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THE EQUATIONS OF STATIONARY, INCOMPRESSIBLE MAGNETOHYDRODYNAMICS WITH MIXED BOUNDARY CONDITIONS

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Abstract—This paper deals with the questions of existence, uniqueness, and finite element approximation of solutions to the equations of steady-state magnetohydrodynamics with mixed boundary conditions, posed on a bounded, three-dimensional domain. The boundary conditions for the velocity equations are of Dirichlet, Neumann, and mixed type. These boundary conditions are important when considering free boundary value problems, problems on artificially truncated domains, and control problems which are governed by these equations.

1. INTRODUCTION

In this paper, we study the equations of stationary, incompressible magnetohydrodynamics with mixed boundary conditions. Namely we consider boundary conditions on the velocity (Dirichlet type boundary conditions), boundary conditions on the stress (Neumann type boundary conditions), and mixed boundary conditions. These are useful when dealing with free boundary value problems, control problems governed by the equations of stationary incompressible magnetohydrodynamics (considered in a forthcoming paper), and when dealing with artificially truncated computational domains where one needs to prescribe boundary conditions on these artificial boundaries (inflow and outflow boundary conditions). We mention that these equations model phenomena such as flow of liquid metals in the presence of magnetic fields, and plasmas. Thus these equations have direct applications to nuclear reactor technology, magnetic propulsion devices, and design of electromagnetic pumps.

In this paper, we are only interested in problems posed on three-dimensional domains, the analysis for lower dimensional domains is similar and simpler.

In Section 2, we introduce some notation, function spaces, and state some preliminary results. Section 3 is devoted to a description of the boundary conditions, and Section 4 is devoted to the description of the weak form employed. The main results of this paper are in Section 5, an existence and uniqueness result, and in Section 6, a finite element analysis.

We also mention that for the Stokes equations some stress type boundary conditions have been considered in [1,2], while for the Navier Stokes equations some stress type boundary conditions have been considered in [3,4]. For a general overview of possible boundary conditions for the Stokes and Navier Stokes equations we refer the reader to [5]. The equations of magnetohydrodynamics with different boundary conditions have been studied in [6], here we detail the analysis only when it is substantially different due to the different boundary conditions.

2. EQUATIONS, AND FUNCTION SPACES

In this section we describe the equations governing the flows under consideration, introduce notation, and give references to some results that will be needed later.

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We consider the equations posed on a bounded, simply connected, domain $\Omega \subset \mathbb{R}^3$ which is of class $C^{1,1}$ or is a convex polyhedron, see [7,8] (we mention that the following analysis can be extended to include multiply connected domains using the techniques in [9]).

The equations governing the flows under consideration are: in the domain Ω,

$$-\frac{1}{M^2}\nabla\cdot\mathcal{D}(\mathbf{u}) + \frac{1}{N}(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \frac{1}{R_m}(\nabla\times\mathbf{B})\times\mathbf{B} = \mathbf{f}, \tag{2.1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{2.2}$$

$$\frac{1}{R_m}\nabla\times(\nabla\times\mathbf{B}) - \nabla\times(\mathbf{u}\times\mathbf{B}) = \mathbf{0}, \tag{2.3}$$

and

$$\nabla\cdot\mathbf{B} = 0. \tag{2.4}$$

Here $\mathbf{u}$ the velocity, p the pressure, and $\mathbf{B}$ the magnetic field, are the unknowns. A body force $\mathbf{f}$ is given, and M, N, and R_m (the Hartmann number, interaction parameter, and magnetic Reynolds number, respectively) are non-dimensional constants that characterize the flow. The notation $\mathcal{D}(\mathbf{u})$ stands for $(\nabla\mathbf{u} + (\nabla\mathbf{u})^t)$ (note that $\nabla\cdot\mathcal{D}(\mathbf{u}) = \Delta\mathbf{u}$ whenever $\nabla\cdot\mathbf{u} = 0$). For additional discussion and derivation of these equations see [6,10], and the references therein. Also let $\mathcal{T}$ denote the stress tensor which is given by $\frac{1}{M^2}\mathcal{D}(\mathbf{u}) - pI$ (here I is the identity tensor).

Let $\partial\Omega = \overline{\Gamma_1}\cup\overline{\Gamma_2}\cup\overline{\Gamma_3}\cup\overline{\Gamma_4}$ where each of the Γ_i's is regular, open, with a finite number of connected components, and $\Gamma_i\cap\Gamma_j = \emptyset$ if $i \neq j$. Also let $\mathbf{n}$ denote the unit, outward pointing, normal vector to Ω.

Boundary conditions for the velocity can be of different types. For instance, boundary conditions can be of Dirichlet type (given velocity) on some part of the boundary Γ_1, i.e.,

$$\mathbf{u}|_{\Gamma_1} = \mathbf{h}_1, \tag{2.5}$$

of Neumann type (given stress) on some other part of the boundary Γ_2, i.e.,

$$(\mathcal{T}\mathbf{n})|_{\Gamma_2} = \mathbf{g}_2, \tag{2.6}$$

or of mixed type (given normal velocity and tangential stress, or tangential velocity and normal stress) on other parts of the boundary Γ_3, and Γ_4, i.e.,

$$(\mathbf{u}\cdot\mathbf{n})|_{\Gamma_3} = h_3 \quad \text{and} \quad [\mathcal{T}\mathbf{n} - (\mathbf{n}\mathcal{T}\mathbf{n})\mathbf{n}]|_{\Gamma_3} = \mathbf{g}_3, \tag{2.7}$$

and

$$[\mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n}]|_{\Gamma_4} = \mathbf{h}_4 \quad \text{and} \quad (\mathbf{n}\mathcal{T}\mathbf{n})|_{\Gamma_4} = g_4. \tag{2.8}$$

If $\partial\Omega = \overline{\Gamma_1}\cup\overline{\Gamma_3}$, then the boundary data for the velocity must satisfy the compatibility condition $\int_{\Gamma_1}\mathbf{h}_1\cdot\mathbf{n}\,dx + \int_{\Gamma_3}h_3\,dx = 0$, moreover, if $\overline{\Gamma_1}$ and $\overline{\Gamma_3}$ or $\overline{\Gamma_1}$ and $\overline{\Gamma_4}$ have common points, then the boundary conditions at these points must be compatible. For the magnetic field we use the boundary conditions

$$(\mathbf{B}\cdot\mathbf{n})|_{\partial\Omega} = l \quad \text{with} \quad \int_{\partial\Omega} l\,d\mathbf{x} = 0, \tag{2.9}$$

and

$$\left[\frac{1}{R_m}(\nabla\times\mathbf{B})\times\mathbf{n} - (\mathbf{u}\times\mathbf{B})\times\mathbf{n}\right]\Big|_{\partial\Omega} = \mathbf{k}. \tag{2.10}$$

For the physical interpretation of these see [6,10]. Moreover $\mathbf{g}_3$, $\mathbf{h}_4$, and $\mathbf{k}$ must satisfy some additional compatibility conditions, which will be detailed later. An explanation of these compatibility conditions, and why they are necessary for existence of a solution will be given later (and may also be found in [6,10]). At this time we also point out that the boundary conditions $\mathbf{h}_1$, h_3, $\mathbf{h}_4$, and l will be the essential boundary conditions while $\mathbf{g}_2$, $\mathbf{g}_3$, g_4, and $\mathbf{k}$ will be the natural boundary conditions.

We now introduce some function spaces and their associated norms, along with some related notation (for details see [7]). Let $H^m(\Omega)$ (m a nonnegative integer) be the usual m^{th} order Sobolev space equipped with the norm $\|\bullet\|_m$, and let $\mathbf{H}^m(\Omega) := \left(H^m(\Omega)\right)^3$ with norm $\|\bullet\|_m$ be its vector-valued counterpart. On $\mathbf{H}^1(\Omega)$ we use the norm

$$\|\mathbf{w}\|_1 = \left(\|\mathbf{w}\|_0^2 + \|\nabla\mathbf{w}\|_0^2\right)^{1/2}.$$

Clearly, $H^0(\Omega) = L^2(\Omega)$. Two particular subspaces of $\mathbf{H}^1(\Omega)$-functions that satisfy specific boundary conditions are needed; they are

$$\mathbf{H}_b^1(\Omega) := \left\{\mathbf{w} \in \mathbf{H}^1(\Omega) : \mathbf{w}|_{\Gamma_1} = \mathbf{0}, \quad (\mathbf{w}\cdot\mathbf{n})|_{\Gamma_3} = 0, \quad [\mathbf{w} - (\mathbf{w}\cdot\mathbf{n})\mathbf{n}]|_{\Gamma_4} = \mathbf{0}\right\},$$
$$\mathbf{H}_n^1(\Omega) := \left\{\mathbf{T} \in \mathbf{H}^1(\Omega) : (\mathbf{T}\cdot\mathbf{n})|_{\partial\Omega} = 0\right\}.$$

Note that $\mathbf{H}_b^1(\Omega)$ and $\mathbf{H}_n^1(\Omega)$ are closed subspaces of $\mathbf{H}^1(\Omega)$ under the usual $\mathbf{H}^1(\Omega)$-norm, thus we use this norm on these subspaces. We will also make use of the product spaces

$$\mathcal{W}(\Omega) := \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega),$$
$$\mathcal{W}_{bn}(\Omega) := \mathbf{H}_b^1(\Omega) \times \mathbf{H}_n^1(\Omega),$$

which we will equip with the usual product norm. Note that $\mathbf{H}_0^1(\Omega) \times \mathbf{H}_n^1(\Omega) \subset \mathcal{W}_{bn}(\Omega)$. We also define a subspace of $L^2(\Omega)$

$$L_0^2(\Omega) := \left\{q \in L^2(\Omega) : \int_\Omega q\, d\mathbf{x} = 0\right\}.$$

On this subspace, and in general on subspaces, we will use norms induced by the original spaces. We also let $\mathcal{S}(\Omega)$ denote $L_0^2(\Omega)$ if $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_3}$, otherwise $\mathcal{S}(\Omega)$ denotes $L^2(\Omega)$.

Certain trace spaces will also be needed. In particular,

$$H^{1/2}(\partial\Omega) := \left\{w|_{\partial\Omega} : w \in H^1(\Omega)\right\},$$
$$\mathbf{H}^{1/2}(\partial\Omega) := \left\{\mathbf{w}|_{\partial\Omega} : w_i \in H^{1/2}(\partial\Omega), \quad \mathbf{w} = (w_1, w_2, w_3)\right\},$$
$$H^{-1/2}(\partial\Omega) := \left(H^{1/2}(\partial\Omega)\right)^*, \quad \text{and} \quad \mathbf{H}^{-1/2}(\partial\Omega) := \left(\mathbf{H}^{1/2}(\partial\Omega)\right)^*,$$

which are equipped with the usual norms.

We end this section with a few results whose proof may be found in [7,8,11]. If Ω is bounded and has a Lipschitz continuous boundary (these are the kinds of domains under consideration here), Sobolev's embedding theorem yields that $H^1(\Omega) \hookrightarrow\hookrightarrow L^4(\Omega)$, where $\hookrightarrow\hookrightarrow$ denotes compact embedding. Obviously a similar result holds for the spaces $\mathbf{H}^1(\Omega)$ and $\mathbf{L}^4(\Omega)$. For domains as above and for functions $\mathbf{w} \in \mathbf{H}_n^1(\Omega)$ we have a Poincaré type inequality, i.e., there exist a constant c such that

$$\|\mathbf{w}\|_0 \le c\left(\|\nabla\times\mathbf{w}\|_0^2 + \|\nabla\cdot\mathbf{w}\|_0^2\right)^{1/2},$$

moreover on $\mathbf{H}_n^1(\Omega)$, $\left(\|\nabla\times(\bullet)\|_0^2 + \|\nabla\cdot(\bullet)\|_0^2 + \|\bullet\|_0^2\right)^{1/2}$ is a norm equivalent to $\|\bullet\|_1$; this implies that on $\mathbf{H}_n^1(\Omega)$, $\left(\|\nabla\times(\bullet)\|_0^2 + \|\nabla\cdot(\bullet)\|_0^2\right)^{1/2}$ is an equivalent norm to $\|\bullet\|_1$. For details see [6,8,10,11].

