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Exponential stability for stochastic differential equation driven by G-Brownian motion

Defei Zhang [a,b,*], Zengjing Chen [a]

[a] Department of Mathematics, Shandong University, Jinan 250100, China

[b] Department of Mathematics, Honghe University, Mengzi 661100, China

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ABSTRACT

Consider a stochastic differential equation driven by G-Brownian motion

$$dX(t) = AX(t)dt + \sigma(t, X(t))dB_t$$

which might be regarded as a stochastic perturbed system of

$$dX(t) = AX(t)dt.$$

Suppose the second equation is quasi surely exponentially stable. In this paper, we investigate the sufficient conditions under which the first equation is still quasi surely exponentially stable.

1. Introduction

An accurate mathematical model of a dynamic system in finance, economy, or control engineering often includes the consideration of additive stochastic elements as well as multiplicative stochastic elements. As we know, Lyapunov [1] first obtained the stability for the nonlinear system $\dot{X} = A(t)X + f(t, X)$ according to the stability of linear system $\dot{X} = A(t)X$. Since then Itô's breakthrough work about Itô calculus, Hasminskii [2] first studied the stability of the origin of the linear Itô equation

$$dX(t) = AX(t)dt + \sum_{i=1}^{m} B_i X \circ dW_i(t), \quad X(0) = x_0 \in R^n, \ t \geq 0, \tag{1.1}$$

which might be regarded as a stochastic perturbed system of $dX(t) = AX(t)dt$. Arnold et al. [3] studied more systematically the almost sure and moment stability for (1.1). For more information about exponential stability, we refer the reader to Mao [4]. Motivated by uncertainty problems, risk measures and the superhedging in finance, Peng [5] introduced the notion of sublinear expectation space in 2006, together with the notion of sublinear expectation, Peng also introduced the related G-normal distribution and G-Brownian motion. The stochastic calculus with respect to the G-Brownian motion has been established by Peng in [5,6]. Since these notions were introduced, many properties of G-Brownian motion have been studied by researchers, for example, Denis et al. [7] and Gao and Jiang [8], et al. In this paper, we will give the sufficient conditions to show that the solution of stochastic differential equation driven by G-Brownian motion is quasi surely (q.s.) exponentially stable.

* Corresponding author at: Department of Mathematics, Shandong University, Jinan 250100, China. Tel.: +86 015169088242; fax: +86 053188365550.
E-mail addresses: zhdefei@163.com (D. Zhang), zjchen@sdu.edu.cn (Z. Chen).

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2. Preliminaries

In this section, we introduce some notations and preliminaries about sublinear expectations and G-Brownian motion, more details concerning this section may be found in [5,6].

Let Ω be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on Ω. We further suppose that $\mathcal{H}$ satisfies $C \in \mathcal{H}$ for each constant C and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$.

Definition 2.1. A sublinear expectation $\mathbb{E}$ is a functional $\mathbb{E} : \mathcal{H} \to R$ satisfying

(i) Monotonicity: $\mathbb{E}[X] \geq \mathbb{E}[Y]$ if $X \geq Y$.
(ii) Constant preserving: $\mathbb{E}[C] = C$ for $C \in R$.
(iii) Sub-additivity: For each $X, Y \in \mathcal{H}, \mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$.
(iv) Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$ for $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sublinear expectation space. If (i) and (ii) are satisfied, $\mathbb{E}[\cdot]$ is called a nonlinear expectation and the triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a nonlinear expectation space.

We omit the notions of G-normal distribution and G-expectation $\hat{\mathbb{E}}[\cdot]$, see [5]. Let $(B_t)_{t\geq 0}$ be a 1-dimensional G-Brownian motion with $G(a) := \frac{1}{2}\hat{\mathbb{E}}[aB_1^2] = \frac{1}{2}(\bar{\sigma}^2 a^+ - \underline{\sigma}^2 a^-)$, where $\bar{\sigma}^2 = \hat{\mathbb{E}}[B_1^2]$, $\underline{\sigma}^2 = -\hat{\mathbb{E}}[-B_1^2]$, $0 \leq \underline{\sigma} \leq \bar{\sigma} < \infty$. Peng defined the Itô integral with G-Brownian motion and space $L_G^2(\Omega_{t_0}; R^n)$, see the Chapter 3 of Peng [6].

