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Characterization of symmetric planes in dimension at most 4

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ABSTRACT

A stable plane is a topological geometry with the properties that (i) any two points are joined by a unique line, and (ii) the operations of join and intersection are continuous and have open domains of definition. A stable plane is called symmetric if its point space is a (differentiable) symmetric space whose symmetries are automorphisms of the plane.

Among locally compact stable planes of positive (topological) dimension ≤ 4, we determine those which admit a reflection at each point (i.e., an involutory automorphism fixing this point linewise), and we list the possible groups containing reflections at all points. Together with an additional, purely geometric condition, this yields a characterization of symmetric planes and, indirectly, of the plane geometries defined by real and complex hermitian forms. No differentiability hypotheses and no algebraic axioms ruling the reflections are needed.

INTRODUCTION

Our aim is to characterize the real and complex hermitian planes in terms of involutory automorphisms. By a *hermitian plane* over a field **F**, we mean a subset of the projective plane $P_2\mathbf{F}$ consisting of those one-dimensional subspaces of $\mathbf{F}^3$ on which a given hermitian form f is positive definite (or nonzero), endowed with the induced incidence structure; cf. [16]. f is allowed to be degenerate or symmetric.

A first approach, relying mainly on differentiability conditions, has been suggested in [16, 17]. There, we started from the notions of a stable plane and of a symmetric space in the sense of differential geometry. A rough definition of a *stable plane* is given in the abstract above; details can be found in [14]. Examples are topological projective planes, and their open subsets. A *sym-*

metric space may be defined, with a little loss of generality but sufficiently general for our considerations, as a conjugation invariant set S of involutions generating a centre free Lie group Σ. The latter is called the *motion group* of the space S. If S is given a differentiable structure as a sum of homogeneous spaces of Σ then conjugation by $s \in S$ induces an involutory diffeomorphism σ_s on S, called the *symmetry* around s. See [13] for details. Combining these notions, we arrived at the concept of a *symmetric plane*: A stable plane is called symmetric if its point space is a symmetric space whose symmetries are automorphisms of the geometry. The symmetry around s is then a *reflection with centre s*; that is, it fixes each line through s. We proved the following.

THEOREM [16, 17]. *The symmetric planes of positive dimension* $\leqq 4$ *are precisely the nondesarguesian* 4*-dimensional affine translation planes and the real and complex hermitian planes.*

It is the aim of the present paper to go one step further. We wish to dispense with the uniqueness and conjugacy conditions imposed on the reflections and with the differentiability hypotheses inherent in the notion of a symmetric plane. The only hypothesis that we are going to maintain concerns existence of reflections; throughout this paper, we make the following

GENERAL ASSUMPTIONS

(★) *$(M, \mathscr{L})$ is a locally compact stable plane with point set M and line set $\mathscr{L}$, satisfying* $0 < \dim M \leqq 4$. *On some nonempty subset $Z \subseteq M$, a function $z \to \sigma_z$ is given which assigns a reflection at z to the point z.*

Σ denotes the closure of the subgroup generated by the reflections σ_z in $\Gamma = \mathrm{Aut}\,(M, \mathscr{L})$.

Our results are summarized by the following theorem. See 'prerequisites' below for unexplained terms such as *coaffine point.*

MAIN THEOREM. *If $Z = M$, then there are three possibilities.*

i) *M has no coaffine points and is not projective. Then M is a symmetric plane and Σ is the motion group.*

ii) *Precisely the points of one line W are coaffine. Then M is the real or complex projective plane with a certain part of one line W' deleted (a 'dual hermitian plane').*

iii) *All points are coaffine or M is projective. Then M is a desarguesian projective or punctured projective plane.*

REMARK. a) In each case, the possible planes M and the possible groups Σ can be explicitly described. This is done in [16] and [17] for case (i) (cf. the previous theorem), and in Theorems 2.2 and 4.4 of the present paper for the cases (ii) and (iii), respectively. The planes in case (ii) are inhomogeneous with respect to their full automorphism groups. In case (iii), the planes are symmetric but Σ need not be the motion group. In fact, Σ can either be larger than the motion group, or it can be the motion group of a smaller symmetric plane, inflated slightly so as to contain reflections at the additional points.

b) The opposite planes (cf. 'prerequisites') of the planes of case (ii) are hermitian. So the Main Theorem may be regarded as a characterization of symmetric planes 'up to duality'. This interpretation of the planes (ii) also provides a satisfactory way of looking at their groups; cf. [16], § 2.

c) In case (i), the choice function selecting a reflection at each point is uniquely determined; see 0.6 below. In case (ii), it turns out that the group Σ is uniquely determined although the reflection at a point is not always unique.

d) The restriction to low dimensions is necessary mainly because we need to know that the isotropy group of a line is a Lie group, and because we rely heavily on some results of [18] (see 0.6 and 0.7 below), which could only be obtained using strong tools of low dimension topology such as Brouwer's *Translationssatz*. Note that the possible dimensions of connected symmetric planes are 2, 4, 8, 16; see [16], 4.3.

The following is an immediate consequence of the Main Theorem.

COROLLARY 1. *A locally compact stable plane of positive dimension $d \leqq 4$ is symmetric if and only if it admits a reflection at each point and does not contain precisely one pointwise coaffine line.* □

REMARK. Call a line projective if it meets all other lines. Then pointwise coaffine lines may be described, purely geometrically, as nonprojective lines which meet only the projective lines; cf. 0.4 and 0.5 below. Since reflections are necessarily continuous in stable planes ([14], 3.2), the corollary characterizes symmetric planes among locally compact stable planes of dimension $\leqq 4$ by geometric conditions alone.

Strictly speaking, Corollary 1 is not a characterization of hermitian planes, since it does not distinguish between the complex affine plane and the other 4-dimensional affine translation planes. However, from Theorem 2.1 and the fact that topological translation planes admit reflections at all points it follows that such a characterization may be obtained simply by adding the hypothesis that reflections exist at many axes (Corollary 2). Note that several non-affine symmetric planes fail to admit reflections at *all* lines. Planes admitting reflections at many lines are studied systematically in a forthcoming paper [20].

