GENERAL FAILURE OF LOGIC PROGRAMS

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▷ A classification of any logic program's failures into different levels of general finite failure is introduced. The general failure is then shown to be the limit of those general finite failures, and its interpretation coincides with the complement of the greatest model of the program. As a consequence of this, the negation-as-failure rule is proved. ◁

1. INTRODUCTION

Since the negation-as-failure rule was introduced by Clark [3], several authors have contributed to Clark's work by justifying this rule in various environments. Apt and van Emden [1] established the soundness of the rule for SLD resolution with respect to finite failure. Blair [2] gave a measure to the incompleteness of the rule. Lassez and Maher [8] proved the rule's soundness and completeness for fair SLD resolution, and Jaffar et al. [5] established its completeness for complete logic programs. Recently, Jaffar et al. [6] (also [7]) went a step further by proving the soundness and completeness of the negation-as-failure rule for complete logic programs incorporating equality axioms of the form of definite clauses. However, the proof of these results requires unification completeness and lengthy argument to extend the equality theory so that subscripted variable substitutions are included in the equality model. As the authors indicated, the need for this complexity was due to their inability to define the general failure inductively. It is also expected that if we work at the ground level instead of at the interpreter level, then there will be no problem of variable replacement, and so unification completeness is not needed. In this paper, we define derivation sequences for a logic program at the ground level, and we classify the program's failures into different categories of general finite failures. We then prove that the general failure set is the limit of those general finite failure sets, and that its interpretation is the complement of the greatest model of the program.

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As a consequence of this, the soundness and completeness of the negation-as-failure rule are proved for complete logic programs with equality theory.

The paper is organized as follows. In Section 2, we summarize the definitions related to logic programs with equality discussed by Jaffar et al. [6,7]. Section 3 contains the definition of ground derivation sequences and some basic properties that are needed in subsequent sections. In Section 4, we define the success, finite failure, and first-order general finite failure sets, both explicitly and inductively. The interpretations of these sets are also characterized. Those definitions and results are extended to higher-order general finite failure sets which are presented in Section 5. In the final section, we present our main results, namely Theorem 1 and Corollary 2. The former states that the general finite failure sets converge to the general failure set, and the latter is its consequence that verifies the negation-as-failure rule.

2. PRELIMINARIES: LOGIC PROGRAMS WITH EQUALITY

Throughout this paper, the symbol P stands for a *definite-clause logic program*, i.e. a finite set of definite clauses (van Emden and Kowalski [4]), and the letter E stands for a *definite-clause equality theory*, i.e. a set of definite equality clauses of the form $e \leftarrow e_1, \dots, e_m$, where $m \geq 0$ and the atoms e, e_i are equations (Selman [9]).

A *logic program* is defined to be a pair (P, E) of a definite clause logic program and a definite equality theory. Let V, Σ, and Π denote, respectively, the sets of variables, function symbols, and nonlogical predicate symbols occurring in (P, E). The symbols $\tau(\Sigma)$ and $\tau(\Sigma \cup V)$ are used respectively to denote the sets of ground terms and terms possibly containing variables. By a *model of E* we mean any set R of pairs of ground terms such that all clauses of E are true in R. Analogous with the existence of a least model for P, a least model for E exists, as shown by Jaffar et al. [6]. We denote by $\tau(\Sigma)/E$ the quotient of $\tau(\Sigma)$ by this least relation, with the obvious functional assignment $f([t_1], \dots, [t_n]) = [f(t_1, \dots, t_n)]$ for all n-ary f in Σ. This relation also partitions the Herbrand base of P into classes $[p(\tilde{t})]$ where $p \in \Pi$ and $\tilde{t} \in (\tau(\Sigma))^n$.

