A Two-Parameter Family of Orthogonal Polynomials with Respect to a Jacobi-Type Weight on the Unit Circle

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A two-parameter family of polynomials is introduced by a recursion formula. The polynomials are orthogonal on the unit circle with respect to the weight \( w_{\alpha, \beta}(z) = |(1 - z)^\alpha (1 + z)^\beta|^2 \), \( \alpha, \beta > -\frac{1}{2}, z = e^{i\theta} \). Explicit representation, norm estimates, shift identities, and explicit connection to Jacobi polynomials on the real interval \([-1, 1]\) is presented. © 1999 Academic Press

1. INTRODUCTION

In his book, *Orthogonal Polynomials*, Szegö noted in the chapter on orthogonal polynomials on the unit circle that the polynomials orthogonal with respect to the weight

\[
\omega_{\alpha, \beta}(\theta) = |(1 - z)^\alpha (1 + z)^\beta|^2
\]

\[
= 2^{\alpha + \beta} (1 - \cos \theta)^\alpha (1 + \cos \theta)^\beta, \quad \alpha, \beta > -\frac{1}{2}, z = e^{i\theta}
\]

are connected to Jacobi polynomials on the real interval \([-1, 1]\), see [6, p. 288]. The author has in a licentiate thesis, see [3], established explicit representations of the polynomials \( \varphi_n^{\alpha, \beta}(z) \) orthogonal on the unit circle with respect to the weight (1.1). Also the explicit connection to Jacobi polynomials was established and some shift identities were derived using the shift identities for Jacobi polynomials. The \( L_\infty \) norm on the unit circle for the \( \varphi_n^{\alpha, \beta} \) polynomials, when at least one of the parameters are
nonnegative, was established. The proofs of many of the presented results are long, so those who are interested in the detailed proofs may consult the report [3], which can be provided by the author. Also, orthogonal expansion and approximation of transfer functions using \( \varphi^k_{n,0} \), \( k = 1, 2, \ldots \), polynomials has been investigated by the author, which is documented in [2, 3].

2. RECURSION FORMULAS, ALGEBRAIC PROPERTIES, AND ORTHOGONALITY

**Definition 2.1.** Define for real \( \alpha, \beta > -\frac{1}{2} \) the monic polynomials \( \varphi^{\alpha, \beta}_n(z) \) by the recursion formula

\[
\varphi^{\alpha, \beta}_{n+1}(z) = z \varphi^{\alpha, \beta}_n(z) + \frac{\alpha + (-1)^{n+1} \beta}{\alpha + \beta + n + 1} z^n \varphi^{\alpha, \beta}_n(z^{-1}), \quad n \geq 0, \quad (2.1)
\]

\[
\varphi^{\alpha, \beta}_0(z) = 1.
\]

Substituting \( z^{-1} \) for \( z \) in (2.1) and multiplying both sides with \( z^{n+1} \) gives us the recursion formula for the reciprocal polynomial \( z^n \varphi^{\alpha, \beta}_n(z^{-1}) \). So,

\[
z^{n+1} \varphi^{\alpha, \beta}_{n+1}(z^{-1}) = z^n \varphi^{\alpha, \beta}_n(z^{-1}) + \frac{\alpha + (-1)^{n+1} \beta}{\alpha + \beta + n + 1} z^n \varphi^{\alpha, \beta}_n(z), \quad n \geq 0, \quad (2.2)
\]

\[
\varphi^{\alpha, \beta}_0(z^{-1}) = 1.
\]

Note that Formula (2.1) in the cases \( \beta = 0, \alpha = 0, \alpha = \beta = 0, \alpha = \beta, \) and \( \beta = -\alpha \) is reduced to

\[
\varphi^{\alpha, 0}_{n+1}(z) = z \varphi^{\alpha, 0}_n(z) \quad \frac{\alpha}{\alpha + n + 1} z^n \varphi^{\alpha, 0}_n(z^{-1}),
\]

\[
\varphi^{0, \beta}_{n+1}(z) = z \varphi^{0, \beta}_n(z) \quad (-1)^{n+1} \frac{\beta}{\beta + n + 1} z^n \varphi^{0, \beta}_n(z^{-1}),
\]

\[
\varphi^{0, 0}_{n+1}(z) = z^{n+1},
\]

\[
\varphi^{\alpha, \alpha}_{n+1}(z) = z \varphi^{\alpha, \alpha}_n(z) \quad \frac{(1 + (-1)^{n+1}) \alpha}{2 \alpha + n + 1} z^n \varphi^{\alpha, \alpha}_n(z^{-1}),
\]

\[
\varphi^{\alpha, -\alpha}_{n+1}(z) = z \varphi^{\alpha, -\alpha}_n(z) \quad \frac{(1 - (-1)^{n+1}) \alpha}{n + 1} z^n \varphi^{\alpha, -\alpha}_n(z^{-1}).
\]
By denoting
\[ \varphi_n^{\alpha, \beta}(z) = \sum_{p=0}^{n} c_n^{\alpha, \beta}(p) z^p, \]  
(2.4)
\[ z^n \varphi_n^{\alpha, \beta}(z^{-1}) = \sum_{p=0}^{n} c_n^{\alpha, \beta}(n-p) z^p, \]

it follows directly from (2.1) and (2.4) that the coefficients of the \( \varphi_n^{\alpha, \beta}(z) \) polynomials satisfy the difference equation
\[
\begin{align*}
  c_n^{\alpha, \beta}(0) &= 1, \\
  c_n^{\alpha, \beta}(n) &= 1, \\
  c_n^{\alpha, \beta}(0) &= \frac{\alpha + (-1)^n \beta}{\alpha + \beta + n}, \quad n \geq 1, \\
  c_n^{\alpha, \beta}(p) &= c_{n-1}^{\alpha, \beta}(p-1) \\
  &\quad + c_n^{\alpha, \beta}(0)c_{n-1}^{\alpha, \beta}(n-p-1), \quad 1 \leq p \leq n-1, n \geq 2.
\end{align*}
\]  
(2.5)

**Definition 2.2.** The notation \([x]\) is used for the integer part of \( x \), that is the greatest integer less than or equal to \( x \). We let \( k! \) stand for \( \Gamma(k + 1) \), when \( k \neq -1, -2, -3, \ldots \).