Proposition 2.1 (*See [7]*)**.** *Let $\hat{\mathbb{E}}$ be G-expectation. Then there exists a weekly compact family of probability measures $\mathbb{P}$ on $(\Omega, \mathbf{B}(\Omega))$ such that for all $X \in \mathcal{H}$, $\hat{\mathbb{E}}[X] = \max_{P\in\mathbb{P}} E_P[X]$, where $E_P[\cdot]$ is the linear expectation with respect to P.*

They defined the associated G-capacity: $\hat{C}(A) := \sup_{P\in\mathbb{P}} P(A)$, $\underline{C}(A) := \inf_{P\in\mathbb{P}} P(A), A \in \mathbf{B}(\Omega)$. Note that $\hat{C}(A) = 1 - \underline{C}(A^c), A \in \mathbf{B}(\Omega)$.

3. The main result

In this section, we consider the exponential stability for G-stochastic differential equations. Firstly, given an exponentially stable stochastic linear system

$$\begin{cases} dX_t = AX_t dt, & t \geq t_0 \geq 0, \\ X_{t_0} = X_0, & t_0 \geq 0, \end{cases} \tag{3.1}$$

where the initial condition $X_0 \in L_G^2(\Omega_{t_0}; R^n)$, $X = (X_1, \ldots, X_n)^T$, A is a constant $n \times n$ matrix. Assume that some parameters are excited or perturbed by G-Brownian motion, and the perturbed system has the form

$$\begin{cases} dX_t = AX_t dt + \sigma(t, X_t)dB_t, & t \geq t_0 \geq 0, \\ X_{t_0} = X_0, & t_0 \geq 0, \end{cases} \tag{3.2}$$

where B_t is a d-dimensional G-Brownian motion, and $\sigma : R^+ \times R^n \times \Omega \to R^{n\times d}$ satisfies the conditions for the existence and uniqueness of the solution, see [6]. Denote by $X(t, t_0, X_0)$ the solution of Eq. (3.2).

Theorem 3.1. *Let λ be the maximum of the real parts of all eigenvalues of $-A$. Assume that there exist constants $\rho \geq 0, \alpha > 0$, and a polynomial $p(t)$ such that*

$$\|\sigma(t, x)\|^2 \leq p(t)e^{(-2\lambda+\rho)t}, \tag{3.3}$$

for all $x \in R^n$ and sufficiently large t, and $\limsup_{t\to\infty} \frac{\log \|e^{At}\|^2}{t} \leq -\alpha$. Then the solution of (3.2) has the property

$$\limsup_{t\to\infty} \frac{\log \|X(t, t_0, X_0)\|^2}{t} \leq -\alpha + \rho \quad q.s., \tag{3.4}$$

for all $t_0 \geq 0$ and any $X_0 \in L_G^2(\Omega_{t_0}; R^n)$.

In order to prove Theorem 3.1 we need the following lemma.

Lemma 3.1. *Let B_t be a one dimensional G-Brownian motion. Suppose that there exist constants $\varepsilon > 0$ and $\delta > 0$ such that $\hat{\mathbb{E}}[\exp\{\frac{\delta^2}{2}(1+\varepsilon)\int_0^T \eta^2 d\langle B\rangle_s\}] < \infty$. Then for any $T > 0$ and $r > 0$,*

$$\hat{C}\left(\sup_{0\leq t\leq T}\left\{\int_0^t \eta_s dB_s - \frac{\delta}{2}\int_0^t \eta_s^2 d\langle B\rangle_s\right\} > r\right) \leq e^{-\delta r}.$$

