Poisson algebras in terms of non-associative algebras

Michel Goze*, Elisabeth Remm

Université de Haute Alsace, 68093 Mulhouse, France

Received 8 October 2007
Available online 13 February 2008
Communicated by E.I. Khukhro

Abstract

Poisson algebra is usually defined to be a commutative algebra together with a Lie bracket, and these operations are required to satisfy the Leibniz rule. We describe Poisson structures in terms of a single bilinear operation. This enables us to explore Poisson algebras in the realm of non-associative algebras. We study their algebraic and cohomological properties, their deformations as non-associative algebras, and settle the classification problem in low dimensions.

© 2008 Elsevier Inc. All rights reserved.

Keywords: Poisson algebras; Cohomology; Classification; Power associative algebras

1. Poisson algebras and non-associative algebras

Let \( \mathbb{K} \) be a commutative field of characteristic different from 2 and 3.

1.1. A bijective correspondence

A Poisson algebra over \( \mathbb{K} \) is a \( \mathbb{K} \)-vector space \( P \) equipped with two bilinear operations:

1. A Lie bracket, referred to as the Poisson bracket, usually denoted by \{,\}.
2. An associative commutative multiplication which we denote it by \( \cdot \).

* Corresponding author.
E-mail address: michel.goze@uha.fr (M. Goze).
These two operations are required to satisfy Leibniz condition:

\[
\{X \bullet Y, Z\} = X \bullet \{Y, Z\} + \{X, Z\} \bullet Y,
\]

for all \(X, Y, Z\) in \(g\). This condition means that, with respect to each of the two variables, the Poisson bracket behaves as a derivation relative to the multiplication. We denote a Poisson algebra by \((P, \{ , \}, \bullet)\).

Let \(\cdot : (X, Y) \to X \cdot Y\) be a bilinear map on the \(\mathbb{K}\)-vector space \(P\). The associator \(A\) of \(\cdot\) is the trilinear map on \(P\) given by

\[
A(X, Y, Z) = (X \cdot Y) \cdot Z - X \cdot (Y \cdot Z).
\]

Throughout the paper we do not assume algebras to be associative. Such an algebra is called a non-associative algebra (then a non-associative algebra can be associative).

**Proposition 1.** Let \((P, \cdot)\) be a \(\mathbb{K}\)-algebra. Define the \(P\)-valued operations \(\{ , \}\) and \(\bullet\) on \(P \times P\) by

\[
\{X, Y\} = \frac{1}{2} (X \cdot Y - Y \cdot X),
\]

\[
X \bullet Y = \frac{1}{2} (X \cdot Y + Y \cdot X).
\]

Then \((P, \{ , \}, \bullet)\) is a Poisson algebra if and only if the operation \(X \cdot Y\) satisfies the identity:

\[
3A(X, Y, Z) = (X \cdot Z) \cdot Y + (Y \cdot Z) \cdot X - (Y \cdot X) \cdot Z - (Z \cdot X) \cdot Y.
\]

**Proof.** See [11]. Let us note that the product \(X \cdot Y\) is Lie-admissible [4]. □

**Definition 2.** We call a non-associative \(\mathbb{K}\)-algebra \((P, \cdot)\) an algebra whose associator satisfies Eq. (4) an admissible Poisson algebra.

Let \((P, \cdot)\) and \((P, \star)\) be admissible Poisson algebras defining the same Poisson algebra \((P, \{ , \}, \bullet)\). Then

\[
X \cdot Y - Y \cdot X = X \star Y - Y \star X = 2\{X, Y\},
\]

\[
X \cdot Y + Y \cdot X = X \star Y + Y \star X = 2X \bullet Y
\]

and \(X \cdot Y = X \star Y\) because the characteristic of \(\mathbb{K}\) is not 2.

**Proposition 3.** Every Poisson algebra \((P, \{ , \}, \bullet)\) is associated to precisely one admissible Poisson algebra \((P, \cdot)\). That is we have a bijective correspondence between admissible Poisson algebras and ordinary Poisson algebras.

**Proof.** From Proposition 1 and the previous remark, we have a correspondence between Poisson algebras and admissible Poisson algebras. This is a one-to-one correspondence because if \((P, \{ , \}, \bullet)\) and \((P, \{ , \}_1, \bullet_1)\) give the same admissible Poisson algebra then
\[ X \cdot Y - Y \cdot X = 2\{X, Y\} = 2[X, Y]_1, \]
\[ X \cdot Y + Y \cdot X = 2 X \bullet Y = 2 X \bullet_1 Y. \]

From Proposition 1, this correspondence is also onto. \(\square\)

Given a Poisson algebra \((\mathcal{P}, \{\cdot\}, \bullet)\), we shall say that \((\mathcal{P}, \cdot)\) where

\[ X \cdot Y = \{X, Y\} + X \bullet Y \]

is the admissible Poisson algebra associated \((\mathcal{P}, \{\cdot\}, \bullet)\). The corresponding Lie algebra \((\mathcal{P}, \{\cdot\})\) will be denoted by \(g_\mathcal{P}\) and the associative commutative algebra \((\mathcal{P}, \bullet)\) by \(A_\mathcal{P}\).

**Notation.** We will call the product \(X \cdot Y\) the admissible Poisson product or the Poisson product (this is justified by Proposition 3) and we will denote (when no confusion is possible) the Poisson product by \(XY\) instead of \(X \cdot Y\).

**Proposition 4.** An admissible Poisson algebra \((\mathcal{P}, \cdot)\) is flexible, that is, the associator satisfies

\[ A(X, Y, X) = 0 \]

for every \(X, Y \in \mathcal{P}\).

**Proof.** From (4) we have

\[ 3A(X, Y, X) = X^2Y + (YX)X - (YX)X - X^2Y = 0 \]

where \(X^2 = XX\). Then \((\mathcal{P}, \cdot)\) is flexible. \(\square\)

We deduce easily that the associator of the multiplication \(-\) satisfies

\[ A(X, Y, Z) + A(Z, Y, X) = 0 \quad \text{(flexibility),} \]
\[ A(X, Y, Z) + A(Y, Z, X) = 0. \]

This last relation is obtained by writing identity (4) for the triples \((X, Y, Z)\), \((Y, Z, X)\) and \((Y, X, Z)\).

**Remark 5.** The system \{(5), (6)\} is equivalent to the equation

\[ 2A(X, Y, Z) + \frac{1}{2}A(Y, X, Z) + A(Z, Y, X) + A(Y, Z, X) + \frac{3}{2}A(Z, X, Y) = 0. \]

Indeed (5) + (6) implies (7). Conversely if (7) is satisfied, then (7) applied to the triple \((X, Y, X)\) gives

\[ 2A(X, Y, X) + A(Y, X, X) + A(X, X, Y) = 0 \]
and to the triple $(X, X, Y)$

$$5A(X, X, Y) + 5A(Y, X, X) + 2A(X, Y, X) = 0.$$ 

We deduce (5) and (6). It is worth noting that a non-associative algebra satisfying (7) is not always an admissible Poisson algebra.

**Proposition 6.** An admissible Poisson algebra $(P, \cdot)$ is a power associative algebra.

**Proof.** Recall that a non-associative algebra is power associative if every element generates an associative subalgebra. Let $X$ be in $(P, \cdot)$. We define the power of $X$ by $X^1 = X$, $X^{i+1} = X \cdot X^i$. We will prove that $X^{i+n} X^{j-n} = X^{i-p} X^p = X^{i+j}$ for all $i, j \geq 1$ and $1 \leq p \leq i, 1 \leq n \leq j$. Since $(P, \cdot)$ is flexible, we have $A(X, X^j, X) = 0$ for any $1 \leq j$. We have $X^j X = XX^j$ for $j = 1$. Suppose that this equation is true for $j$, then $A(X, X^j, X) = 0$ and $X^{j+1} X = X (X^j X) = X (XX^j) = XX^{j+1}$. So for any $j \geq 1, X^j X = XX^j$. Now we shall use induction over $i$ to prove that, for any $j \geq 1, X^i X^j = X^j X^i$. This identity is trivial for $i = 1$. Suppose that it is satisfied for $i \geq 1$. Then relation (4) gives

$$3A(X, X^i, X^j) - (XX^j) X^i - (X^i X^j) X + (X^i X) X^j + (X^j X) X^i = 0$$

and as $X^i X^j = X^j X^i$, we obtain

$$4X^{i+1} X^j = 3X(X^i X^j) + (X^i X^j) X.$$

Similarly, (4) applied to the triple $(X, X^j, X^i)$ gives

$$4X^{j+1} X^i = 3X(X^j X^i) + (X^j X^i) X.$$

By assumption $X^i X^j = X^j X^i$, we obtain $X^i X^{j+1} = X^j X^{i+1}$. By (4), this implies $A(X^i, X, X^j) = 0$. Thus,

$$X^{i+1} X^j = X^i X^{j+1} = X^j X^{i+1}$$

and $X^i X^j = X^j X^i$ for all $i, j$. Finally, we prove that for any $i$ the relation $X^{i-p} X^p = X^i$ is satisfied for any $1 \leq p < i$. It is evident for $i = 1$. Suppose that these relations are satisfied for a fixed $i$. Then

$$3A(X^{i-p}, X, X^p) = X^{p+1} X^{i-p} - X^{i-p+1} X^p$$

implies $X^{i-p+1} X^p = X^{i-p} X^{p+1}$ and

$$3A(X^{i-p}, X^p, X) = X^{p+1} X^{i-p} - X^{i+1}$$

implies $X^{i+1} = X^{p+1} X^{i-p}$. Thus $X^{i+1-p} X^p = X^{i+1}$ and the algebra $(P, \cdot)$ is power associative. □

**Remark 7.** Poisson algebras as $\mathbb{K}[\Sigma_3]$-associative algebras.
In [6], large classes of non-associative algebras were studied. In this section we show that admissible Poisson algebras belong to this category of algebras.

Let $\Sigma_3$ be the order three symmetric group and $K[\Sigma_3]$ its $K$-group algebra. A (non-associative) $K$-algebra $(A, \mu)$ is called a $K[\Sigma_3]$-associative algebra if there exists $v \in K[\Sigma_3]$, $v \neq 0$, such that

$$A_\mu \circ \Phi_v = 0,$$

where $A_\mu = \mu \circ (\mu \otimes \text{Id}) - \mu \circ (\text{Id} \otimes \mu)$ is the associator of the algebra $A$ and $\Phi_v : A^{\otimes 3} \to A^{\otimes 3}$ is defined by

$$\Phi_\sigma (v_1, v_2, v_3) = (v_{\sigma^{-1}(1)}, v_{\sigma^{-1}(2)}, v_{\sigma^{-1}(3)})$$

for all $\sigma \in \Sigma_3$.