COROLLARY 2. *A locally compact stable plane of positive dimension $d \leqq 4$ is a real or complex hermitian plane if and only if it has the following three properties.*

i) *Every point is the centre of a reflection*
ii) *The set of all axes of reflections has nonempty interior*
iii) *The number of pointwise coaffine lines is different from one.* □

The *proof of the Main Theorem* is given in a sequence of separate theorems. Theorem 1.5b shows that M must be symmetric if M is not projective and has no coaffine points. One observes from the explicit description of all motion groups of symmetric planes of dimension at most 4 given in [16], § 2 and [17], § 6 that then the motion group is closed in the group of those automorphisms of M which extend to the desarguesian projective plane. By [14], 2.8, the latter group is closed in $\Gamma = \mathrm{Aut}(M, \mathscr{L})$. Since reflections are unique (0.6), this implies that the motion group coincides with Σ.

By 0.5, M contains a pointwise coaffine line W if M has a coaffine point. Theorem 2.2 describes M and Σ in the case where W contains all coaffine points. Theorem 3.6 shows that in all other cases M is projective or coaffine, and Theorem 4.4 describes M and Σ in this situation.

The proof of the Main Theorem uses the classification theorem of [16, 17] on two occasions; see 2.2, 3.2. In fact, proofs of 3.6 and 4.4 may be based entirely on the existence of open orbits (1.4) and the classification; see the Corollary in [19].

PREREQUISITES

For detailed information on stable planes, the reader may consult [14]. We mention a few facts, which will be used frequently. Let $(E, \mathscr{E})$ be a stable plane with point set E and line set $\mathscr{E}$. E is always assumed to be *locally compact* and of *positive topological dimension*. The line joining $x, y \in E$ is denoted by $x \cup y$, and the point of intersection of $K, L \in \mathscr{E}$ is written $K \cap L$. Lines are thought of as sets of points.

If $\dim E \leqq 4$ then the pencil $\mathscr{E}_x$ of all lines through a point x is a sphere S_l of dimension $l = 1$ or 2, and each line is homeomorphic to an open subset of S_l ([14], 1.13, 1.19, 1.20). E and $\mathscr{E}$ are manifolds of dimension $2l$. The continuous collineations of $(E, \mathscr{E})$ form a group Γ, which is locally compact in its compact-open topology; the isotropy groups Γ_x, Γ_L of points and lines are Lie groups ([14], § 2). A collineation γ is said to be *central* with *centre* x if γ fixes each line through x; we write $\gamma \in \Gamma_{[x]}$. *Axial* collineations are defined dually. Central and axial collineations are continuous ([14], 3.2). A collineation $\gamma \in \Gamma_{[x,A]}$ is called an *elation* if $x \in A$, and a *homology* otherwise. $\gamma \in \Gamma_{[x]}$ is called a *reflection* at x if γ is not an elation and $\gamma^2 = 1 \neq \gamma$. The former condition is redundant in stable planes ([18], 3.2; cf. also [15], 1.4).

The behaviour of reflections of projective planes is well understood; see, for example, [11], 4.21, 4.22:

0.1. *In any projective plane, the product of two reflections α, β with the same axis and different centres is an elation. The centres of α, β and $\alpha\beta$ are collinear.*

0.2. *In a locally compact projective plane of positive dimension at most* 4, *a reflection is uniquely determined by its centre and axis* ([27], 4.8, [30], 2.1).

0.3. *Let π be a (not necessarily continuous) polarity of a compact projective* *plane of dimension at most* 4, *and let* $\sigma \in \Gamma_{[z,A]}$ *be a reflection. If* $A = z^{\pi}$ *then* σ *centralizes* π.

This is a consequence of 0.2 Indeed, σ^{π} is a (continuous) reflection at z with axis A and hence coincides with σ.

For stable planes, results similar to 0.1 and 0.2 are needed in order to prove our theorem; they have been provided in [18]. Regarding these questions, the following simple notions are crucial; cf. [18], § 1.

The *opposite plane* $(E, \mathscr{E})^*$ of a stable plane $(E, \mathscr{E})$ is defined as the geometry $(\mathscr{K}, K)$, where $\mathscr{K} \subseteq \mathscr{E}$ is the set of compact lines and $K = \bigcup \mathscr{K}$ is the set of points covered by $\mathscr{K}$. A point x is *coaffine* if (i) x lies in K and (ii) x considered as a line of $(E, \mathscr{E})^*$ is affine. Here, a line L is called *affine* if each point outside L is on a unique line not meeting L. These notions become valuable owing to the following facts (0.4, 0.5).

0.4. *The set* $\mathscr{K}$ *of compact lines is open in* $\mathscr{E}$, *and a line belongs to* $\mathscr{K}$ *if and only if it meets each other line* ([14], 1.15, 1.16).

Consequently, a plane is projective if and only if it is compact ([14]), 1.27); *another equivalent condition is that some pencil* $\mathscr{E}_x$ *is contained in* $\mathscr{K}$([18], 1.3).

In particular, 0.4 shows that $(E, \mathscr{E})^*$ is a stable plane. Coaffine points may be characterized in topological terms, see [18], 1.6:

0.5. *A point* x *is coaffine if and only if* x *lies on a unique noncompact line* L, *which is then pointwise coaffine*; *i.e., all points of* L *are coaffine. The set of coaffine points is closed in* E ([18], 1.9).

We shall always denote the sets of coaffine points and pointwise coaffine lines by C and $\mathscr{C}$, respectively. If $C = E$ then E is called *coaffine.* This happens precisely if E is punctured projective, that is, a projective plane P with one point ∞ removed. P is then called the *projective hull* of E.

The results on reflections can now be stated. Let $(E, \mathscr{E})$ be a stable plane of positive *dimension at most* 4. Assume that the plane is *not projective* and that $z \in E$ is *not coaffine. Then assertions* 0.6 *and* 0.7 *hold*; *cf. also* 3.3.

0.6. *There is at most one reflection at* z ([18], 3.21).

