A Hilbert-type integral inequality whose kernel is a homogeneous form of degree $-3$

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Abstract

In this paper, by estimating the weight function, we give a new Hilbert-type integral inequality whose kernel is a homogeneous form of degree $-3$ with the best constant factor and the reverse form is considered.

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1. Introduction

If $f(x), g(x) \geq 0$, such that $0 < \int_0^\infty f^2(x) \, dx < \infty$ and $0 < \int_0^\infty g^2(x) \, dx < \infty$, then [1]

$$\int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x+y} \, dx \, dy < \pi \left\{ \int_0^\infty f^2(x) \, dx \int_0^\infty g^2(x) \, dx \right\}^{1/2},$$

(1.1)

where the constant factor $\pi$ is the best possible. Inequality (1.1) is well known as Hilbert’s integral inequality which has been extended by Hardy and Riesz [2].

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(x) \geq 0$, such that $0 < \int_0^\infty f^p(x) \, dx < \infty$ and $0 < \int_0^\infty g^q(x) \, dx < \infty$, then we have the following Hardy–Hilbert’s integral inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} \, dx \, dy < \frac{\pi}{\sin(\pi/p)} \left\{ \int_0^\infty f^p(x) \, dx \right\}^{1/p} \left\{ \int_0^\infty g^q(x) \, dx \right\}^{1/q},$$

(1.2)

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible.

Hardy–Hilbert’s integral inequality is important in analysis and its applications [3]. In recent years, Yang [4,5] gave some generalizations and the reverse form of (1.2) as

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(1) If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, r > 1, \frac{1}{r} + \frac{1}{s} = 1 \) and \( \lambda > 0 \), \( f, g \) are non-negative functions such that \( 0 < \int_0^\infty x^{p(1-\lambda/r)-1} f^p(x) \, dx < \infty \), \( 0 < \int_0^\infty x^{q(1-\lambda/s)-1} g^q(x) \, dx < \infty \), then we have
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x^{\lambda} + y^{\lambda}} \, dx \, dy < \frac{\pi \lambda}{\lambda \sin(\pi/r)} \left\{ \int_0^\infty x^{p(1-\lambda/r)-1} f^p(x) \, dx \right\}^{1/p} \left\{ \int_0^\infty x^{q(1-\lambda/s)-1} g^q(x) \, dx \right\}^{1/q},
\]
where the constant factor \( \frac{\pi \lambda}{\lambda \sin(\pi/r)} \) is the best possible.

(2) If \( f, g > 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda > 0 \) and \( 0 < \int_0^\infty x^{p-1-\lambda} f^p(x) \, dx < \infty \), \( 0 < \int_0^\infty x^{q-1-\lambda} g^q(x) \, dx < \infty \), then we have
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{\lambda}} \, dx \, dy < B\left( \frac{\lambda}{p}, \frac{\lambda}{p} \right) \left\{ \int_0^\infty x^{p-1-\lambda} f^p(x) \, dx \right\}^{1/p} \left\{ \int_0^\infty x^{q-1-\lambda} g^q(x) \, dx \right\}^{1/q},
\]
where the constant factor \( B\left( \frac{\lambda}{p}, \frac{\lambda}{p} \right) \) is the best possible (\( B(u, v) \) is the \( \beta \)-function).

In this paper, by obtaining the weight function as those in [4,5], we give a new Hilbert-type integral inequality whose kernel is a homogeneous form of degree \(-3\) with a best constant factor. The reverse form and some applications are considered.

2. Some lemmas

Lemma 2.1. If \( a > 0, b > 0, c > 0 \), define the weight functions as follows:
\[
\omega(y) = \int_0^\infty \frac{x^{1/2}}{(x+a^2 y)(x+b^2 y)(x+c^2 y)} \, dx, \quad \sigma(x) = \int_0^\infty \frac{y^{1/2}}{(x+a^2 y)(x+b^2 y)(x+c^2 y)} \, dy,
\]
then
\[
\omega(y) = \frac{K}{y^{3/2}}, \quad \sigma(x) = \frac{\tilde{K}}{x^{3/2}},
\]
where
\[
K = \int_0^\infty \frac{u^{1/2}}{(u+a^2)(u+b^2)(u+c^2)} \, du, \quad \tilde{K} = \int_0^\infty \frac{u^{1/2}}{(1+a^2 u)(1+b^2 u)(1+c^2 u)} \, du
\]
and
\[
K = \tilde{K} = \frac{\pi}{(a+b)(a+c)(b+c)}.
\]