Now suppose that $(P, \cdot)$ is an admissible Poisson algebra. From (4) we see that the associator of the multiplication satisfies

$$A_\mu \circ \Phi_v = 0$$

for $v_1 = \text{Id} - \tau_{12} + c_1$, where $\tau_{ij}$ interchanges elements $i$ and $j$ and $c_1(1, 2, 3) = (2, 3, 1)$. The flexibility identity (5) can be written as $A_\mu \circ \Phi_v = 0$ for $v_2 = \text{Id} + \tau_{13}$. Recalling the classification of [6], we deduce that any Poisson algebra is an algebra of type (IV$_1$) for $\alpha = -\frac{1}{2}$ (we have $v = 2\text{Id} + \frac{1}{2}\tau_{12} + \tau_{13} + c_1 + \frac{3}{2}c_2$ and $F_v$ is 4-dimensional).

1.2. Pierce decomposition

We say that a power associative algebra $P$ is a nilalgebra if any element $X$ is nilpotent, i.e.

$$\forall X \in P, \exists r \in \mathbb{N} \text{ such that } X^r = 0.$$ 

**Proposition 8.** Any finite-dimensional admissible Poisson algebra which is not a nilalgebra contains a non-zero idempotent element.

This is a consequence of the power associativity of a Poisson algebra.

Let $e$ be a non-zero idempotent, i.e. $e^2 = e$. Eq. (3) implies $e \bullet e = e$, thus $e$ is an idempotent of the associative algebra $AP$. The Leibniz identity implies

$$\{e, x\} = \{e \bullet e, x\} = 2e \bullet \{e, x\}.$$ 

Therefore, $\{e, x\}$ is either zero or an eigenvector of the operator

$$L_e^*: x \to e \bullet x$$

in $AP$ associated to the eigenvalue $\frac{1}{2}$. Since $e$ is an idempotent, the eigenvalues associated to $L_e^*$ are 1 or 0. It follows that $\{e, x\} = 0$ which implies that $e \in Z(gP)$ and $e \cdot x = e \bullet x = x \bullet e = x \cdot e$. 

Proposition 9. Let \((\mathcal{P}, \cdot)\) be an admissible Poisson algebra such that the center of the associated Lie algebra \(g\mathcal{P}\) is zero. Then \((\mathcal{P}, \cdot)\) has no idempotent different from zero. If \(\mathcal{P}\) is of finite dimension then it is a nilalgebra.

Suppose that there exists an idempotent \(e \neq 0\). Since \(\mathcal{P}\) is flexible, the operators \(L^*_e\) and \(R^*_e\) defined by \(L^*_e(x) = e \bullet x\) and \(R^*_e(x) = x \bullet e\) commute and \(L^*_e = L_e, R^*_e = R_e\). Then \(\mathcal{P}\) decomposes as

\[
\mathcal{P} = \mathcal{P}_{0,0} \oplus \mathcal{P}_{0,1} \oplus \mathcal{P}_{1,0} \oplus \mathcal{P}_{1,1}
\]

with \(\mathcal{P}_{i,j} = \{x_{i,j} \in \mathcal{P}\) such that \(ex_{i,j} = ix_{i,j}, x_{i,j}e = jx_{i,j}\), \(i, j \in \{0, 1\}\). From Proposition 8, \(e \in Z(g\mathcal{P})\). So \(\{e, x\} = 0\) for any \(x\), that is, \(ex = xe\) and \(\mathcal{P}_{0,1} = \mathcal{P}_{1,0} = \{0\}\).

Proposition 10. If the admissible Poisson algebra \((\mathcal{P}, \cdot)\) has a non-zero idempotent, it admits the Pierce decomposition

\[
\mathcal{P} = \mathcal{P}_{0,0} \oplus \mathcal{P}_{1,1},
\]

where \(\mathcal{P}_{0,0}\) and \(\mathcal{P}_{1,1}\) are admissible Poisson algebras with the induced product.

Proof. We have to show that \(\mathcal{P}_{0,0}\) and \(\mathcal{P}_{1,1}\) are Poisson subalgebras. Let \(x, y \in \mathcal{P}_{0,0}\), then \(ex = ey = xe = ye = 0\). From (4), we obtain

\[
\begin{align*}
-3e(xy) &= (xy)e, \\
0 &= (xy)e - (yx)e, \\
3(xy)e &= -(yx)e.
\end{align*}
\]

So \((xy)e = -3e(xy) = (yx)e = -3(xy)e\) and \((xy)e = e(xy) = 0\). Then \(xy \in \mathcal{P}_{0,0}\). Similarly if \(x, y \in \mathcal{P}_{1,1}\), then (4) applied to the triple \((e, x, y)\) gives \(xy = e(xy)\). The same equation applied to \((x, e, y)\) and \((x, y, e)\) gives

\[
\begin{align*}
(xy)e + yx - xy - (yx)e &= 0, \\
3(xy)e - 3xy - yx + (yx)e &= 0.
\end{align*}
\]

Thus, \(4(xy)e - 4xy = 0\) which means that \((xy)e = xy\) and \(\mathcal{P}_{1,1}\) is a Poisson subalgebra of \((\mathcal{P}, \cdot)\). \(\square\)

Remark 11. Poisson algebras are Lie-admissible power-associative algebras. In [9] Kosier gave examples of simple Lie-admissible power-associative finite-dimensional algebras called anti-flexible algebras. These algebras also have the property \(A = A_{00} \oplus A_{11}\) in every Pierce decomposition.

1.3. Pierce decomposition associated to orthogonal idempotents

Let \(e_1\) and \(e_2\) be non-zero orthogonal idempotents, \(e_1e_2 = e_2e_1 = 0\). Let \(\mathcal{P} = \mathcal{P}^1_{0,0} \oplus \mathcal{P}^1_{1,1} = \mathcal{P}^2_{0,0} \oplus \mathcal{P}^2_{1,1}\) be the corresponding Pierce decompositions. Let us suppose that \(x \in \mathcal{P}^1_{0,0}\). Applying (4) to the triples associated to the elements \([e_1, e_2, x]\), we obtain the condition

\[
(xe_2)e_1 = (e_2x)e_1 = e_1(e_2x) = e_1(xe_2) = 0
\]
for the elements $xe_2$ and $e_2x$ in $P^1_{0,0}$. In other words,

$$L_{e_2}(P^1_{0,0}) \subset P^1_{0,0}, \quad R_{e_2}(P^1_{0,0}) \subset P^1_{0,0},$$

where $L_{e_2}(x) = e_2x$ and $R_{e_2}x = xe_2$. So, $e_2$ is an idempotent of the Poisson algebra $(P^1_{0,0}, \cdot)$. Thus we have

$$P^1_{0,0} = P^2_{0,0} \cap P^1_{0,0} \oplus P^1_{0,0} \cap P^2_{1,1}.$$ 

Using the same reasonings, we can show that if $x \in P^1_{1,1}$ then, $e_2x = xe_2 = 0$ and

$$P^1_{1,1} \subset P^2_{0,0}.$$ 

Thus, $P^1_{0,0} = P^1_{0,0} \cap P^2_{0,0} \oplus P^2_{1,1}$. Observe that $P^1_{1,1}$ cannot be further decomposed using the spaces $P^2_{0,0}$ and $P^2_{1,1}$ associated to $e_2$ as we have

$$P^1_{1,1} = P^1_{1,1} \cap P^2_{0,0} \oplus P^1_{1,1} \cap P^1_{1,1}.$$ 

But $P^2_{1,1} \subset P^1_{0,0}$ so that $P^1_{1,1} \cap P^2_{1,1} = \{0\}$ and $P^1_{1,1} \cap P^2_{0,0} = P^1_{1,1}$. Then,

$$P = P^1_{0,0} \cap P^2_{0,0} \oplus P^1_{1,1} \oplus P^2_{1,1}.$$ 

**Proposition 12.** If $e_1$ and $e_2$ are non-zero orthogonal idempotents, then $P$ decomposes into a direct sum of Poisson subalgebras,

$$P = P^1_{0,0} \cap P^2_{0,0} \oplus P^1_{1,1} \oplus P^2_{1,1}.$$ 

Proposition 12 can be easily generalized to a family of orthogonal idempotents $\{e_1, \ldots, e_k\}$. The corresponding decomposition can then be written as

$$P = \bigcap_{i=1}^{k} P^i_{0,0} \bigoplus_{j=1}^{k} P^j_{1,1}.$$ 

### 1.4. Radical of a Poisson algebra

We already know that a Poisson algebra $(P, \cdot)$ is power associative. Recall that an element $x \in P$ is nilpotent if there is an integer $r$ such that $x^r = 0$. An algebra (two-sided ideal) consisting only of nilpotent elements is called a nilalgebra (nilideal). If $P$ is a finite-dimensional Poisson algebra, then there is a unique maximal nilalgebra $N(P)$ called the nilradical. Let $A_P$ be the commutative associative algebra associated to $(P, \cdot)$. Then, the Jacobson radical $J(A_P)$ of $A_P$ contains $N(P)$. Since $N(P)$ is a two-sided ideal of $(P, \cdot)$, it is also a Lie ideal of $g_P$. One can easily prove:

**Proposition 13.** The nilradical $N(P)$ of $(P, \cdot)$ coincides with the maximal Lie ideal of $g_P$ contained in $J(A_P)$. 

Remark 14. In the category of associative algebras, or more generally, of alternative algebras, any nilalgebra is nilpotent. This is no longer true in the category of Poisson algebras as the following example shows.

Let \((P, \cdot)\) be the 3-dimensional algebra defined by

\[
\begin{align*}
e_1^2 &= 0, \\
e_1 e_2 &= -e_2 e_1 = e_2, \\
e_1 e_3 &= -e_3 e_1 = -e_3, \\
e_2 e_3 &= -e_3 e_2 = e_1.
\end{align*}
\]

The corresponding algebra \(A_P\) is abelian and any element of \(P\) is nilpotent. The Poisson algebra \(P\) is a nilalgebra. But \(P^2 = P\) so \(P\) is not a nilpotent algebra. This algebra is an example of simple nilalgebra.