0.7. *Each sequence of reflections* σ_n *at points* $z_n \to z$ *converges to a reflection at* z ([18], 3.24).

1. EXISTENCE OF OPEN ORBITS, AND NONCOMPACT PLANES WITHOUT COAFFINE POINTS

PROPOSITION 1.1: *Let* Δ *be a second countable Lie group and let* $\mathscr{T}$ *be the set of involutions in* Δ. *Then each* Δ^1*-conjugacy class in* $\mathscr{T}$ *is open in* $\mathscr{T}$, *and there* *are at most countably many such classes.* (Δ^1 *denotes the connected component of* Δ.)

PROOF. It suffices to to prove that the classes are open, and this follows from a well known theorem on conjugacy of neighbouring compact subgroups in a Lie group ([21], p. 216).

For the special case of that theorem required here, a more elementary proof is easily available. Indeed, given $\alpha \in \mathscr{X}$, choose an exponential neighbourhood $U \subseteq \Delta^1$ such that $\log U$ is star shaped and such that on $\log U$, the inner automorphism induced by α commutes with the exponential map. Given a different involution $\beta \in \mathscr{X}$ such that $\beta\alpha \in U$, we prove that α is conjugate to β under an element $\delta \in U$. Indeed, the one-parameter group Φ determined by $\beta\alpha$ contains an element δ such that $\delta^2 = \beta\alpha$. Since $(\beta\alpha)^\alpha = (\beta\alpha)^{-1}$, the involution α induces inversion on Φ. Thus, $\delta = \alpha\delta^{-1}\alpha$ and

$$\alpha\delta = \delta^{-1}\alpha = \delta^{-1}\beta\delta^2.$$

Hence, $\alpha = \delta^{-1}\beta\delta$.

DEFINITION 1.2: A set-valued mapping $f: X \to 2^Y \setminus \{\emptyset\}$ between topological spaces X, Y is called *weakly continuous* if for each convergent sequence $x_n \to x$ in X there is a subsequence x_k and a convergent sequence $y_k \in x_k^f$ in Y such that $\lim y_k \in x^f$.

Our hypothesis in the next theorem is a slight modification of the general assumption (★).

THEOREM 1.3: *Assume that Z is a connected open subset of a line L and that there is a weakly continuous map $s: Z \to 2^\Gamma$ sending each z to a nonempty set of reflections at z.*

Then the connected component of the closed subgroup $\Delta \leqq \Gamma = \mathrm{Aut}(M, \mathscr{L})$ generated by $\bigcup Z^s$ is transitive on $U = Z^{\Delta^1}$.

PROOF. We may assume that $Z = U$. Since $\dim M \leqq 4$, we know that $\Delta \leqq \Gamma_L$ is a second countable Lie group ([14], § 2). If two reflections in Δ are conjugate under Δ^1 then their centres are in the same Δ^1-orbit. By 1.1, the number of these orbits in U is at most countable. Each orbit is a continuous image of some coset space Δ^1/Φ, which is a σ-compact manifold. Thus, an orbit with empty interior is a countable union of nowhere dense subsets of U. According to Baire's theorem, there must be an orbit B containing an open set. Then B is open.

We complete the proof by showing that B is also closed in U. Indeed, if the points $b_n \in B$ converge to $u \in U$, select reflections $\sigma_k \in b_k^s$ and $\sigma \in u^s$ such that $\sigma_k \to \sigma$. Since the conjugacy classes of involutions are open, σ_k lies in the class of σ for k large. Then u lies in the orbit B of b_k.

REMARK 1.4: The first paragraph of the preceding proof does not use weak continuity of s. This yields the following more general result.

Let $(E, \mathscr{E})$ be a stable plane of arbitrary positive dimension. *If* $\Delta \leqq \mathrm{Aut}\ (E, \mathscr{E})$

is a Lie group and contains a reflection at each point of some open set in E then Δ has an open orbit of points. If, for example, E is a 4-dimensional projective plane then Aut $(E, \mathscr{E})$ itself is known to be a Lie group ([27], 3.9).

The same remark applies when Δ fixes a line L and contains reflections at all points of some open set in that line. In this case, Δ has an open orbit in L.

Part (b) of the following corollary proves (i) of the Main Theorem and will also be useful later on. From now on, ($\bigstar$) will be assumed.

COROLLARY 1.5: *Assume that M is not projective and consider the set $N = M \setminus C$ of non-coaffine points.*

a) *Let L be a line. If $Z = L \cap N$ then Σ^1 is transitive on each connected component of Z.*
b) *If $Z = N$ then N is a symmetric plane, whose motion group is contained in* Aut $(M, \mathscr{L})$.

PROOF. Recall from 0.5 that N is open in M, hence is a stable plane (non-empty if $Z \subseteq N$). The reflection σ_x at a point $x \in N$ is unique and depends continuously on x, see 0.6 and 0.7. Thus, (a) follows from the theorem.

If $Z = N$ then (a) implies that the orbits of Σ^1 are open in N. Indeed, in the plane N, the connected components of a point p in all lines through p cover a neighbourhood of p; see [16], 4.1. Note that the theorem cannot be improved so as to apply directly, since Aut $(M, \mathscr{L})$ is not in general known to be a Lie group.

We infer that Σ^1 is transitive on each connected component of N and that the reflections σ_z form an invariant subset of Σ. According to [16], Theorem A, this implies that the components of N, and N itself, are symmetric planes.

2. PLANES WITH FEW COAFFINE POINTS

If M contains precisely one pointwise coaffine line W then Σ fixes W. As *examples* of groups generated by reflections and fixing a line, we consider several groups of automorphisms of the desarguesian affine plane over $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$. Theorems 2.1 and 2.2 below will show that the examples are typical. Let (G i),..., (G vii) be the extensions of the translation group $\mathbf{F}^2$ by the following linear groups.

