency of corresponding Hamiltonian cycles in $Q_{n}$ is currently studied. Attempts to use these results on the pedigree polytope to design a new algorithm to solve STSP problem is underway.

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Appendix

Proof of Theorem 4.2. Since $X \in \text{conv}(P_{n})$ we have

$$x_{l}(e) = \sum_{s \in I(\lambda)} \lambda^{s} x_{l}^{s}(e), \quad e \in E_{l-1}, \quad l \in V_{n-1} \setminus V_{3},$$

where $I(\lambda)$ is the index set for some $\lambda \in A_{k}(X)$. To show that $X$ is such that FAT are all feasible, we proceed as follows:

First partition $I(\lambda)$ according to the pedigree, $X^{x} \in P_{k}$,

$$S^{x}_{O} = \{ s \mid s \in I(\lambda) \text{ and } X^{x} \text{ is a descendant of } X^{x} \in P_{k} \}. \quad (11)$$

Secondly, partition $I(\lambda)$ according to the edge $e_{\beta} \in E_{k}$,

$$S^{\beta}_{D} = \{ s \mid s \in I(\lambda) \text{ and } X^{x} \text{ is such that } x_{k+1}^{\beta}(e_{\beta}) = 1 \}. \quad (12)$$

$O$ and $D$ in the suffices refer to origins and destinations in the FAT problem. Here, we say a pedigree in $X \in P_{n}$ is a descendant of a pedigree in $Y \in P_{k}$ in case $X/k = Y$.

Let $a_{x} = \sum_{s \in S^{x}_{O}} \lambda^{s}; \ b_{\beta} = \sum_{s \in S^{\beta}_{D}} \lambda^{s} = x_{k+1}(e_{\beta})$ and let the set of arcs be given by

$$\mathcal{A} = \{ (x, \beta) \mid S^{x}_{O} \cap S^{\beta}_{D} \neq \emptyset \}.$$

Now for $k = 4$ we have $(X/4) \in P_{4}$, as $a_{x}$ are positive and add up to 1. Applying Lemma 2.1, we have a feasible $f$ for this problem given by $f_{x} = x_{k+1}(e_{\beta}) x_{k+1}(e_{\beta})$. Thus FAT is feasible. And so $X/5$ is in $\text{conv}(P_{5})$. Also notice that the origins corresponding to FAT are precisely the pedigrees in $P_{5}$ with $f_{x}$ positive, here $\lambda$ is given by the feasible flow for FAT given by Lemma 2.1.

In general, let $\lambda_{k}$ be defined as $\lambda_{k}(X) = a_{x}, \ X^{x} \in P_{k}$ and $S^{x}_{O} \neq \emptyset$. In the FAT problem, we have an origin for each pedigree in $P_{k}$ which receives a positive weight according to $\lambda_{k}$. We have a FAT problem, corresponding to $k$ and $\lambda_{k}$. The feasibility of FAT for every $k \in V_{n-1} \setminus V_{3}$ then follows from an application of Lemma 2.1. Hence the theorem. \□
Proof of Theorem 4.3. Consider \( X^\alpha, \alpha \in I(\lambda) \) and an edge \( e_\beta \in E_k \) such that \((\alpha, \beta)\) is not forbidden. So \( e_\beta \in H^x \in \mathcal{H}_k \). So \( e_\beta \) is one of the edges available for insertion of \( k + 1 \). As noticed earlier, every \((\alpha, \beta)\) not forbidden corresponds to a pedigree \( X^{\alpha\beta} \in P_{k+1} \) as defined below:
\[
X^{\alpha\beta} = (X^\alpha, y^\beta) \quad \text{where} \quad y^\beta(e) = \begin{cases} 1 & \text{if } e = e_\beta, \\ 0 & \text{otherwise.} \end{cases}
\]
\( (y^\beta \) is the indicator of \( e_\beta. \) Therefore, from the feasibility of \( \text{FAT}_k(\lambda) \) we have a flow with \( f_{x\beta} \geq 0 \), and
\[
\sum_{\beta \ni (\alpha, \beta) \in \mathcal{A}} f_{x\beta} = \lambda_\alpha, \quad \alpha \in I(\lambda),
\]
\[
\sum_{\alpha \ni (\alpha, \beta) \in \mathcal{A}} f_{x\beta} = x_{k+1}(e_\beta), \quad e_\beta \in E_k, \quad x_{k+1}(e_\beta) > 0.
\]
We shall show that
\[
\sum_{\alpha, \beta} X^{\alpha\beta} f_{x\beta} = X/k + 1.
\]
Substituting \( X^{\alpha\beta} = (X^\alpha, y^\beta) \) in the above equation and simplifying we get
\[
\sum_{\alpha} \sum_{\beta} X^{\alpha\beta} f_{x\beta} = \left( \sum_{\alpha} X^\alpha \sum_{\beta} f_{x\beta}, \sum_{\beta} y^\beta \sum_{\alpha} f_{x\beta} \right)
\]
\[
= \left( \sum_{\alpha} \lambda_\alpha X^\alpha, \sum_{\beta} x_{k+1}(e_\beta) y^\beta \right)
\]
\[
= (X/k, x_{k+1})
\]
\[
= X/k + 1.
\]
Hence the result. \(\square\)

References