Remark 15. An element \(x \in P\) is properly nilpotent if it is nilpotent and \(xy\) and \(yx\) are nilpotent for any \(y \in P\). The Jacobson radical of \(A_P\) coincides with the set of properly nilpotent elements of \(A_P\). Let \(x\) be a properly nilpotent element of \(P\) and suppose that \(x/ \in N(P)\). We know that \(x \in J(A_P)\). By Proposition 13, there exists \(y \in P\) such that \(\{x, y\} \notin N(P)\). We have \(x \cdot y \in J(A_P)\). This implies that \(\{x, y\} \notin J(A_P)\), otherwise \(xy \in J(A_P)\) and \(N(P)\) would not be maximal. But \(x \in J(A_P)\), so \(xy\) is nilpotent and \(xy \in J(A_P)\). This is a contradiction and the nilradical coincides with the set of properly nilpotent elements. Zorn’s theorem concerning nilalgebra still holds in the framework of Poisson algebras.

Remark 16. We have seen that any finite-dimensional Poisson algebra which is not a nilalgebra contains a non-zero idempotent. An idempotent \(e\) is principal if there is no idempotent \(u\) orthogonal to \(e\) (i.e. \(ue = eu = 0\) with \(u^2 = u \neq 0\)). If \((P, \cdot)\) is not a nilalgebra, \(A_P\) is not a nilalgebra and it has a principal idempotent element. Let \(e\) be such an element. As \(e^2 = e \cdot e = e\), it is an idempotent element of \(P\). If one can find \(u\) such that \(u^2 = u \cdot u = u\) with \(ue = eu = 0\), then \(u \cdot e = e \cdot u = 0\) which is impossible. Therefore we have:

**Proposition 17.** Any finite-dimensional admissible Poisson algebra which is not a nilalgebra contains a principal idempotent element.

Remark 18. Let us assume that \(P\) is a unitary algebra. If \(x\) is an invertible element of \(P\), there exists \(x^{-1} \in P\) such that \(xx^{-1} = x^{-1}x = 1\). In particular \(x \cdot x^{-1} = x^{-1} \cdot x = 1\) and \(x^{-1}\) is the inverse of \(x\) in \(A_P\). Thus the inverse of an invertible element of \(P\) is unique. Let us note that if \(P\) is unitary, finite-dimensional and if the unit is the only idempotent element, any non-nilpotent element is invertible. Indeed, such an element \(x\) generates an associative algebra which admits an idempotent. Then \(1 \in P\), which turns out to be the only idempotent and can be expressed as

\[
1 = \sum \alpha_i x^i = x \left( \sum \alpha_i x^{i-1} \right).
\]

It follows that \(\sum \alpha_i x^{i-1}\) is the inverse of \(x\).
1.5. Simple Poisson algebras

An admissible Poisson algebra \((\mathcal{P}, \cdot)\) is simple if it has not some proper ideal and if \(\mathcal{P}^2 \neq \{0\}\). Let \(L_x\) and \(R_x\) be the left and right translations by \(x \in \mathcal{P}\). Let \(\mathcal{M}(\mathcal{P})\) be the associative subalgebra of \(\text{End}(\mathcal{P})\) generated by \(L_x, R_x\) for \(x \in \mathcal{P}\). In this algebra, we have the following relations

\[
\begin{cases}
L_x \cdot R_x = R_x \cdot L_x, \\
4L_{x^2} = 3(L_x)^2 - (R_x)^2 + 2R_x \cdot L_x, \\
4R_{x^2} = 3(R_x)^2 - (L_x)^2 + 2R_x \cdot L_x.
\end{cases}
\]

The algebra \(\mathcal{P}\) is simple if and only if \(\mathcal{P}\) is a non-trivial irreducible \(\mathcal{M}(\mathcal{P})\)-module.

One can consider the centralizer \(\tilde{C}\) of \(\mathcal{M}(\mathcal{P})\) in \(\text{End}(\mathcal{P})\). If \(\mathcal{P}\) is simple and if \(\tilde{C}\) is non-trivial, then \(\tilde{C}\) is a field which is a central simple Poisson algebra over itself.

Remark 19. We saw in Remark 14 that there are admissible Poisson algebras which are nilalgebras. In this case \(\mathcal{N}(\mathcal{P})\) is non-zero. We can consider the Albert radical \(\mathcal{R}(\mathcal{P})\) defined as the intersection of all maximal ideals \(\mathcal{M}\) of \(\mathcal{P}\) such that \(\mathcal{P}^2 \not\subseteq \mathcal{M}\). In the algebra defined in Remark 14, \(\mathcal{P}^2 = \mathcal{P}\). If \(\mathcal{M}\) is maximal and satisfies \(\mathcal{M} \subseteq \mathcal{P}^2\) and \(\mathcal{M} \neq \mathcal{P}^2\), then \(\mathcal{M} = \{0\}\). The Albert radical is \(\{0\}\) which implies the semi-simplicity of \(\mathcal{P}\).

Proposition 20. If \((\mathcal{P}, \cdot)\) is a simple nilalgebra such that \(x^2 = 0\) for all \(x \in \mathcal{P}\) then \(\mathcal{A}_\mathcal{P}\) is an associative nilalgebra satisfying \((\mathcal{A}_\mathcal{P})^2 = 0\).

Proof. The subalgebra \(\mathcal{P}^2 = \{xy, x, y \in \mathcal{P}\}\) is an ideal of \(\mathcal{P}\), so \(\mathcal{P}^2 = \mathcal{P}\). By the hypothesis, for every \(x \in \mathcal{P}\) we have \(x^2 = 0\). Then

\[
(x + y)^2 = x^2 + y^2 + xy + yx = xy + yx = 0
\]

for all \(x, y \in \mathcal{P}^2\). This implies

\[
x \cdot y = \frac{1}{2}(xy + yx) = 0
\]

thus the associative algebra \(\mathcal{A}_\mathcal{P}\) is trivial. \(\Box\)

We can also consider simple admissible Poisson algebras which are not nilalgebras. In this case the Albert radical is \(\{0\}\) and \(\mathcal{P}^2 \neq 0\).

Proposition 21. Let \((\mathcal{P}, \cdot)\) be a finite-dimensional simple admissible Poisson algebra which is not a nilalgebra. Then it has a unit element.

Proof. Indeed \(\mathcal{P}\) has a principal idempotent \(e\). Its Pierce decomposition \(\mathcal{P} = \mathcal{P}_{0,0} \oplus \mathcal{P}_{1,1}\) is such that \(\mathcal{P}_{0,0} \subset \mathcal{R}(\mathcal{P})\). Then \(\mathcal{P}_{0,0} = \{0\}\) and \(\mathcal{P} = \mathcal{P}_{1,1}\). Therefore, \(e = 1\). \(\Box\)

1.6. Classification of simple complex Poisson algebras such that \(\mathfrak{g} \mathcal{P}\) is simple

Lemma 22. Let \((\mathcal{P}, \cdot)\) be an admissible Poisson algebra. If \(\mathfrak{g} \mathcal{P}\) is a simple Lie algebra then \(\mathcal{P}\) is a simple algebra.
Proof. If $I \not\subseteq \mathcal{P}$ is an ideal of $\mathcal{P}$, then $I$ is also an ideal of $g_\mathcal{P}$ so $I$ must be trivial. \qed

**Proposition 23.** If $g_\mathcal{P}$ is a simple complex Lie algebra, then $UV = \{U, V\}$ for all $U, V \in \mathcal{P}$, that is, the associative algebra $A_\mathcal{P}$ satisfies $A^2_\mathcal{P} = \{0\}$.

**Proof.** Let $g_\mathcal{P}$ be a simple complex Lie algebra of rank $r$. Let $n_- \oplus \mathfrak{h} \oplus n_+$ be its root-decomposition, where $\mathfrak{h}$ is a Cartan subalgebra. Let $\{Y_j, H_i, X_j\}$ be the corresponding Weyl basis. Since $\{H^2_k, H_i\} = 0$ for all $i = 1, \ldots, r$ we deduce that

$$H^2_k \in \mathfrak{h}, \quad k = 1, \ldots, r.$$ 

Thus, $H^2_k = \sum_{i=1}^r \alpha^i_k H_i$. Let us put $\{H_k, X_j\} = \rho_{k,j} X_j$. We obtain

$$\{H_k, X^2_j\} = 2\rho_{k,j} X^2_j$$

for all $k = 1, \ldots, r$. Thus $2\rho_{k,j}$ are also roots of $g_\mathcal{P}$, but this is impossible so $X^2_j = 0$ for every $j$. Similarly we have for all $k = 1, \ldots, r$,

$$\{H_k, X_j \cdot X_i\} = (\rho_{k,j} + \rho_{k,j}) X_j \cdot X_i$$

so $(\rho_{k,j} + \rho_{k,j})$ are roots. This implies

$$X_j \cdot X_i = 0.$$

In the same way we have

$$Y^0_j = Y_j \cdot Y_i = 0$$

for all $i, j$. It turns out that

$$\{H^2_k, X_j\} = 2H_k \cdot \{H_k, X_j\} = 2\rho_{k,j} H_k \cdot X_j = \sum_{i=1}^r \alpha^i_k \rho_{i,j} X_j$$

and

$$\rho_{k,i} H_k \cdot X_j = \frac{1}{2} \left( \sum_{i=1}^r \alpha^i_k \rho_{i,j} \right) X_j.$$ 

For any $j$ there is $k$ such that $\rho_{k,j} \neq 0$. Thus

$$\{H_k \cdot X_j, X_j\} = 0 = H_k \cdot \{X_j, x_j\} + \{H_k, X_j\} \cdot X_j = \rho_{k,j} X^2_j$$ 

and

$$X^2_j = 0, \quad \forall j.$$
By similar arguments, the identities $Y_j^2 = 0$ hold. For $i = 1, \ldots, r$ we have
\[
\{ X_i^2, Y_i \} = 0 = 2X_i \cdot \{ X_i, Y_i \} = -4X_i \cdot H_i.
\]
Thus, $\sum \alpha_i^j \rho_{j,i} = 0$. As the matrix $(\rho_{j,i})$ is non-singular, we deduce that $\alpha_i^j = 0$, i.e.,
\[
H_i^2 = 0, \quad \forall i = 1, \ldots, r.
\]

The Poisson algebra $\mathcal{P}$ is a nilalgebra. Moreover, $H_i \cdot X_j = H_i \cdot Y_j = 0$ and we conclude that $U \cdot V = 0$ for all $U, V \in \mathcal{A}_\mathcal{P}$. \hfill \Box

2. On the classification of finite-dimensional complex Poisson algebras

Let $\mathcal{P}$ be a finite-dimensional complex Poisson algebra.