Proof. Assume first that \( (a-b)(b-c)(c-a) \neq 0 \), setting \( u = y/x \) we have
\[
\sigma(x) = \int_0^\infty \frac{y^{1/2}}{(x+a^2 y)(x+b^2 y)(x+c^2 y)} \, dy = \frac{1}{x^{3/2}} \int_0^\infty \frac{u^{1/2}}{(1+a^2 u)(1+b^2 u)(1+c^2 u)} \, du = \frac{\tilde{K}}{x^{3/2}}.
\]
Similarly, setting \( u = x/y \), we have
\[
\omega(y) = \int_0^\infty \frac{x^{1/2}}{(x+a^2 y)(x+b^2 y)(x+c^2 y)} \, dx = \frac{K}{y^{3/2}},
\]
then we obtain (2.1). On the other hand,
Lemma 2.2. If

$$\lim_{b \to +\infty} (a \to_b - u)(b = K) \int_0^\infty (x + a^2)(x + b^2)(x + c^2) \, dx = 2 \int_0^\infty \frac{x^2}{(x^2 + a^2)(x^2 + b^2)(x^2 + c^2)} \, dx$$

then write

$$\tilde{K} = \frac{2a^2}{(a^2 - b^2)(c^2 - a^2)} \int_0^\infty \frac{dx}{x^2 + a^2} + \frac{2b^2}{(b^2 - a^2)(c^2 - b^2)} \int_0^\infty \frac{dx}{x^2 + b^2} + \frac{2c^2}{(c^2 - a^2)(b^2 - c^2)} \int_0^\infty \frac{dx}{x^2 + c^2}$$

$$= \frac{(a + b)(a + c)(b + c)}{(a + b)(a + c)(b + c)}.$$  

The lemma is proved.

Lemma 2.3. Setting $\varepsilon = \frac{1}{a^2 b^2 c^2}$, we obtain (2.2). The lemma is proved. □

Lemma 2.2. For $0 < \varepsilon < p$, we have

$$\int_0^\infty \frac{u^{1/2}}{(u + a^2)(u + b^2)(u + c^2)} \, du = K + o(1) \quad (\varepsilon \to 0^+). \quad (2.3)$$

Proof. Since $F(u) = \frac{u^{1/2}}{(u + a^2)(u + b^2)(u + c^2)}$, then $F(x)$ has a maximum $h_1$ on $[0, 1]$; and the limit relation

$$\lim_{u \to +\infty} u^{5/2} F(u) = 1$$

shows that $u^{5/2} F(u)$ has a maximum $h_2 > 0$ on the interval $[1, +\infty),$

$$\int_0^1 \frac{u^{1/2}(1 - u^{-\varepsilon/p})}{(u + a^2)(u + b^2)(u + c^2)} \, du + \int_1^\infty \frac{u^{1/2}(1 - u^{-\varepsilon/p})}{(u + a^2)(u + b^2)(u + c^2)} \, du$$

$$\leq h_1 \int_0^1 (u^{-\varepsilon/p} - 1) \, du + h_2 \int_1^\infty (u^{-2} - u^{-2-\varepsilon/p}) \, du$$

$$= h_1 \left( \frac{1}{1 - \varepsilon/p} - 1 \right) + h_2 \left( 1 - \frac{1}{1 + \varepsilon/p} \right) \to 0 \quad \text{for} \ \varepsilon \to 0^+. \quad (2.4)$$

The lemma is proved. □

Lemma 2.3. For $p > 1$ (or $0 < p < 1$), $\frac{1}{p} + \frac{1}{q} = 1$ and $0 < \varepsilon < p$, set

$$I = \int_1^\infty \left( \int_1^\infty \frac{x^{1/2 - \varepsilon/p}}{(x + a^2 y)(x + b^2 y)(x + c^2 y)} \, dx \right) \frac{1}{y^{1/2 - \varepsilon/q}} \, dy.$$
then we have
\[ \frac{1}{\varepsilon}(K + o(1)) - O(1) \leq I \leq \frac{1}{\varepsilon}(K + o(1)), \quad \varepsilon \to 0^+. \] (2.5)