Proof. By the representation theorem of G-expectation, we have

$$\hat{C}\left(\sup_{0\le t\le T}\left\{\int_0^t \eta_s dB_s - \frac{\delta}{2}\int_0^t \eta_s^2 d\langle B\rangle_s\right\} > r\right) = \hat{C}\left(\sup_{0\le t\le T}\exp\left\{\delta\int_0^t \eta_s dB_s - \frac{\delta^2}{2}\int_0^t \eta_s^2 d\langle B\rangle_s\right\} > e^{\delta r}\right)$$
$$\le \frac{\hat{\mathbb{E}}\left[\exp\left\{\delta\int_0^T \eta_s dB_s - \frac{\delta^2}{2}\int_0^T \eta_s^2 d\langle B\rangle_s\right\}\right]}{e^{\delta r}} = e^{-\delta r}.$$

The last inequality is because $\exp(\delta\int_0^T \eta_s dB_s - \frac{\delta^2}{2}\int_0^T \eta_s^2 d\langle B\rangle_s)$ is exponential martingale under $\hat{\mathbb{E}}$. Since

$$E_P\left[\exp\left\{\frac{\delta^2}{2}\int_0^T \eta^2 d\langle B\rangle_s\right\}\right] < \hat{\mathbb{E}}\left[\exp\left\{\frac{\delta^2}{2}(1+\varepsilon)\int_0^T \eta^2 d\langle B\rangle_s\right\}\right] < \infty,$$

from the Novikov's condition, we know that $\exp\{\delta\int_0^t \eta_s dB_s - \frac{\delta^2}{2}\int_0^t \eta_s^2 d\langle B\rangle_s\}$ is a martingale under each E_P and $E_P[\exp\{\delta\int_0^t \eta_s dB_s - \frac{\delta^2}{2}\int_0^t \eta_s^2 d\langle B\rangle_s\}] = 1$, thus $\hat{\mathbb{E}}[\exp\{\delta\int_0^t \eta_s dB_s - \frac{\delta^2}{2}\int_0^t \eta_s^2 d\langle B\rangle_s\}] = 1$. □

Proof of Theorem 3.1. Fix $\epsilon > 0$ arbitrarily, and there exists a positive constant $c = c(\epsilon)$ such that

$$\|e^{-At}\| \le ce^{(2\lambda+\epsilon)t}, \qquad p(t) \le ce^{\epsilon t}, \quad t \ge 0. \tag{3.5}$$

By G-Itô's formula (see Theorem 6.5 in [6]), we have $d(e^{-At}X(t)) = e^{-At}\sigma(t, X_t)dB_t$. Define $u(t) := \|e^{-At}X(t)\|^2$. By G-Itô's formula again we get

$$u(t) = u(t_0) + 2\int_{t_0}^t X_s^T e^{-A^T s}e^{-As}\sigma(s, X_s)dB_s + \int_{t_0}^t \mathrm{trace}\{e^{-As}\sigma(X_s, s)\sigma(X_s, s)^T e^{-A^T s}\}d\langle B\rangle_s \tag{3.6}$$

for all $t \ge t_0$. It follows from Lemma 3.1 that for any $\delta > 0$, $t_1 > t_0$,

$$\hat{C}\left(\sup_{t_0\le t\le t_1}\left\{\int_{t_0}^t X_s^T e^{-A^T s}e^{-As}\sigma(s, X_s)dB_s - \frac{r}{2}\int_{t_0}^t X_s^T e^{-A^T s}e^{-As}\sigma(s, X_s)\sigma(s, X_s)^T e^{-A^T s}e^{-As}X_s d\langle B\rangle_s\right\} > \delta\right) \le e^{-r\delta}. \tag{3.7}$$