**Lemma 24.** If there is a non-zero vector $X \in \mathfrak{g}_\mathcal{P}$ such that $\text{ad} X$ is diagonalizable with 0 as a simple root, then $\mathcal{A}^2_{\mathcal{P}} = \{0\}$.

**Proof.** Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathfrak{g}_\mathcal{P}$ such that $\text{ad} e_1$ is diagonal with respect to this basis. By assumption, $\{e_1, e_i \} = \lambda_i e_i$ with $\lambda_i \neq 0$ for $i \geq 2$. Since $\{e_1^2, e_1\} = 2e_1 \cdot \{e_1, e_1\} = 0$, it follows that $e_1^2 = a e_1$. But for any $i \neq 1$, $\{e_i^2, e_1\} = 2e_1 \cdot \{e_1, e_i\} = 2\lambda_i e_1 \cdot e_i$ and $\{e_i^2, e_i\} = a \lambda_i e_i$, thus $e_1 \cdot e_i = \frac{a}{\lambda_i} e_i$. The associativity of the product $\cdot$ implies that $(e_1 \cdot e_1) \cdot e_i = a e_1 \cdot e_i = \frac{a^2}{\lambda_i} e_i = e_1 \cdot (e_1 \cdot e_i) = \frac{a^2}{\lambda_i} e_i$. Therefore $a = 0$ and $e_1^2 = 0 = e_1 \cdot e_i$ for any $i$. Finally, $0 = \{e_1 \cdot e_j, e_i\} = e_1 \cdot \{e_j, e_i\} + e_j \cdot \{e_1, e_i\} = \lambda_i e_j \cdot e_i$, which implies $e_i \cdot e_j = 0, \forall i, j \geq 1$. \hfill \Box

2.1. Classification of 2-dimensional Poisson algebras

- If $\mathfrak{g}_\mathcal{P}$ is abelian then $\mathcal{A}_\mathcal{P}$ can be any complex associative commutative algebra and $XY = X \cdot Y$. In this case the classification of Poisson algebras boils down to the classification of commutative associative complex algebras [2].
- If $\mathfrak{g}_\mathcal{P}$ is not abelian, it is solvable and isomorphic to the Lie algebra given by $\{e_1, e_2\} = e_2$. From Lemma 24 we know that $\mathcal{A}_\mathcal{P}$ is trivial and $e_i e_j = \{e_i, e_j\}$ for $i, j = 1, 2$.

2.2. Classification of 3-dimensional Poisson algebras

- If $\mathfrak{g}_\mathcal{P}$ is abelian then $\mathcal{A}_\mathcal{P}$ can be an arbitrary associative commutative algebra and $XY = X \cdot Y$. In this case the classification is given in [2].
- If $\mathfrak{g}_\mathcal{P}$ is nilpotent but not abelian it is isomorphic to the Heisenberg algebra. Let us consider a basis $\{e_i\}_{i=1,2,3}$ of $\mathfrak{g}_\mathcal{P}$ such that $\{e_1, e_2\} = e_3$. It follows from the Leibniz identities that $e_1^2 = a e_1 + b e_3$. But $\{e_1^2, e_2\} = 2e_1 \cdot e_3 = a e_3$ and $\{e_1 \cdot e_3, e_2\} = e_3 \cdot e_3 = [ae_3, e_2] = 0$. The associativity of $\cdot$ implies that $a = 0$. We see that
\[
e_1^2 = a e_3, \quad e_1 \cdot e_3 = e_3^2 = 0.
\]
Similarly,
\[
e_2^2 = b e_3, \quad e_2 \cdot e_3 = 0.
\]
Finally, \( \{e_1 \cdot e_2, e_i\} = 0 \) for \( i = 1, 2, 3 \) implies \( e_1 \cdot e_2 = \gamma e_3 \). Thus \( \mathcal{A}_P \) is isomorphic to the algebra:

\[
\begin{align*}
  e_1^2 &= \alpha e_3, \\
  e_2^2 &= \beta e_3, \\
  e_1 \cdot e_2 &= e_2 \cdot e_1 = \gamma e_3.
\end{align*}
\]

We obtain the following Poisson algebra

\[
\begin{align*}
  e_1^2 &= \alpha e_3, \\
  e_2^2 &= \beta e_3, \\
  e_1 \cdot e_2 &= (\gamma + 1) e_3, \\
  e_2 \cdot e_1 &= (\gamma - 1) e_3.
\end{align*}
\]

The base change

\[
\begin{align*}
  e'_1 &= a e_1 + b e_2, \\
  e'_2 &= c e_1 + d e_2
\end{align*}
\]

gives

\[
\begin{align*}
  (e'_1)^2 &= (a^2 \alpha + 2ab \gamma + b^2 \beta) e_3, \\
  (e'_2)^2 &= (c^2 \alpha + 2cd \gamma + d^2 \beta) e_3.
\end{align*}
\]

If \( \gamma^2 - \alpha \beta \neq 0 \), the equation \( \alpha + 2x \gamma + x^2 \beta = 0 \) has two distinct roots and we can assume that \( e'_1 \) and \( e'_2 \) are linearly independent such that \( (e'_1)^2 = (e'_2)^2 = 0 \). In this case the only possible values of parameters \( \alpha \) and \( \beta \) are \( \alpha = \beta = 0 \). We obtain the one-parametric family

\[
\mathcal{P}_{3,1}(\gamma) = \begin{cases}
  e_1^2 = e_2^2 = e_3^2 = 0, \\
  e_1 \cdot e_2 = (1 + \gamma) e_3, \\
  e_2 \cdot e_1 = (-1 + \gamma) e_3, \\
  e_1 \cdot e_3 = e_3 \cdot e_1 = e_3 \cdot e_2 = e_2 \cdot e_3 = 0.
\end{cases}
\]

If \( \gamma^2 - \alpha \beta = 0 \) and if \( \beta \neq 0 \), we can always choose \( c \) and \( d \) such that \( e_2^2 = 0 \). Then we can suppose that \( \beta = 0 \). This implies \( \gamma = 0 \). If \( \alpha = 0 \) we obtain \( \mathcal{P}_{3,1}(0) \). If \( \alpha \neq 0 \), we can assume \( \alpha = 1 \) which gives the algebra:

\[
\mathcal{P}_{3,2} = \begin{cases}
  e_1^2 = e_3, \\
  e_2^2 = e_3^2 = 0, \\
  e_1 \cdot e_2 = e_3, \\
  e_2 \cdot e_1 = -e_3, \\
  e_1 \cdot e_3 = e_3 \cdot e_1 = e_3 \cdot e_2 = e_2 \cdot e_3 = 0.
\end{cases}
\]

Suppose that \( g_P \) is solvable but not nilpotent. Then the following three cases may happen.
(i) The multiplication is defined by \( \{e_1, e_2\} = e_2 \). Then \((\mathcal{P}, \cdot)\) is isomorphic to one of the following Poisson algebras:

\[
\begin{aligned}
\mathcal{P}_{3,3}(\alpha) &= \left\{ 
\begin{array}{l}
\alpha e_3
\end{array}
\right. \\
& \left\{ 
\begin{array}{l}
e_1 \cdot e_2 = -e_2 \cdot e_1 = e_2, \\
e_1 \cdot e_3 = e_3 \cdot e_1 = \alpha e_3, \\
e_2 = 0, \\
e_2 \cdot e_3 = e_3 \cdot e_2 = 0, \\
e_3 = e_3,
\end{array}
\right.
\end{aligned}
\]

\[
\begin{aligned}
\mathcal{P}_{3,4} &= \left\{ 
\begin{array}{l}
e_1^2 = e_3, \\
\alpha e_3
\end{array}
\right. \\
& \left\{ 
\begin{array}{l}
e_1 \cdot e_2 = -e_2 \cdot e_1 = e_2, \\
e_1 \cdot e_3 = e_3 \cdot e_1 = 0, \\
e_2 = 0, \\
e_2 \cdot e_3 = e_3 \cdot e_2 = 0,
\end{array}
\right.
\end{aligned}
\]

The first family give the Poisson algebras

\[
\mathcal{P}_{3,5} = \left\{ 
\begin{array}{l}
e_1^2 = e_2^2 = e_3^2 = 0, \\
\alpha e_3
\end{array}
\right. \\
& \left\{ 
\begin{array}{l}
e_1 \cdot e_2 = -e_2 \cdot e_1 = e_2, \\
e_1 \cdot e_3 = e_3 \cdot e_1 = 0, \\
e_2 \cdot e_3 = e_3 \cdot e_2 = 0.
\end{array}
\right.
\]

The second family reduces to

\[
\mathcal{P}_{3,6} = \left\{ 
\begin{array}{l}
e_1^2 = e_2^2 = 0, \\
\alpha e_3
\end{array}
\right. \\
& \left\{ 
\begin{array}{l}
e_1 \cdot e_2 = -e_2 \cdot e_1 = e_2, \\
e_1 \cdot e_3 = e_3 \cdot e_1 = 0, \\
e_2 \cdot e_3 = e_3 \cdot e_2 = e_2.
\end{array}
\right.
\]

(ii) The multiplication is given by \( \{e_1, e_2\} = e_2 \) and \( \{e_1, e_3\} = \alpha e_3 \) with \( \alpha \neq 0 \). From Lemma 14, \((\mathcal{P}, \cdot)\) is isomorphic to

\[
\mathcal{P}_{3,7}(\alpha) = \left\{ 
\begin{array}{l}
e_1^2 = e_2^2 = e_3^2 = 0, \\
\alpha e_3
\end{array}
\right. \\
& \left\{ 
\begin{array}{l}
e_1 \cdot e_2 = -e_2 \cdot e_1 = e_2, \\
e_1 \cdot e_3 = -e_3 \cdot e_1 = \alpha e_3, \\
e_2 \cdot e_3 = e_3 \cdot e_2 = 0,
\end{array}
\right. \quad \alpha \neq 0.
\]
(iii) The multiplication is given by \(\{e_1, e_2\} = e_2 + e_3\) and \(\{e_1, e_3\} = e_3\). As 1 is an eigenvalue of \(\text{ad}_{e_1}\) with multiplicity 2, by adapting the proof of Lemma 24, we can conclude that \(\mathcal{A}\mathcal{P}\) is trivial. We get the Poisson algebra:

\[
\mathcal{P}_{3,8} = \begin{cases} 
  e_1^2 = e_2^2 = e_3^2 = 0, \\
  e_1 \cdot e_2 = -e_2 \cdot e_1 = e_2 + e_3, \\
  e_1 \cdot e_3 = -e_3 \cdot e_1 = e_3, \\
  e_2 \cdot e_3 = e_3 \cdot e_2 = 0.
\end{cases}
\]