(G i) $\quad \langle \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \rangle \cdot \{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}; 0 \neq a \in \mathbf{F} \}$

(G ii) $\quad \langle \begin{pmatrix} & -i \\ i & \end{pmatrix} \rangle \cdot \mathrm{SL}_2\mathbf{R} \quad (\mathbf{F} = \mathbf{C},\ i^2 = -1)$

(G iii) $\quad$ one of the groups (G i) or (G ii) extended by $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

(G iv) $\quad \langle \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \rangle \cdot \{ \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}; a \in \mathbf{F} \}$

(*G* v) $\langle\begin{pmatrix}1 & \\ & -1\end{pmatrix}\rangle \cdot SO_2\mathbf{R}$ $(\mathbf{F}=\mathbf{R})$

(*G* vi) $\langle\begin{pmatrix}1 & \\ & -1\end{pmatrix}\rangle \cdot SU_2\mathbf{C}$ $(\mathbf{F}=\mathbf{C})$

(*G* vii) $\langle\begin{pmatrix}1 & \\ & -1\end{pmatrix}\rangle \cdot SL_2\mathbf{F}$

THEOREM 2.1: *In addition to* (★), *assume that* (i) *M is a projective translation plane with respect to an axis W fixed by* Σ *and* (ii) *Z contains* $M \setminus W$ *and a nonempty open subset of W.*

Then M is desarguesian, and Σ *is equivalent to one of the groups* (*G* i) *through* (*G* vii).

REMARK. Affine translation planes are symmetric, and their motion groups contain all translations; cf. [16], 2.2. The nontrivial part of the hypothesis is that we require existence of reflections at the points of $U = Z \cap W$.

PROOF. 1) For basic information on topological translation planes, the reader is referred to [25], § 7, and [1]. There are no two-dimensional proper translation planes [25]. Thus, the first part of the assertion need only be proved in the four-dimensional case.

2) By 1.4, Σ has an open orbit $X \subseteq W$ meeting U. This implies that the dimension of $\Gamma = \mathrm{Aut}\,(M, \mathscr{L})$ is at least $3 \cdot \dim W + 1$ because, apart from the translations, $\Gamma_{[W]}$ contains a one-parameter group of homologies; see, for example, [9], 2.4 and 3.2.

3) If the connected component Σ^1 fixes a unique point $w \in W$ then also Σ fixes w. Transitivity of the translation group allows us to replace the reflections σ_u at the points $u \in U$ by suitable conjugates in such a way that all have the same axis $A \in \mathscr{L}_w \setminus \{W\}$. By (2), Σ contains a transitive group $\Sigma_{[w,A]}$ of shears and is a group of Lenz type V. The plane is coordinatized by a distributive quasi-field (= division ring) ([11], 6.9), and is desarguesian ([27], 1.3'). The group is (*G* iv).

4) In order to prove the first part of the assertion, we use the determination of all four-dimensional translation planes with $\dim\Gamma \geqq 7$, which has been obtained by D. Betten [1–6]. If $\dim \Gamma \geqq 9$ then M is desarguesian [1]. Consider now a point $o \notin W$. If $\dim\Gamma = 7$ or 8 then the action of $\Delta = \Gamma_o^1$ on W must be of one of the following three types ([2], Satz 1; [5], Lemma 1): (A) Δ fixes precisely one point on W or (B) $\dim \Gamma = 7$ and Δ fixes precisely two points on W or (C) $\dim \Gamma = 8$ and Δ fixes no point on W.

If in our situation Δ fixes precisely one point w then Σ fixes w, and (3) applies. If $\dim \Gamma = 7$ and Δ fixes precisely two points $v, w \in W$ then the reflection at a point of $U \setminus \{v, w\}$ interchanges v and w. Hence, the actions of Δ on the lines $o \cup v$ and $o \cup w$ are equivalent. By [5], Lemma 2, the plane is one of those described in [5], Satz 3 and Satz 4. In case (C), the plane is as described in [1], Satz 5 or [3], Satz.

In each case, Betten has determined the full automorphism group Γ. Inspection shows that the points of an open orbit of Γ on W never are the centres of reflections.

5) It remains to determine the group Σ. We may replace each reflection σ_u by a suitable conjugate $\sigma'_u \in \Gamma_{[u],o}$ fixing o. It follows that Σ is the product of the group $\langle -1 \rangle \cdot \Gamma_{[W,W]}$ generated by the affine reflections with a group $\Phi \leqq \Gamma_{o,W}$ generated by reflections at points at infinity. Φ operates linearly on the group $\Gamma_{[W,W]} \cong \mathbf{F}^2$, and this action is equivalent to that on $M \setminus W$. In order to identify Φ, we begin by determining the normal subgroup $\mathrm{S}\Phi = \Phi \cap \mathrm{SL}_2\mathbf{F}$ of index 2, using its action on the line at infinity.

The group $\mathrm{S}\Phi$ is easily determined if its component $\mathrm{S}\Phi^1$ fixes precisely one point; see (3). Suppose now that the group $\mathrm{S}\Phi^1$ fixes two points. For dimension reasons, it is then the identity component of the isotropy group of those points in $\mathrm{SL}_2\mathbf{F}$. If $\mathrm{S}\Phi$ is disconnected then $\mathrm{S}\Phi = Ns(\mathrm{S}\Phi^1)$ is the extension of $\mathrm{S}\Phi^1$ by a rotation of order 4 which interchanges the fixed points; this happens if U contains fixed points of $\mathrm{S}\Phi^1$.

If $\mathbf{F} = \mathbf{R}$ then the only fixed point free connected subgroups of $\mathrm{SL}_2\mathbf{F}$ are $\mathrm{SO}_2\mathbf{R}$ and $\mathrm{SL}_2\mathbf{R}$ itself. If $\mathbf{F} = \mathbf{C}$ and the component $\mathrm{S}\Phi^1$ fixes no point then it has a simple Lie algebra ([31], 4.10), and is one of the groups $\mathrm{SL}_2\mathbf{C}$, $\mathrm{SL}_2\mathbf{R}$, $\mathrm{SU}_2\mathbf{C}$. Note that each of them has a unique faithful representation on $\mathbf{C}^2$. If $\mathrm{S}\Phi$ is disconnected then $\mathrm{S}\Phi = Ns(\mathrm{S}\Phi^1)$ is the extension of $\mathrm{SL}_2\mathbf{R}$ by a transformation with eigenvalues i, $-i$ and real eigenvectors; this happens if U contains points with real coordinates.