The *E-base* is defined to be the set of all classes $[p(\tilde{t})]$. Any subset of the E-base is called an *E-interpretation*. If G is a set of ground atoms, we write $[G]$ to mean $\{[g] : g \in G\}$. For any $\tilde{t}, \tilde{u} \in (\tau(\Sigma \cup V))^n$, we say that $\tilde{t}$ *E-unifies with* $\tilde{u}$ iff there exists a substitution θ such that $[\tilde{t}\theta] = [\tilde{u}\theta]$.

3. DERIVATION SEQUENCES

Let $\mathscr{G}$ be the set of all finite lists $(p_1, \dots, p_m)$ of ground atoms $p_i(\tilde{t})$ where p_i is an n-ary predicate symbol and $\tilde{t} \in (\tau(\Sigma))^n$; the empty list is denoted by $[\,]$. Let $\mathscr{C}$ be the set of all finite lists $(C_1, \dots, C_k)$ where each C_j is a ground instance of a clause in P.

We define a set-valued function ϕ from $\mathscr{G}$ into $\mathscr{C}$ as follows:

$$\phi(p(\tilde{t})) = \{C : C \text{ is a ground instance of a clause in } P,$$
$$p(\tilde{u}) \leftarrow q_1, \dots, q_m, \quad m \geq 0,$$
$$\text{such that } \tilde{t} \text{ } E\text{-unifies with } \tilde{u}\},$$
$$\phi(p_1, \dots, p_n) = \phi(p_1) \times \cdots \times \phi(p_n),$$
$$\phi(G) = \bigcup\{\phi(g) : g \in G\} \qquad \text{for any} \quad G \subseteq \mathscr{G}.$$

We also define a function ψ from $\mathscr{C}$ to $\mathscr{G}$ by

$$\psi(p \leftarrow q_1,\dots,q_m) = (q_1,\dots,q_m),$$
$$\psi(C_1,\dots,C_k) = (\psi(C_1),\dots,\psi(C_k)),$$
$$\psi(\Gamma) = \bigcup\{\psi(C) : C \in \Gamma\} \qquad \text{for any} \quad \Gamma \subseteq \mathscr{C}.$$

Thus if C is a ground instance of a fact in P, then $\psi(C) = [\,]$. For convenience we add to $\mathscr{C}$ an extra member denoted by $\phi([\,])$ and assume that $\psi(\phi([\,])) = [\,]$.

Now for any set $G \subseteq \mathscr{G}$, define

$$S(G) = \phi(G)$$
$$S^n(G) = \phi\psi(S^{n-1}(G)) \qquad \text{for} \quad n > 1.$$

The sequence $\{S^n(G) : n = 1,2,\dots\}$ is called the *(P, E)-derivation sequence* of the goal G. We define a *single derivation sequence from $p(\tilde{t})$* as any sequence $\{C_n\}$ such that $C_1 \in S(p(\tilde{t}))$ and $C_n \in S(\psi(C_{n-1}))$ for all $n > 1$. It may be desirable to attach to each $\phi(p_1,\dots,p_n)$ the associated substitution from which it results, as for a single derivation sequence $\{C_n\}$, the composition of the associated substitutions $\{\theta_n\}$ may represent a solution, if it exists, to the problem in hand. In this paper, however, we have no need to refer to those substitutions, as we place our emphasis on the success or failure of the program, and not on the solution to the problem.

We shall need the following results.

Lemma 1. For any $n \geq 1$ and any list $(p_1,\dots,p_m) \in \mathscr{G}$,

$$S^n(p_1,\dots,p_m) = S^n(p_1) \times \cdots \times S^n(p_m).$$

The proof by induction is quite straightforward.

Lemma 2. For any $n \geq 1$ and $p(\tilde{t}) \in \mathscr{G}$, we have $A \in S^{n+1}(p(\tilde{t}))$ iff there is a ground instance of a clause in P, $p(\tilde{u}) \leftarrow q_1,\dots,q_m$, such that $\tilde{t}$ E-unifies with $\tilde{u}$ and $A \in S^n(q_1,\dots,q_m)$.