- If \(g_{\mathcal{P}}\) is simple, it is isomorphic to \(sl(2)\). Therefore, it is rigid. We have already studied this case in the previous section. We deduce that \(\mathcal{P}\) is isomorphic to

\[
\mathcal{P}_{3,9} = \begin{cases} 
  e_1^2 = e_2^2 = e_3^2 = 0, \\
  e_1 \cdot e_2 = -e_2 \cdot e_1 = 2e_2, \\
  e_1 \cdot e_3 = -e_3 \cdot e_1 = -2e_3, \\
  e_2 \cdot e_3 = -e_3 \cdot e_2 = e_1.
\end{cases}
\]

3. Cohomology of Poisson algebras

In [10], A. Lichnerowicz introduced a cohomology for Poisson algebras (see also [14]). The \(k\)-cochains are skew-symmetric \(k\)-linear maps that are derivatives in each of their arguments. The coboundary operator denoted by \(\delta_{LP}\) is given by

\[
\delta_{LP}\varphi(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i \left[ X_i, \varphi(X_0, \ldots, \hat{X}_i, \ldots, X_k) \right] + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \varphi([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k)
\]

where \(\hat{X}_i\) means that the term \(X_i\) is omitted and \([,\) is the Lie bracket of the Poisson multiplication. Note that if \(f: \mathcal{P}_1 \to \mathcal{P}_2\) is a morphism of Poisson algebras, then \(f\) does not lead, in general, to a non-trivial functorial morphism between the cohomology groups. The functoriality question for Poisson cohomology has been addressed in the literature for instance in [8].

Since the Lichnerowicz cohomology pays attention only to the Lie part of a Poisson algebra, we need a better definition of cohomology that would govern general deformations of Poisson algebras. Such a definition is provided by theory of quadratic Koszul operads. We describe it in details only in degrees 0, 1, 2 and 3. Our approach will be based on the definition of admissible Poisson algebra.

3.1. The operad \(\text{Poiss}\)

The operad \(\text{Poiss}\) has already been studied in [12]. We will give an alternative description based on the definition of Poisson algebras. Let \(E = \mathbb{K}[\Sigma_2]\) be the \(\mathbb{K}\)-group algebra of the symmetric group on two elements. The basis of the free \(\mathbb{K}[\Sigma_n]\)-module \(\mathcal{F}(E)(n)\) consists of the “parenthesized products” of \(n\) variables \(\{x_1, \ldots, x_n\}\). Let \(R\) be the \(\mathbb{K}[\Sigma_3]\)-submodule of \(\mathcal{F}(E)(3)\) generated by the vector

\[
u = 3x_1(x_2x_3) - 3(x_1x_2)x_3 + (x_1x_3)x_2 + (x_2x_3)x_1 - (x_2x_1)x_3 - (x_3x_1)x_2.
\]
Then \( \mathcal{P}_{\text{Poiss}} \) is the binary quadratic operad with generators \( E \) and relations \( R \). It is given by

\[
\mathcal{P}_{\text{Poiss}}(n) = \left( \frac{\mathcal{F}(E)}{R} \right)(n) = \frac{\mathcal{F}(E)(n)}{R(n)}
\]

where \( R \) is the operadic ideal of \( \mathcal{F}(E) \) generated by \( R \) satisfying \( R(1) = R(2) = 0, R(3) = R \).

The dual operad \( \mathcal{P}_{\text{Poiss}}^! \) is equal to \( \mathcal{P}_{\text{Poiss}} \), that is, \( \mathcal{P}_{\text{Poiss}} \) is self-dual. In [7] we defined, for a binary quadratic operad \( \mathcal{E} \), an associated quadratic operad \( \tilde{\mathcal{E}} \) which gives a functor

\[
\mathcal{E} \otimes \tilde{\mathcal{E}} \to \mathcal{E}.
\]

In the case \( \mathcal{E} = \mathcal{P}_{\text{Poiss}} \), we have \( \tilde{\mathcal{E}} = \mathcal{P}_{\text{Poiss}}^! = \mathcal{P}_{\text{Poiss}} \).

### 3.2. The \( k \)-cochains

We proved in [6] that for any \( \mathbb{K}[\Sigma_3]^2 \)-associative algebra \( (\mathcal{A}, \mu) \) defined by the relation

\[
A_\mu^v \circ \phi_v - A_\mu^w \circ \phi_w = 0,
\]

with \( v, w \in \mathbb{K}[\Sigma_3] \), the cochains \( \phi \in C^i(\mathcal{A}, \mathcal{A}) \) can be chosen invariant under \( F_v^\perp \cap F_w^\perp \) (for the notations see [6]). For a Poisson algebra we have \( v = Id, w = 3ld - \tau_2 + \tau_1 - c_1 + c_2 \). Then \( F_v^\perp \cap F_w^\perp = \{0\} \) and if \( C^k(\mathcal{P}, \mathcal{P}) \) is the space of \( k \)-cochains of \( \mathcal{P} \), we obtain

\[
C^k(\mathcal{P}, \mathcal{P}) = \text{End}(\mathcal{P} \otimes^k, \mathcal{P}).
\]

**Remark.** In [12] an explicit presentation of the space of cochains is given using operads. More precisely, we have

\[
C^k(\mathcal{P}, \mathcal{P}) = \text{Lin}(\mathcal{P}_{\text{Poiss}}(n)^! \otimes_{\Sigma_n} V^{\otimes^n}, V)
\]

where \( V \) is the underlying vector space (here \( C'' \)). We can see that \( \text{End}(\mathcal{P} \otimes^k, \mathcal{P}) \) is isomorphic to \( \text{Lin}(\mathcal{P}_{\text{Poiss}}(n)^! \otimes_{\Sigma_n} V^{\otimes^n}, V) \).

### 3.3. The coboundary operators \( \delta^k_{\mathcal{P}} \) \((k = 0, 1, 2)\)

**Notation.** Let \( (\mathcal{P}, \cdot) \) be a Poisson algebra, \( g_{\mathcal{P}} \) and \( A_{\mathcal{P}} \) its corresponding Lie and associative algebras. We denote by

\[
H_C^*(g_{\mathcal{P}}, g_{\mathcal{P}}) = Z_C^*(g_{\mathcal{P}}, g_{\mathcal{P}}) / B_C^*(g_{\mathcal{P}}, g_{\mathcal{P}})
\]

the Chevalley cohomology of \( g_{\mathcal{P}} \) and by \( H_H^*(A_{\mathcal{P}}, A_{\mathcal{P}}) \) the Harrison cohomology of \( A_{\mathcal{P}} \). We will define coboundary operators \( \delta^k_{\mathcal{P}} \) on \( C^k(\mathcal{P}, \mathcal{P}) \).

(i) \( k = 0 \).

We put

\[
H^0(\mathcal{P}, \mathcal{P}) = \{ X \in \mathcal{P} \text{ such that } \forall Y \in \mathcal{P}, X \cdot Y = 0 \}.
\]
(ii) \( k = 1 \).

For \( f \in \text{End}(\mathcal{P}, \mathcal{P}) \), we put
\[
\delta^1_P f(X, Y) = f(X) \cdot Y + X \cdot f(Y) - f(X \cdot Y)
\]
for any \( X, Y \in \mathcal{P} \). Then we have
\[
H^1(\mathcal{P}, \mathcal{P}) = H_C^1(g_\mathcal{P}, g_\mathcal{P}) \cap H_H^1(A_\mathcal{P}, A_\mathcal{P}).
\]

(iii) \( k = 2 \).

For \( \varphi \in C^2(\mathcal{P}, \mathcal{P}) \) we define
\[
\delta^2_P \varphi(X, Y, Z) = 3\varphi(X \cdot Y, Z) - 3\varphi(X, Y \cdot Z) - \varphi(X \cdot Z, Y) - \varphi(Y \cdot Z, X)
+ \varphi(Y \cdot X, Z) + \varphi(Z \cdot X, Y) + 3\varphi(X, Y) \cdot Z - 3X \cdot \varphi(Y, Z)
- \varphi(X, Z) \cdot Y - \varphi(Y, Z) \cdot X + \varphi(Y, X) \cdot Z + \varphi(Z, X) \cdot Y.
\]

The space \( H^2(\mathcal{P}, \mathcal{P}) \) parametrizes deformations of the multiplication of \( \mathcal{P} \). We saw in the previous sections that deformations of \( (\mathcal{P}, \cdot) \) induce deformations of \( g_\mathcal{P} \) and of \( A_\mathcal{P} \). In contrast to \( H^*(\mathcal{P}, \mathcal{P}) \), the Lichnerowicz–Poisson cohomology reflects deformations of the bracket only.

Suppose that the Poisson product satisfies \( X \cdot Y = -Y \cdot X \). Then \( \{X, Y\} = X \cdot Y \) and \( X \cdot Y = 0 \). If \( \varphi \in C^2(\mathcal{P}, \mathcal{P}) \) is also skew-symmetric, then
\[
\delta^2_P \varphi(X, Y, Z) = 2\varphi(X \cdot Y, Z) + 2\varphi(Y \cdot Z, X) - 2\varphi(X, Z, Y)
+ 2\varphi(X, Y) \cdot Z + 2\varphi(Y, Z) \cdot X - 2\varphi(X, Z) \cdot Y
= \delta^2_L \varphi(X, Y, Z).
\]

We recognize the formula of Lichnerowicz–Poisson differential.