**Proof.** For fixed \( y \), setting \( u = x/y \), we obtain
\[
I = \int_1^\infty y^{-1-\varepsilon} \left( \int_{y^{-1}}^\infty \frac{u^{1-\frac{\varepsilon}{p}}}{(u+a^2)(u+b^2)(u+c^2)} \, du \right) \, dy
\]
\[
= \int_1^\infty y^{-1-\varepsilon} \left( \int_0^\infty \frac{u^{2-\varepsilon}}{(u+a^2)(u+b^2)(u+c^2)} \, du \right) \, dy - \int_1^\infty y^{-1-\varepsilon} \left( \int_0^{y^{-1}} \frac{u^{2-\varepsilon}}{(u+a^2)(u+b^2)(u+c^2)} \, du \right) \, dy
\]
\[
\geq \frac{1}{\varepsilon}(K + o(1)) - \int_1^\infty y^{-1} \left( h_1 \int_0 \frac{u^{-\varepsilon}}{u-\frac{\varepsilon}{p}} \, du \right) \, dy
\]
\[
= \frac{1}{\varepsilon}(K + o(1)) - h_1(1 - \varepsilon/p)^{-2} = \frac{1}{\varepsilon}(K + o(1)) - O(1).
\]
On the other hand, we get
\[
I \leq \int_1^\infty \left( \int_0^\infty \frac{x^{1-\frac{\varepsilon}{p}}}{(x+a^2 y)(x+b^2 y)(x+c^2 y)} \, dx \right) \frac{y^{1-\frac{\varepsilon}{q}}}{q} \, dy = \frac{1}{\varepsilon}(K + o(1)).
\]
The lemma is proved. \( \square \)

3. **Main results**

**Theorem 3.1.** If \( p > 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \), \( f(x), g(x) \geq 0 \) such that \( 0 < \int_0^\infty x^{-1-p/2} f^p(x) \, dx < \infty \) and \( 0 < \int_0^\infty x^{-1-q/2} g^q(x) \, dx < \infty \), then
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+a^2 y)(x+b^2 y)(x+c^2 y)} \, dx \, dy < K \left\{ \int_0^\infty x^{-1-p/2} f^p(x) \, dx \right\}^{1/p} \left\{ \int_0^\infty x^{-1-q/2} g^q(x) \, dx \right\}^{1/q}, \tag{3.1}
\]
where the constant factor \( K \) defined by Lemma 2.1 is the best possible.

**Theorem 3.2.** If \( 0 < p < 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \), \( f(x), g(x) \geq 0 \) such that \( 0 < \int_0^\infty x^{-1-p/2} f^p(x) \, dx < \infty \) and \( 0 < \int_0^\infty x^{-1-q/2} g^q(x) \, dx < \infty \), then
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+a^2 y)(x+b^2 y)(x+c^2 y)} \, dx \, dy > K \left\{ \int_0^\infty x^{-1-p/2} f^p(x) \, dx \right\}^{1/p} \left\{ \int_0^\infty x^{-1-q/2} g^q(x) \, dx \right\}^{1/q}, \tag{3.2}
\]
where the constant factor \( K \) is the best possible.

**Proof of Theorem 3.1.** By Hölder’s inequality and (2.1), (2.2), we have
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+a^2 y)(x+b^2 y)(x+c^2 y)} \, dx \, dy
\]
\[
= \int_0^\infty \int_0^\infty \left( \frac{1}{(x+a^2 y)(x+b^2 y)(x+c^2 y)} \right)^{1/p} x^{-1/(2q)} y^{1/(2p)} f(x)
\]
that (3.2) is still valid if we replace $K$.

Proof of Theorem 3.2.

Possible. The theorem is proved.

By the reverse Hölder’s inequality and the same way, we have (3.2).

If (3.3) takes the form of equality, then there exist constants $M$ and $N$, which are not all zero such that

$$M \frac{x^{-(p-1)/2}y^{1/2}f^p(x)}{(x+a^2y)(x+b^2y)(x+c^2y)} = N \frac{y^{-(q-1)/2}g^q(y)}{(x+a^2y)(x+b^2y)(x+c^2y)},$$

$$Mx^{-p/2}f^p(x) = Ny^{-q/2}g^q(y) \quad \text{a.e. in } [0, \infty) \times [0, \infty).$$

Hence, there exists a constant $C$, such that

$$Mx^{-p/2}f^p(x) = Ny^{-q/2}g^q(y) = C \quad \text{a.e. in } [0, \infty).$$

We claim that $M = 0$. In fact, if $M \neq 0$, then $x^{-(p-1)/2}f^p(x) = C/M$ a.e. in $(0, \infty)$ which contradicts the fact that $0 < \int_0^\infty x^{-(p-1)/2}f^p(x) \, dx < \infty$. By the same way, we claim that $N = 0$. This is a contradiction. Hence by (3.3), we have (3.1).