PROOF. The lemma is obviously true when $n = 1$. For $n > 1$, it follows from the fact that

$$S^{n+1}(p(\tilde{t})) = (\phi\psi)^n(\phi(p(\tilde{t}))) = (\phi\psi)^{n-1}(\phi(\psi(\phi(p(\tilde{t})))))$$
$$= S^n(\psi(\phi(p(\tilde{t})))). \qquad \square$$

4. SUCCESS AND FINITE FAILURE SETS

We now define the success, finite failure, and general finite failure sets, denoted respectively by SS, FF, and GF, as follows:

$$\mathrm{SS}(P,E) = \{\, p(\tilde{t}) : \tilde{t} \text{ is ground and } \exists n : [\,] \in \psi(S^n(p(\tilde{t})))\}$$
$$\mathrm{FF}(P,E) = \{\, p(\tilde{t}) : \tilde{t} \text{ is ground and } \exists n : S^n(p(\tilde{t})) = \{\,\}\,\}$$
$$\mathrm{GF}(P,E) = \{\, p(\tilde{t}) : \tilde{t} \text{ is ground and } \exists n : \text{ either } S^n(p(\tilde{t})) = \{\,\}$$
$$\text{or } \forall A \in S^n(p(\tilde{t})), \exists k : S^k(\psi(A)) = \{\,\}\,\}.$$

Thus FF is the set of all ground atoms from which the derivation sequence fails at some stage (i.e., all single derivation sequences fail uniformly). The set GF is more general: it covers all ground atoms such that at a certain stage of the unification process, the subsequent derivation sequence from each instance of this stage fails finitely (but possibly nonuniformly).

The success and finite failure sets have also been defined inductively. We recall those definitions here, and we also give an inductive definition for the general finite failure set, which serves as a pattern for the definition of higher-order general finite failure sets to be presented in Section 5:

$$\mathrm{SS}_0(P,E)=\{\},$$

$$\begin{aligned}\mathrm{SS}_{n+1}(P,E)=\{\,p(\tilde{t}):\tilde{t}\text{ is ground and}\\&\text{there is a ground instance of a clause in } P,\\&p(\tilde{u})\leftarrow q_1,\dots,q_m,\quad m\geq 0,\\&\text{such that } \tilde{t}\ E\text{-unifies with } \tilde{u} \text{ and}\\&q_i\in \mathrm{SS}_n(P,E) \text{ for all } 1\leq i\leq m\};\end{aligned}$$

$$\mathrm{FF}_0(P,E)=\{\},$$

$$\begin{aligned}\mathrm{FF}_{n+1}(P,E)=\{\,p(\tilde{t}):\tilde{t}\text{ is ground and}\\&\text{for each ground instance of a clause in } P,\\&p(\tilde{u})\leftarrow q_1,\dots,q_m,\quad m\geq 0,\\&\text{either } \tilde{t} \text{ does not } E\text{-unify with } \tilde{u}\\&\text{or } q_i\in \mathrm{FF}_n(P,E) \text{ for some } 1\leq i\leq m\};\end{aligned}$$

$$\mathrm{GF}_0(P,E)=\mathrm{FF}(P,E),$$

$$\begin{aligned}\mathrm{GF}_{n+1}(P,E)=\{\,p(\tilde{t}):\tilde{t}\text{ is ground and}\\&\text{for each ground instance of a clause in } P,\\&p(\tilde{u})\leftarrow q_1,\dots,q_m,\quad m\geq 0,\\&\text{either } \tilde{t} \text{ does not } E\text{-unify with } \tilde{u}\\&\text{or } q_i\in \mathrm{GF}_n(P,E) \text{ for some } 1\leq i\leq m\}.\end{aligned}$$

We then have the following equivalence:

Lemma 3. For any $n\geq 1$ and any ground atom $p(\tilde{t})$,

(a) *$p(\tilde{t})\in \mathrm{SS}_n(P,E)$ iff $[\,]\in\psi(S^n(p(\tilde{t})))$;*
(b) *$p(\tilde{t})\in \mathrm{FF}_n(P,E)$ iff $S^n(p(\tilde{t}))=\{\}$;*
(c) *$p(\tilde{t})\in \mathrm{GF}_n(P,E)$ iff either $S^n(p(\tilde{t}))=\{\}$ or for every $A\in S^n(p(\tilde{t}))$ there exists k such that $S^k(\psi(A))=\{\}$.*

PROOF. Parts (a) and (b) follow immediately from the definitions and Lemmas 1 and 2. We now prove part (c). In the case $n=1$ we have $p(\tilde{t})\in \mathrm{GF}_1$ iff either $S(p(\tilde{t}))=\{\}$ or for each $A\in S(p(\tilde{t}))$ of the form $p(\tilde{u})\leftarrow q_1,\dots,q_m$, there exists $q_i\in \mathrm{FF}(P,E)$, that is, $S^k(q_i)=\{\}$ for some k. This means that $S^k(q_1)\times\cdots\times S^k(q_m)=\{\}$. That is, by Lemma 1, $S^k(\psi(A))=S^k(q_1,\dots,q_m)=\{\}$. Thus part (c) is true in the case $n=1$. Suppose that (c) is true for n. Let $p(\tilde{t}) \in \mathrm{GF}_{n+1}(P, E)$, and assume that $S^{n+1}(p(\tilde{t}))$ is nonempty. Let $A \in S^{n+1}(p(\tilde{t}))$; then by Lemma 2, there is a ground instance of a clause in P, $p(\tilde{u}) \leftarrow q_1, \ldots, q_m$, such that $\tilde{t}$ E-unifies with $\tilde{u}$ and $A \in S^n(q_1, \ldots, q_m) = S^n(q_1) \times \cdots \times S^n(q_m)$. Thus $A = (A_1, \ldots, A_m)$ with $A_i \in S^n(q_i)$ for $1 \le i \le m$. Now by definition of GF_{n+1}, there exists $q_i \in \mathrm{GF}_n$. Then by our assumption, there exists k such that $S^k(\psi(A_i)) = \{\}$. Take that k; we have $S^k(\psi(A)) = S^k(\psi(A_1)) \times \cdots \times S^k(\psi(A_m)) = \{\}$.

Conversely, assume that the right-hand side of (c) is true for $n+1$. We prove that $p(\tilde{t}) \in \mathrm{GF}_{n+1}(P, E)$. Let $p(\tilde{u}) \leftarrow q_1, \ldots, q_m$ be a ground instance of a clause in P such that $\tilde{t}$ E-unifies with $\tilde{u}$; we claim that $q_i \in \mathrm{GF}_n$ for some $1 \le i \le m$. If not, then all $S^n(q_i)$ are nonempty, and for each i, there exists $A_i \in S^n(q_i)$ such that $S^k(\psi(A_i))$ is nonempty for every k. Let $A = (A_1, \ldots, A_m) \in S^n(q_1, \ldots, q_m)$. Then, by Lemma 2, $A \in S^{n+1}(p(\tilde{t}))$, and $S^k(\psi(A)) = S^k(\psi(A_1)) \times \cdots \times S^k(\psi(A_m))$ is nonempty for every k. This contradicts the above assumption and hence completes the proof. □

From Lemma 3, we have

Proposition 1.

(a) $\mathrm{SS}(P, E) = \bigcup_{n=0}^{\infty} \mathrm{SS}_n(P, E)$,

(b) $\mathrm{FF}(P, E) = \bigcup_{n=0}^{\infty} \mathrm{FF}_n(P, E)$,

(c) $\mathrm{GF}(P, E) = \bigcup_{n=0}^{\infty} \mathrm{GF}_n(P, E)$.