Choose an arbitrary $\theta > 1$ and let k be an integer large enough so that $k \ge t_0$. Set $t_1 = k$, $r = e^{-\rho k}$, $\delta = \theta e^{\rho k}\log k$. We then obtain

$$\hat{C}\left(\sup_{t_0\le t\le k}\left\{\int_{t_0}^t X_s^T e^{-A^T s}e^{-As}\sigma(s, X_s)dB_s - \frac{e^{-\rho k}}{2}\int_{t_0}^t X_s^T e^{-A^T s}e^{-As}\sigma(s, X_s)\sigma(s, X_s)^T e^{-A^T s}e^{-As}X_s d\langle B\rangle_s\right\} > \theta e^{\rho k}\log k\right) \le \frac{1}{k^\theta}. \tag{3.8}$$

Using the Borel–Cantelli lemma for capacity (see Lemma 2 in [9]), we deduce that there exists a k_0 such that

$$\int_{t_0}^t X_s^T e^{-A^T s}e^{-As}\sigma(s, X_s)dB_s \le \frac{e^{-\rho k}}{2}\int_{t_0}^t X_s^T e^{-A^T s}e^{-As}\sigma(s, X_s)\sigma(s, X_s)^T e^{-A^T s}e^{-As}X_s d\langle B\rangle_s + \theta e^{\rho k}\log k \tag{3.9}$$

for all $k \ge k_0$, $t_0 \le t \le k$. From assumption (3.3), we know that

$$\begin{aligned}\int_{t_0}^t & X_s^T e^{-A^T s}e^{-As}\sigma(s, X_s)dB_s \\ &\le \frac{e^{-\rho k}}{2}\int_{t_0}^t X_s^T e^{-A^T s}e^{-As}\sigma(s, X_s)\sigma(s, X_s)^T e^{-A^T s}e^{-As}X_s d\langle B\rangle_s + \theta e^{\rho k}\log k \\ &\le \frac{e^{-\rho k}}{2}c^2\int_{t_0}^t u(s)e^{\rho s}d\langle B\rangle_s + \theta e^{\rho k}\log k.\end{aligned} \tag{3.10}$$

It follows from (3.6) and (3.10) that

$$\begin{aligned}u(t) &\le u(t_0) + e^{-\rho k}c^2\int_{t_0}^t u(s)e^{\rho s}d\langle B\rangle_s + 2\theta e^{\rho k}\log k + \int_{t_0}^t \mathrm{trace}\{e^{-As}\sigma(X_s, s)\sigma(X_s, s)^T e^{-A^T s}\}d\langle B\rangle_s \\ &\le u(t_0) + e^{-\rho k}c^2\int_{t_0}^t u(s)e^{\rho s}d\langle B\rangle_s + 2\theta e^{\rho k}\log k + nc^2\int_{t_0}^t e^{\rho s}d\langle B\rangle_s.\end{aligned} \tag{3.11}$$

Then $u(t) \le u(t_0) + e^{-\rho k}c^2\bar{\sigma}^2 \int_{t_0}^t u(s)e^{\rho s}ds + 2\theta e^{\rho k}\log k + nc^2\bar{\sigma}^2 \int_{t_0}^t e^{\rho s}ds$. From the generalization of Gronwall–Bellman inequality (see Theorem 2.1 in [10]), we have

$$\begin{aligned} u(t) &\le (u(t_0) + 2\theta e^{\rho k}\log k)\exp\left[e^{-\rho k}c^2\bar{\sigma}^2\int_{t_0}^t e^{\rho s}ds\right] + nc^2\bar{\sigma}^2\int_{t_0}^t \exp\left[c^2\bar{\sigma}^2 e^{-\rho k}\int_s^t e^{\rho r}dr\right]e^{\rho s}ds \\ &\le (u(t_0) + 2\theta e^{\rho k}\log k + ne^{\rho k})\exp\left[e^{-\rho k}c^2\bar{\sigma}^2\int_{t_0}^t e^{\rho s}ds\right] \\ &\le (u(t_0) + 2\theta e^{\rho k}\log k + ne^{\rho k})\exp\left\{\frac{c^2\bar{\sigma}^2}{\rho}\right\}, \quad t_0 \le t \le k,\ k \ge k_0,\ \text{q.s.} \end{aligned} \tag{3.12}$$