Given $\mathrm{S}\Phi$, the extension Φ by an element of determinant -1 is computed easily, since $\mathrm{S}\Phi$ has index at most 4 in its normalizer in the group of $\mathbf{F}$-linear transformations of determinant ± 1. In fact, Φ is uniquely determined by $\mathrm{S}\Phi$. □

As *examples* of stable planes admitting a reflection at each point and containing precisely one pointwise coaffine line, we consider several subplanes of the desarguesian plane $P_2\mathbf{F}$, $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$, which will be called *dual hermitian planes*. They are obtained by removing some subset X of the line L_∞ at infinity. Specifically, X may consist of one of the following:

(P i) two points ($\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$)
(P i′) in the real case, a closed segment on the line L_∞
(P ii) in the complex case, all points on L_∞ with real homogeneous coordinates
(P ii′) one of the closed discs bounded by the set of real points.

COROLLARY 2.2: *If $Z = M$ and M contains precisely one pointwise coaffine line then M is one of the dual hermitian planes* (P i) *through* (P ii′), *and accordingly Σ is the group* (G i) *or* (G ii).

PROOF. 1) We shall show that the subplane $M \setminus W$ is an affine translation plane. Then [18], 2.7 asserts that M may be embedded into the projective hull P of $M \setminus W$, in such a way that W is mapped onto an open part of the line W' at infinity. Automorphisms of M extend to P, and P satisfies the hypothesis of Theorem 2.1. Since M is not pointwise coaffine, $X = W' \setminus W$ contains more than one point. Since Σ leaves X invariant and is generated by the reflections at the points of the complement, Σ can only be (G i) or (G ii), and the corollary follows.

2) Since W is the only pointwise coaffine line, 1.5b shows that $M \setminus W$ is a symmetric plane. By [18], 1.8, the complementary plane of a pointwise coaffine line has a point set homeomorphic to $\mathbf{R}^{2l}$ and lines homeomorphic to $\mathbf{R}^l$, where $l = 1$ or 2. Apart from translation planes, there are four symmetric planes with this property ([16], Theorem F and [17], Theorem A); namely, the real and complex cylindrical and interior hyperbolic planes. Now, each line of M meeting W is an affine line of $M \setminus W$, whereas the only affine lines in the four planes mentioned above are the 'vertical' lines in the cylinder planes. This shows that $M \setminus W$ is a translation plane.

3. PLANES WITH MANY COAFFINE POINTS

Assume that $Z = M$ and that M possesses more than one pointwise coaffine line. The aim of this section is to show that then M is coaffine; in other words, that M is a projective plane with one point deleted. This means that we have to exclude planes such as the example constructed in [18], 5.4. The proof will be divided up into a sequence of lemmas.

We continue to denote by C, $\mathscr{C}$ and $\mathscr{K}$ the sets of all coaffine points, pointwise coaffine lines and compact lines, respectively. Since $\mathscr{K} \neq \emptyset$, each reflection σ_p has an axis ([18], 3.5, 3.4), which we denote by A_p. We have $p \notin A_p$ by [18], 3.2. Throughout this section, we shall frequently use the results 0.4 and 0.5 on compact lines and coaffine points, as well as the following topological consequences of the existence of several pointwise coaffine lines ([18], 1.8): The lines of M are homeomorphic to euclidean space $\mathbf{R}^l$ or to the sphere S_l ($l = 1$ or 2); moreover, the complement of a pointwise coaffine line and the opposite plane $(M, \mathscr{L})^* = (\mathscr{K}, M)$ are $(\mathbf{R}^{2l}, \mathbf{R}^l)$-planes, which means that they have a point set homeomorphic to $\mathbf{R}^{2l}$ and lines homeomorphic to $\mathbf{R}^l$.

3.1. a) *The axis A_p of σ_p is not compact.*

b) *A line through p is compact if and only if it meets A_p.*

PROOF. On $L \in \mathscr{L}_p$, the reflection σ_p has precisely one or two fixed points, according as $L \cap A_p = \emptyset$ or not. Hence, (b) follows from the classification of the involutory homeomorphisms of the sphere S_l, cf. [7, 12]; they have either no fixed points or two or infinitely many fixed points in S_l.

If the axis A_p is compact then it meets each line through p, see 0.4. Then, (b) implies that $\mathscr{L}_p \subseteq \mathscr{K}$, and 0.4 shows that M is projective. This proves (a). □

3.2 *If the opposite plane $(M, \mathscr{L})^* = (\mathscr{K}, M)$ admits a reflection at each point $K \in \mathscr{K}$ then $(M, \mathscr{L})$ is coaffine.*

PROOF. The set C of coaffine points of $(M, \mathscr{L})$ is by definition the same as the set of affine lines of $(M, \mathscr{L})^*$. So we have to show that $(M, \mathscr{L})^*$ is an affine plane. We know that $(M, \mathscr{L})^*$ is an $(\mathbf{R}^{2l}, \mathbf{R}^l)$-plane. Hence, 1.5b shows that it is symmetric. The only symmetric $(\mathbf{R}^{2l}, \mathbf{R}^l)$-planes are the affine translation planes and the real and complex cylindrical and interior hyperbolic planes ([16], Theorem F, [17], Theorem A).

A hyperbolic plane contains no affine lines. In a cylinder plane, the affine lines are precisely the 'vertical' lines; they form a set homeomorphic to $\mathbf{R}^l$. On the other hand, C contains several lines, which are closed and homeomorphic to $\mathbf{R}^l$. Hence, C cannot be homeomorphic to $\mathbf{R}^l$. □

The following lemma is taken from [18], 3.22.