**Proposition 25.** Let \( \varphi \) be in \( C^2(\mathcal{P}, \mathcal{P}) \). If \( \varphi_a \) and \( \varphi_s \) are respectively the skew-symmetric and the symmetric parts of \( \varphi \) then we have:

\[
12\delta^2_C \varphi_a(X, Y, Z) = \delta^2_P \varphi(X, Y, Z) - \delta^2_P \varphi(Y, X, Z) - \delta^2_P \varphi(Z, Y, X)
- \delta^2_P \varphi(X, Z, Y) + \delta^2_P \varphi(Y, Z, X) + \delta^2_P \varphi(Z, X, Y),
\]

\[
12\delta^2_H \varphi_s(X, Y, Z) = \delta^2_P \varphi(X, Y, Z) - \delta^2_P \varphi(Z, Y, X) + \delta^2_P \varphi(Z, X, Y)
- \delta^2_P \varphi(Z, X, Y).
\]

**Proof.** The proof is a straightforward calculation. Recall that if \( \varphi \) is a skew-symmetric bilinear map then the Chevalley coboundary operator is given by
\[
\delta_C(\varphi)(X, Y, Z) = \{\varphi(X, Y), Z\} + \{\varphi(Y, Z), X\} + \{\varphi(Z, X), Y\}
+ \varphi(\{X, Y\}, Z) + \varphi(\{Y, Z\}, X) + \varphi(\{Z, X\}, Y).
\]
and if \( \varphi \) is a symmetric bilinear map then the Harrison coboundary operator is given by

\[
\delta_H(\varphi)(X, Y, Z) = \varphi(X, Y) \cdot Z - X \cdot \varphi(Y, Z) + \varphi(X \cdot Y, Z) - \varphi(X, Y \cdot Z).
\]

Now, to compute \( \delta_C^2 \varphi_a \) we replace \( \varphi_a(X, Y) \) by \( (\varphi(X, Y) - \varphi(Y, X))/2 \) and \( \{X, Y\} \) by \( (X \cdot Y - Y \cdot X)/2 \) in the expression of \( \delta_C^2 \varphi_a(X, Y, Z) \). We leave it to the reader.  

**Corollary 26.** Let \( \varphi_s \) and \( \varphi_a \) be the symmetric and skew-symmetric parts of \( \varphi \in C^2(\mathcal{P}, \mathcal{P}) \). If \( \varphi \in Z^2(\mathcal{P}, \mathcal{P}) \), then \( \varphi_s \in Z^2_H(\mathcal{A} \mathcal{P}, \mathcal{A} \mathcal{P}) \) and \( \varphi_a \in Z^2_C(\mathfrak{g} \mathfrak{p}, \mathfrak{g} \mathfrak{p}) \).

**3.4. Relation between** \( Z^2(\mathcal{P}, \mathcal{P}) \) and \( Z^2_H(\mathcal{A} \mathcal{P}, \mathcal{A} \mathcal{P}), Z^2_C(\mathfrak{g} \mathfrak{p}, \mathfrak{g} \mathfrak{p}) \)

To show the relation between \( Z^2(\mathcal{P}, \mathcal{P}) \) and the classical Chevalley and Harrison cohomological spaces, we have to introduce the following operators

\[
L_1, L_2 : C^2(\mathcal{P}, \mathcal{P}) \rightarrow C^3(\mathcal{P}, \mathcal{P}).
\]

They are given by

\[
L_1(\varphi)(X, Y, Z) = \varphi(X \cdot Y, Z) - \varphi(X, Z) \cdot Y - X \cdot \varphi(Y, Z)
\]

and

\[
L_2(\varphi)(X, Y, Z) = -3\varphi(X, \{Y, Z\}) + \{\varphi(X, Y), Z\} - \{\varphi(X, Z), Y\}.
\]

**Lemma 27.** Let \( \varphi \in C^2(\mathcal{P}, \mathcal{P}) \). If \( \varphi_s \) and \( \varphi_a \) are the symmetric and skew-symmetric parts of \( \varphi \), we have

\[
\delta_C^2 \varphi = \delta_C^2 \varphi_a + 2\delta_H^2 \varphi_s + \delta_C^2 \varphi_s + \delta_H^2 \varphi_a + L_1(\varphi_a) + L_2(\varphi_s)
\]

where \( \delta_C \) and \( \delta_H \) are the linear maps \( C^2(\mathcal{P}, \mathcal{P}) \rightarrow C^3(\mathcal{P}, \mathcal{P}) \) extending naturally \( \delta_C \) and \( \delta_H \).

**Proof.** Starting from \( \varphi = \varphi_a + \varphi_s \) and \( X \cdot Y = \{X, Y\} + X \cdot Y \) we obtain

\[
\delta_C^2 \varphi(X, Y, Z) = 3\varphi_s(\{X, Y\}, Z) - 3\varphi_a(X, \{Y, Z\}) - \varphi_a([X, Z], Y)
\]

\[
- \varphi_a([Y, Z], X) + \varphi_a([Y, X], Z) + \varphi_a([Z, X], Y) + 3\varphi_a(X, Y, Z)
\]

\[
- 3\{X, \varphi_a(Y, Z)\} - \{\varphi_a(X, Z), Y\} - \{\varphi_a(Y, X), Z\} + \varphi_a(Y, X, Z)
\]

\[
+ \{\varphi_a(Z, X), Y\} + 3\varphi_a(X \cdot Y, Z) - 3\varphi_a(X, Y \cdot Z) - \varphi_a(X \cdot Z, Y)
\]

\[
- \varphi_a(Y \cdot Z, X) + \varphi_a(Y \cdot X, Z) + \varphi_a(Z \cdot X, Y) + 3\varphi_a(X, Y) \cdot Z
\]

\[
- 3X \cdot \varphi_a(Y, Z) - \varphi_a(X, Z) \cdot Y - \varphi_a(Y, Z) \cdot X + \varphi_a(Y, X) \cdot Z
\]

\[
+ \varphi_a(Z, X) \cdot Y + 3\varphi_a([X, Y], Z) - 3\varphi_a([X, Y], Z) - \varphi_a([X, Z], Y)
\]

\[
- \varphi_s([Y, Z], X) + \varphi_s([Y, X], Z) + \varphi_s([Z, X], Y) + 3\varphi_s(X, Y, Z)
\]

\[
- 3\{X, \varphi_s(Y, Z)\} - \{\varphi_s(X, Z), Y\} - \{\varphi_s(Y, Z), X\} + \{\varphi_s(Y, X), Z\}
\]
\[ + \{ \varphi_s(Z, X), Y \} + 3\varphi_s(X \bullet Y, Z) - 3\varphi_s(X, Y \bullet Z) - \varphi_s(X \bullet Z, Y) \\
- \varphi_s(Y \bullet Z, X) + \varphi_s(Y \bullet X, Z) + \varphi_s(Z \bullet X, Y) + 3\varphi_s(X, Y) \bullet Z \\
- 3X \bullet \varphi_s(Y, Z) - \varphi_s(X, Z) \bullet Y - \varphi_s(Y, Z) \bullet X + \varphi_s(Y, X) \bullet Z \\
+ \varphi_s(Z, X) \bullet Y. \]

As \( \varphi_a \) is skew-symmetric and \( \varphi_s \) symmetric, this relation gives

\[
\delta_2^P \varphi(X, Y, Z) = 2\varphi_a(\{ X, Y \}, Z) - 2\varphi_a(X, \{ Y, Z \}) - 2\varphi_a(\{ X, Z \}, Y) \\
+ 2\varphi_a(X, Y), Z] - 2\{ X, \varphi_a(Y, Z) \} - 2\{ \varphi_a(X, Z), Y \} + 4\varphi_a(X \bullet Y, Z) \\
- 2\varphi_a(X, Y \bullet Z) + 2\varphi_a(X \bullet Y) \bullet Z - 4X \bullet \varphi_a(Y, Z) - 2\varphi_a(X, Z) \bullet Y \\
+ 2\varphi_s(\{ X, Y \}, Z) - 4\varphi_s(\{ X, \{ Y, Z \} \} - 2\varphi_s(\{ X, Z \}, Y) + 4\{ \varphi_s(X, Y), Z \} \\
- 2\{ X, \varphi_s(Y, Z) \} + 4\varphi_s(X \bullet Y, Z) - 4\varphi_s(X, Y \bullet Z) + 4\varphi_s(X, Y) \bullet Z \\
- 4X \bullet \varphi_s(Y, Z) \\
\]

that is

\[
\delta_2^P \varphi(X, Y, Z) = 2\delta_C \varphi_a(X, Y, Z) + 2\tilde{\delta}_H \varphi_a(X, Y, Z) + 2\mathcal{L}_1(\varphi_a)(X, Y, Z) \\
+ 4\delta_H \varphi_s(X, Y, Z) + 2\tilde{\delta}_C \varphi_s(X, Y, Z) + 2\mathcal{L}_2(\varphi_s)(X, Y, Z) \\
\]

this gives the lemma. \( \square \)

**Theorem 28.** Let \( \varphi \) be in \( C^2(\mathcal{P}, \mathcal{P}) \) and let \( \varphi_s, \varphi_a \) be its symmetric and skew-symmetric parts. Then the following propositions are equivalent:

1. \( \delta_2^P \varphi = 0. \)
2. \( \begin{cases} 
\delta_2^C \varphi_a = 0, & \delta_2^H \varphi_s = 0, \\
\delta_2^C \varphi_s + \tilde{\delta}_H^2 \varphi_a + \mathcal{L}_1(\varphi_a) + \mathcal{L}_2(\varphi_s) = 0. 
\end{cases} \)

**Proof.** (2) \( \Rightarrow \) (1) is a consequence of Corollary 26. (1) \( \Rightarrow \) (2) is a consequence of Corollary 26 and Lemma 27. \( \square \)

**Applications.** Suppose that \( \varphi \) is skew-symmetric. Then \( \varphi = \varphi_a \) and \( \varphi_s = 0. \) Then \( \delta_2^P \varphi = 0 \) if and only if \( \delta_2^C \varphi = 0 \) and \( \delta_2^H \varphi_s + \mathcal{L}_1(\varphi) = 0. \) Moreless if we suppose than \( \varphi \) is a biderivation on each argument, that is \( \mathcal{L}_1(\varphi) = 0, \) then Theorem 28 implies that \( \delta_2^P \varphi = 0 \) if and only if \( \delta_2^H \varphi = 0. \) But

\[
\delta_2^H \varphi(X, Y, Z) = \varphi(X, Y) \bullet Z - X \bullet \varphi(Y, Z) + \varphi(X \bullet Y, Z) - \varphi(X, Y \bullet Z) \\
= \mathcal{L}_1(\varphi)(X, Y, Z) + \mathcal{L}_1(\varphi)(Y, Z, X). 
\]

Thus \( \delta_2^H \varphi = 0 \) as soon as \( \mathcal{L}_1(\varphi) = 0. \)
Proposition 29. Let \( \varphi \) be a skew-symmetric map which is a biderivation, that is \( \varphi \) is a Lichnerowicz–Poisson 2-cochain. Then \( \varphi \in Z^2_{\text{LP}}(P, P) \) if and only if \( \varphi \in Z^2_P(P, P) \).