The following definition is made for the sake of convenience.

**Definition 2.3.** Define \( l(\alpha, \beta, n) \) for \( n = 0, 1, 2, \ldots \), and \( \alpha, \beta > -\frac{1}{2} \) by
\[ l(\alpha, \beta, n) = 2\pi \frac{[n/2]!(2\alpha + n)!(2\beta + n)!(\alpha + \beta + [n/2])!}{((\alpha + \beta + n)!)^2(\alpha + [n/2])!(\beta + [n/2])!}. \]  
(2.6)

Five algebraic properties are grouped together into a proposition.

**Proposition 2.1.** For \( n = 0, 1, 2, \ldots \), and \( \alpha, \beta > -\frac{1}{2} \) the following identities hold
\[
\begin{align*}
  \varphi_n^{\alpha, \beta}(0) &= \frac{\alpha + (-1)^{n+1} \beta}{\alpha + \beta + n + 1}, \\
  \varphi_n^{\alpha, \beta}(1) &= \frac{\alpha!(2\alpha + n)!(\alpha + \beta + [n/2])!}{(2\alpha)!(\alpha + \beta + n)!(\alpha + [n/2])!}, \\
  \varphi_n^{\alpha, \beta}(-1) &= (-1)^n \frac{\beta!(2\beta + n)!(\alpha + \beta + [n/2])!}{(2\beta)!(\alpha + \beta + n)!(\beta + [n/2])!}.
\end{align*}
\]  
(2.7a, 2.7b, 2.7c)
The proof of the proposition is based on induction. Details can be found in [3].

We can obtain three-term recurrence formulas with or without reciprocal polynomials.

**Proposition 2.2.** For \( n \geq 0 \) and \( \alpha, \beta > -\frac{1}{2} \) the following three-term recursion formulas hold

\[
\varphi_{n+1}^{\alpha, \beta}(0) \varphi_{n+2}^{\alpha, \beta}(z) = \left( \varphi_{n+1}^{\alpha, \beta}(0) + \varphi_{n+2}^{\alpha, \beta}(0) \right) \varphi_{n+1}^{\alpha, \beta}(z) - \varphi_{n+2}^{\alpha, \beta}(0) \left[ 1 - \left( \varphi_{n+1}^{\alpha, \beta}(0) \right)^2 \right] z \varphi_{n+1}^{\alpha, \beta}(z),
\]

\[
\varphi_{0}^{\alpha, \beta}(z) = 1,
\]

\[
\varphi_{n+1}^{\alpha, \beta}(0) z^{n+2} \varphi_{n+2}^{\alpha, \beta}(z^{-1}) = \left( \varphi_{n+1}^{\alpha, \beta}(0) + \varphi_{n+2}^{\alpha, \beta}(0) z \right) z^{n+1} \varphi_{n+1}^{\alpha, \beta}(z^{-1}) - \varphi_{n+2}^{\alpha, \beta}(0) \left[ 1 - \left( \varphi_{n+1}^{\alpha, \beta}(0) \right)^2 \right] z^{n+1} \varphi_{n+1}^{\alpha, \beta}(z^{-1}),
\]

\[
\varphi_{0}^{\alpha, \beta}(z^{-1}) = 1.
\]

**Proof.** Multiply (2.1), where \( n \) is replaced by \( n + 1 \), with \( \varphi_{n+1}^{\alpha, \beta}(0) \) and make use of (2.2) and (2.1) to obtain (2.8). Substituting \( z^{-1} \) for \( z \) in (2.8) and multiplying both sides by \( z^{n+2} \) gives (2.9). The proof is complete.

**Definition 2.4.** We define on the interval \([0, 2\pi]\) the nonnegative weight function \( \omega_{\alpha, \beta}(\theta) \) by

\[
\omega_{\alpha, \beta}(\theta) = \left| (1 - e^{i\theta})^\alpha (1 + e^{i\theta})^\beta \right|^2
= 2^{\alpha+\beta} (1 - \cos(\theta))^\alpha (1 + \cos(\theta))^\beta, \quad \alpha, \beta > -\frac{1}{2}.
\]

The inner product \((f, g)\) for the functions \( f(z) \) and \( g(z) \) is defined by

\[
(f, g) = \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \omega_{\alpha, \beta}(\theta) d\theta,
\]

if the integral exists. The bar in (2.11) means complex conjugation.

**Theorem 2.1.** The polynomials \( \{\varphi_n^{\alpha, \beta}\}_{n=0}^{\infty} \) generated by recursion formula (2.1) are orthogonal polynomials with respect to the inner product (2.11) if
\[ \alpha, \beta > - \frac{1}{2}. \] Furthermore,

\[ \left( \varphi_n^{\alpha, \beta}, \varphi_n^{\alpha, \beta} \right) = l(\alpha, \beta, n). \] (2.12)

The proof of the theorem is presented in detail in [3]. First it is easy to show that \( \varphi_0^{\alpha, \beta}, \varphi_1^{\alpha, \beta} \) are orthogonal and that (2.12) holds for \( n = 0, 1 \). The induction hypothesis is that \( \varphi_0^{\alpha, \beta}, \varphi_1^{\alpha, \beta}, \ldots, \varphi_k^{\alpha, \beta} \) are orthogonal and that (2.12) holds for \( n = 0, 1, \ldots, k \). The unique monic polynomial \( P_{k+1}(z) \) orthogonal to \( \varphi_0^{\alpha, \beta}, \varphi_1^{\alpha, \beta}, \ldots, \varphi_k^{\alpha, \beta} \) can be written in the form \( P_{k+1}(z) = z \varphi_k^{\alpha, \beta}(z) + \sum_{j=0}^k c_j \varphi_j^{\alpha, \beta}(z) \). The coefficients \( c_j \) are then determined using the induction hypothesis, the algebraic properties of Proposition 2.1, the Christoffel–Darboux formula for orthogonal polynomials on the unit circle, see [1, p. 81], and the fact that \( P_{k+1}(z) \) is the unique polynomial \( \pi_{k+1} \) of the form \( z^{k+1} + a_k z^k + \cdots + a_0 \) that minimizes the inner product \( (\pi_{k+1}, \pi_{k+1}) \), see Szegö [5, p. 198]. When the \( c_j \)'s are determined we have \( P_{k+1}(z) = \varphi_{k+1}(z) \), and also (2.12) is satisfied for \( n = k + 1 \), which completes the proof.