3. BOUNDARY CONDITIONS

In this section we state the precise conditions (and regularity) which the boundary conditions, and right hand side must satisfy, in order to guarantee existence of a solution to our problem. We also explain why these boundary conditions must satisfy some compatibility conditions alluded to earlier. We assume that $\mathbf{f} \in \mathbf{H}^1(\Omega)^*$. In the case that the domain Ω is of class $C^{1,1}$ we assume the given boundary conditions for the velocity and the stress, $\mathbf{h}_1$, $\mathbf{g}_2$, h_3, $\mathbf{g}_3$, $\mathbf{h}_4$, and g_4 satisfy

$$\mathbf{h}_1 \in \mathbf{H}^{1/2}(\Gamma_1), \tag{3.1}$$

$$\mathbf{g}_2 \in \mathbf{H}^{-1/2}(\Gamma_2), \tag{3.2}$$

$$h_3 \in H^{1/2}(\Gamma_3), \tag{3.3}$$

$$\mathbf{g}_3 \in \mathbf{H}^{-1/2}(\Gamma_3) \quad \text{with} \quad \mathbf{g}_3 \cdot \mathbf{n} = 0 \quad \text{a.e. on} \quad \Gamma_3, \tag{3.4}$$

$$\mathbf{h}_4 \in \mathbf{H}^{1/2}(\Gamma_4) \quad \text{with} \quad \mathbf{h}_4 \cdot \mathbf{n} = 0 \quad \text{a.e. on} \quad \Gamma_4, \tag{3.5}$$

and

$$g_4\mathbf{n} \in \mathbf{H}^{-1/2}(\Gamma_4). \tag{3.6}$$

The compatibility condition on $\mathbf{g}_3$ and $\mathbf{h}_4$ arises from the fact that both $\mathbf{g}_3$ and $\mathbf{h}_4$ should be tangent to the boundary. As stated earlier, if $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_3}$, then the velocity boundary data must satisfy the additional compatibility condition

$$\int_{\Gamma_1} \mathbf{h}_1 \cdot \mathbf{n}\, d\mathbf{x} + \int_{\Gamma_3} h_3\, d\mathbf{x} = 0, \tag{3.7}$$

which arises from the fact that the velocity is solenoidal (divergence free). In addition, if $\overline{\Gamma_1} \cap \overline{\Gamma_3} \neq \emptyset$, or $\overline{\Gamma_1} \cap \overline{\Gamma_4} \neq \emptyset$, then $\mathbf{h}_1$ and h_3, or $\mathbf{h}_1$ and $\mathbf{h}_4$ need to be compatible so that they are a trace of an $\mathbf{H}^1(\Omega)$-function.

The magnetic field boundary conditions l and $\mathbf{k}$ are assumed to satisfy

$$l \in H^{1/2}(\partial\Omega) \quad \text{with} \quad \int_{\partial\Omega} l\, d\mathbf{x} = 0; \tag{3.8}$$

here the compatibility condition arises from the fact that the magnetic field is solenoidal, and

$$\begin{aligned} &\mathbf{k} \in \mathbf{H}^{-1/2}(\partial\Omega) \quad \text{with} \quad \mathbf{k} \cdot \mathbf{n} = 0 \quad \text{a.e. on} \quad \partial\Omega, \\ &\langle \mathbf{k}, \mathbf{1} \rangle_{\partial\Omega} = 0, \quad \text{and} \quad \langle \mathbf{k}, \nabla\phi|_{\partial\Omega} \rangle_{\partial\Omega} = 0 \quad \forall \phi \in H^2(\Omega). \end{aligned} \tag{3.9}$$

Here $\langle \bullet, \bullet \rangle$ denotes duality pairing. The compatibility conditions on $\mathbf{k}$ arise from the fact that equation (2.3) must be satisfied, and that for $\mathbf{T} \in \mathbf{H}^1(\Omega)$ and $\phi \in H^2(\Omega)$, we have the following identity

$$\int_\Omega (\nabla \times \mathbf{T}) \cdot \nabla\phi\, d\mathbf{x} = -\langle (\mathbf{T} \times \mathbf{n})|_{\partial\Omega}, \nabla\phi|_{\partial\Omega} \rangle_{\partial\Omega},$$

and if $\nabla \times \mathbf{T} = \mathbf{0}$ (i.e., if $\mathbf{T}$ is irrotational) then

$$0 = \int_\Omega (\nabla \times \mathbf{T}) \cdot \mathbf{1}\, d\mathbf{x} = -\langle (\mathbf{T} \times \mathbf{n})|_{\partial\Omega}, \mathbf{1} \rangle_{\partial\Omega}.$$

If the domain Ω is of class $C^{1,1}$ and the given boundary data satisfies (3.1)–(3.9), there exists an extension $(\mathbf{u}_0, \mathbf{B}_0) \in \mathcal{W}(\Omega)$, of the essential boundary data into the domain Ω; moreover one may choose this extension so that $\nabla \cdot \mathbf{u}_0 = 0$. In case the domain Ω is only a convex polyhedron we must require that the data satisfy additional conditions in order to guarantee that one may find an extension $(\mathbf{u}_0, \mathbf{B}_0)$ as above. One such condition is that $h_3\mathbf{n} \in \mathbf{H}^{1/2}(\Gamma_3)$ and that $l\mathbf{n} \in \mathbf{H}^{1/2}(\partial\Omega)$ (this condition guarantees compatibility of the boundary data along the edges and at the vertices of the domain). The existence of these extensions can be shown using the methods of [6,10] and also cf. Section 5.

4. WEAK FORMULATION

We introduce the following forms. For $(\mathbf{u},\mathbf{B})$, $(\mathbf{v},\mathbf{L})$, $(\mathbf{w},\mathbf{T}) \in \mathcal{W}(\Omega)$, $q \in \mathcal{S}(\Omega)$

$$
\begin{aligned}
a\big((\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{L})\big) &:= \int_\Omega \left\{ \frac{1}{2M^2}\mathcal{D}(\mathbf{u}) : \mathcal{D}(\mathbf{v}) + \frac{1}{R_m^2}\left[(\nabla\times\mathbf{B})\cdot(\nabla\times\mathbf{L}) + (\nabla\cdot\mathbf{B})(\nabla\cdot\mathbf{L})\right] \right\} d\mathbf{x}, \\
c\big((\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{L}),(\mathbf{w},\mathbf{T})\big) &:= \int_\Omega \frac{1}{N}(\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\,d\mathbf{x} \\
&\quad - \int_\Omega \frac{1}{R_m}\left[(\nabla\times\mathbf{L})\times\mathbf{B}\cdot\mathbf{w} - (\nabla\times\mathbf{T})\times\mathbf{B}\cdot\mathbf{v}\right] d\mathbf{x}, \\
b\big((\mathbf{v},\mathbf{L}),q\big) &:= -\int_\Omega q\nabla\cdot\mathbf{v}\,d\mathbf{x},
\end{aligned}
$$

and

$$
\begin{aligned}
F\big((\mathbf{v},\mathbf{L})\big) &:= \langle \mathbf{f},\mathbf{v}\rangle_\Omega + \langle \mathbf{g}_2,\mathbf{v}|_{\Gamma_2}\rangle_{\Gamma_2} + \langle \mathbf{g}_3,\mathbf{v}|_{\Gamma_3}\rangle_{\Gamma_3} \\
&\quad + \langle g_4\mathbf{n},\mathbf{v}|_{\Gamma_4}\rangle_{\Gamma_4} + \frac{1}{R_m}\langle \mathbf{k},\mathbf{L}|_{\partial\Omega}\rangle_{\partial\Omega},
\end{aligned}
$$

where : denotes the scalar product on $\mathbb{R}^{3\times3}$, $\cdot$ denotes the scalar product on $\mathbb{R}^3$, and $\times$ denotes the vector product on $\mathbb{R}^3$.

We are now prepared to introduce the weak form of the equations with mixed boundary conditions: find

$$\big((\mathbf{u},\mathbf{B}),p\big) \in \mathcal{W}(\Omega)\times\mathcal{S}(\Omega)$$

such that

$$(\mathbf{u},\mathbf{B}) - (\mathbf{u}_0,\mathbf{B}_0) \in \mathcal{W}_{bn}(\Omega), \tag{4.1}$$

$$a\big((\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{L})\big) + c\big((\mathbf{u},\mathbf{B}),(\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{L})\big) + b\big((\mathbf{v},\mathbf{L}),p\big) = F\big((\mathbf{v},\mathbf{L})\big), \quad \forall(\mathbf{v},\mathbf{L}) \in \mathcal{W}_{bn}(\Omega), \tag{4.2}$$

and

$$b\big((\mathbf{u},\mathbf{B}),q\big) = 0 \qquad \forall q \in \mathcal{S}(\Omega). \tag{4.3}$$

PROPOSITION 4.1. *Equations (4.1)–(4.3) are a weak formulation of equations (2.1)–(2.4) and boundary conditions (2.5)–(2.10).*

PROOF. This proposition is proved in a standard fashion, i.e., using integration by parts and a judicious choice of test functions, and the identity

$$
\begin{aligned}
&\int_\Omega \left\{ \left[-\frac{1}{M^2}\nabla\cdot\mathcal{D}(\mathbf{u}) + \frac{1}{N}(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \frac{1}{R_m}(\nabla\times\mathbf{B})\times\mathbf{B}\right]\cdot\mathbf{v}\right\} d\mathbf{x} \\
&\quad + \int_\Omega \left\{ \frac{1}{R_m}\left[\frac{1}{R_m}\nabla\times(\nabla\times\mathbf{B}) - \nabla\times(\mathbf{u}\times\mathbf{B})\right]\cdot\mathbf{L}\right\} d\mathbf{x} + \int_\Omega \frac{1}{R_m^2}(\nabla\cdot\mathbf{B})(\nabla\cdot\mathbf{L})\,d\mathbf{x} \\
&= \int_\Omega \left\{ \frac{1}{2M^2}\mathcal{D}(\mathbf{u}) : \mathcal{D}(\mathbf{v}) + \frac{1}{R_m^2}\left[(\nabla\times\mathbf{B})\cdot(\nabla\times\mathbf{L}) + (\nabla\cdot\mathbf{B})(\nabla\cdot\mathbf{L})\right]\right\} d\mathbf{x} \\
&\quad + \int_\Omega \frac{1}{N}(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\,d\mathbf{x} - \int_\Omega \frac{1}{R_m}\left[(\nabla\times\mathbf{B})\times\mathbf{B}\cdot\mathbf{v} - (\nabla\times\mathbf{L})\times\mathbf{B}\cdot\mathbf{u}\right] d\mathbf{x} - \int_\Omega p\nabla\cdot\mathbf{v}\,d\mathbf{x} \\
&\quad - \langle (\mathcal{T}\mathbf{n})|_{\partial\Omega},\mathbf{v}|_{\partial\Omega}\rangle_{\partial\Omega} - \frac{1}{R_m}\langle [\frac{1}{R_m}(\nabla\times\mathbf{B})\times\mathbf{n} - (\mathbf{u}\times\mathbf{B})\times\mathbf{n}]|_{\partial\Omega},\mathbf{L}|_{\partial\Omega}\rangle_{\partial\Omega};
\end{aligned}
$$

for details see [5,6,10]. We point out that, since $\mathbf{v}|_{\Gamma_1} = \mathbf{0}$, $(\mathbf{v}\cdot\mathbf{n})|_{\Gamma_3} = 0$, and $[\mathbf{v} - (\mathbf{v}\cdot\mathbf{n})\mathbf{n}]|_{\Gamma_4} = \mathbf{0}$, we get