3.3. *Let $a \in A_z$. Then there are two possibilities:*
i) If $z \in A_a$, *then $\sigma_a\sigma_z$ is an involution with axis $a \cup z$;*
ii) otherwise, $(\sigma_a\sigma_z)^2 \in \Gamma_{[a,A_z]}$ *is a nontrivial elation, and $A_z \in \mathscr{C}$.* □

3.4. *For $z \in M \setminus C$, the axis A_z lies in $\mathscr{C}$.*

PROOF. Assume that $A_z \notin \mathscr{C}$ for some $z \notin C$. Using 3.3, we shall show that then each compact line is the axis of some involution. By 3.2, it will follow that M is coaffine. But then our assumption is absurd since there is no $z \notin C$.

For $a \in A_z$ we must have $z \in A_a$, and $\sigma_a\sigma_z$ is a reflection at the compact line $a \cup z$, cf. 3.1. Since $z \in A_a \setminus C$, the axis A_a cannot be pointwise coaffine. Repeating our argument we find that $\sigma_a\sigma_b$ is a reflection at $a \cup b$ whenever $b \in A_a$. Now, given $K \in \mathscr{K}$, let $a = K \cap A_z$ and $b = K \cap A_a$. Then $\sigma_a\sigma_b$ is a reflection at K. Note that A_a and A_z are disjoint by 3.1.

3.5. *For each $z \in M$, the axis A_z lies in $\mathscr{C}$.*

PROOF. 1) Assume that $A_z \notin \mathscr{C}$ is a counterexample. By the previous lemma we have $z \in C$. *Let $A \in \mathscr{C}$* be the noncompact line containing z. By 3.3 and 3.1, A is the axis A_a for each point $a \in A_z$. The group $\Phi \leq \Gamma_{[A]}$ generated by the corresponding reflections σ_a operates transitively on $A_z \subseteq M \setminus C$; see 1.5a. For $x \in A$, it is also transitive on

$$\mathscr{K}_x = \mathscr{L}_x \setminus \{A\} = \{a \cup x;\ a \in A_z\};$$

note that x is coaffine and that A_z is noncompact and hence disjoint from A. Observe that we get a reflection σ_K at each line $K \in \mathscr{K}_z$ by defining $\sigma_K = \sigma_a\sigma_z$, where $a = K \cap A_z$; see 3.3.

2) The line A might be fixed by Σ. In that case, apply 1.5b to show that Σ is transitive on the $(\mathbf{R}^{2l}, \mathbf{R}^l)$-plane $M \setminus A$. Since C is not contained in A, the plane M is coaffine, and our assumption becomes absurd in view of 3.1.

3) Assume now that some element $\sigma \in \Sigma$ moves A to a different line $B \in \mathscr{C}$. Then $A \cap B = \emptyset$ by 0.5. For $b \in B$, the group Φ^σ operates transitively on $\mathscr{K}_b$. Since each compact line K meets B, we may transform K into $\mathscr{K}_z$ using Φ^σ. By

(1), there is a reflection at K, and 3.2 shows that again M is coaffine, contrary to our assumption.

THEOREM 3.6: *If M possesses more than one pointwise coaffine line and $Z=M$ then M is coaffine.*

PROOF. 1) Assume that $p \in M$ is not coaffine, and let K be any compact line. p lies on some noncompact line L by 0.4, and 0.5 shows that $L \cap K$ is not coaffine. Thus, the open set $K \setminus C$ (cf. 0.5) is nonempty. Let V be a connected component of $K \setminus C$. By 1.5a, the closed group Δ generated by all reflections σ_v at points of V operates transitively on V. By 0.7, σ_v depends continuously on $v \in V$, and we have $\Delta/\Delta^1 \leqq \mathbf{Z}_2$.

2) V must be simply connected. Indeed, V is a homogeneous surface which can be embedded into the sphere K. Hence, the only other possibility is $V = \mathbf{R} \times S_1$, a cylinder. In that case, 3.5 shows that the reflection σ_v at $v \in V$ fixes both components C_1 and C_2 of $K \setminus V$. Thus, σ_v induces an involution with precisely three fixed points on the sphere $K/C_1, C_2$ obtained from K by shrinking each C_i to a point. This is impossible by [7, 12].

3) The boundary $\dot{V}$ is contained in the closure of each Δ-orbit $D \subseteq K \setminus V$. Indeed, assume that $x \in \dot{V} \setminus \bar{D}$, and consider a sequence $v_n \rightarrow x$ of points in V. Assume that the axes A_{v_n} converge to a line $A \in \mathscr{L}_x$. Since the image sequence $\{d^{\sigma_n}\}$ of $d \in D$ is bounded in $K \setminus \{x\}$, we may assume that the reflections σ_n at v_n converge to an elation $\delta \in \Gamma_{[x,A]}$; see [18], 3.8. By a theorem of M.H.A. Newman [22], δ is not the identity, but we have $\delta^2 = 1$, in contradiction to [18], 3.2.

Now assume that the sequence $\{A_{v_n}\}$ does not accumulate at $\mathscr{L}_x$. Then [18], 3.24b shows that the reflections σ_{v_n} accumulate at some reflection $\sigma \in \Gamma_{[x]}$. The compact set $\bar{V}$ is invariant under σ. Furthermore, $V^\sigma \subseteq \bar{V}$ is a connected component of $K \setminus C$, so that $V^\sigma = V$. By [7, 12], σ fixes a point in V, contrary to 3.5.

4) By a theorem of Halder [10], Δ has a closed orbit B in K. Since V is not closed, B must be contained in $K \setminus V$, so that $B = \dot{V}$ by (3); in particular, $\dim B < l = \dim K$, and B is connected by (2). If B is a point then $B = K \setminus V = K \cap C$; but each pointwise coaffine line meets K, and $|K \cap C| = |\mathscr{C}| > 1$. This shows that B must be a circle. V is then one of the complementary components of B in $K \cong S_2$.

5) The reflection σ_b at $b \in B$ cannot leave V invariant, because it fixes no point outside C. The image $W = V^{\sigma_b}$ is a connected component of $K \setminus C \subseteq K \setminus \dot{V}$. Therefore, it is contained in $K \setminus \bar{V}$. By (3), the Δ-invariant disc $\bar{W} = \bar{V}^{\sigma_b}$ is equal to $K \setminus V$, and $B = K \cap C$.