3. NORMS AND EXPlict REPRESENTATIONS

We define the norms \( \| \cdot \|_2 \) and \( \| \cdot \|_{\infty} \) for \( f(z) \) by \( \| f \|_2 = \sqrt{\langle f, f \rangle} \) and \( \| f \|_{\infty} = \max_{|z| = 1} |f(z)| \), respectively. Note that \( \| \varphi_n^{\alpha, \beta} \|_2^2 = l(\alpha, \beta, n) \).

**Proposition 3.1.** If \( n \geq 0 \) and \( \alpha, \beta > - \frac{1}{2} \) then

\[ \| \varphi_n^{\alpha, \beta} \|_2^2 \leq 2\pi \frac{(2\alpha)!(2\beta)!}{(\alpha + \beta)!\alpha!\beta!}. \] (3.1)

**Proof.** The sequence \( \| \varphi_n^{\alpha, \beta} \|_2^2 \) is decreasing and it decreases strictly if \( \alpha \neq \pm \beta \). Using (2.6) we get

\[ \| \varphi_0^{\alpha, \beta} \|_2^2 = l(\alpha, \beta, 0) = 2\pi \frac{(2\alpha)!(2\beta)!}{(\alpha + \beta)!\alpha!\beta!}. \]

So the inequality (3.1) holds.

**Proposition 3.2.** If \( n \geq 0 \) and \( \alpha, \beta > - \frac{1}{2} \) then the following equalities hold

\[ \| \varphi_n^{\alpha, \beta} \|_{\infty} = \varphi_n^{\alpha, \beta}(1), \quad \alpha \geq \beta > - \frac{1}{2} \text{ and } \alpha \geq 0, \]

\[ \| \varphi_n^{\alpha, \beta} \|_{\infty} = (-1)^n \varphi_n^{\alpha, \beta}(-1), \quad \beta \geq \alpha > - \frac{1}{2} \text{ and } \beta \geq 0. \] (3.2)
Proof. Suppose that $\alpha \geq \beta > -\frac{1}{2}$ and $\alpha \geq 0$. We have $\|\varphi_{0}^{\alpha, \beta}\|_{\infty} = 1 = \varphi_{0}^{\alpha, \beta}(1)$. Suppose that $\|\varphi_{n}^{\alpha, \beta}\|_{\infty} = \varphi_{n}^{\alpha, \beta}(1)$ for fixed $n$, $n \geq 0$. By (2.1), the induction hypothesis and the conditions on $\alpha$ and $\beta$ we get

$$\|\varphi_{n+1}^{\alpha, \beta}\|_{\infty} = \max_{|z|=1} \left| z \varphi_{n}^{\alpha, \beta}(z) + \frac{\alpha + (-1)^{n+1} \beta}{\alpha + \beta + n + 1} z^{n} \varphi_{n}^{\alpha, \beta}(z^{-1}) \right| $$

$$\leq \max_{|z|=1} \left| z \varphi_{n}^{\alpha, \beta}(z) \right| + \max_{|z|=1} \left| \frac{\alpha + (-1)^{n+1} \beta}{\alpha + \beta + n + 1} z^{n} \varphi_{n}^{\alpha, \beta}(z^{-1}) \right| $$

$$= 1 \cdot \varphi_{n}^{\alpha, \beta}(1) + \frac{\alpha + (-1)^{n+1} \beta}{\alpha + \beta + n + 1} \cdot 1^{n} \varphi_{n}^{\alpha, \beta}(1^{-1}) $$

$$= \varphi_{n+1}^{\alpha, \beta}(1). \quad (3.3)$$

So $\|\varphi_{n+1}^{\alpha, \beta}\|_{\infty} = \varphi_{n+1}^{\alpha, \beta}(1)$. By induction the first identity in (3.2) holds. Suppose now that $\beta \geq \alpha > -\frac{1}{2}$ and $\beta \geq 0$. We have $\|\varphi_{0}^{\alpha, \beta}\|_{\infty} = 1 = (-1)^{0} \varphi_{0}^{\alpha, \beta}(-1)$. Suppose that $\|\varphi_{n}^{\alpha, \beta}\|_{\infty} = (-1)^{n} \varphi_{n}^{\alpha, \beta}(-1)$ for fixed $n$, $n \geq 0$. By (2.1), the induction hypothesis and the conditions on $\alpha$ and $\beta$ we get

$$\|\varphi_{n+1}^{\alpha, \beta}\|_{\infty} = \max_{|z|=1} \left| z \varphi_{n}^{\alpha, \beta}(z) + \frac{\alpha + (-1)^{n+1} \beta}{\alpha + \beta + n + 1} z^{n} \varphi_{n}^{\alpha, \beta}(z^{-1}) \right| $$

$$\leq \max_{|z|=1} \left| z \varphi_{n}^{\alpha, \beta}(z) \right| $$

$$+ (-1)^{n+1} \frac{\alpha + (-1)^{n+1} \beta}{\alpha + \beta + n + 1} \max_{|z|=1} \left| z^{n} \varphi_{n}^{\alpha, \beta}(z^{-1}) \right| $$