$$\begin{aligned}
\langle (\mathcal{T}\mathbf{n})|_{\partial\Omega}, \mathbf{v}|_{\partial\Omega} \rangle_{\partial\Omega} &= \langle (\mathcal{T}\mathbf{n})|_{\Gamma_2}, \mathbf{v}|_{\Gamma_2} \rangle_{\Gamma_2} + \langle (\mathcal{T}\mathbf{n})|_{\Gamma_3}, \mathbf{v}|_{\Gamma_3} \rangle_{\Gamma_3} + \langle (\mathcal{T}\mathbf{n})|_{\Gamma_4}, \mathbf{v}|_{\Gamma_4} \rangle_{\Gamma_4} \\
&= \langle (\mathcal{T}\mathbf{n})|_{\Gamma_2}, \mathbf{v}|_{\Gamma_2} \rangle_{\Gamma_2} + \langle (\mathcal{T}\mathbf{n})|_{\Gamma_3}, [\mathbf{v} - (\mathbf{v}\cdot\mathbf{n})\mathbf{n}]|_{\Gamma_3} \rangle_{\Gamma_3} \\
&\quad + \langle (\mathcal{T}\mathbf{n})|_{\Gamma_4}, [\mathbf{v} - (\mathbf{v}\cdot\mathbf{n})\mathbf{n} + (\mathbf{v}\cdot\mathbf{n})\mathbf{n}]|_{\Gamma_4} \rangle_{\Gamma_4} \\
&= \langle (\mathcal{T}\mathbf{n})|_{\Gamma_2}, \mathbf{v}|_{\Gamma_2} \rangle_{\Gamma_2} + \langle (\mathcal{T}\mathbf{n})|_{\Gamma_3}, \mathbf{v}|_{\Gamma_3} \rangle_{\Gamma_3} - \langle (\mathbf{n}\mathcal{T}\mathbf{n})|_{\Gamma_3}, (\mathbf{v}\cdot\mathbf{n})|_{\Gamma_3} \rangle_{\Gamma_3} \\
&\quad + \langle (\mathbf{n}\mathcal{T}\mathbf{n})|_{\Gamma_4}, (\mathbf{v}\cdot\mathbf{n})|_{\Gamma_4} \rangle_{\Gamma_4} \\
&= \langle (\mathcal{T}\mathbf{n})|_{\Gamma_2}, \mathbf{v}|_{\Gamma_2} \rangle_{\Gamma_2} + \langle [\mathcal{T}\mathbf{n} - (\mathbf{n}\mathcal{T}\mathbf{n})\mathbf{n}]|_{\Gamma_3}, \mathbf{v}|_{\Gamma_3} \rangle_{\Gamma_3} \\
&\quad + \langle [(\mathbf{n}\mathcal{T}\mathbf{n})\mathbf{n}]|_{\Gamma_4}, \mathbf{v}|_{\Gamma_4} \rangle_{\Gamma_4}. \qquad \blacksquare
\end{aligned}$$

5. EXISTENCE AND UNIQUENESS

We are now in a position to consider existence and uniqueness of a solution to problem (4.1)–(4.3). We begin by stating some preliminary results, then we turn our attention to a linear problem (associated with the original nonlinear problem) and finally we treat the nonlinear problem (4.1)–(4.3). Throughout this section we assume that $\mathbf{f}$ and the boundary conditions satisfy the hypotheses and conditions stated in Section 3.

For $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^3$, define $\mathbf{w}_{\mathbf{a},\mathbf{b}} : \Omega \mapsto \mathbb{R}^3$ by $\mathbf{w}_{\mathbf{a},\mathbf{b}}(\mathbf{x}) := \mathbf{a} + \mathbf{b} \times \mathbf{x}$. Let

$$\mathcal{V}(\Omega) := \operatorname{span}\left\{\mathbf{w}_{\mathbf{a},\mathbf{b}} : \mathbf{a}, \mathbf{b} \in \mathbb{R}^3, |\mathbf{a}| = |\mathbf{b}| = 1\right\}.$$

LEMMA 5.1. *There exists a constant c such that for all $\mathbf{w} \in \mathbf{H}^1(\Omega)/\mathcal{V}(\Omega)$*

$$\|\mathbf{w}\|_1^2 \le c \int_\Omega \frac{1}{2M^2}\mathcal{D}(\mathbf{w}) : \mathcal{D}(\mathbf{w})\, d\mathbf{x}. \tag{5.1}$$

PROOF. The result follows from Korn's second inequality (see [12,13]), i.e., the existence of a constant c such that

$$\|\mathbf{w}\|_1^2 \le c \int_\Omega \left\{\frac{1}{2M^2}\mathcal{D}(\mathbf{w}) : \mathcal{D}(\mathbf{w}) + \mathbf{w}\cdot\mathbf{w}\right\} d\mathbf{x}, \qquad \forall \mathbf{w} \in \mathbf{H}^1(\Omega),$$

from the fact that if $\mathcal{D}(\mathbf{w}) = 0$, then $\mathbf{w} \in \mathcal{V}(\Omega)$, and from the Peetre Tartar Lemma (see [8] for the Peetre Tartar Lemma). ∎

In the analysis that follows we assume, for simplicity, that Ω and the partition of $\partial\Omega$ into the Γ_is is such that $\mathcal{V}(\Omega) \cap \mathbf{H}^1_b(\Omega) = \{\mathbf{0}\}$. Conditions that guarantee that $\mathcal{V}(\Omega) \cap \mathbf{H}^1_b(\Omega) = \{\mathbf{0}\}$ are that $\Gamma_1 \ne \emptyset$, or that $\Gamma_3 = \partial\Omega$ and Ω does not have an axis of symmetry (see [2]). If this is not the case the following analysis carries over with $\mathbf{H}^1_b(\Omega)$ replaced by $\mathbf{H}^1_b(\Omega)/\mathcal{V}(\Omega)$.

LEMMA 5.2. *The forms $a(\bullet,\bullet)$, $c(\bullet,\bullet,\bullet)$, $b(\bullet,\bullet)$, and $F(\bullet)$ are continuous,* i.e., *there exist constants κ_a, κ_c, κ_b, and κ_F such that*

$$\big|a((\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{L}))\big| \le \kappa_a \|(\mathbf{u},\mathbf{B})\|_{\mathcal{W}} \|(\mathbf{v},\mathbf{L})\|_{\mathcal{W}}, \qquad \forall (\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{L}) \in \mathcal{W}(\Omega), \tag{5.2}$$

$$\begin{aligned}\big|c((\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{L}),(\mathbf{w},\mathbf{T}))\big| &\le \kappa_c \|(\mathbf{u},\mathbf{B})\|_{\mathcal{W}} \|(\mathbf{v},\mathbf{L})\|_{\mathcal{W}} \|(\mathbf{w},\mathbf{T})\|_{\mathcal{W}}, \\ &\qquad \forall (\mathbf{u},\mathbf{B}),(\mathbf{v},\mathbf{L}),(\mathbf{w},\mathbf{T}) \in \mathcal{W}(\Omega),\end{aligned} \tag{5.3}$$

$$\big|b((\mathbf{v},\mathbf{L}),q)\big| \le \kappa_b \|(\mathbf{v},\mathbf{L})\|_{\mathcal{W}} \|q\|_0, \qquad \forall (\mathbf{v},\mathbf{L}) \in \mathcal{W}(\Omega), q \in \mathcal{S}(\Omega), \tag{5.4}$$

and

$$\big|F((\mathbf{v},\mathbf{L}))\big| \le \kappa_F \|(\mathbf{v},\mathbf{L})\|_{\mathcal{W}}, \qquad \forall (\mathbf{v},\mathbf{L}) \in \mathcal{W}(\Omega). \tag{5.5}$$

PROOF. The proof follows from the definition of the forms and is based on an application of Hölders inequality and the fact that $\mathbf{H}^1(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^4(\Omega)$; see [6,10] for details. ∎

LEMMA 5.3. *The bilinear form $a(\bullet,\bullet)$ is coercive on $\mathcal{W}_{bn}(\Omega)$ and the form $b(\bullet,\bullet)$ satisfies the* inf-sup *condition (Ladyzhenskaya-Babuška-Brezzi Condition, see [5,8,14]) on $\mathcal{W}_{bn}(\Omega)\times\mathcal{S}(\Omega)$,* i.e., *there exist positive constants α and β such that*

$$a\big((\hat{\mathbf{u}},\hat{\mathbf{B}}),(\hat{\mathbf{u}},\hat{\mathbf{B}})\big) \ge \alpha\|(\hat{\mathbf{u}},\hat{\mathbf{B}})\|^2_{\mathcal{W}}, \qquad \forall(\hat{\mathbf{u}},\hat{\mathbf{B}}) \in \mathcal{W}_{bn}(\Omega), \tag{5.6}$$

and

$$\inf_{p\in\mathcal{S}(\Omega)} \sup_{(\hat{\mathbf{u}},\hat{\mathbf{B}})\in\mathcal{W}_{bn}(\Omega)} \frac{b\big((\hat{\mathbf{u}},\hat{\mathbf{B}}),p\big)}{\|(\hat{\mathbf{u}},\hat{\mathbf{B}})\|_{\mathcal{W}}\|p\|_0} \ge \beta. \tag{5.7}$$

PROOF. The first inequality follows from Lemma 5.1 and from the existence of a Poincaré type inequality for the functions in $\mathbf{H}^1_n(\Omega)$ (mentioned in Section 2). In the case $\mathcal{S}(\Omega) = L^2_0(\Omega)$ (i.e., if $\partial\Omega = \overline{\Gamma_1}\cup\overline{\Gamma_3}$) the inf-sup condition, (5.7) follows from the classical result that the divergence operator maps $\mathbf{H}^1_0(\Omega)$ onto $L^2_0(\Omega)$, (see [8]), i.e., there exists a β such that

$$\inf_{p\in L^2_0(\Omega)} \sup_{(\hat{\mathbf{u}},\hat{\mathbf{B}})\in\mathbf{H}^1_0(\Omega)\times\{\mathbf{0}\}} \frac{b\big((\hat{\mathbf{u}},\hat{\mathbf{B}}),p\big)}{\|(\hat{\mathbf{u}},\hat{\mathbf{B}})\|_{\mathcal{W}}\|p\|_0} \ge \beta,$$

and since $\mathbf{H}^1_0(\Omega)\times\{\mathbf{0}\}\subset\mathcal{W}_{bn}(\Omega)$, we have that

$$\begin{aligned}\inf_{p\in L^2_0(\Omega)} \sup_{(\hat{\mathbf{u}},\hat{\mathbf{B}})\in\mathcal{W}_{bn}(\Omega)} \frac{b\big((\hat{\mathbf{u}},\hat{\mathbf{B}}),p\big)}{\|(\hat{\mathbf{u}},\hat{\mathbf{B}})\|_{\mathcal{W}}\|p\|_0} &\ge \inf_{p\in L^2_0(\Omega)} \sup_{(\hat{\mathbf{u}},\hat{\mathbf{B}})\in\mathbf{H}^1_0(\Omega)\times\{\mathbf{0}\}} \frac{b\big((\hat{\mathbf{u}},\hat{\mathbf{B}}),p\big)}{\|(\hat{\mathbf{u}},\hat{\mathbf{B}})\|_{\mathcal{W}}\|p\|_0}\\ &\ge \beta.\end{aligned}$$

In the case $\mathcal{S}(\Omega) = L^2(\Omega)$ (i.e., if $\partial\Omega \ne \overline{\Gamma_1}\cup\overline{\Gamma_3}$), using a slight modification to the arguments used in [1], we can show that

$$\inf_{p\in L^2(\Omega)} \sup_{(\hat{\mathbf{u}},\hat{\mathbf{B}})\in\mathbf{H}^1_b(\Omega)\times\{\mathbf{0}\}} \frac{b\big((\hat{\mathbf{u}},\hat{\mathbf{B}}),p\big)}{\|(\hat{\mathbf{u}},\hat{\mathbf{B}})\|_{\mathcal{W}}\|p\|_0} \ge \beta,$$

and since $\mathbf{H}^1_b(\Omega)\times\{\mathbf{0}\}\subset\mathcal{W}_{bn}(\Omega)$, we have that

$$\begin{aligned}\inf_{p\in L^2(\Omega)} \sup_{(\hat{\mathbf{u}},\hat{\mathbf{B}})\in\mathcal{W}_{bn}(\Omega)} \frac{b\big((\hat{\mathbf{u}},\hat{\mathbf{B}}),p\big)}{\|(\hat{\mathbf{u}},\hat{\mathbf{B}})\|_{\mathcal{W}}\|p\|_0} &\ge \inf_{p\in L^2(\Omega)} \sup_{(\hat{\mathbf{u}},\hat{\mathbf{B}})\in\mathbf{H}^1_b(\Omega)\times\{\mathbf{0}\}} \frac{b\big((\hat{\mathbf{u}},\hat{\mathbf{B}}),p\big)}{\|(\hat{\mathbf{u}},\hat{\mathbf{B}})\|_{\mathcal{W}}\|p\|_0}\\ &\ge \beta.\end{aligned}$$

∎

Next we turn our attention to a linear version of the problem. We first prove an existence and uniqueness result for this linear problem; this in turn will allow us to prove an existence and uniqueness result for the original nonlinear problem under consideration.