Following van Emden and Kowalski [4] and Jaffar et al. [6, 7], we define the function T from and into E-interpretations of a logic program (P, E) as follows

$$T(I) = \{[p(\tilde{t})] : \text{there is a ground instance of a clause in } P,\ p(\tilde{u}) \leftarrow q_1, \ldots, q_m,\ m \ge 0,\ \text{such that } \tilde{t}\ E\text{-unifies with } \tilde{u} \text{ and } [q_i] \in I \text{ for all } 1 \le i \le m\}.$$

We also use the following notation, in which ω denotes the first limit ordinal:

$$T \uparrow \omega = \bigcup_{n=0}^{\infty} T^n(\{\}),$$

$$T \downarrow \omega = \bigcap_{n=0}^{\infty} T^n(E\text{-base}),$$

$$T \downarrow \omega + \omega = \bigcap_{n=0}^{\infty} T^n(T \downarrow \omega).$$

Jaffar et al. [6] have proved the following results, in which the bar indicates the complement of a set:

(a) $p(\tilde{t}) \in \mathrm{SS}(P, E)$ iff $[p(\tilde{t})] \in T \uparrow \omega$,

(b) $p(\tilde{t}) \in \mathrm{FF}(P, E)$ iff $[p(\tilde{t})] \in \overline{T \downarrow \omega}$.

In order to establish an analogous correspondence between the general finite failure set and $T \downarrow \omega + \omega$, we prove the following lemma.

Lemma 4. For any $n \geq 0$ and ground atom $p(\tilde{t})$,

$$p(\tilde{t}) \in \mathrm{GF}_n(P, E) \quad \textit{iff} \quad [p(\tilde{t})] \in \overline{T^n(T \downarrow \omega)}.$$

PROOF. The equivalence is clearly true for $n = 0$. Assume that it is true for n. Then $[p(\tilde{t})] \notin T^{n+1}(T \downarrow \omega)$ iff for every ground instance of a clause in P, $p(\tilde{u}) \leftarrow q_1, \ldots, q_m$, such that $\tilde{t}$ E-unifies with $\tilde{u}$, there exists $[q_i] \notin T^n(T \downarrow \omega)$ or equivalently $q_i \in \mathrm{GF}_n$, and this means $p(\tilde{t}) \in \mathrm{GF}_{n+1}$. □

The following proposition follows immediately from the above lemma.

Proposition 2.

$$[\mathrm{GF}(P, E)] = \overline{T \downarrow \omega + \omega}.$$

5. HIGHER-ORDER GENERAL FINITE FAILURE SETS

We now extend the definition and related results on the general finite failure set to higher-order general finite failure sets. For abbreviation we write $\mathscr{F}(B, n_i, A_i, k)$ to mean the following failure property:

$$\begin{aligned}
&\exists n_0 : \text{either} \quad S^{n_0}(B) = \{\} \\
&\qquad \text{or} \quad \forall A_1 \in S^{n_0}(B), \quad \exists n_1 : \text{either} \quad S^{n_1}(\psi(A_1)) = \{\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{or} \quad \forall A_2 \in S^{n_1}(\psi(A_1)), \quad \exists n_2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
&\qquad\qquad\qquad\qquad\qquad\qquad \exists n_k : S^{n_k}(\psi(A_k)) = \{\}.
\end{aligned}$$

If $k = \omega$ the sentence is infinite. Note that FF and GF are sets of ground atoms $p(\tilde{t})$ satisfying $\mathscr{F}(p(\tilde{t}), n_i, A_i, k)$ with $k = 0$ and $k = 1$ respectively.

Now let Ω denote the least nonconstructive ordinal. For ordinals $\alpha \leq \Omega$, define

$$\begin{aligned}
&\mathrm{GGF}_0(P, E) = \{\} \\
&\mathrm{GGF}_\alpha(P, E) = \quad \textit{if} \quad (\alpha \text{ is a nonlimit ordinal}) \\
&\qquad \textit{then} \quad \{ p(\tilde{t}) : \tilde{t} \text{ is ground and} \\
&\qquad\qquad \text{for each ground instance of a clause in } P, \\
&\qquad\qquad p(\tilde{u}) \leftarrow q_1, \ldots, q_m, \quad m \geq 0, \\
&\qquad\qquad \text{either } \tilde{t} \text{ does not } E\text{-unify with } \tilde{u} \\
&\qquad\qquad \text{or } q_i \in \mathrm{GGF}_{\alpha-1}(P, E) \text{ for some } 1 \leq i \leq m \} \\
&\qquad \textit{else} \quad \bigcup_{\beta < \alpha} \mathrm{GGF}_\beta(P, E).
\end{aligned}$$

The following result is an extension of Lemma 3, part (c).