$$= (-1)^{n} \varphi_{n}^{\alpha, \beta}(-1) $$

$$+ (-1)^{n+1} \frac{\alpha + (-1)^{n+1} \beta}{\alpha + \beta + n + 1} (-1)^{n} \varphi_{n}^{\alpha, \beta}((-1)^{-1}) $$

$$= (-1)^{n+1} \left((-1)^{n} \varphi_{n}^{\alpha, \beta}(-1) $$

$$+ \frac{\alpha + (-1)^{n+1} \beta}{\alpha + \beta + n + 1} (-1)^{n} \varphi_{n}^{\alpha, \beta}((-1)^{-1}) \right) $$

$$= (-1)^{n+1} \varphi_{n+1}^{\alpha, \beta}(-1). \quad (3.4)$$

Then $\|\varphi_{n+1}^{\alpha, \beta}\|_{\infty} = (-1)^{n+1} \varphi_{n+1}^{\alpha, \beta}(-1) = |\varphi_{n+1}^{\alpha, \beta}(-1)|$. By induction the second identity in (3.2) is proved. The proof is completed.
In the cases $\alpha < 0$ and $\beta < 0$ it seems hard to determine the supremum norm. Compare this with the Jacobi polynomials, where $\max_{-1 \leq \alpha, \beta \leq 1} |P_n^{\alpha, \beta}(x)|$ can be determined easily if $\max(\alpha, \beta) \geq -\frac{1}{2}$, see Szegő [6, p. 163].

**Definition 3.1.** We define binomial coefficients for real $a, b$ using the gamma function. Then,

$$\binom{a}{b} := \frac{\Gamma(a + 1)}{\Gamma(b + 1)\Gamma(a - b + 1)} = \frac{a!}{b!(a - b)!},$$

$$a + 1, b + 1, a - b + 1 \neq 0, -1, -2, \ldots. \quad (3.5)$$

Explicit representations for the polynomials can be obtained in some special cases, which are demonstrated in the next proposition. In Section 4 we establish the connection to Jacobi polynomials which make it possible to obtain a representation for all $\alpha, \beta > -\frac{1}{2}$ using the representation formula for Jacobi polynomials.

**Proposition 3.3.** If $n \geq 0$ and $\alpha, \beta > -\frac{1}{2}$ then we have the following explicit representations. So,

$$\varphi_n^{0,0}(z) = z^n, \quad (3.6a)$$

$$\varphi_n^{\alpha,0}(z) = \frac{n!}{(n + \alpha)!} \sum_{j=0}^{n} \binom{j + \alpha}{j} \binom{n + \alpha - j - 1}{n - j} z^j, \quad \alpha \neq 0, \quad (3.6b)$$

$$\varphi_n^{0,\beta}(z) = \frac{n!}{(n + \beta)!} \sum_{j=0}^{n} (-1)^{n+j} \binom{j + \beta}{j} \binom{n + \beta - j - 1}{n - j} z^j, \quad \beta \neq 0, \quad (3.6c)$$

$$\varphi_{2n}^{\alpha,0}(z) = \frac{n!}{(n + \alpha)!} \sum_{j=0}^{n} \binom{j + \alpha}{j} \binom{n + \alpha - j - 1}{n - j} z^{2j} = \varphi_n^{\alpha,0}(z^2), \quad \alpha \neq 0, \quad (3.6d)$$

$$\varphi_{2n+1}^{\alpha,0}(z) = \frac{n!}{(n + \alpha)!} \sum_{j=0}^{n} \binom{j + \alpha}{j} \binom{n + \alpha - j - 1}{n - j} z^{2j+1}$$

$$= z \varphi_n^{\alpha,0}(z^2) = z \varphi_{2n}^{\alpha,0}(z), \quad \alpha \neq 0. \quad (3.6e)$$

**Proof.** It can be verified that the coefficients of the polynomials satisfy the recursion formula (2.1) and the difference equation (2.5). For details see [3].
4. THE CONNECTION TO JACOBI POLYNOMIALS

**Definition 4.1.** Jacobi polynomials, $P_n^{(\alpha, \beta)}(x)$, $\alpha, \beta > -1$, are the orthogonal polynomials of a real variable on the real interval $[-1, 1]$ that satisfy the orthogonality property

$$\int_{-1}^{1} P_n^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(x) (1-x)^\alpha (1+x)^\beta \, dx = h_n^{(\alpha, \beta)} \delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker delta, and where the product $(2n + \alpha + \beta + 1)(n + \alpha + \beta)!$ in the denominator of $h_n^{(\alpha, \beta)}$ must be changed to $(\alpha + \beta + 1)!$ if $n = 0$, see Szegö [6, p. 67].

We are going to use results from Szegö [6, Chap. XI], where the theory of orthogonal polynomials on the unit circle is presented. Szegö defines the inner product by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \omega(\theta) \, d\theta. \quad (4.2)$$

By Szegö [6, Theorem 11.5] the following holds: let $v(x)$ be a weight function on the interval $-1 \leq x \leq 1$, and let

$$\omega(\theta) = v(\cos(\theta))|\sin(\theta)|. \quad (4.3)$$

Furthermore let $\{p_n(x)\}$ and $\{q_n(x)\}$ be orthonormal polynomials with respect to the weights $v(x)$ and $(1-x^2)v(x)$, respectively, on the interval $-1 \leq x \leq 1$, and let $\{\phi_n(z)\}$ be orthonormal polynomials with respect to the weight $\omega(\theta)$, $z = e^{i\theta}$, on the unit circle. Then using the notation $x = \frac{1}{2}(z + z^{-1})$, it holds for $n \geq 1$ that