6) We have now arrived at an absurdity. By 3.5, the reflection σ_v at $v \in V$ has a fixed point in the circle $B = K \cap C$. Hence, it fixes two points in $B = B^{\sigma_v}$ and one in W, which gives a total of 4 fixed points in K.

4. COAFFINE AND PROJECTIVE PLANES

We are now left with the situation where $Z = M$ is projective or coaffine. We shall treat these cases simultaneously. So let P be a compact projective plane such that $M=P$ or $M=P\setminus\{\infty\}$ for some point $\infty\in P$. Denote the axis of the reflection σ_z by A_z, as before, and let π be the map assigning A_z to z. We proceed along the following lines. If π is injective and if, moreover, each σ_z commutes with each reflection σ_a at a point of A_z then π defines a polarity, and $M=P$. The group Σ is then the elliptic motion group or the full projective linear group of the classical projective plane $P_2\mathbf{F}=P$. The two possibilities arise according as π is continuous or not.

The cases where for some choice of z and $a\in A_z$ the reflections σ_a and σ_z do not commute or where π is not injective are treated beforehand in two separate lemmas. Up to duality, this leads to the situations considered in § 2, where Σ fixes a line; however, also the possibility $\Sigma=\mathrm{PSL}_3\mathbf{F}$ arises here again.

4.1. *If Σ fixes a line W then P is desarguesian, and $\infty\in W$ or $M=P$. Σ is one of the groups* (G iii) *through* (G vii) *of* § 2. *Only* (G iv) *can act on $M=P\setminus\{\infty\}$.*

PROOF. The affine plane $M\setminus W$ is symmetric by 1.5b and, hence, is a translation plane ([23], p. 213; cf. also [16], Theorem D). The assertion follows directly from 2.1; note that the groups (G i) and (G ii) do not contain reflections at the points of their low dimensional orbits.

4.2. *Suppose that Σ fixes no line and that σ_a and σ_z do not commute for some choice of $z\in M$ and $a\in A_z$. There are two possibilities:*

i) *Σ fixes a point $q\in P$, where $q=\infty$ if $M\neq P$, and the dual plane $(\mathscr{L},P)$ is a translation plane with axis q, or*

ii) *P is desarguesian over $\mathbf{F}=\mathbf{R}$ or $\mathbf{C}$, and $\Sigma=\mathrm{PSL}_3\mathbf{F}$.*

PROOF. 1) Suppose that Σ contains a transitive elation group $\Sigma_{[w,W]}$, where $w\in M\cup\{\infty\}$. We show that this implies the assertion.

Note first that $M=P$ or $\infty\in W$. The hypothesis implies that some reflection σ_z moves W. Then $\Sigma_{[w',W']}$ is transitive for $w'=w^{\sigma_z}$ and $W'=W^{\sigma_z}$. If $w=W\cap W'$ then $w=w'$, and the dual translation group $\Sigma_{[w,w]}$ is transitive. If Σ fixes w, (i) follows. Otherwise, w^{Σ} contains a triangle since Σ fixes no line. By [11], 4.20, M is a Moufang plane, and (ii) follows by [25], 7.27.

If $w\neq w'$, let $v=W\cap W'$. Then all elation groups $\Sigma_{[u,U]}$ are transitive for $u\in V:=w\cup w'$ and $U=u\cup v$. In other words, the group $\Sigma_{v,V}$ is of Lenz type III. Since Σ does not fix V, the plane is of Lenz type at least VI, and then even Moufang; see [29] and [8], p. 130.

2) The line $W=a\cup z$ might be the axis A_b for each $b\in A_z\setminus\{a\}$. In that case, $\Sigma_{[a,W]}$ is transitive as one sees, for example, by applying 1.5 to the plane $M\setminus W$; alternatively, the result of [28] may be used. The assertion follows by (1).

3) The hypothesis implies that the elation

$$\gamma=\sigma_a\sigma_z\sigma_a^{-1}\sigma_z^{-1}\in\Sigma_{[a,A_z]}$$

is nontrivial. Now assume that $W = a \cup z$ is moved by the reflection σ_b at some point $b \in A_z \setminus \{a\}$. Then A_z is the axis for each of the reflections σ_x, where

$$x \in X := \{z, z^\gamma, z^{\sigma_b}, z^{\gamma\sigma_b}\}.$$

X contains a triangle. Hence, no reflection at a point $c \in A_z$ can commute with each σ_x. By 0.1, the group $\Lambda = \Sigma_{[A_z, A_z]}$ contains nontrivial elations for each centre $c \in A_z$. This fact alone does not imply transitivity of Λ. We can, however, prove transitivity in at least one direction, using that Σ is generated by many reflections. This is done in (4), employing a method of H. Salzmann [28]. By (1), the proof will then be complete.

4) The group Λ is an uncountable Lie group with a second countable topology, and its connected component Φ is a nontrivial normal subgroup of Σ_{A_z}. Each one-parameter subgroup $\Psi \leqq \Phi$ contains a locally cyclic dense subgroup and, hence, lies in some subgroup $\Sigma_{[c, A_z]}$. Thus, assigning the centre c to the subgroup Ψ we obtain a continuous map α from the space $\mathscr{X}$ of one-parameter subgroups of Φ into A_z. The space $\mathscr{X}$ is here considered as the projective space $P(F)$ of the Lie algebra F of Φ.

If $d = \dim M = 2$, or if $d = 4$ and α is not injective, the group $\Sigma_{[c, A_z]}$ is transitive for some $c \in A_z$ and the assertion follows by (1).

If α is injective and $d = 4$ then $\dim \Phi \leqq 3$. If $\dim \Phi = 3$ then α is an embedding of the real projective plane $\mathscr{X}$ into the 2-sphere A_z; this is impossible. In the case $\dim \Phi = 2$, consider the reflection σ_c at a point $c \in A_z \setminus \mathscr{X}^\alpha$. Its action on F fixes at least two elements of $\mathscr{X}$, and σ_c fixes the corresponding points in $\mathscr{X}^\alpha$, a contradiction. If $\dim \Phi = 1$ then Λ is the union of countably many compact arcs, and $A_z = \Lambda^\beta$ is a countable union of nowhere dense sets, contrary to Baire's theorem; here, β is the map sending an elation to its centre.