Since $\theta > 1$ is arbitrary and $\frac{u(t)}{e^{\rho t}\log t} \le \frac{u(t)}{e^{\rho(k-1)}\log(k-1)}$, $k-1 \le t \le k$. By (3.12), we have

$$\begin{aligned} \limsup_{t\to\infty}\frac{u(t)}{e^{\rho t}\log t} &\le \limsup_{k\to\infty}\frac{(u(t_0) + 2\theta e^{\rho k}\log k + ne^{\rho k})\exp\left\{\frac{c^2\bar{\sigma}^2}{\rho}\right\}}{e^{\rho(k-1)}\log(k-1)} \\ &\le 2\exp\left\{\rho + \frac{c^2\bar{\sigma}^2}{\rho}\right\} \quad \text{q.s.} \end{aligned} \tag{3.13}$$

Since

$$\limsup_{t\to\infty}\frac{\log\|X(t,t_0,X_0)\|^2}{t} \le \limsup_{t\to\infty}\frac{\log\|e^{At}\|^2}{t} + \limsup_{t\to\infty}\frac{\log\|e^{-At}X(t,t_0,X_0)\|^2}{t}, \tag{3.14}$$

from the inequality (3.13) and assumption, we know that $\limsup_{t\to\infty}\frac{\log\|X(t,t_0,X_0)\|^2}{t} \le -\alpha + \rho$ q.s. □

In the last, we give some examples to illustrate our results.

Example 1. Let $\alpha > 0$, $\rho \ge 0$ and $p(t)$ be a polynomial of t. Suppose that $B(t)$ is one-dimensional G-Brownian motion. Consider G-SDE with initial condition $X(t_0) = x_0$ $(t_0 \ge 0)$:

$$dX(t) = -\alpha X(t)dt + p(t)e^{(-2\alpha+\rho)t}\frac{X(t)}{1+\sqrt{X^2(t)}}dB(t), \quad t \ge t_0. \tag{3.15}$$

By Theorem 3.1, we deduce that the solution of (3.15) has the property $\limsup_{t\to\infty}\frac{\log\|X(t,t_0,x_0)\|^2}{t} \le -\alpha+\rho$ q.s. for all $t_0 \ge 0$ and $x_0 \in L_G^2(\Omega_{t_0}; R)$.

Example 2. Let $B(t) = (B^1(t), B^2(t))$ be a two-dimensional G-Brownian motion, consider a two-dimensional G-SDE:

$$\begin{cases} dX(t) = \begin{pmatrix} -4 & -2 \\ 3 & 1 \end{pmatrix} X(t)dt + e^{-1.5t}\begin{pmatrix} t^2\sin x_1 & t\cos(x_1+x_2) \\ t^3\cos x_2 & t^2\sin(x_1-x_2) \end{pmatrix} dB(t), \quad t \ge t_0, \\ X(t_0) = x_0 \quad (t_0 \ge 0). \end{cases} \tag{3.16}$$

Note that A and $-A$ have the eigenvalues $-1, -2$ and $1, 2$, respectively. Therefore, from the Theorem 3.1, we deduce that the solution of (3.16) has the property $\limsup_{t\to\infty}\frac{\log\|X(t,t_0,x_0)\|^2}{t} \le -\frac{1}{2}$ q.s. for all $t_0 \ge 0$ and $x_0 \in L_G^2(\Omega_{t_0}; R^2)$.

Example 3. Assume that $\rho < \frac{1}{2}$ and $B(t)$ is one-dimensional G-Brownian motion. We consider the following stochastic differential equation driven by G-Brownian motion:

$$d\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ p(t)e^{-\frac{1}{2}\rho t} \end{pmatrix} dB(t),$$

where the initial conditions are $X_1(0) = 0$ and $X_2(0) = 0$. From the Theorem 3.1, we have

$$\limsup_{t\to\infty}\frac{\log(X_1^2(t) + X_2^2(t))}{t} \le -\rho \quad \text{q.s.}$$

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