$$p_n(x) = \frac{1}{\sqrt{2\pi}} \left(1 + \frac{\phi_{2n}(0)}{k_{2n}}\right)^{-1/2} \left(z^{-n}\phi_{2n}(z) + z^n\phi_{2n}(z^{-1})\right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(1 - \frac{\phi_{2n}(0)}{k_{2n}}\right)^{-1/2} \left(z^{-n+1}\phi_{2n-1}(z) + z^{n-1}\phi_{2n-1}(z^{-1})\right),$$
\[ q_n(x) = \sqrt{\frac{2}{\pi}} \left(1 - \frac{\phi_{2n+2}(0)}{k_{2n+2}} \right)^{-1/2} \times \frac{z^{-n-1}\phi_{2n+2}(z) - z^{n+1}\phi_{2n+2}(z^{-1})}{z - z^{-1}} \]

\[ = \sqrt{\frac{2}{\pi}} \left(1 + \frac{\phi_{2n+2}(0)}{k_{2n+2}} \right)^{-1/2} \frac{z^{-n}\phi_{2n+1}(z) - z^n\phi_{2n+1}(z^{-1})}{z - z^{-1}}, \]

where \( k_n \) denotes the leading coefficient of \( \phi_n(z) \). The formulas in (4.4), except the second one, hold for \( n = 0 \).

Let \( \nu(x) \) be the Jacobi weight that is given by

\[ \nu(x) = 2^{\alpha+\beta} (1 - x)^{\alpha-1/2} (1 + x)^{\beta-1/2}, \quad \alpha, \beta > -\frac{1}{2}, \quad -1 \leq x \leq 1. \]  

(4.5)

Then using (4.3) we get the corresponding weight \( \omega(\theta) \) on the unit circle. So

\[ \omega(\theta) = 2^{\alpha+\beta} (1 - \cos(\theta))^{\alpha-1/2} (1 + \cos(\theta))^{\beta-1/2} |\sin(\theta)| \]

\[ = 2^{\alpha+\beta} (1 - \cos(\theta))^{\alpha-1/2} (1 + \cos(\theta))^{\beta-1/2} (1 - \cos^2(\theta))^{1/2} \]

\[ = 2^{\alpha+\beta} (1 - \cos(\theta))^{\alpha} (1 + \cos(\theta))^{\beta} \]

\[ = \omega_{\alpha, \beta}(\theta), \quad \alpha, \beta > -\frac{1}{2}. \]  

(4.6)

Using (4.1), (4.5), and the fact that the orthonormal polynomials \( p_n(x) \) and \( q_n(x) \) correspond to the weights \( \nu(x) \) and \( (1 - x^2)\nu(x) \), respectively, we get for \( n \geq 0 \) that

\[ p_n(x) = (2^{\alpha+\beta} h_n^{(\alpha-1/2, \beta-1/2)})^{-1/2} \times P_n^{(\alpha-1/2, \beta-1/2)}(x), \quad \alpha, \beta > -\frac{1}{2}, \]  

(4.7)

\[ q_n(x) = (2^{\alpha+\beta} h_n^{(\alpha+1/2, \beta+1/2)})^{-1/2} \times P_n^{(\alpha+1/2, \beta+1/2)}(x), \quad \alpha, \beta > -\frac{1}{2}. \]
By (2.6), (2.11), and (4.2) we get for $n \geq 0$ that

$$\phi_n(z) = \left(\frac{2\pi}{l(\alpha, \beta, n)}\right)^{1/2} \varphi_n^{\alpha, \beta}(z), \quad \alpha, \beta > -\frac{1}{2},$$

$$k_n = \left(\frac{2\pi}{l(\alpha, \beta, n)}\right)^{1/2}, \quad \alpha, \beta > -\frac{1}{2}. \quad (4.8)$$

The following theorem gives the connection between the Jacobi polynomials and the $\varphi_n^{\alpha, \beta}$ polynomials.

**Theorem 4.1.** If $\alpha, \beta > -\frac{1}{2}$ and $x = \frac{1}{2}(z + z^{-1})$ then the following holds

$$z^n p_n^{(\alpha-1/2, \beta-1/2)}(x)$$

$$= 2^{-(2n+1)} \binom{2n + \alpha + \beta}{n} \left(\varphi_n^{\alpha, \beta}(z) + z^{2n} \varphi_n^{\alpha, \beta}(z^{-1})\right), \quad n \geq 0,$$

$$= 2^{-2n} \binom{2n + \alpha + \beta - 1}{n} \times \left(\varphi_{2n-1}^{\alpha, \beta}(z) + z^{2n-1} \varphi_{2n-1}^{\alpha, \beta}(z^{-1})\right), \quad n \geq 1,$$

$$(z^2 - 1) z^n p_n^{(\alpha+1/2, \beta+1/2)}(x)$$

$$= 2^{-(2n+1)} \binom{2n + \alpha + \beta + 2}{n + 1} \times \left(\varphi_{2n+2}^{\alpha, \beta}(z) - z^{2n+2} \varphi_{2n+2}^{\alpha, \beta}(z^{-1})\right), \quad n \geq 0,$$

$$= 2^{-2n} \binom{2n + \alpha + \beta + 1}{n} \times \left(\varphi_{2n+1}^{\alpha, \beta}(z) - z^{2n+1} \varphi_{2n+1}^{\alpha, \beta}(z^{-1})\right), \quad n \geq 0. \quad (4.9)$$