4.3. *Suppose that σ_a and σ_z commute whenever $a \in A_z$. If π is not injective then the conclusion* (i) *of* 4.2 *holds, and $\infty \notin M$.*

PROOF. The hypothesis implies that for $x, y \in M$ the conditions $x \in A_y$ and $y \in A_x$ are equivalent. Now assume that $A_z = A_w = A$ for some pair of points $z \neq w$. For $a \in A$ we have $A_a = z \cup w =: B$. In particular, $A \cap B = \infty \notin M$; otherwise, the reflection at $A \cap B$ would be an elation. The same argument now applies to $z', w' \in A$; it shows that $A = A_b$ for each $b \in B$. Thus, the elation groups $\Sigma_{[\infty, A]}$ and $\Sigma_{[\infty, B]}$ are transitive.

THEOREM 4.4: *Let M be projective, $M = P$, or punctured projective, $M = P \setminus \{\infty\}$, and assume that $Z = M$.*

Then P is the desarguesian plane over $\mathbf{F} = \mathbf{R}$ *or* $\mathbf{C}$. *The group Σ and the map $\pi: z \to A_z$ sending $z \in M$ to the axis of σ_z answer one of the following descriptions.*

i) *π is not a polarity, and*

ia) *up to duality, Σ is one of the affine groups* (G iii) *through* (G vii) *of* § 2, *or*

ib) *$M = P$ and $\Sigma =$* $\mathrm{PSL}_3\mathbf{F}$.

ii) *π is a polarity and $M = P$. If π is discontinuous then* $\mathbf{F} = \mathbf{C}$ *and* $\Sigma = \mathrm{PSL}_3\mathbf{C}$. *If π is continuous then Σ is the elliptic motion group* $\mathrm{PSO}_3\mathbf{R}$ ($\mathbf{F} = \mathbf{R}$) *or* $\mathrm{PSU}_3\mathbf{C}$ ($\mathbf{F} = \mathbf{C}$).

PROOF. 1) Consider the map π. If π is injective and each reflection σ_z commutes with the reflections σ_a at all points $a \in A_z$ then M is projective and π defines a polarity of M. Indeed, Σ cannot fix ∞, or π would be constant on each line x^π; furthermore, the conditions $x \in A_y$ and $y \in A_x$ are equivalent for each pair $x, y \in M$, and $\pi : M \to \mathscr{L}$ is surjective since $x = z^\pi \cap w^\pi$ is mapped to $L = z \cup w$.

If π is continuous then [18], 3.24b shows that the map $z \to \sigma_z$ is continuous as well. By 1.3, Σ is transitive on P. (Applying 1.3 again to the action of Σ_x on $\mathscr{L}_x$ one can see that Σ is even flag transitive.) By 0.3, the group Σ centralizes π. Hence, it contains a unique reflection at each point (0.2), and Theorem A of [16] shows that P is a symmetric plane. According to the classification [16, 17], P is desarguesian and Σ is the elliptic motion group. This follows also from Salzmann's results on homogeneous (or flag homogeneous) projective planes, see [25], 5.1 and [27], 5.4.

If π is not continuous then $\dim M = 4$; see [24]. A beautiful theorem of Salzmann [26] asserts that π then is extremely discontinuous; in fact, the image of each open set in P is dense in $\mathscr{L}$. This implies that the group Σ contains reflections with centres and axes arbitrarily close to any given pair $(p, L) \in P \times \mathscr{L}$ with $p \notin L$. By [18], 3.24b, it contains also a reflection at p with axis L. Therefore, P admits all possible elations ([23], p. 213; cf. 0.1 and 1.4) and Σ is the little projective group. P is desarguesian by [25], 7.27.

2) If Σ fixes a line W then our assertion has been proved (4.1). Assume next that the hypotheses of 4.2 or 4.3 are satisfied. If the assertion (ii) of 4.2 holds then nothing is left to prove. In case (i), the group Σ and the plane P remain to be identified. By dualizing we obtain a projective translation plane E with axis W, and on the affine plane $E \setminus W$ we get a preassigned reflection σ_L at each line $L \neq W$. We use an indirect approach to determine the closed group $\Sigma^* \cong \Sigma$ generated by these reflections. First, we add to our set of generators the set of all affine reflections at points of $E \setminus W$. Note that these exist in each locally compact translation plane of positive dimension. Later, we show that the resulting group $\Sigma^\# \geqq \Sigma^*$ actually coincides with Σ^*.

3) $\Sigma^\#$ satisfies the hypotheses of 2.1; indeed, multiplying σ_L by the affine reflection at a point $o \in L$ we get a reflection at $L \cap W$. Consequently, P is desarguesian. $\Sigma^\#$ must be one of groups (G iii) through (G vii), since (G i) and (G ii) do not contain reflections at all points of W.

4) Let $\Sigma^\#$ be one of (G iii), (G v), (G vi). Then $\Sigma^\#$ contains a unique reflection $\sigma = \sigma_L$ at each affine line L, and σ must be contained in Σ^*. Let $\sigma_o \in \Sigma^\#$ be the reflection at $o \in L$. Then $\sigma_L \cdot \sigma_o$ is a reflection at a line. Σ^* contains $\sigma_L \cdot \sigma_o$ and $\sigma_o = \sigma_L \cdot (\sigma_L \cdot \sigma_o)$. Thus, Σ^* contains all generators of $\Sigma^\#$.

5) Let $\Sigma^\#$ be (G vii). Then $\Sigma^\#$ and Σ^* act on W doubly transitively. Hence, Σ^* contains all possible reflections at affine lines; they generate $\Sigma^\#$.

6) Let $\Sigma^\#$ be (G iv), and let v be the fixed point of $\Sigma^\#$ on W. Let w be the centre of the reflection σ_L at an affine line in the direction of v. Then each reflection σ_K at a line in the direction of w fixes v and, hence, commutes with σ_L. Again, Σ^* contains the reflections at the affine points. This completes the proof of the theorem.

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