The regularity of the essential boundary conditions and the compatibility on these, stated in Section 3 ensure the existence of an extension $(\mathbf{u}_0,\mathbf{B}_0)\in\mathcal{W}(\Omega)$, with $\nabla\cdot\mathbf{u}_0 = 0$, of the essential boundary data into the domain Ω. These conditions guarantee the existence of $\mathbf{u}_1\in\mathbf{H}^1(\Omega)$ such that

$$\mathbf{u}_1|_{\Gamma_1} = \mathbf{h}_1, \quad (\mathbf{u}_1\cdot\mathbf{n})|_{\Gamma_3} = h_3, \quad \text{and} \quad [\mathbf{u}_1 - (\mathbf{u}_1\cdot\mathbf{n})\mathbf{n}]|_{\Gamma_4} = \mathbf{h}_4.$$

Let $\mathbf{u}_2\in\mathbf{H}^1_b(\Omega)$ satisfy

$$\nabla\cdot\mathbf{u}_2 = -\nabla\cdot\mathbf{u}_1.$$

The existence of $\mathbf{u}_2$ follows from (5.7), see [8]; then $\mathbf{u}_0 = \mathbf{u}_1 + \mathbf{u}_2$. Likewise the above mentioned conditions guarantee the existence of $\mathbf{B}_0\in\mathbf{H}^1(\Omega)$ such that

$$(\mathbf{B}_0\cdot\mathbf{n})|_{\partial\Omega} = l.$$

Using the extensions of the essential boundary data into the domain $(\mathbf{u}_0, \mathbf{B}_0)$, we can write

$$(\mathbf{u}, \mathbf{B}) = (\mathbf{u}_0, \mathbf{B}_0) + (\hat{\mathbf{u}}, \hat{\mathbf{B}}),$$

where $(\hat{\mathbf{u}}, \hat{\mathbf{B}}) \in \mathcal{W}_{bn}(\Omega)$.

Consider the linear problem: find

$$\big((\mathbf{u}, \mathbf{B}), p\big) \in \mathcal{W}(\Omega) \times \mathcal{S}(\Omega),$$

such that

$$(\mathbf{u}, \mathbf{B}) - (\mathbf{u}_0, \mathbf{B}_0) \in \mathcal{W}_{bn}(\Omega), \tag{5.8}$$

$$a\big((\mathbf{u}, \mathbf{B}), (\mathbf{v}, \mathbf{L})\big) + b\big((\mathbf{v}, \mathbf{L}), p\big) = F\big((\mathbf{v}, \mathbf{L})\big), \qquad \forall (\mathbf{v}, \mathbf{L}) \in \mathcal{W}_{bn}(\Omega), \tag{5.9}$$

and

$$b\big((\mathbf{u}, \mathbf{B}), q\big) = 0, \qquad \forall q \in \mathcal{S}(\Omega). \tag{5.10}$$

Since $\nabla \cdot \mathbf{u}_0 = 0$, this problem is obviously equivalent to the problem: find

$$((\hat{\mathbf{u}}, \hat{\mathbf{B}}), p) \in \mathcal{W}_{bn}(\Omega) \times \mathcal{S}(\Omega), \tag{5.11}$$

such that

$$\begin{aligned} a\big((\hat{\mathbf{u}}, \hat{\mathbf{B}}), (\mathbf{v}, \mathbf{L})\big) + b\big((\mathbf{v}, \mathbf{L}), p\big) = F\big((\mathbf{v}, \mathbf{L})\big) - a\big((\mathbf{u}_0, \mathbf{B}_0), (\mathbf{v}, \mathbf{L})\big), \\ \forall (\mathbf{v}, \mathbf{L}) \in \mathcal{W}_{bn}(\Omega), \end{aligned} \tag{5.12}$$

and

$$b\big((\hat{\mathbf{u}}, \hat{\mathbf{B}}), q\big) = 0, \qquad \forall q \in \mathcal{S}(\Omega). \tag{5.13}$$

THEOREM 5.4. *The linear problem (5.11)–(5.13) has a unique solution* $\big((\hat{\mathbf{u}}, \hat{\mathbf{B}}), p\big) \in \mathcal{W}_{bn}(\Omega) \times \mathcal{S}(\Omega)$. *Moreover,*

$$\|(\hat{\mathbf{u}}, \hat{\mathbf{B}})\|_{\mathcal{W}} \le \frac{\kappa_F}{\alpha} + \frac{\kappa_a}{\alpha} \|(\mathbf{u}_0, \mathbf{B}_0)\|_{\mathcal{W}}, \tag{5.14}$$

and therefore

$$\|(\mathbf{u}, \mathbf{B})\|_{\mathcal{W}} \le \frac{\kappa_F}{\alpha} + \left(1 + \frac{\kappa_a}{\alpha}\right) \|(\mathbf{u}_0, \mathbf{B}_0)\|_{\mathcal{W}}, \tag{5.15}$$

and

$$\|p\|_0 \le \frac{\kappa_F}{\beta} + \frac{\kappa_a}{\beta} \|(\mathbf{u}, \mathbf{B})\|_{\mathcal{W}}, \tag{5.16}$$

where $(\mathbf{u}, \mathbf{B}) = (\mathbf{u}_0, \mathbf{B}_0) + (\hat{\mathbf{u}}, \hat{\mathbf{B}})$.

PROOF. The proof is just an application of the well known Babuška-Brezzi theory (for details see [5,8,14]) using the estimates established in Lemma 5.2 and Lemma 5.3. Estimate (5.14) is obtained by setting $(\mathbf{v}, \mathbf{L}) = (\hat{\mathbf{u}}, \hat{\mathbf{B}})$ in (5.12), and using (5.2), (5.5), (5.6), and (5.13). Estimate (5.15) follows from (5.14) and the triangle inequality, and estimate (5.16) follows easily from (5.2), (5.5), (5.7), and (5.9). ∎

We now turn our attention to the nonlinear problem. Define

$$\gamma := \kappa_F + \Big(\kappa_a + 2\kappa_c \|(\mathbf{u}_0, \mathbf{B}_0)\|_{\mathcal{W}}\Big) \|(\mathbf{u}_0, \mathbf{B}_0)\|_{\mathcal{W}} \tag{5.17}$$

and $\mathcal{B}\big((\mathbf{u}_0, \mathbf{B}_0), \frac{2\gamma}{\alpha}\big)$ to be the ball in $\mathcal{W}(\Omega)$ of radius $\frac{2\gamma}{\alpha}$ centered at $(\mathbf{u}_0, \mathbf{B}_0)$.

THEOREM 5.5. *If the data is sufficiently small, more precisely if $8\gamma\kappa_c < \alpha^2$, then the nonlinear problem (4.1)–(4.3) has a unique solution $\big((\mathbf{u}, \mathbf{B}), p\big) \in \mathcal{W}(\Omega) \times \mathcal{S}(\Omega)$, with $(\mathbf{u}, \mathbf{B}) \in \mathcal{B}\big((\mathbf{u}_0, \mathbf{B}_0), \frac{2\gamma}{\alpha}\big)$.*

PROOF. The proof is based on a fixed point argument using the contraction mapping principle. Define the mapping G by $G\big((\mathbf{w}, \mathbf{T})\big) = (\mathbf{u}, \mathbf{B})$, where $(\mathbf{u}, \mathbf{B})$ is the unique solution of the following linear problem: find

$$\big((\mathbf{u}, \mathbf{B}), p\big) \in \mathcal{W}(\Omega) \times \mathcal{S}(\Omega),$$

such that

$$(\mathbf{u}, \mathbf{B}) - (\mathbf{u}_0, \mathbf{B}_0) \in \mathcal{W}_{bn}(\Omega), \tag{5.18}$$

$$\begin{aligned} a\big((\mathbf{u}, \mathbf{B}), (\mathbf{v}, \mathbf{L})\big) + b\big((\mathbf{v}, \mathbf{L}), p\big) = F\big((\mathbf{v}, \mathbf{L})\big) - c\big((\mathbf{w}, \mathbf{T}), (\mathbf{w}, \mathbf{T}), (\mathbf{v}, \mathbf{L})\big),& \\ \forall (\mathbf{v}, \mathbf{L}) \in \mathcal{W}_{bn}(\Omega),& \end{aligned} \tag{5.19}$$

and

$$b\big((\mathbf{u}, \mathbf{B}), q\big) = 0, \qquad \forall q \in \mathcal{S}(\Omega) \tag{5.20}$$

(the existence and uniqueness of a solution to this problem follows from Theorem 5.4). We first show that under the hypotheses of the theorem G maps the ball $\mathcal{B}\big((\mathbf{u}_0, \mathbf{B}_0), \frac{2\gamma}{\alpha}\big)$ into itself, and that this mapping is a strict contraction. Taking $(\mathbf{v}, \mathbf{L}) = (\mathbf{u}, \mathbf{B}) - (\mathbf{u}_0, \mathbf{B}_0)$ in (5.19) we get

$$\begin{aligned} a\big((\mathbf{u}, \mathbf{B}) - (\mathbf{u}_0, \mathbf{B}_0), (\mathbf{u}, \mathbf{B}) - (\mathbf{u}_0, \mathbf{B}_0)\big) + b\big((\mathbf{u}, \mathbf{B}) - (\mathbf{u}_0, \mathbf{B}_0), p\big) = F\big((\mathbf{u}, \mathbf{B}) - (\mathbf{u}_0, \mathbf{B}_0)\big) \\ - a\big((\mathbf{u}_0, \mathbf{B}_0), (\mathbf{u}, \mathbf{B}) - (\mathbf{u}_0, \mathbf{B}_0)\big) - c\big((\mathbf{w}, \mathbf{T}), (\mathbf{w}, \mathbf{T}), (\mathbf{u}, \mathbf{B}) - (\mathbf{u}_0, \mathbf{B}_0)\big), \end{aligned}$$

and from (5.20) and the fact that $\nabla \cdot \mathbf{u}_0 = 0$ we get that

$$\begin{aligned} a\big((\mathbf{u}, \mathbf{B}) - (\mathbf{u}_0, \mathbf{B}_0), (\mathbf{u}, \mathbf{B}) - (\mathbf{u}_0, \mathbf{B}_0)\big) = F\big((\mathbf{u}, \mathbf{B}) - (\mathbf{u}_0, \mathbf{B}_0)\big) - a\big((\mathbf{u}_0, \mathbf{B}_0), (\mathbf{u}, \mathbf{B}) - (\mathbf{u}_0, \mathbf{B}_0)\big) \\ - c\big((\mathbf{w}, \mathbf{T}), (\mathbf{w}, \mathbf{T}), (\mathbf{u}, \mathbf{B}) - (\mathbf{u}_0, \mathbf{B}_0)\big). \end{aligned}$$

Now using (5.2), (5.3), (5.5), and (5.6) we deduce that

$$\alpha \|(\mathbf{u}, \mathbf{B}) - (\mathbf{u}_0, \mathbf{B}_0)\|_{\mathcal{W}}^2 \le \Big(\kappa_F + \kappa_a \|(\mathbf{u}_0, \mathbf{B}_0)\|_{\mathcal{W}} + \kappa_c \|(\mathbf{w}, \mathbf{T})\|_{\mathcal{W}}^2\Big) \|(\mathbf{u}, \mathbf{B}) - (\mathbf{u}_0, \mathbf{B}_0)\|_{\mathcal{W}}.$$

Using

$$\|(\mathbf{w}, \mathbf{T})\|_{\mathcal{W}}^2 \le 2\|(\mathbf{w}, \mathbf{T}) - (\mathbf{u}_0, \mathbf{B}_0)\|_{\mathcal{W}}^2 + 2\|(\mathbf{u}_0, \mathbf{B}_0)\|_{\mathcal{W}}^2,$$

and the definition of γ, (5.17), we get that

$$\alpha \|(\mathbf{u}, \mathbf{B}) - (\mathbf{u}_0, \mathbf{B}_0)\|_{\mathcal{W}} \le \gamma + 2\kappa_c \|(\mathbf{w}, \mathbf{T}) - (\mathbf{u}_0, \mathbf{B}_0)\|_{\mathcal{W}}^2.$$

Now since $(\mathbf{w}, \mathbf{T})$ is in the ball $\mathcal{B}\big((\mathbf{u}_0, \mathbf{B}_0), \frac{2\gamma}{\alpha}\big)$, it is obvious that $\|(\mathbf{w}, \mathbf{T}) - (\mathbf{u}_0, \mathbf{B}_0)\|_{\mathcal{W}}^2 \le \frac{4\gamma^2}{\alpha^2}$, and employing the hypothesis $8\gamma\kappa_c < \alpha^2$, we get that $(\mathbf{u}, \mathbf{B}) \in \mathcal{B}\big((\mathbf{u}_0, \mathbf{B}_0), \frac{2\gamma}{\alpha}\big)$.