**Proof.** We make use of (4.1), (4.4), (4.7), (4.8), (2.6), and (2.7a) to prove (4.9). The first identity is obtained by the following calculations

$$z^n p_n^{(\alpha-1/2, \beta-1/2)}(x)$$

$$= \left(2^{\alpha + \beta} h_n^{(\alpha-1/2, \beta-1/2)}\right)^{1/2} \times \frac{1}{\sqrt{2\pi}} \left(1 + \frac{\phi_{2n}(0)}{k_{2n}}\right)^{-1/2} \left(\phi_{2n}(z) + z^{2n} \phi_{2n}(z^{-1})\right)$$
\[
\left(2^{\alpha+\beta} \frac{2^{\alpha+\beta}(n + \alpha - \frac{1}{2})!(n + \beta - \frac{1}{2})!}{(2n + \alpha + \beta) n!(n + \alpha + \beta - 1)!}\right)^{1/2} \\
\times \frac{1}{\sqrt{2\pi}} (1 + \varphi_{2n}^{\alpha, \beta}(0))^{-1/2} \left(\frac{2\pi}{l(\alpha, \beta, 2n)}\right)^{1/2} \\
\times \left(\varphi_{2n}^{\alpha, \beta}(z) + z^{2n} \varphi_{2n}^{\alpha, \beta}(z^{-1})\right) \\
= \frac{2^{\alpha+\beta-1/2}}{\sqrt{\pi}} \left(\frac{\sqrt{\pi} (2(n + \alpha))! \sqrt{\pi} (2(n + \beta))!}{2^{2(n+\alpha)}(n + \alpha)! 2^{2(n+\beta)}(n + \beta)!} \right)^{1/2} \\
\times \left(\frac{\alpha + \beta + 2n}{2(\alpha + \beta + n)}\right)^{1/2} \left(\frac{(\alpha + \beta + 2n)! (n + \alpha)!(n + \beta)!}{n!(2n+2\alpha)!(2n+2\beta)! (\alpha + \beta + n)!}\right)^{1/2} \\
\times \left(\varphi_{2n}^{\alpha, \beta}(z) + z^{2n} \varphi_{2n}^{\alpha, \beta}(z^{-1})\right) \\
= 2^{-(2n+1)} \binom{2n + \alpha + \beta}{n} \left(\varphi_{2n}^{\alpha, \beta}(z) + z^{2n} \varphi_{2n}^{\alpha, \beta}(z^{-1})\right). \tag{4.10}
\]

The three remaining identities are proved in a similar manner, see [3]. Then the proof is completed.

**Theorem 4.2.** If \( n \geq 1, \ \alpha, \beta > -\frac{1}{2} \), and \( x = \frac{1}{2}(z + z^{-1}) \) then the following holds

\[
\varphi_{2n}^{\alpha, \beta}(z) = 2^{2n} \binom{2n + \alpha + \beta}{n}^{-1} \\
\times z^{n-1} \left( zP_{n}^{(\alpha-1/2, \beta-1/2)}(x) + z^{2} - 1 \right) \frac{1}{4} P_{n-1}^{(\alpha+1/2, \beta+1/2)}(x),
\]

\[
\varphi_{2n-1}^{\alpha, \beta}(z) = 2^{2n-1} \binom{2n + \alpha + \beta - 1}{n}^{-1} \\
\times z^{n-2} \left( zP_{n}^{(\alpha-1/2, \beta-1/2)}(x) + \frac{n + \alpha + \beta}{n} \right) \frac{1}{4} P_{n-1}^{(\alpha+1/2, \beta+1/2)}(x), \tag{4.11}
\]
\[ z^n P_n^{(\alpha-1/2, \beta-1/2)}(x) \]
\[ = 2^{-2n} \frac{n + \alpha + \beta}{\alpha + \beta} \binom{2n + \alpha + \beta}{n} \]
\[ \times \left( \varphi_{2n}^{\alpha, \beta}(z) - \frac{2n}{2n + \alpha + \beta} z \varphi_{2n-1}^{\alpha, \beta}(z) \right), \]

\[ (z^2 - 1) z^n P_{n-1}^{(\alpha+1/2, \beta+1/2)}(x) \]
\[ = -2^{-(2n-2)} \frac{n}{\alpha + \beta} \binom{2n + \alpha + \beta}{n} \]
\[ \times \left( \varphi_{2n}^{\alpha, \beta}(z) - \frac{2(n + \alpha + \beta)}{2n + \alpha + \beta} z \varphi_{2n-1}^{\alpha, \beta}(z) \right), \]

where \( \alpha \neq -\beta \) in the last two identities.

**Proof.** The first identity in (4.9) and the third identity in (4.9) with \( n \) exchanged with \( n - 1 \) gives the following system of equations,

\[ z^n P_n^{(\alpha-1/2, \beta-1/2)}(x) \]
\[ = 2^{-(2n+1)} \binom{2n + \alpha + \beta}{n} \left( \varphi_{2n}^{\alpha, \beta}(z) + z^{2n} \varphi_{2n}^{\alpha, \beta}(z^{-1}) \right), \]

\[ (z^2 - 1) z^n P_{n-1}^{(\alpha+1/2, \beta+1/2)}(x) \]  \hspace{1cm} (4.12)
\[ = 2^{-(2n-1)} \binom{2n + \alpha + \beta}{n} \left( \varphi_{2n}^{\alpha, \beta}(z) - z^{2n} \varphi_{2n}^{\alpha, \beta}(z^{-1}) \right), \]

from which the first identity in (4.11) is obtained. The second identity in (4.9) and the fourth identity in (4.9) with \( n \) exchanged with \( n - 1 \) gives the following system of equations

\[ z^n P_n^{(\alpha-1/2, \beta-1/2)}(x) \]
\[ = 2^{-2n} \binom{2n + \alpha + \beta - 1}{n} \left( z \varphi_{2n-1}^{\alpha, \beta}(z) + z^{2n-1} \varphi_{2n-1}^{\alpha, \beta}(z^{-1}) \right), \]

\[ (z^2 - 1) z^n P_{n-1}^{(\alpha+1/2, \beta+1/2)}(x) \]  \hspace{1cm} (4.13)
\[ = 2^{-(2n-2)} \binom{2n + \alpha + \beta - 1}{n-1} \left( z \varphi_{2n-1}^{\alpha, \beta}(z) - z^{2n-1} \varphi_{2n-1}^{\alpha, \beta}(z^{-1}) \right). \]

From (4.13) the second identity in (4.11) is obtained. The system of equations which is given by the first two identities in (4.11), where we require that \( \alpha \neq -\beta \), gives the last two identities in (4.11). The proposition is proved.
5. EXPLICIT REPRESENTATION AND SHIFT IDENTITIES

In Proposition 3.3 we presented explicit representations for the $\varphi_n^{\alpha, \beta}$ polynomial in the cases $\alpha = \beta = 0$, $\beta = 0$, $\alpha = 0$, and $\alpha = \beta$. Now we use the representation formula for the Jacobi polynomials, see Szegö [6, p. 67], and obtain a representation formula for the $\varphi_n^{\alpha, \beta}$ polynomials.