If the constant factor $K$ in (3.1) is not the best possible, then there exists a positive constant $H$ (with $H < K$), hence (3.1) is still valid if we replace $K$ by $H$. For $0 < \varepsilon < p$ small enough, set $f_\varepsilon$ and $g_\varepsilon$ as $f_\varepsilon(x) = g_\varepsilon(x) = 0$, for $x \in (0, 1)$; $f_\varepsilon(x) = x^{1/2-\varepsilon/p}$, $g_\varepsilon(x) = x^{1/2-\varepsilon/q}$, for $x \in [1, \infty)$, then we have

$$H \left\{ \int_0^\infty x^{-(p-1)/2}f_\varepsilon^p(x) \, dx \right\}^{1/p} \left\{ \int_0^\infty x^{-q/2}g_\varepsilon^q(x) \, dx \right\}^{1/q} = H \left\{ \int_1^\infty x^{-1} \, dx \right\}^{1/p} \left\{ \int_1^\infty x^{-1-\varepsilon} \, dx \right\}^{1/q} \geq \frac{H}{\varepsilon}.$$

By using (2.5), we have

$$\int_0^\infty \int_0^\infty \frac{f_\varepsilon(x)g_\varepsilon(y)}{(x+a^2y)(x+b^2y)(x+c^2y)} \, dx \, dy = \int_1^\infty \left( \int_1^\infty \frac{x^{1/2-\varepsilon}}{(x+a^2y)(x+b^2y)(x+c^2y)} \, dx \right) \frac{1}{y^{1-\varepsilon}} \, dy \geq \frac{1}{\varepsilon} (K + o(1)) - O(1).$$

Hence we find

$$\frac{1}{\varepsilon} (K + o(1)) - O(1) \leq \frac{H}{\varepsilon} \quad \text{or} \quad (K + o(1)) - \varepsilon O(1) \leq H.$$

For $\varepsilon \to 0^+$, it follows that $K \leq H$, which contradicts the fact that $H < K$. Hence the constant $K$ in (3.1) is the best possible. The theorem is proved.

Proof of Theorem 3.2. By the reverse Hölder’s inequality and the same way, we have (3.2).

If the constant factor $K$ in (3.2) is not the best possible, then there exists a positive constant $H$ (with $H > K$), such that (3.2) is still valid if we replace $K$ by $H$. For $0 < \varepsilon < p$ small enough, set $f_\varepsilon$ and $g_\varepsilon$ as $f_\varepsilon(x) = g_\varepsilon(x) = 0$, for $x \in (0, 1)$; $f_\varepsilon(x) = x^{1/2-\varepsilon/p}$, $g_\varepsilon(x) = x^{1/2-\varepsilon/q}$, for $x \in [1, \infty)$, then we have
By (2.5), we have

\[
\int_0^\infty \int_0^\infty \frac{f_\varepsilon(x)g_\varepsilon(y)}{(x + a^2y)(x + b^2y)(x + c^2y)} dx dy = \int_0^\infty \left( \int_0^1 \frac{1}{(x + a^2y)(x + b^2y)(x + c^2y)} x^{-\varepsilon/2} \frac{1}{p} dx \right) y^{-\varepsilon/4} dy
\]

\[
\leq \frac{1}{\varepsilon} (K + o(1)).
\]

Hence we find

\[
\frac{1}{\varepsilon} (K + o(1)) \geq \frac{H}{\varepsilon} \quad \text{or} \quad \left( K + o(1) \right) \geq H.
\]

For \( \varepsilon \to 0^+ \), it follows that \( K \geq H \), which contradicts the fact that \( H > K \). Hence the constant \( K \) in (3.2) is the best possible. The theorem is proved. \( \square \)

**Theorem 3.3.** Under the assumption of Theorem 3.1. we have

\[
\int_0^\infty \frac{f(x)}{(x + a^2y)(x + b^2y)(x + c^2y)} dx^p < K^p \int_0^\infty x^{-p/2} f^p(x) dx,
\]

where the constant factor \( K^p \) is the best possible. Inequalities (3.5) and (3.1) are equivalent.

**Theorem 3.4.** Under the assumption of Theorem 3.2, we have

\[
\int_0^\infty \frac{f(x)}{(x + a^2y)(x + b^2y)(x + c^2y)} dx^p > K^p \int_0^\infty x^{-p/2} f^p(x) dx,
\]

where the constant factor \( K^p \) is the best possible.

Inequalities (3.6) and (3.2) are equivalent. We prove only Theorem 3.3, since the proof of Theorem 3.4 is similar.