We now show that G is a strict contraction. Let $(\mathbf{w}, \mathbf{T})_1$, $(\mathbf{w}, \mathbf{T})_2 \in \mathcal{B}\big((\mathbf{u}_0, \mathbf{B}_0), \frac{2\gamma}{\alpha}\big)$ and let $(\mathbf{u}, \mathbf{B})_1 = G\big((\mathbf{w}, \mathbf{T})_1\big)$ and $(\mathbf{u}, \mathbf{B})_2 = G\big((\mathbf{w}, \mathbf{T})_2\big)$. Writing (5.19) for each of these and taking the difference of the two equations we get

$$\begin{aligned} a\big((\mathbf{u}, \mathbf{B})_1 - (\mathbf{u}, \mathbf{B})_2, (\mathbf{v}, \mathbf{L})\big) + b\big((\mathbf{v}, \mathbf{L}), p_1 - p_2\big) \\ = -c\big((\mathbf{w}, \mathbf{T})_1, (\mathbf{w}, \mathbf{T})_1, (\mathbf{v}, \mathbf{L})\big) + c\big((\mathbf{w}, \mathbf{T})_2, (\mathbf{w}, \mathbf{T})_2, (\mathbf{v}, \mathbf{L})\big) \end{aligned}$$

(where p_1 and p_2 denote the pressures corresponding to $(\mathbf{u},\mathbf{B})_1$ and $(\mathbf{u},\mathbf{B})_2$, respectively). Taking $(\mathbf{v},\mathbf{L}) = (\mathbf{u},\mathbf{B})_1 - (\mathbf{u},\mathbf{B})_2$ and using (5.20) we get

$$\begin{aligned} a\big((\mathbf{u},\mathbf{B})_1 - (\mathbf{u},\mathbf{B})_2, (\mathbf{u},\mathbf{B})_1 - (\mathbf{u},\mathbf{B})_2\big) = &- c\big((\mathbf{w},\mathbf{T})_1, (\mathbf{w},\mathbf{T})_1, (\mathbf{u},\mathbf{B})_1 - (\mathbf{u},\mathbf{B})_2\big) \\ &+ c\big((\mathbf{w},\mathbf{T})_2, (\mathbf{w},\mathbf{T})_2, (\mathbf{u},\mathbf{B})_1 - (\mathbf{u},\mathbf{B})_2\big), \end{aligned}$$

so

$$\begin{aligned} a\big((\mathbf{u},\mathbf{B})_1 - (\mathbf{u},\mathbf{B})_2, (\mathbf{u},\mathbf{B})_1 - (\mathbf{u},\mathbf{B})_2\big) = &-c\big((\mathbf{w},\mathbf{T})_1 - (\mathbf{w},\mathbf{T})_2, (\mathbf{w},\mathbf{T})_1, (\mathbf{u},\mathbf{B})_1 - (\mathbf{u},\mathbf{B})_2\big) \\ &+ c\big((\mathbf{w},\mathbf{T})_2, (\mathbf{w},\mathbf{T})_2 - (\mathbf{w},\mathbf{T})_1, (\mathbf{u},\mathbf{B})_1 - (\mathbf{u},\mathbf{B})_2\big), \end{aligned}$$

and using (5.3) and (5.6) we get

$$\begin{aligned} &\alpha\|(\mathbf{u},\mathbf{B})_1 - (\mathbf{u},\mathbf{B})_2\|_{\mathcal{W}}^2 \\ &\qquad \le \kappa_c\Big(\|(\mathbf{w},\mathbf{T})_1\|_{\mathcal{W}} + \|(\mathbf{w},\mathbf{T})_2\|_{\mathcal{W}}\Big)\|(\mathbf{w},\mathbf{T})_1 - (\mathbf{w},\mathbf{T})_2\|_{\mathcal{W}}\,\|(\mathbf{u},\mathbf{B})_1 - (\mathbf{u},\mathbf{B})_2\|_{\mathcal{W}}. \end{aligned}$$

Since $(\mathbf{w},\mathbf{T})_1$, $(\mathbf{w},\mathbf{T})_2 \in \mathcal{B}\big((\mathbf{u}_0,\mathbf{B}_0), \frac{2\gamma}{\alpha}\big)$, it is also obvious that

$$\|(\mathbf{w},\mathbf{T})_1\|_{\mathcal{W}} + \|(\mathbf{w},\mathbf{T})_2\|_{\mathcal{W}} \le \frac{4\gamma}{\alpha} + 2\|(\mathbf{u}_0,\mathbf{B}_0)\|_{\mathcal{W}},$$

so we get that

$$\alpha\|(\mathbf{u},\mathbf{B})_1 - (\mathbf{u},\mathbf{B})_2\|_{\mathcal{W}} \le \Big(\frac{4\gamma\kappa_c}{\alpha} + 2\kappa_c\|(\mathbf{u}_0,\mathbf{B}_0)\|_{\mathcal{W}}\Big)\|(\mathbf{w},\mathbf{T})_1 - (\mathbf{w},\mathbf{T})_2\|_{\mathcal{W}}. \tag{5.21}$$

From the hypothesis $8\gamma\kappa_c < \alpha^2$, we have that $\frac{4\gamma\kappa_c}{\alpha} < \frac{\alpha}{2}$ and from the definition of γ, (5.17), $2\kappa_c\|(\mathbf{u}_0,\mathbf{B}_0)\|_{\mathcal{W}}^2 \le \gamma$, so that

$$\|(\mathbf{u}_0,\mathbf{B}_0)\|_{\mathcal{W}}^2 \le \frac{\gamma}{2\kappa_c} < \frac{\alpha^2}{16\kappa_c^2},$$

thus $2\kappa_c\|(\mathbf{u}_0,\mathbf{B}_0)\|_{\mathcal{W}} < \frac{\alpha}{2}$, or $\frac{4\gamma\kappa_c}{\alpha} + 2\kappa_c\|(\mathbf{u}_0,\mathbf{B}_0)\|_{\mathcal{W}} < \alpha$; hence from (5.21) we conclude that

$$\|(\mathbf{u},\mathbf{B})_1 - (\mathbf{u},\mathbf{B})_2\|_{\mathcal{W}} \le c\|(\mathbf{w},\mathbf{T})_1 - (\mathbf{w},\mathbf{T})_2\|_{\mathcal{W}},$$

for some constant $c < 1$. Thus we have proved the existence of a unique $(\mathbf{u},\mathbf{B}) \in \mathcal{B}\big((\mathbf{u}_0,\mathbf{B}_0), \frac{2\gamma}{\alpha}\big)$, and a unique $p \in \mathcal{S}(\Omega)$ which satisfy (5.18)–(5.20) with $(\mathbf{w},\mathbf{T}) = (\mathbf{u},\mathbf{B})$. Thus (4.1)–(4.3) have a unique solution $\big((\mathbf{u},\mathbf{B}),p\big) \in \mathcal{W}(\Omega) \times \mathcal{S}(\Omega)$ with $(\mathbf{u},\mathbf{B}) \in \mathcal{B}\big((\mathbf{u}_0,\mathbf{B}_0), \frac{2\gamma}{\alpha}\big)$. ∎

We remark that the condition for existence and uniqueness in Theorem 5.5 is a condition on the smallness of the data relative to the Hartmann number, and the Magnetic Reynolds number. More explicitly α, which depends on the domain, the Hartmann number, and the Magnetic Reynolds number must be such that $8\gamma\kappa_c < \alpha^2$ (where κ_c depends on the domain and interaction parameter, and γ depends on the body force $\mathbf{f}$, the domain, and the extension $(\mathbf{u}_0,\mathbf{B}_0)$). For fixed flow parameters the boundary data must be such as to have an extension into the domain $(\mathbf{u}_0,\mathbf{B}_0)$ with sufficiently small norm so that $8\gamma\kappa_c < \alpha^2$.

Assuming $8\gamma\kappa_c < \alpha^2$, one may easily derive the estimate

$$\begin{aligned} \|(\mathbf{u},\mathbf{B})\|_{\mathcal{W}} &\le \|(\mathbf{u}_0,\mathbf{B}_0)\|_{\mathcal{W}} + \frac{2\gamma}{\alpha} \\ &< \frac{\alpha}{2\kappa_c}, \end{aligned} \tag{5.22}$$

for the solution $\big((\mathbf{u},\mathbf{B}),p\big)$ to (4.1)–(4.3) guaranteed by Theorem 5.5, and using a similar argument to that used to obtain (5.16), we get

$$\begin{aligned} \|p\|_0 &\le \frac{\kappa_F}{\beta} + \frac{\kappa_a}{\beta}\|(\mathbf{u},\mathbf{B})\|_{\mathcal{W}} + \frac{\kappa_c}{\beta}\|(\mathbf{u},\mathbf{B})\|_{\mathcal{W}}^2 \\ &< \frac{1}{\beta}\Big(\kappa_F + \frac{\alpha\kappa_a}{2\kappa_c} + \frac{\alpha^2}{4\kappa_c}\Big). \end{aligned} \tag{5.23}$$

The uniqueness result of Theorem 5.5 can be improved slightly; in fact without any smallness assumptions on the data, there exists at most one solution $((\mathbf{u},\mathbf{B}),p) \in \mathcal{W}(\Omega) \times \mathcal{S}(\Omega)$ to (4.1)–(4.3) such that

$$\|(\mathbf{u},\mathbf{B})\|_{\mathcal{W}} < \frac{\alpha}{2\kappa_c}.$$

To prove this, assume there are two such solutions $(\mathbf{u},\mathbf{B})_1$ and $(\mathbf{u},\mathbf{B})_2$, write (5.19) for each and subtract the two equations; setting $(\mathbf{v},\mathbf{L}) = (\mathbf{u},\mathbf{B})_1 - (\mathbf{u},\mathbf{B})_2$ and using (5.20) we get

$$\begin{aligned} a\big((\mathbf{u},\mathbf{B})_1 - (\mathbf{u},\mathbf{B})_2, (\mathbf{u},\mathbf{B})_1 - (\mathbf{u},\mathbf{B})_2\big) = &-c\big((\mathbf{u},\mathbf{B})_1 - (\mathbf{u},\mathbf{B})_2, (\mathbf{u},\mathbf{B})_1, (\mathbf{u},\mathbf{B})_1 - (\mathbf{u},\mathbf{B})_2\big) \\ &+ c\big((\mathbf{u},\mathbf{B})_2, (\mathbf{u},\mathbf{B})_2 - (\mathbf{u},\mathbf{B})_1, (\mathbf{u},\mathbf{B})_1 - (\mathbf{u},\mathbf{B})_2\big), \end{aligned}$$

and using (5.3) and (5.6) we get

$$\alpha\|(\mathbf{u},\mathbf{B})_1 - (\mathbf{u},\mathbf{B})_2\|_{\mathcal{W}}^2 \le \kappa_c\Big(\|(\mathbf{u},\mathbf{B})_1\|_{\mathcal{W}} + \|(\mathbf{u},\mathbf{B})_2\|_{\mathcal{W}}\Big)\|(\mathbf{u},\mathbf{B})_1 - (\mathbf{u},\mathbf{B})_2\|_{\mathcal{W}}^2,$$

or

$$\Big[\alpha - \kappa_c\Big(\|(\mathbf{u},\mathbf{B})_1\|_{\mathcal{W}} + \|(\mathbf{u},\mathbf{B})_2\|_{\mathcal{W}}\Big)\Big]\|(\mathbf{u},\mathbf{B})_1 - (\mathbf{u},\mathbf{B})_2\|_{\mathcal{W}}^2 \le 0.$$

The result then follows from the fact that the hypothesis implies that

$$0 < \Big[\alpha - \kappa_c\Big(\|(\mathbf{u},\mathbf{B})_1\|_{\mathcal{W}} + \|(\mathbf{u},\mathbf{B})_2\|_{\mathcal{W}}\Big)\Big]. \qquad \blacksquare$$

6. FINITE ELEMENT APPROXIMATION AND ERROR ESTIMATES

We now turn to the question of finite element approximations. For simplicity, we assume Ω is a convex polyhedron. We point out that there may be a loss of accuracy, due to a Babuška-type paradox if the domain Ω has curved boundaries and it is approximated by a polyhedron, see [3,4] for details and ways to overcome this problem.