**Proposition 5.1.** If $n \geq 1$ and $\alpha, \beta > -\frac{1}{2}$ then the following formulas hold

$$P_n^{(\alpha, \beta)}(x) = \sum_{j=0}^{n} \binom{n+\alpha}{n-j} \binom{n+\beta}{j} \left( \frac{x-1}{2} \right)^j \left( \frac{x+1}{2} \right)^{n-j}, \quad \alpha, \beta > -1,$$

(5.1)

see Szegö [6, p. 67], and obtain a representation formula for the $\varphi_n^{\alpha, \beta}$ polynomials.

$$\varphi_{2n}^{\alpha, \beta}(z) = \left( \frac{2n + \alpha + \beta}{n} \right)^{-1} \times \left( \sum_{j=0}^{n} \binom{n+\alpha-\frac{1}{2}}{n-j} \binom{n+\beta-\frac{1}{2}}{j} (z-1)^{2j} (z+1)^{2(n-j)} \right) + (z^2 - 1) \sum_{j=0}^{n-1} \binom{n+\alpha-\frac{1}{2}}{n-j-1} \binom{n+\beta-\frac{1}{2}}{j} (z-1)^{2j} (z+1)^{2(n-j-1)}.$$  

(5.2)

$$2z \varphi_{2n-1}^{\alpha, \beta}(z) = \left( \frac{2n + \alpha + \beta - 1}{n} \right)^{-1} \times \left( \sum_{j=0}^{n} \binom{n+\alpha-\frac{1}{2}}{n-j} \binom{n+\beta-\frac{1}{2}}{j} \right) \times (z-1)^{2j} (z+1)^{2(n-j)} + \frac{n + \alpha + \beta}{n} (z^2 - 1) \sum_{j=0}^{n-1} \binom{n+\alpha-\frac{1}{2}}{n-j-1} \binom{n+\beta-\frac{1}{2}}{j} (z-1)^{2j} (z+1)^{2(n-j-1)}.$$  

(5.3)
Proof. Let \( x = \frac{1}{2}(z + z^{-1}) \). Then (5.1) can be put in

\[
P_n^{(\alpha, \beta)}(x) = \sum_{j=0}^{n} \binom{n + \alpha}{n - j} \binom{n + \beta}{j} \left( \frac{x - 1}{2} \right)^j \left( \frac{x + 1}{2} \right)^{n-j}.
\]

Then (5.1) can be put in

\[
P_n^{(\alpha, \beta)}(x) = \sum_{j=0}^{n} \binom{n + \alpha}{n - j} \binom{n + \beta}{j} \left( \frac{z - 1}{4z} \right)^j \left( \frac{z + 1}{4z} \right)^{n-j}.
\]

If we combine the last identity in (5.3), where \( \alpha, \beta \) are shifted with \( \pm \frac{1}{2} \), with the two first identities in (4.11), then we get (5.2).

The connection to Jacobi polynomials make it possible to derive identities involving \( \varphi_n^{\alpha, \beta} \) polynomials with shifted parameters. Here we present identities without proofs. The reader may consult [3] for proofs and further shift identities.

First a symmetry property is derived. Jacobi polynomials obey

\[
P_n^{(\alpha, \beta)}(x) = (-1)^n P_n^{(\beta, \alpha)}(-x),
\]

see Szegö [6, p. 58], or Gradshteyn [4, p. 1035].

**Proposition 5.2.** If \( n \geq 0 \) and \( \alpha, \beta > -\frac{1}{2} \) then the following symmetry property holds

\[
\varphi_n^{x, \beta}(z) = (-1)^n \varphi_n^{\beta, \alpha}(-z).
\]

**Proof.** See [3].

The following holds for Jacobi polynomials.

\[
P_n^{(\alpha, \beta-1)}(x) - P_n^{(\alpha-1, \beta)}(x) = P_n^{(\alpha, \beta)}(x), \quad \alpha, \beta > 0
\]

see Gradshteyn [4, p. 1036]. We get analogous formulas for the \( \varphi_n^{\alpha, \beta} \) polynomials.

**Proposition 5.3.** If \( \alpha, \beta > \frac{1}{2} \) then the following identities hold

\[
\varphi_{2n+2}^{\alpha, \beta-1}(z) - \varphi_{2n+2}^{\alpha-1, \beta}(z) = \frac{4(n + 1)}{2n + \alpha + \beta + 1} z \varphi_{2n}^{\alpha, \beta}(z), \quad n \geq 0.
\]
\[
\phi_{2n+1}^{\alpha,\beta-1}(z) - \phi_{2n+1}^{\alpha-1,\beta}(z) = \frac{2}{\alpha + \beta} \phi_{2n}^{\alpha,\beta}(z) + \frac{4n(\alpha + \beta - 1)}{(\alpha + \beta)(2n + \alpha + \beta)} x z \phi_{2n-1}^{\alpha,\beta}(z), \quad n \geq 1,
\]

\[
\phi_1^{\alpha,\beta-1}(z) - \phi_1^{\alpha-1,\beta}(z) = \frac{2}{\alpha + \beta}.
\]

Proof. See [3].

We can also “go the other way” and obtain results for Jacobi polynomials starting from \(\phi_n^{\alpha,\beta}\) polynomials, which are demonstrated in the following proposition.