Proposition 3. For any ordinal $\alpha < \omega^2$ *and ground atom* $p(\tilde{t})$, *we have* $p(\tilde{t}) \in \mathrm{GGF}_\alpha(P, E)$ *iff* $\mathscr{F}(p(\tilde{t}), n_i, A_i, k_\alpha)$, *where* k_α *is the number of limit ordinals less than* α.

PROOF. The proposition has been proved for $1 \leq \alpha \leq 2\omega$ [parts (b) and (c) of Lemma 3]. Assume that the proposition holds for all $\alpha \leq k\omega$, where k is an integer ≥ 2. We prove that it holds for $\alpha = k\omega + h$, where h is any positive integer. We apply the same technique used in the proof of Lemma 3, part (c). Note first that if $A_i = (A_i^{(1)}, \dots, A_i^{(m)})$, then

$$S^{n_i}(\psi(A_i)) = \{\} \quad \text{iff} \quad S^{n_i}\big(\psi\big(A_i^{(j)}\big)\big) = \{\} \quad \text{for some } 1 \leq j \leq m,$$

$$A_{i+1} \in S^{n_i}(\psi(A_i)) \quad \text{iff} \quad A_{i+1}^{(j)} \in S^{n_i}\big(\psi\big(A_i^{(j)}\big)\big) \quad \text{for all } 1 \leq j \leq m.$$

Now for the case $h = 1$, $p(\tilde{t}) \in \mathrm{GGF}_{k\omega+1}$ iff either $S(p(\tilde{t})) = \{\}$ or for every $A \in S(p(\tilde{t}))$ with $\psi(A) = (q_1, \dots, q_m)$, we have $q_j \in \mathrm{GGF}_{k\omega}$ for some $1 \leq j \leq m$. This, by our assumption, means that $\mathscr{F}(q_j, n_i, A_i^{(j)}, k-1)$ for some $1 \leq j \leq m$, and by the notes mentioned above, it is equivalent to $\mathscr{F}(\psi(A), n_i, A_i, k-1)$. That is, $\mathscr{F}(p(\tilde{t}), m_i, A_i, k)$ with $m_0 = 1$. Thus the proposition holds for $\alpha = k\omega + 1$. Suppose that it holds for $\alpha = k\omega + h$, we extend the result to the case $\alpha = k\omega + h + 1$ in the same way. □

Thus for each ordinal $\alpha = k\omega + h$, where k and h are integers, $k \geq 0$, and $h \geq 1$, the set GGF_α is called a *kth order general finite failure set*; the integer h indicates the first stage of possible failure. The result of Proposition 3 is now further extended to all ordinals $\leq \Omega$ as follows.

Proposition 4. For any ordinal $\alpha \leq \Omega$ *and ground atom* $p(\tilde{t})$, *if* $p(\tilde{t}) \in \mathrm{GGF}_\alpha(P, E)$ *then* $\mathscr{F}(p(\tilde{t}), n_i, A_i, k)$ *for some* $k \leq \omega$.

PROOF. The proof is similar to that of Proposition 3. □

Intuitively, the number k in Proposition 4 is the number of limit ordinals encountered when going backward in the inductive definition from GGF_α to GGF_ω. This number depends on α and $p(\tilde{t})$ as well. It may be unclear at this stage whether Proposition 4 includes the case $k = \omega$. Theorem 1 of the next section will confirm that it does.