We start by choosing families of finite dimensional spaces $\mathcal{X}^\hbar(\Omega) \subset \mathbf{H}^1(\Omega)$, $\mathcal{Y}^\hbar(\Omega) \subset \mathbf{H}^1(\Omega)$ and $\mathcal{S}^\hbar(\Omega) \subset \mathcal{S}(\Omega)$ parametrized by a parameter $\hbar$ which satisfies $0 < \hbar < 1$. We also define

$$\mathcal{X}_b^\hbar(\Omega) := \mathcal{X}^\hbar(\Omega) \cap \mathbf{H}_b^1(\Omega) \quad \text{and} \quad \mathcal{Y}_n^\hbar(\Omega) := \mathcal{Y}^\hbar(\Omega) \cap \mathbf{H}_n^1(\Omega).$$

On these spaces we use the norms induced by the norms on $\mathbf{H}^1(\Omega)$ and $\mathcal{S}(\Omega)$. Next, we define the product spaces

$$\mathcal{W}^\hbar(\Omega) := \mathcal{X}^\hbar(\Omega) \times \mathcal{Y}^\hbar(\Omega) \quad \text{and} \quad \mathcal{W}_{bn}^\hbar(\Omega) := \mathcal{X}_b^\hbar(\Omega) \times \mathcal{Y}_n^\hbar(\Omega),$$

with norms induced by the norm on $\mathcal{W}(\Omega)$.

As in the previous section we assume throughout this section that $\mathbf{f}$ and the boundary conditions satisfy the hypotheses and conditions stated in Section 3.

We approximate the essential boundary conditions $\mathbf{h}_1$, h_3, $\mathbf{h}_4$, and l by $\mathbf{h}_1^\hbar$, $h_3^\hbar$, $\mathbf{h}_4^\hbar$, and $l^\hbar$ which belong to the restriction to Γ_1 of elements of $\mathcal{X}^\hbar(\Omega)$, to the restriction to Γ_3 of normal components of $\mathcal{X}^\hbar(\Omega)$, to the restriction to Γ_4 of tangential components of $\mathcal{X}^\hbar(\Omega)$, and to the restriction to $\partial\Omega$ of normal components of $\mathcal{Y}^\hbar(\Omega)$, respectively. There are several ways of choosing these approximate boundary conditions. For example these may be chosen as the interpolants of the given functions in the appropriate boundary spaces, or these may be chosen to be some projection of the given boundary conditions onto the appropriate boundary spaces. We assume that we have available

$$\begin{aligned} &\mathbf{h}_1^\hbar \in \mathcal{X}^\hbar(\Omega)|_{\Gamma_1}, \quad h_3^\hbar \in \Big\{(\mathbf{w}^\hbar \cdot \mathbf{n})|_{\Gamma_3} : \mathbf{w}^\hbar \in \mathcal{X}^\hbar(\Omega)\Big\}, \\ &\mathbf{h}_4^\hbar \in \Big\{[\mathbf{w}^\hbar - (\mathbf{w}^\hbar \cdot \mathbf{n})\mathbf{n}]|_{\Gamma_4} : \mathbf{w}^\hbar \in \mathcal{X}^\hbar(\Omega)\Big\}, \quad \text{and} \quad l^\hbar \in \Big\{(\mathbf{w}^\hbar \cdot \mathbf{n})|_\Omega : \mathbf{w}^\hbar \in \mathcal{Y}^\hbar(\Omega)\Big\}. \end{aligned}$$

These are the approximations to $\mathbf{h}_1$, h_3, $\mathbf{h}_4$, and l, respectively. As in the continuous case one may find an extension into the domain of the essential boundary data $(\mathbf{u_0}^\hbar, \mathbf{B_0}^\hbar)$. We emphasize that these extensions are not needed in order to compute the approximate solution and therefore are never constructed in practice, but are only used to derive the following theoretical results.

The approximate (discrete) problem we consider is: find

$$((\mathbf{u}^\hbar, \mathbf{B}^\hbar), p^\hbar) \in \mathcal{W}^\hbar(\Omega) \times \mathcal{S}^\hbar(\Omega)$$

such that

$$(\mathbf{u}^\hbar, \mathbf{B}^\hbar) - (\mathbf{u_0}^\hbar, \mathbf{B_0}^\hbar) \in \mathcal{W}_{bn}(\Omega), \tag{6.1}$$

$$a\big((\mathbf{u}^\hbar, \mathbf{B}^\hbar), (\mathbf{v}^\hbar, \mathbf{L}^\hbar)\big) + c\big((\mathbf{u}^\hbar, \mathbf{B}^\hbar), (\mathbf{u}^\hbar, \mathbf{B}^\hbar), (\mathbf{v}^\hbar, \mathbf{L}^\hbar)\big) + b\big((\mathbf{v}^\hbar, \mathbf{L}^\hbar), p^\hbar\big) = F\big((\mathbf{v}^\hbar, \mathbf{L}^\hbar)\big) \quad \forall (\mathbf{v}^\hbar, \mathbf{L}^\hbar) \in \mathcal{W}_{bn}^\hbar(\Omega), \tag{6.2}$$

and

$$b\big((\mathbf{u}^\hbar, \mathbf{B}^\hbar), q^\hbar\big) = 0, \qquad \forall q^\hbar \in \mathcal{S}^\hbar(\Omega). \tag{6.3}$$

We quickly derive existence and uniqueness results for solutions of the discrete problem, since these closely mimic those of Section 5 for the continuous problem.

Of course, the forms $a(\bullet,\bullet)$, $c(\bullet,\bullet,\bullet)$, $b(\bullet,\bullet)$, and $F(\bullet)$ are continuous on the appropriate finite-dimensional subspaces, and $a(\bullet,\bullet)$ is coercive on $\mathcal{W}_{bn}^\hbar(\Omega)$.

The inf-sup condition, condition (5.7), is not automatically satisfied over the subspaces $\mathcal{W}_{bn}^\hbar(\Omega)$ and $\mathcal{S}^\hbar(\Omega)$; in fact this condition turns out to be a constraint on the finite element pairs $\mathcal{X}^\hbar(\Omega)$ and $\mathcal{S}^\hbar(\Omega)$ that can be used to obtain a stable and accurate approximation. We therefore require that $\mathcal{X}^\hbar(\Omega)$ and $\mathcal{S}^\hbar(\Omega)$ are chosen so that the inf-sup condition is satisfied, i.e., so that

$$\inf_{p^\hbar \in \mathcal{S}^\hbar(\Omega)} \sup_{(\hat{\mathbf{u}}^\hbar, \hat{\mathbf{B}}^\hbar) \in \mathcal{W}_{bn}^\hbar(\Omega)} \frac{b\big((\hat{\mathbf{u}}^\hbar, \hat{\mathbf{B}}^\hbar), p^\hbar\big)}{\|(\hat{\mathbf{u}}^\hbar, \hat{\mathbf{B}}^\hbar)\|_{\mathcal{W}} \|p^\hbar\|_0} \geq \beta^\hbar, \tag{6.4}$$

for some positive constant $\beta^\hbar$. Many finite element pairs that satisfy this requirement have been devised for the approximation of the usual Navier-Stokes equations (since this is exactly the condition necessary for the analogous discretization of the Navier-Stokes equations to yield meaningful approximations); see, e.g., [5] and the references therein. For simplicity and in view of the error estimates it is convenient to choose $\mathcal{Y}^\hbar(\Omega) = \mathcal{X}^\hbar(\Omega)$.

As stated above we denote $(\mathbf{u_0}^\hbar, \mathbf{B_0}^\hbar)$ the extension into the domain of the data (analogous to the one in the continuous case). Given $\mathbf{h}_1^\hbar$, $h_3^\hbar$, $\mathbf{h}_4^\hbar$, and $l^\hbar$, the existence of $\mathbf{u_0}^\hbar \in \mathcal{X}^\hbar(\Omega)$ such that

$$\mathbf{u_0}^\hbar|_{\Gamma_1} = \mathbf{h}_1^\hbar, \quad (\mathbf{u_0}^\hbar \cdot \mathbf{n})|_{\Gamma_3} = h_3^\hbar, \quad \text{and} \quad [\mathbf{u_0}^\hbar - (\mathbf{u_0}^\hbar \cdot \mathbf{n})\mathbf{n}]|_{\Gamma_4} = \mathbf{h}_4^\hbar,$$

and $\mathbf{B}_0 \in \mathcal{Y}^\hbar(\Omega)$ such that

$$(\mathbf{B_0}^\hbar \cdot \mathbf{n})|_{\partial\Omega} = l^\hbar,$$

is obvious from the way the approximations to the boundary conditions were derived. Moreover condition (6.4) guarantees that $\mathbf{u_0}^\hbar$ may be chosen in such a way so that

$$\int_\Omega q^\hbar \nabla \cdot \mathbf{u_0}^\hbar \, d\mathbf{x} = 0, \qquad \forall q^\hbar \in \mathcal{S}^\hbar(\Omega).$$

The proof of this is exactly the same as that in the continuous case.

It should be noted that for reasonable choices of finite element spaces and sufficiently small $\hbar$, $\beta^\hbar$ can be bounded from below uniformly in $\hbar$ and thus we may essentially let $\beta^\hbar = \beta$ (we therefore omit the superscript in the discussion below).

As in the continuous case, define

$$\gamma^\hbar := \kappa_F + \Big(\kappa_a + 2\kappa_c \|(\mathbf{u_0}^\hbar, \mathbf{B_0}^\hbar)\|_{\mathcal{W}}\Big) \|(\mathbf{u_0}^\hbar, \mathbf{B_0}^\hbar)\|_{\mathcal{W}}.$$

It should be noted that by choosing $\hbar$ sufficiently small, $\gamma^\hbar$ can be made arbitrarily close to γ.

THEOREM 6.1. *If the data is sufficiently small, more precisely if $8\gamma^\hbar\kappa_c < \alpha^2$, then the discrete problem (6.1)–(6.3) has a unique solution $((\mathbf{u}^\hbar,\mathbf{B}^\hbar),p^\hbar) \in \mathcal{W}^\hbar(\Omega) \times \mathcal{S}^\hbar(\Omega)$ with $(\mathbf{u}^\hbar,\mathbf{B}^\hbar) \in \mathcal{B}((\mathbf{u}_0{}^\hbar,\mathbf{B}_0{}^\hbar),\frac{2\gamma^\hbar}{\alpha})$.*

PROOF. The proof proceeds exactly as that for Theorem 5.5. ∎

Without any smallness assumption on the data, there exists at most one solution such that $\|(\mathbf{u}^\hbar,\mathbf{B}^\hbar)\|_{\mathcal{W}} < \frac{\alpha}{2\kappa_c}$ and the estimates (5.22) and (5.23) hold with $(\mathbf{u},\mathbf{B})$ and p replaced by $(\mathbf{u}^\hbar,\mathbf{B}^\hbar)$ and $p^\hbar$, respectively.

We now turn our attention to obtaining an error estimate, i.e., an estimate for the difference between the solution of the approximate problem (6.1)–(6.3) and that of the continuous problem (4.1)–(4.3). We only consider the case where both the continuous problem and the discrete problem have a unique solutions in $\mathcal{B}((\mathbf{u}_0,\mathbf{B}_0),\frac{2\gamma}{\alpha})$ and $\mathcal{B}((\mathbf{u}_0{}^\hbar,\mathbf{B}_0{}^\hbar),\frac{2\gamma^\hbar}{\alpha})$, respectively.