**Proposition 5.4.** If \(n \geq 2\), \(\alpha, \beta > -1\), \(\alpha + \beta \neq -1\), and \(x = \frac{1}{2}(z + z^{-1})\) then the following holds

\[
2z(n + \alpha)(n + \beta) \left( z P_{n-1}^{(\alpha,\beta)}(x) + \frac{z^2 - 1}{4} P_{n-2}^{(\alpha+1,\beta+1)}(x) \right)
= nz(2n + \alpha + \beta - (\alpha - \beta)z) P_n^{(\alpha,\beta)}(x)
+ (n + \alpha + \beta + 1) \frac{z^2 - 1}{4} (2n + \alpha + \beta + (\alpha - \beta)z)
\times P_{n-1}^{(\alpha+1,\beta+1)}(x).
\]

Proof. The recursion formula (2.8) gives

\[
\phi_{2n}^{\alpha+1/2,\beta+1/2}(0) \phi_{2n}^{\alpha+1/2,\beta+1/2}(z)
= (\phi_{2n-1}^{\alpha+1/2,\beta+1/2}(0) z + \phi_{2n}^{\alpha+1/2,\beta+1/2}(0)) \phi_{2n-1}^{\alpha+1/2,\beta+1/2}(z)
- \phi_{2n}^{\alpha+1/2,\beta+1/2}(0) \left( 1 - (\phi_{2n-1}^{\alpha+1/2,\beta+1/2}(0))^2 \right) z \phi_{2n-2}^{\alpha+1/2,\beta+1/2}(z).
\]
By using \((2.7a)\) and the two first identities in \((4.11)\) we obtain
\[
\frac{\alpha - \beta}{\alpha + \beta + 2n} 2^{2n} \left( 2n + \frac{\alpha + \frac{1}{2} + \beta + \frac{1}{2}}{n} \right)^{-1}
\times z^{n-1} \left( z P_n^{(\alpha, \beta)}(x) + \frac{z^2 - 1}{4} P_{n-1}^{(\alpha + 1, \beta + 1)}(x) \right)
= \left( \frac{\alpha - \beta}{2n + \alpha + \beta} + \frac{\alpha + \beta + 1}{\alpha + \beta + 2n + 1} \right) 2^{2n-1} \left( 2n + \frac{\alpha + \beta}{n} \right)^{-1}
\times z^{n-2} \left( z P_n^{(\alpha, \beta)}(x) + \frac{n + \alpha + \beta + 1}{n} \cdot \frac{z^2 - 1}{4} P_{n-1}^{(\alpha + 1, \beta + 1)}(x) \right)
- \frac{\alpha + \beta + 1}{\alpha + \beta + 2n + 1} \left( 1 - \left( \frac{\alpha - \beta}{\alpha + \beta + 2n} \right)^2 \right)
\times z^{2n-2} \left( 2n + \frac{\alpha + \beta - 1}{n - 1} \right)^{-1} z^{n-2}
\times \left( z P_{n-1}^{(\alpha, \beta)}(x) + \frac{z^2 - 1}{4} P_{n-2}^{(\alpha + 1, \beta + 1)}(x) \right).
\] \( (5.10) \)

If \(\alpha + \beta = -1\) then both sides in \((5.10)\) are identically equal, so we disregard this case. Then \((5.10)\) can be put in the form \((5.8)\), and the proof is complete.

We note some special cases of \((5.8)\). \(P_n^{(\alpha, \beta)}(x)\) corresponds to Chebyshev polynomials of the first kind, Chebyshev polynomials of the second kind, Legendre polynomials, and Gegenbauer polynomials if \(\alpha = \beta = -\frac{1}{2}, \alpha = \frac{1}{2}, \alpha = \beta = 0,\) and \(\alpha = \beta,\) respectively. The case \(\alpha = \beta = -\frac{1}{2}\) does not apply to \((5.9)\) because \(\alpha + \beta = -1.\) In the case \(\alpha = \beta = 0\) we get the following identity using \((5.8)\). So,
\[
z P_n^{(0,0)}(x) + \frac{n + 1}{n} \cdot \frac{z^2 - 1}{4} P_n^{(1,1)}(x) = z^2 P_{n-1}^{(0,0)}(x) + \frac{z^2 - 1}{4} z P_{n-2}^{(1,1)}(x).
\] \( (5.11) \)

The case \(\alpha = \beta = \frac{1}{2}\) gives us the identity
\[
z P_n^{(1/2,1/2)}(x) + \frac{n + 2}{n} \cdot \frac{z^2 - 1}{4} P_n^{(3/2,3/2)}(x)
= z \frac{n + \frac{1}{2}}{n} \left( z P_{n-1}^{(1/2,1/2)}(x) + \frac{z^2 - 1}{4} P_{n-2}^{(3/2,3/2)}(x) \right).
\] \( (5.12) \)
In the case $\alpha = \beta \neq -\frac{1}{2}$ we get

$$zP_n^{(\alpha, \alpha)}(x) + \frac{n + 2\alpha + 1}{n} \cdot \frac{z^2 - 1}{4} P_n^{(\alpha + 1, \alpha + 1)}(x)$$

$$= z \frac{n + \alpha}{n} \left(zP_{n-1}^{(\alpha, \alpha)}(x) + \frac{z^2 - 1}{4} P_{n-2}^{(\alpha + 1, \alpha + 1)}(x)\right). \quad (5.13)$$

6. CONCLUSION

We have studied a family of orthogonal polynomials on the unit circle associated with a weight of “Jacobi type.” Recursion formulas, explicit representations, some algebraic properties, and norm properties have been presented. An explicit connection to the Jacobi polynomials has been established with the aid of Szegő’s theory for orthogonal polynomials on the unit circle. Identities holding for Jacobi polynomials are then used to obtain analogous identities for the polynomials on the unit circle. We have not dealt with asymptotic properties, orthogonal expansion, or norm estimates on arcs of the unit circle. Perhaps there could be something of interest in those subjects too, considering the connection to Jacobi polynomials and possible consequences for them.

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REFERENCES