Similarly, if \( \varphi \) is symmetric, then \( \delta^2_P \varphi = 0 \) if and only if \( \delta^2_H \varphi = 0 \) and \( \delta^2_C \varphi + \mathcal{L}_2(\varphi) = 0 \).

3.5. The case \( k = 3 \)

We need to define \( \delta^3_P \psi \) for \( \psi \in C^3(P, P) \) so that \( H^3(P, P) \) represents obstructions to integrability of infinitesimal deformations of the Poisson algebra \( P \). For each \( \psi \in C^3(P, P) \) we consider

\[
\hat{\psi}(Z, T, X \cdot Y) = \psi(Z, T, X \cdot Y) - \psi(Z, T \cdot X, Y) + \frac{1}{3} \psi(Z, T \cdot Y, X)
+ \frac{1}{3} \psi(Z, X \cdot Y, T) - \frac{1}{3} \psi(Z, X \cdot T, Y) - \frac{1}{3} \psi(Z, Y \cdot T, X).
\]

Suppose that \( X \cdot \psi(Y, Z, T) \) appears in \( \delta^3_P \psi(X, Y, Z, T) \). Since \( \delta^3_P \circ \delta^2_P \varphi = 0 \), we see that the term \( X \cdot \varphi(Y, Z, T) \) occurs in \( X \cdot \delta^2_P \varphi(Y, Z, T) \). This term appears only once if \( \varphi \) is not skew-symmetric. Thus, in the general case, \( \delta^3_P \psi(X, Y, Z, T) \) cannot contain terms as \( X \cdot \psi(Y, Z, T) \).

We conclude that \( \delta^3_P \psi(X, Y, Z, T) \) can be written as:

\[
\delta^3_P \psi(X, Y, Z, T) = \alpha_1 \hat{\psi}(Z, T, X \cdot Y) + \alpha_2 \hat{\psi}(Y, T, X \cdot Y) + \alpha_3 \hat{\psi}(Y, Z, X \cdot T)
+ \alpha_4 \hat{\psi}(X, T, Y \cdot Z) + \alpha_5 \hat{\psi}(X, Z, Y \cdot T) + \alpha_6 \hat{\psi}(X, Y, Z \cdot T).
\]

From the relations between \( Z^2(P, P) \) and \( Z^2_H(A_P, A_P), Z^2_C(g_P, g_P) \), we have to assume that \( \delta^3_P \psi(X, Y, Z, T) = 0 \) as soon as \( \psi \) is Lichnerowicz–Poisson cochain. This permits to compute the constants \( \alpha_i \). We will go in detail on this computation in a forthcoming paper.

4. Deformations of complex Poisson algebras

4.1. Generalities

By a deformation we understand a formal deformation in Gerstenhaber’s sense. It turns out that formal deformations are equivalent to perturbations in the sense of [5].

Let \( P = (V, \mu) \) be a Poisson algebra with multiplication \( \mu \) and \( V \) the underlying complex vector space. Let \( \mathbb{C}[t] \) be the ring of complex formal power series. A deformation of \( \mu \) (or \( P \)) is a \( \mathbb{C} \)-bilinear map:

\[
\mu' : V \times V \longrightarrow V \otimes \mathbb{C}[t]
\]

given by

\[
\mu'(X, Y) = \mu(X, Y) + t \varphi_1(X, Y) + t^2 \varphi_2(X, Y) + \cdots + t^n \varphi_n(X, Y) + \cdots
\]
for all $X, Y \in V$ such that $\varphi_i$ are bilinear maps satisfying, for $k \geq 1$,

$$\begin{aligned}
\sum_{i+j=2k+1} \varphi_i \circ \varphi_j + \varphi_j \circ \varphi_i + \delta(\varphi_{k+1}) &= 0, \\
\sum_{i+j=2k, i < j} \varphi_i \circ \varphi_j + \varphi_j \circ \varphi_i + \delta(\varphi_k) + \varphi_k \circ \varphi_k &= 0,
\end{aligned}$$

with

$$\varphi_i \circ \varphi_j(X, Y, Z) = \varphi_i(\varphi_j(X, Y), Z) - \varphi_i(X, \varphi_j(Y, Z)) - \frac{1}{3} \varphi_i(\varphi_j(X, Z), Y)$$

and $\delta \varphi_i$ the coboundary operator of the Poisson cohomology defined in the previous section.

**Definition 30.** A Poisson algebra $P = (V, \mu)$ is rigid if every deformation $\mu'$ is isomorphic to $\mu$, i.e., if there exists $f \in Gl(V \otimes \mathbb{C}[t])$ such that

$$f^{-1}(\mu(f(X), f(Y))) = \mu'(X, Y)$$

for all $X, Y \in V$.

As for Lie or associative algebras, one can show, using similar arguments:

**Proposition 31.** If $H^2(P, P) = 0$, then $P = (V, \mu)$ is rigid.

The converse is not true. A rigid complex $n$-dimensional Poisson algebra with $H^2(P, P) \neq 0$ corresponds to a point $\mu$ of the algebraic variety of Poisson structures on $\mathbb{C}^n$ such that the corresponding affine schema is not reduced at this point. We will see an example in the following section.

### 4.2. Finite-dimensional complex rigid Poisson algebras

Let $P = (\mathbb{C}^n, \mu)$ be an $n$-dimensional complex Poisson algebra and suppose that the associated Lie algebra $g_P$ is a finite-dimensional rigid solvable Lie algebra. It follows from [1] that $g_P$ can be written as $g_P = t \oplus n$, where $n$ is the nilradical and $t$ a maximal abelian subalgebra such that the operators $ad X$ are semi-simple for all $X$ in $t$. The subalgebra $t$ is called the maximal exterior torus and its dimension the rank of $g_P$.

Suppose that $\dim t = 1$ and for $X \in g_P, X \neq 0$, the restriction of the operator $ad X$ on $n$ is invertible (all known solvable rigid Lie algebras satisfy this hypothesis). By Lemma 14, the associated algebra $A_P$ satisfies $A^2_P = \{0\}$.

**Theorem 32.** Let $P$ a complex Poisson algebra such that $g_P$ is rigid solvable of rank 1 (i.e. $\dim t = 1$) with non-zero roots. Then $P$ is a rigid Poisson algebra.

**Proof.** If $\mu'$ is a deformation of $\mu$, then the corresponding Lie bracket $\{.,.\}_{\mu'}$ is a deformation of the Lie bracket $\{.,.\}_\mu$ of $g_P$. Since $(g_P, \{.,.\}_\mu)$ is rigid, then $\{.,.\}_{\mu'}$ is isomorphic to $\{.,.\}_\mu$. If we denote by $P' = (\mathbb{C}^n, \mu')$ the deformation of $P = (\mathbb{C}^n, \mu)$, then $A_{P'}$ satisfies also $A^2_{P'} = \{0\}$. So, $\mu'$ is isomorphic to $\mu$ and $P$ is rigid. $\square$
Theorem 32 can be used to construct rigid Poisson algebras.

**Proposition 33.** Let \( g \) be a rigid solvable Lie algebra of rank 1 with non-zero roots. Then there is only one Poisson algebra \((P, \cdot)\) such that \( g_P = g \). It is defined by

\[ X_i \cdot X_j = \{X_i, X_j\}. \]

**Example.** The Poisson algebra \( P_{2,6} \) is rigid with \( \dim H^2(P, P) = 0 \). Indeed

\[ Z^2(P, P) = \{ \varphi \in C^2(P, P), \varphi(e_1, e_1) = \varphi(e_2, e_2) = 0, \varphi(e_1, e_2) = -\varphi(e_2, e_1) \} \]

and for every \( f \in \text{End}(P) \) we have \( \delta f (e_1, e_1) = 0 = \delta f (e_2, e_2) \) and \( \delta f (e_1, e_2) = -\delta f (e_2, e_1) = ae_1 + be_2 \). We observe that \( H^2_C(g_P, g_P) \neq 0 \).

We can generalize the previous result to rigid solvable Lie algebras \((g_P, \{\,,\} \mu)\) of rank \( r \). In this case the nilradical \( n \) is graded by the roots of \( t \) [1]. If none of the roots is zero, then using the same arguments as in Lemma 14, we prove that \( A^2_P = \{0\} \) and \( P \) is rigid. Then we have

**Proposition 34.** Let \((P, \mu)\) be an \( n \)-dimensional complex Poisson algebra such that \( g_P \) is a solvable rigid Lie algebra of rank \( r \). If the roots are non-zero, then \((P, \mu)\) is rigid and \( A^2_P = \{0\} \).

**Remark 35.** We show how a rigid Lie algebra with \( H^2_C(g_P, g_P) \neq 0 \) leads to a rigid Poisson algebra with the same property. Consider an admissible Poisson algebra satisfying the hypothesis of Proposition 34. Thus \( \mu = \{\,,\} \mu \) and if \( \varphi \in Z^2(P, P) \) is the first term of a deformation of \( \mu \), then \( \varphi \) is a skew-symmetric map and \( \delta \varphi(X, Y, Z) = (2/3) \delta_{c} \varphi(X, Y, Z) \). In particular, if \( g_P \) is rigid with \( H^2_C(g_P, g_P) \neq 0 \) then \( P \) is rigid with \( H^2(P, P) \neq 0 \). This gives examples of rigid Poisson algebras with non-trivial cohomology based on the constructions [3].

**Remark 36.** It may happen that a Poisson algebra \( P \) is rigid although \( g_P \) is not. An example is the Poisson algebra \( P_{3,6} \) of Section 2.

**Remark 37.** We can consider deformations of \( P \) which leave the associated product of \( A_P \) unchanged. This means that \( \varphi \) is a skew-bilinear map and, as in Remark 35, cocycles of the Poisson cohomology are also cocycles of the Lichnerowicz–Poisson cohomology. In this case \( H^2(P, P) = H^2_C(g_P, g_P) \).