Define the subspaces

$$\mathcal{Z}(\Omega) := \left\{\mathbf{w} \in \mathbf{H}^1_b(\Omega) : \nabla\cdot\mathbf{w} = 0\right\}$$

and

$$\mathcal{Z}^\hbar(\Omega) := \left\{\mathbf{w}^\hbar \in \mathcal{X}^\hbar_b(\Omega) : \int_\Omega q^\hbar \nabla\cdot\mathbf{w}^\hbar \, d\mathbf{x} = 0 \quad \forall q^\hbar \in \mathcal{S}^\hbar(\Omega)\right\}.$$

The space $\mathcal{Z}(\Omega)$ is the subspace of functions of $\mathbf{H}^1_b(\Omega)$ which are solenoidal (divergence free) and $\mathcal{Z}^\hbar(\Omega)$is the subspace of functions of $\mathcal{X}^\hbar_b(\Omega)$ which are discretely solenoidal. Note that in general $\mathcal{Z}^\hbar(\Omega) \not\subset \mathcal{Z}(\Omega)$, in fact a measure of the "angle" between the two spaces $\mathcal{Z}(\Omega)$ and $\mathcal{Z}^\hbar(\Omega)$ (see [5] and the references therein) is given by

$$\Theta := \sup_{\substack{\mathbf{z}^\hbar \in \mathcal{Z}^\hbar(\Omega)\\ \|\mathbf{z}^\hbar\|_1 = 1}} \inf_{\mathbf{z}\in\mathcal{Z}(\Omega)} \|\mathbf{z} - \mathbf{z}^\hbar\|_1 .$$

Also note that $0 \le \Theta \le 1$. We also define the affine spaces

$$\mathcal{X}^\hbar_1(\Omega) := \left\{\mathbf{w}^\hbar \in \mathcal{X}^\hbar(\Omega) : \mathbf{w}^\hbar|_{\Gamma_1} = \mathbf{h}^\hbar_1, \quad (\mathbf{w}^\hbar\cdot\mathbf{n})|_{\Gamma_3} = h^\hbar_3, \quad \text{and} \quad [\mathbf{w}^\hbar - (\mathbf{w}^\hbar\cdot\mathbf{n})\mathbf{n}]|_{\Gamma_4} = \mathbf{h}^\hbar_4\right\},$$

$$\mathcal{Y}^\hbar_1(\Omega) := \left\{\mathbf{T}^\hbar \in \mathcal{Y}^\hbar(\Omega) : (\mathbf{T}^\hbar\cdot\mathbf{n})|_{\partial\Omega} = l^\hbar\right\},$$

and their product

$$\mathcal{W}^\hbar_1(\Omega) := \mathcal{X}^\hbar_1(\Omega) \times \mathcal{Y}^\hbar_1(\Omega),$$

and the spaces

$$\mathcal{X}^\hbar_2(\Omega) := \left\{\mathbf{w}^\hbar \in \mathcal{X}^\hbar_1(\Omega) : \int_\Omega q^\hbar\nabla\cdot\mathbf{w}^\hbar\, d\mathbf{x} = 0, \qquad \forall q^\hbar \in \mathcal{S}^\hbar(\Omega)\right\},$$

and

$$\mathcal{W}^\hbar_2(\Omega) := \mathcal{X}^\hbar_2(\Omega) \times \mathcal{Y}^\hbar_1(\Omega).$$

The basic error estimate is given by the following theorem.

THEOREM 6.2. *Let the hypotheses of Theorem 5.5 and Theorem 6.1 be valid. These theorems guarantee existence of solutions to problems (4.1)–(4.3) and (6.1)–(6.3), denoted $((\mathbf{u},\mathbf{B}),p)$ and $((\mathbf{u}^\hbar,\mathbf{B}^\hbar),p^\hbar)$, respectively. Then there exist positive constants $\kappa_i < \infty$, $i = 1,\dots,4$, such that*

$$\|(\mathbf{u},\mathbf{B}) - (\mathbf{u}^\hbar,\mathbf{B}^\hbar)\|_{\mathcal{W}} \le \kappa_1 \inf_{(\mathbf{v}^\hbar,\mathbf{L}^\hbar)\in\mathcal{W}^\hbar_1(\Omega)} \|(\mathbf{u},\mathbf{B}) - (\mathbf{v}^\hbar,\mathbf{L}^\hbar)\|_{\mathcal{W}} + \kappa_2\Theta \inf_{q^\hbar\in\mathcal{S}^\hbar(\Omega)} \|p - q^\hbar\|_0, \tag{6.5}$$

and

$$\|p-p^h\|_0 \le \kappa_3 \inf_{(\mathbf{v}^h,\mathbf{L}^h)\in\mathcal{W}_1^h(\Omega)} \|(\mathbf{u},\mathbf{B})-(\mathbf{v}^h,\mathbf{L}^h)\|_{\mathcal{W}} + \kappa_4 \inf_{q^h\in\mathcal{S}^h(\Omega)} \|p-q^h\|_0. \tag{6.6}$$

PROOF. Let $(\mathbf{w}^h,\mathbf{T}^h)$ be an arbitrary element of $\mathcal{W}_2^h(\Omega)$, then clearly, $(\mathbf{u}^h,\mathbf{B}^h)-(\mathbf{w}^h,\mathbf{T}^h)\in\mathcal{W}_{bn}^h(\Omega)$, in fact

$$\begin{aligned} b\big((\mathbf{u}^h,\mathbf{B}^h)-(\mathbf{w}^h,\mathbf{T}^h),q^h\big) &= b\big((\mathbf{u}^h,\mathbf{B}^h),q^h\big)-b\big((\mathbf{w}^h,\mathbf{T}^h),q^h\big)\\ &=0, \qquad \forall q^h\in\mathcal{S}^h(\Omega),\end{aligned}$$

therefore $(\mathbf{u}^h,\mathbf{B}^h)-(\mathbf{w}^h,\mathbf{T}^h)\in\mathcal{Z}^h(\Omega)\times\mathcal{Y}_n^h(\Omega)$. For the exact solution we have (4.2), i.e.,

$$\begin{aligned} a\big((\mathbf{u},\mathbf{B}),(\mathbf{v}^h,\mathbf{L}^h)\big)+c\big((\mathbf{u},\mathbf{B}),(\mathbf{u},\mathbf{B}),(\mathbf{v}^h,\mathbf{L}^h)\big)+b\big((\mathbf{v}^h,\mathbf{L}^h),p\big) &= F\big((\mathbf{v}^h,\mathbf{L}^h)\big)\\ \forall(\mathbf{v}^h,\mathbf{L}^h)\in\mathcal{W}_{bn}^h(\Omega),& \end{aligned} \tag{6.7}$$

and for the approximate solution we have (6.2), i.e.,

$$\begin{aligned} a\big((\mathbf{u}^h,\mathbf{B}^h),(\mathbf{v}^h,\mathbf{L}^h)\big)+c\big((\mathbf{u}^h,\mathbf{B}^h),(\mathbf{u}^h,\mathbf{B}^h),(\mathbf{v}^h,\mathbf{L}^h)\big)+b\big((\mathbf{v}^h,\mathbf{L}^h),p^h\big) &= F\big((\mathbf{v}^h,\mathbf{L}^h)\big)\\ \forall(\mathbf{v}^h,\mathbf{L}^h)\in\mathcal{W}_{bn}^h(\Omega),& \end{aligned} \tag{6.8}$$

Subtracting (6.8) from (6.7) yields

$$\begin{aligned} &a\big((\mathbf{u},\mathbf{B})-(\mathbf{u}^h,\mathbf{B}^h),(\mathbf{v}^h,\mathbf{L}^h)\big)+c\big((\mathbf{u},\mathbf{B})-(\mathbf{u}^h,\mathbf{B}^h),(\mathbf{u},\mathbf{B}),(\mathbf{v}^h,\mathbf{L}^h)\big)\\ &\quad+c\big((\mathbf{u}^h,\mathbf{B}^h),(\mathbf{u},\mathbf{B})-(\mathbf{u}^h,\mathbf{B}^h),(\mathbf{v}^h,\mathbf{L}^h)\big)+b\big((\mathbf{v}^h,\mathbf{L}^h),p-p^h\big)=0\\ &\qquad\forall(\mathbf{v}^h,\mathbf{L}^h)\in\mathcal{W}_{bn}^h(\Omega).\end{aligned}$$

Letting q^h be an arbitrary element of $\mathcal{S}^h(\Omega)$, we have

$$\begin{aligned} &a\big((\mathbf{w}^h,\mathbf{T}^h)-(\mathbf{u}^h,\mathbf{B}^h),(\mathbf{v}^h,\mathbf{L}^h)\big)+c\big((\mathbf{w}^h,\mathbf{T}^h)-(\mathbf{u}^h,\mathbf{B}^h),(\mathbf{u},\mathbf{B}),(\mathbf{v}^h,\mathbf{L}^h)\big)\\ &\quad+c\big((\mathbf{u}^h,\mathbf{B}^h),(\mathbf{w}^h,\mathbf{T}^h)-(\mathbf{u}^h,\mathbf{B}^h),(\mathbf{v}^h,\mathbf{L}^h)\big)+b\big((\mathbf{v}^h,\mathbf{L}^h),q^h-p^h\big)\\ &=a\big((\mathbf{w}^h,\mathbf{T}^h)-(\mathbf{u},\mathbf{B}),(\mathbf{v}^h,\mathbf{L}^h)\big)+c\big((\mathbf{w}^h,\mathbf{T}^h)-(\mathbf{u},\mathbf{B}),(\mathbf{u},\mathbf{B}),(\mathbf{v}^h,\mathbf{L}^h)\big)\\ &\quad+c\big((\mathbf{u}^h,\mathbf{B}^h),(\mathbf{w}^h,\mathbf{T}^h)-(\mathbf{u},\mathbf{B}),(\mathbf{v}^h,\mathbf{L}^h)\big)+b\big((\mathbf{v}^h,\mathbf{L}^h),q^h-p\big),\\ &\qquad\forall(\mathbf{v}^h,\mathbf{L}^h)\in\mathcal{W}_{bn}^h(\Omega),(\mathbf{w}^h,\mathbf{T}^h)\in\mathcal{W}_2^h(\Omega),q^h\in\mathcal{S}^h(\Omega).\end{aligned} \tag{6.9}$$

Setting $(\mathbf{v}^h,\mathbf{L}^h)=(\mathbf{w}^h,\mathbf{T}^h)-(\mathbf{u}^h,\mathbf{B}^h)$, letting $\mathbf{z}$ be an arbitrary element of $\mathcal{Z}(\Omega)$, and using the definition of the spaces $\mathcal{Z}(\Omega)$ and $\mathcal{Z}^h(\Omega)$, we have from (6.9)

$$\begin{aligned} &a\big((\mathbf{w}^h,\mathbf{T}^h)-(\mathbf{u}^h,\mathbf{B}^h),(\mathbf{w}^h,\mathbf{T}^h)-(\mathbf{u}^h,\mathbf{B}^h)\big)\\ &\quad+c\big((\mathbf{w}^h,\mathbf{T}^h)-(\mathbf{u}^h,\mathbf{B}^h),(\mathbf{u},\mathbf{B}),(\mathbf{w}^h,\mathbf{T}^h)-(\mathbf{u}^h,\mathbf{B}^h)\big)\\ &\quad+c\big((\mathbf{u}^h,\mathbf{B}^h),(\mathbf{w}^h,\mathbf{T}^h)-(\mathbf{u}^h,\mathbf{B}^h),(\mathbf{w}^h,\mathbf{T}^h)-(\mathbf{u}^h,\mathbf{B}^h)\big)\\ &=a\big((\mathbf{w}^h,\mathbf{T}^h)-(\mathbf{u},\mathbf{B}),(\mathbf{w}^h,\mathbf{T}^h)-(\mathbf{u}^h,\mathbf{B}^h)\big)\\ &\quad+c\big((\mathbf{w}^h,\mathbf{T}^h)-(\mathbf{u},\mathbf{B}),(\mathbf{u},\mathbf{B}),(\mathbf{w}^h,\mathbf{T}^h)-(\mathbf{u}^h,\mathbf{B}^h)\big)\\ &\quad+c\big((\mathbf{u}^h,\mathbf{B}^h),(\mathbf{w}^h,\mathbf{T}^h)-(\mathbf{u},\mathbf{B}),(\mathbf{w}^h,\mathbf{T}^h)-(\mathbf{u}^h,\mathbf{B}^h)\big)\\ &\quad+b\big((\mathbf{w}^h,\mathbf{T}^h)-(\mathbf{u}^h,\mathbf{B}^h)-(\mathbf{z},\mathbf{0}),q^h-p\big),\\ &\qquad\forall(\mathbf{w}^h,\mathbf{T}^h)\in\mathcal{W}_2^h(\Omega),q^h\in\mathcal{S}^h(\Omega),\mathbf{z}\in\mathcal{Z}(\Omega).\end{aligned} \tag{6.10}$$