**Proof of Theorem 3.3.** Setting \( g(y) = y^{3p/2 - 1} \left( \int_0^\infty \frac{f(x)}{(x + a^2y)(x + b^2y)(x + c^2y)} dx \right)^{p-1} \), by (3.1), we have

\[
\int_0^\infty y^{-1-q/2} g^q(y) dy
\]

\[
= \int_0^\infty y^{3p/2 - 1} \left( \int_0^\infty \frac{f(x)}{(x + a^2y)(x + b^2y)(x + c^2y)} dx \right)^p dy
\]

\[
= \int_0^\infty \left( \int_0^\infty \frac{f(x)}{(x + a^2y)(x + b^2y)(x + c^2y)} dx \right) y^{3p/2 - 1} \left( \int_0^\infty \frac{f(x)}{(x + a^2y)(x + b^2y)(x + c^2y)} dx \right)^{p-1} dy
\]

\[
= \int_0^\infty \frac{f(x)g(y)}{(x + a^2y)(x + b^2y)(x + c^2y)} dx dy
\]

\[
\leq K \left\{ \int_0^\infty x^{-1/2 - p} f^p(x) dx \right\}^{1/p} \left\{ \int_0^\infty x^{-1/2 - q} g^q(x) dx \right\}^{1/q},
\]

(3.7)
0 < \left\{ \int_0^\infty y^{-1/2-q} g^q(y) \, dy \right\}^{1/p} \leq K \left\{ \int_0^\infty x^{-1-p/2} f^p(x) \, dx \right\}^{1/p} < \infty. \tag{3.8}

Hence by (3.1), both (3.7) and (3.8) keep the form of strict inequalities, then we have (3.5).

By Hölder’s inequality, we have

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+a^2 y)(x+b^2 y)(x+c^2 y)} \, dx \, dy
\]

\[
= \int_0^\infty \left( \int_0^\infty \frac{f(x)}{(x+a^2 y)(x+b^2 y)(x+c^2 y)} \, dx \right) (y^{-1/2-q} g(y)) \, dy
\]

\[
\leq \left\{ \int_0^\infty y^{3/2-1} \left( \int_0^\infty \frac{f(x)}{(x+a^2 y)(x+b^2 y)(x+c^2 y)} \, dx \right)^p \, dy \right\}^{1/p} \left\{ \int_0^\infty y^{-1-\frac{q}{2}} g^q(y) \, dy \right\}^{1/q}
\]

\[
= \left\{ \int_0^\infty y^{3/2-1} \left( \int_0^\infty \frac{f(x)}{(x+a^2 y)(x+b^2 y)(x+c^2 y)} \, dx \right)^p \, dy \right\}^{1/p} \left\{ \int_0^\infty y^{-1-\frac{q}{2}} g^q(y) \, dy \right\}^{1/q}. \tag{3.9}
\]

Hence by (3.5), we have (3.1), and inequalities (3.1) and (3.5) are equivalent. If the constant factor in (3.5) is not the best possible, then by (3.9), we can get a contradiction that the constant factor in (3.1) is not the best possible. The theorem is proved. \qed

**Remarks.** (1) Replacing \( x, y \) by \( x^\mu, y^\mu \) (\( \mu > 0 \)), and \( x^{\mu-1} f(x^\mu), y^{\mu-1} g(y^\mu) \) by \( f(x), g(y) \) in (3.1), we have

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+a^2 y^\mu)(x+b^2 y^\mu)(x+c^2 y^\mu)} \, dx \, dy
\]

\[
< \frac{K}{\mu} \left\{ \int_0^\infty x^{-1+p-(3\mu p/2)} f^p(x) \, dx \right\}^{1/p} \left\{ \int_0^\infty x^{-1+q-(3q\mu/2)} g^q(x) \, dx \right\}^{1/q}. \tag{3.10}
\]

(2) Similarly, if both \( \varphi(x) \) and \( \psi(x) \) are differentiable and strict increasing functions and \( \varphi(0) = \psi(0) = 0 \), \( \varphi(\infty) = \infty, \psi(\infty) = \infty \), replacing \( x, y, f(x) \) and \( g(y) \) by \( \varphi(x), \psi(x), f(\varphi(x))\varphi'(x) \) and \( g(\psi(y))\psi'(y) \), respectively, we obtain

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(\varphi(x)+a^2 \psi(y))(\varphi(x)+b^2 \psi(y))(\varphi(x)+c^2 \psi(y))} \, dx \, dy
\]

\[
< K \left\{ \int_0^\infty [\varphi(x)]^{-1-p/2} \left[ \varphi'(x) \right]^{1-p} f^p(x) \, dx \right\}^{1/p} \left\{ \int_0^\infty [\psi(x)]^{-1-q/2} \left[ \psi'(x) \right]^{1-q} g^q(x) \, dx \right\}^{1/q}. \tag{3.11}
\]

where the constant factor \( K \) defined by Lemma 2.1 is the best possible.

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