### 4.3. The Poisson algebra \( S(g) \)

Let \( g \) be a finite-dimensional complex Lie algebra. We denote by \( S(g) \) the symmetric algebra on the vector space \( g \). It is an associative commutative algebra. Let \( \{e_1, \ldots, e_n\} \) be a fixed basis of \( g \) and \( \{e_i, e_j\} = \sum_k C^k_{ij} e_k \) its structure constants. We define on \( S(g) \) a structure of Lie algebra by

\[
P_0(p, q) = \sum_{i,j,k=1}^{n} C^k_{ij} e_k \left( \frac{\partial p}{\partial e_i} \frac{\partial q}{\partial e_j} - \frac{\partial p}{\partial e_j} \frac{\partial q}{\partial e_i} \right),
\]
where \( p = p(e_1, \ldots, e_n) \) and \( q = q(e_1, \ldots, e_n) \in S(\mathfrak{g}) = \mathbb{C}[e_1, \ldots, e_n] \). Let \( p \cdot q \) be the ordinary associative product of the polynomials \( p \) and \( q \). The Lie bracket satisfies the Leibniz rule with respect to this product. If

\[
\tilde{P}_0(p, q) = P_0(p, q) + p \cdot q
\]

then \((S(\mathfrak{g}), \tilde{P}_0)\) is a Poisson algebra. This structure is usually called the linear Poisson structure on \( S(\mathfrak{g}) \).

In this subsection we will be interested in deformations \( \tilde{P} \) of \( P_0 \) on \( S(\mathfrak{g}) \) which leave the associated structure \((A(S(\mathfrak{g})), \bullet)\) unchanged. We call such deformations Lie deformations of the Poisson algebra \((S(\mathfrak{g}), \tilde{P}_0)\). Any deformation of the bracket \( P_0 \) can be expanded into

\[
P = P_0 + t\phi_1 + \cdots + t^k\phi_k + \cdots
\]

and the corresponding Lie deformation of \( \tilde{P}_0 \) is

\[
\tilde{P} = \tilde{P}_0 + t\phi_1 + \cdots + t^k\phi_k + \cdots.
\]

Then \( \phi_1 \in Z^2_L,\rho((S(\mathfrak{g}), \tilde{P}_0), (S(\mathfrak{g}), \tilde{P}_0)) \).

Suppose now that \( \mathfrak{g} = t \oplus \mathfrak{n} \) is a complex solvable rigid Lie algebra.

**Proposition 38.** If \( \mathfrak{g} \) is a complex solvable rigid Lie algebra with \( \dim t \geq 2 \), then the Lie algebra \((S(\mathfrak{g}), P_0)\) is not rigid.

**Proof.** Let \( \phi: S(\mathfrak{g}) \times S(\mathfrak{g}) \to S(\mathfrak{g}) \) be a skew-bilinear map given by \( \phi(X_1, X_2) = \alpha_{12} \cdot 1 \) when \( X_1, X_2 \in t \) and \( \phi(Y_1, Y_2) = 0 \) when \( Y_1, Y_2 \in \mathfrak{g} \) but \( Y_1 \) or \( Y_2 \) is not in \( t \). By the assumption, \( \phi \) is a derivation in each argument, so \( \phi \) can be extended onto \( S(\mathfrak{g}) \). It is easy to see that \( \phi \in Z^2_C(S(\mathfrak{g}), S(\mathfrak{g})) \). Since \( P_0 + t\phi \) is not isomorphic to \( P_0 \), we have obtained a non-trivial deformation. \( \square \)

**Corollary 39.** (See [13].) If \( \mathfrak{g} \) is a complex solvable rigid Lie algebra with \( \dim t \geq 2 \), then the Poisson algebra \((S(\mathfrak{g}), \tilde{P}_0)\) is not rigid.

Now we consider the case \( \dim t = 1 \).

**Lemma 40.** The maximal exterior torus \( t \) is a Cartan subalgebra of \((S(\mathfrak{g}), P_0)\).

**Proof.** We denote by \( \{X, Y_1, \ldots, Y_{n-1}\} \) a basis of \( \mathfrak{g} = t \oplus \mathfrak{n} \) adapted to this decomposition. By definition of \( t \) we have \( \{X, Y_i\} = \lambda_i Y_i \). Then

\[
\begin{align*}
P_0(X^i, X^j) &= 0 \quad \text{for any } i, j, \\
P_0(X^i, Y_j) &= i\lambda_j X^{i-1} Y_j, \\
P_0(X, XY_j) &= \lambda_j XY_j, \\
P_0(X, Y_i Y_j) &= (\lambda_i + \lambda_j) Y_i Y_j
\end{align*}
\]

so that \( \text{ad}_{P_0} X \) is a diagonal derivation of \( S(\mathfrak{g}) \). \( \square \)
We conclude that the Lie algebra \((S(\mathfrak{g}), P_0)\) is graded by the eigenvalues of \(\text{ad}_{P_0} X\). In [3] families of rigid Lie algebras of rank 1 were classified. This classification can be used to study \(S(\mathfrak{g})\) for a general rigid Lie algebra. We illustrate it on the case where the eigenvalues of \(\text{ad}_\mathfrak{g} X\) are

\[1, 2, \ldots, n - 1.\]

It follows from [1] that,

- If \(3 \leq n \leq 6\) or \(9 \leq n \leq 12\) then \(\mathfrak{g}\) is not rigid.
- In the remaining cases, \(\mathfrak{g}\) is rigid.

We consider a deformation of \(\tilde{P}_0\) given as \(\tilde{P} = \tilde{P}_0 + t\phi_1 + \cdots\) with \(\phi_1 \in Z^2_\mathfrak{L}_p \langle S(\mathfrak{g}), \tilde{P}_0 \rangle,\) \((S(\mathfrak{g}), \tilde{P}_0)\). It is clear that if \(\phi_1(Y, Z) = 0\) for every \(Y, Z \in \mathfrak{g}\) then \(\phi_1 = 0\). Let \(I_p\) be the Lie ideal of \(S(\mathfrak{g})\) whose elements are polynomials of degree greater than or equal to \(p\). If we denote by \(S_p(\mathfrak{g})\) the quotient Lie algebra \(S(\mathfrak{g})/I_{p+1}\), then \(S_p(\mathfrak{g}) = \mathbb{C}\{1\} \oplus K_p(\mathfrak{g})\) where \(K_p(\mathfrak{g})\) is generated by polynomials of degree greater than or equal to 1. As \((\mathcal{P}, \cdot, \cdot)\) is a Lie deformation it preserves this decomposition. Thus we need to study the Lie algebra \(K_p(\mathfrak{g})\). The Lie subalgebra generated by \([X]\) is a maximal exterior torus of \(K_p(\mathfrak{g})\). The vector \(X\) is in the terminology of [1] a regular vector. The eigenvalues of \(\text{ad}_{K_p(\mathfrak{g})} X\) are \((1, 2, \ldots, n - 1, n, \ldots, p(n - 1))\). Let \((S(X))\) be the corresponding root system [1]. It is easy to see that its rank is equal to \(\dim(n) - 2\). This proves that \(K_p(\mathfrak{g})\) is not rigid. But since we suppose that \(\phi_1\) is a derivation in each argument, this implies that \(\phi_1(X, X^2) = 0\) and the rank of \((S(X))\) is \(\dim(n) - 1\). The grading of \(K_p(\mathfrak{g})\) by the roots of \(\text{ad}_{K_p(\mathfrak{g})} X\) is preserved by such a deformation.

The cocycle \(\phi_1\) leaves invariant each of the eigenspaces of \(\text{ad} X\). Let \(k, k \leq n - 1\), be the smallest index such that \(\phi_1\) restricted to the eigenspace associated to the eigenvalue \(k\) of \(\text{ad} X\) is non-zero. Then \(H_k(\mathfrak{g})\) is a non-rigid Lie algebra such that \(\phi_1\) is a cocycle determined by a deformation. Conversely, let \(\phi_1\) be a 2-cocycle of the Lie algebra \(K_p(\mathfrak{g})\) which is a derivation in each argument such that there exists \(i\) with \(\phi_1(Y_i, Y_{p-i}) \neq 0\). Then we can extend \(\phi_1\) to \(S(\mathfrak{g})\) to obtain a deformation of \(S(\mathfrak{g})\).

Examples.

1. Let us suppose that \(\mathfrak{g}\) is the two-dimensional non-abelian rigid solvable Lie algebra with the bracket defined by \([X, Y] = Y\). Let \((S(\mathfrak{g}), P_0)\) be the corresponding Poisson algebra. Then \(P_0(X, Y) = Y\). If \(P\) is a deformation of \(P_0\), since \(\dim(n) = 1\), \(P = P_0\) and \((S(\mathfrak{g}), P_0)\) is rigid.

2. Let us suppose that \(\mathfrak{g}\) is the decomposable 3-dimensional solvable Lie algebra whose brackets are in the basis \([X, Y_1, Y_2]\) given by:

\[\[X, Y_i\] = i Y_i, \quad i = 1, 2.\]

This Lie algebra is not rigid but, as we argued in Section 2.2, there exists only one Poisson algebra structure whose corresponding Lie algebra is \(\mathfrak{g}\). This Poisson algebra is \(P_{3,7}(2)\) and it can be deformed into \(P_{3,7}(2 + t)\). The corresponding cocycle of deformation is given by \(\phi(X, Y_2) = Y_2\). It defines a deformation of \((S(\mathfrak{g}), P_0)\). The cases \(n = 4, 5\) can be discussed in the same manner.
3. If \( n = 6 \), then \( g \) is rigid. Its structure constants are given by

\[
\begin{align*}
[X, Y_i] &= iY_i, \quad i = 1, \ldots, 5, \\
[Y_1, Y_i] &= Y_{i+1}, \quad i = 2, 3, 4, \\
[Y_2, Y_3] &= Y_5.
\end{align*}
\]

The Lie algebra \( K_2(g) \) can be deformed using the cocycle \( \phi_1(Y_1, Y_3) = Y_2^2 \). Then \((S(g), P_0)\) is not rigid.

More generally, if we suppose \( n > 12 \), then \( g \) is rigid. Taking \( \phi_1(Y_1, Y_2) = Y_1 Y_2 \) then the base change defined by \( Z_1 = Y_1, Z_2 = Y_2, Z_3 = Y_3 + t Y_1 Y_2, Z_i = [Y_1, Z_{i-1}] \) for \( i \leq n - 2 \) shows that the deformation of \((S(g), P_0)\) given by \( \phi_1 \) is isomorphic to a Poisson algebra which satisfies in particular

\[
\begin{align*}
P(Y_1, Y_i) &= Y_{i+1}, \quad i = 2, \ldots, n - 2, \\
P(Y_2, Y_3) &= Y_5 + t(Y_1 Y_4) + t^2(Y_1^2 Y_3 + Y_1 Y_2^2) + t^3(Y_1^3 Y_2)
\end{align*}
\]

and that \((S(g), P_0)\) is not rigid.

References