The right hand side of (6.10) may be bounded from above using the continuity properties (5.2)–(5.4), and the left hand side of (6.10) may be bounded from below using (5.3) and (5.6). And so we get

$$\begin{aligned} &\Big(\alpha-\kappa_c\big[\|(\mathbf{u},\mathbf{B})\|_{\mathcal{W}}+\|(\mathbf{u}^h,\mathbf{B}^h)\|_{\mathcal{W}}\big]\Big)\|(\mathbf{w}^h,\mathbf{T}^h)-(\mathbf{u}^h,\mathbf{B}^h)\|_{\mathcal{W}}\\ &\le\Big(\kappa_a+\kappa_c\big[\|(\mathbf{u},\mathbf{B})\|_{\mathcal{W}}+\|(\mathbf{u}^h,\mathbf{B}^h)\|_{\mathcal{W}}\big]\Big)\|(\mathbf{w}^h,\mathbf{T}^h)-(\mathbf{u},\mathbf{B})\|_{\mathcal{W}}\\ &\quad+\kappa_b\frac{\|\mathbf{w}^h-\mathbf{u}^h-\mathbf{z}\|_1}{\|\mathbf{w}^h-\mathbf{u}^h\|_1}\|q^h-p\|_0.\end{aligned} \tag{6.11}$$

Using the definition of Θ, since $\mathbf{w}^h - \mathbf{u}^h \in \mathcal{Z}^h(\Omega)$, we have

$$\inf_{\mathbf{z}\in\mathcal{Z}(\Omega)} \frac{\|\mathbf{w}^h - \mathbf{u}^h - \mathbf{z}\|_1}{\|\mathbf{w}^h - \mathbf{u}^h\|_1} \leq \Theta. \tag{6.12}$$

From (5.22) and its discrete analog, we have that

$$\alpha_1 := \alpha - \kappa_c\Big(\|(\mathbf{u},\mathbf{B})\|_{\mathcal{W}} + \|(\mathbf{u}^h,\mathbf{B}^h)\|_{\mathcal{W}}\Big) > 0. \tag{6.13}$$

Taking the infimum of (6.11) over $\mathcal{Z}(\Omega)$ and using (6.12), (6.13), and the triangle inequality yields

$$\begin{aligned}\|(\mathbf{u},\mathbf{B}) - (\mathbf{u}^h,\mathbf{B}^h)\|_{\mathcal{W}} &\leq (1+\kappa_0)\|(\mathbf{u},\mathbf{B}) - (\mathbf{w}^h,\mathbf{T}^h)\|_{\mathcal{W}} + \frac{\kappa_b}{\alpha_1}\Theta\|p - q^h\|_0 \\ &\forall(\mathbf{w}^h,\mathbf{T}^h) \in \mathcal{W}_2^h(\Omega), q^h \in \mathcal{S}^h(\Omega),\end{aligned} \tag{6.14}$$

with

$$\kappa_0 = \frac{1}{\alpha_1}\Big[\kappa_a + \kappa_c\Big(\|(\mathbf{u},\mathbf{B})\|_{\mathcal{W}} + \|(\mathbf{u}^h,\mathbf{B}^h)\|_{\mathcal{W}}\Big)\Big].$$

Now taking the infimum of (6.14) over $\mathcal{W}_2^h(\Omega)$ and $\mathcal{S}^h(\Omega)$ we have

$$\begin{aligned}\|(\mathbf{u},\mathbf{B}) - (\mathbf{u}^h,\mathbf{B}^h)\|_{\mathcal{W}} \leq (1+\kappa_0) &\inf_{(\mathbf{w}^h,\mathbf{T}^h)\in\mathcal{W}_2^h(\Omega)} \|(\mathbf{u},\mathbf{B}) - (\mathbf{w}^h,\mathbf{T}^h)\|_{\mathcal{W}} \\ &+ \frac{\kappa_b}{\alpha_1}\Theta \inf_{q^h\in\mathcal{S}^h(\Omega)} \|p - q^h\|_0.\end{aligned} \tag{6.15}$$

If (6.4) is satisfied, we can show that

$$\inf_{(\mathbf{w}^h,\mathbf{T}^h)\in\mathcal{W}_2^h(\Omega)} \|(\mathbf{u},\mathbf{B}) - (\mathbf{w}^h,\mathbf{T}^h)\|_{\mathcal{W}} \leq \Big(1 + \frac{\kappa_b}{\beta}\Big) \inf_{(\mathbf{v}^h,\mathbf{L}^h)\in\mathcal{W}_1^h(\Omega)} \|(\mathbf{u},\mathbf{B}) - (\mathbf{v}^h,\mathbf{L}^h)\|_{\mathcal{W}}. \tag{6.16}$$

To this end, let $\mathbf{v}^h$ be an arbitrary element of $\mathcal{X}_1^h(\Omega)$; (6.4) implies that there exists $\hat{\mathbf{z}}^h \in \mathcal{Z}^h(\Omega)^\perp$ such that

$$b\big((\hat{\mathbf{z}}^h,\mathbf{0}),q^h\big) = b\big((\mathbf{u}-\mathbf{v}^h,\mathbf{0}),q^h\big) \quad \forall q^h \in \mathcal{S}^h(\Omega),$$

and

$$\|\hat{\mathbf{z}}^h\|_1 \leq \frac{\kappa_b}{\beta}\|\mathbf{u}-\mathbf{v}^h\|_1,$$

see [8]. Let $\mathbf{w}^h = \hat{\mathbf{z}}^h + \mathbf{v}^h$ then $\mathbf{w}^h \in \mathcal{X}^h(\Omega)$, moreover $\mathbf{w}^h - \mathbf{v}^h \in \mathcal{X}_b^h(\Omega)$ and

$$\begin{aligned} b\big((\mathbf{w}^h,\mathbf{T}^h),q^h\big) = b\big((\mathbf{w}^h,\mathbf{0}),q^h\big) &= b\big((\hat{\mathbf{z}}^h+\mathbf{v}^h,\mathbf{0}),q^h\big) \\ &= b\big((\mathbf{u},\mathbf{0}),q^h\big) = 0, \qquad \forall q^h \in \mathcal{S}^h(\Omega),\end{aligned}$$

so $\mathbf{w}^h \in \mathcal{X}_2^h(\Omega)$ and

$$\begin{aligned}\|\mathbf{u}-\mathbf{w}^h\|_1 &\leq \|\mathbf{u}-\mathbf{v}^h\|_1 + \|\hat{\mathbf{z}}^h\|_1 \\ &\leq \Big(1+\frac{\kappa_b}{\beta}\Big)\|\mathbf{u}-\mathbf{v}^h\|_1.\end{aligned}$$

Since $\mathbf{v}^h$ was arbitrary

$$\inf_{\mathbf{w}^h\in\mathcal{X}_2^h(\Omega)} \|\mathbf{u}-\mathbf{w}^h\|_1 \leq \Big(1+\frac{\kappa_b}{\beta}\Big) \inf_{\mathbf{v}^h\in\mathcal{X}_1^h(\Omega)} \|\mathbf{u}-\mathbf{v}^h\|_1.$$

Thus,

$$\begin{aligned}\inf_{(\mathbf{w}^h,\mathbf{T}^h)\in\mathcal{W}_2^h(\Omega)}\|(\mathbf{u},\mathbf{B})-(\mathbf{w}^h,\mathbf{T}^h)\|_{\mathcal{W}}^2 &= \inf_{\mathbf{w}^h\in\mathcal{X}_2^h(\Omega)}\|\mathbf{u}-\mathbf{w}^h\|_1^2+\inf_{\mathbf{T}^h\in\mathcal{Y}_1^h(\Omega)}\|\mathbf{B}-\mathbf{T}^h\|_1^2\\ &\le\left(1+\frac{\kappa_b}{\beta}\right)^2\left[\inf_{\mathbf{v}^h\in\mathcal{X}_1^h(\Omega)}\|\mathbf{u}-\mathbf{v}^h\|_1^2+\inf_{\mathbf{L}^h\in\mathcal{Y}_1^h(\Omega)}\|\mathbf{B}-\mathbf{L}^h\|_1^2\right]\\ &=\left(1+\frac{\kappa_b}{\beta}\right)^2\inf_{(\mathbf{v}^h,\mathbf{L}^h)\in\mathcal{W}_1^h(\Omega)}\|(\mathbf{u},\mathbf{B})-(\mathbf{v}^h,\mathbf{L}^h)\|_{\mathcal{W}}^2,\end{aligned}$$

from which the result follows. Substituting (6.16) into (6.15) then yields (6.5) with

$$\kappa_1=(1+\kappa_0)\left(1+\frac{\kappa_b}{\beta}\right)\quad\text{and}\quad\kappa_2=\frac{\kappa_b}{\alpha_1}.$$

Now let q^h be an arbitrary element of $\mathcal{S}^h(\Omega)$; we have from (6.7) and (6.8),

$$\begin{aligned}b\big((\mathbf{v}^h,\mathbf{L}^h),q^h-p^h\big)&=b\big((\mathbf{v}^h,\mathbf{L}^h),q^h-p\big)-a\big((\mathbf{u},\mathbf{B})-(\mathbf{u}^h,\mathbf{B}^h),(\mathbf{v}^h,\mathbf{L}^h)\big)\\ &\quad-c\big((\mathbf{u},\mathbf{B})-(\mathbf{u}^h,\mathbf{B}^h),(\mathbf{u},\mathbf{B}),(\mathbf{v}^h,\mathbf{L}^h)\big)\\ &\quad-c\big((\mathbf{u}^h,\mathbf{B}^h),(\mathbf{u},\mathbf{B})-(\mathbf{u}^h,\mathbf{B}^h),(\mathbf{v}^h,\mathbf{L}^h)\big)\\ &\quad\forall(\mathbf{v}^h,\mathbf{L}^h)\in\mathcal{W}_{bn}^h(\Omega),q^h\in\mathcal{S}^h(\Omega).\end{aligned}$$

Using (5.2)–(5.4) and the inf-sup condition (6.4) yields

$$\|q^h-p^h\|_0\le\frac{1}{\beta}\left[\kappa_b\|p-q^h\|_0+\left(\kappa_a+\kappa_c\left[\|(\mathbf{u},\mathbf{B})\|_{\mathcal{W}}+\|(\mathbf{u}^h,\mathbf{B}^h)\|_{\mathcal{W}}\right]\right)\|(\mathbf{u},\mathbf{B})-(\mathbf{u}^h,\mathbf{B}^h)\|_{\mathcal{W}}\right].$$

Now using the triangle inequality, (6.5), and taking the infimum over $\mathcal{S}^h(\Omega)$, we arrive at (6.6) with

$$\kappa_3=\frac{\kappa_1}{\beta}\left(\kappa_a+\kappa_c\left[\|(\mathbf{u},\mathbf{B})\|_{\mathcal{W}}+\|(\mathbf{u}^h,\mathbf{B}^h)\|_{\mathcal{W}}\right]\right),$$

and

$$\kappa_4=1+\frac{\kappa_b}{\beta}+\frac{\kappa_2\Theta}{\beta}\left(\kappa_a+\kappa_c\left[\|(\mathbf{u},\mathbf{B})\|_{\mathcal{W}}+\|(\mathbf{u}^h,\mathbf{B}^h)\|_{\mathcal{W}}\right]\right).$$

Obviously we may obtain an upper bound on κ_3 and κ_4 through the use of (5.22) and the corresponding bound for $(\mathbf{u}^h,\mathbf{B}^h)$. ∎

We remark that the error estimates (6.5) and (6.6) are optimal with respect to the product norm employed since right hand sides of these estimates involve only approximation-theoretic terms.

From a computational view point, (6.1)–(6.3) is apparently a formidable system to solve. We briefly mention that iterative solution techniques that decouple the calculation of the velocity field from the calculation of the magnetic field may be employed. Several solution schemes for these equations have been suggested and analyzed in [6,10].

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