Corresponding to the sets GGF_α defined above are the E-interpretations $T \downarrow \alpha$ defined as follows:

$$T \downarrow 0 = E\text{-base}$$

$$T \downarrow \alpha = \textit{if} \quad (\alpha \text{ is a nonlimit ordinal}) \quad \textit{then} \quad T(T \downarrow (\alpha - 1)) \quad \textit{else} \quad \bigcap_{\beta < \alpha} T \downarrow \beta.$$

We then have the following proposition, the proof of which is exactly the same as that of Lemma 4.

Proposition 5. For any ordinal $\alpha \leq \Omega$ *and ground atom* $p(\tilde{t})$, *we have* $p(\tilde{t}) \in \mathrm{GGF}_\alpha(P, E)$ *iff* $[p(\tilde{t})] \in \overline{T \downarrow \alpha}$.

Note that $T\downarrow\Omega$ is a fixed point of T; in fact it is the greatest fixed point of T (see for example Blair [2]).

6. THE GENERAL FAILURE SET AND NEGATION AS FAILURE

We now define the general failure set GGF as follows

$$\mathrm{GGF}(P, E) = \{\, p(\tilde{t}) : \quad \tilde{t} \text{ is ground and } \mathscr{F}(p(\tilde{t}), n_i, A_i, k) \text{ for some } k \leq \omega \}.$$

Certainly, GGF contains all ground atoms from which every single derivation sequence finitely fails. In fact, if $p(\tilde{t})$ is such an atom, let n_0 be the minimum length of all single derivation sequences from $p(\tilde{t})$. Then for each $A_1 \in S^{n_0}(p(\tilde{t}))$, if A_1 is the last member of a single derivation sequence from $p(\tilde{t})$, then take $n_1 = 1$; otherwise take n_1 to be the minimum length of all single derivation sequences from $\psi(A_1)$. Thus either $S^{n_1}(\psi(A_1)) = \{\}$ or for each $A_2 \in S^{n_1}(\psi(A_1))$, if A_2 is the last member of a single derivation sequence from $\psi(A_1)$, then take $n_2 = 1$, and so on. Thus $p(\tilde{t})$ satisfies the failure property $\mathscr{F}(p(\tilde{t}), n_i, A_i, \omega)$.

It is also clear that GGF does not contain the ground atoms from which there are infinite (successful or not) single derivation sequences. (Recall that by our convention, a successful single derivation sequence has all but a finite number of members equal to $\phi([\,])$.)

The following theorem shows that the general failure set GGF is the limit of the general finite failure sets GGF_α defined in the previous section.

Theorem 1.

$$\mathrm{GGF}(P, E) = \bigcup_{\alpha \leq \Omega} \mathrm{GGF}_\alpha(P, E).$$

PROOF. By Propositions 3 and 4, GGF contains GGF_α for all $\alpha \leq \Omega$. Now let $B \notin \cup \mathrm{GGF}_\alpha$. Then, by Proposition 5, $[B] \in T\downarrow\Omega$. Since $T\downarrow\Omega$ is a fixed point of T, there exists a single derivation sequence from B which is either successful or actually infinite. Hence $B \notin \mathrm{GGF}$. □

From this theorem and Proposition 5 we have

Corollary 1. The greatest fixed point of T is $\overline{[\mathrm{GGF}(P, E)]}$.

Now let P^* be the *complete logic program* obtained from P in the way described by Clark [3]. Jaffar et al. [6] have proved that if I is an E-interpretation, then I is a fixed point of T iff I is a model for (P^*, E). Thus, by Corollary 1, $\overline{[\mathrm{GGF}(P, E)]}$ is the greatest model for (P^*, E). Hence we have the following corollary that justifies Clark's negation-as-failure rule.

Corollary 2.

$$(P^*, E) \models \neg p(\tilde{t}) \quad \textit{iff} \quad p(\tilde{t}) \in \mathrm{GGF}(P, E).$$

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