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INTEGRAL EQUATION ANALYSIS OF THE DRIVEN-CAVITY BOUNDARY SINGULARITY

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Abstract—A specially-modified boundary integral equation (BIE) method is used to investigate the viability of the singular boundary conditions of the well known driven-cavity Stokes flow problem, a bench-mark problem of computational fluid dynamics. We introduce small 'leaks' to replace the singularities, thus creating a perturbed, physically realizable problem. We make two discoveries, namely: (i) unexpectedly, the introduction of the leaks affects the flow field at considerably greater distances from the leaks than one might perhaps intuitively predict; and (ii) the full, numerical BIE solution reveals that the far field, asymptotic, closed-form solution for the flowfield of the perturbed problem is a surprisingly accurate representation of the flow even in the near field.

1. INTRODUCTION

The driven cavity problem, a classical bench-mark problem of computational fluid dynamics, is remarkable in that it corresponds to a physically unrealizable flow. The impossibility of the flow lies in the inherent boundary singularities, first noted by Taylor [1] in his 'scraper problem', across which the boundary stresses are non-integrable; this implies that an infinite force is required to drive the lid with any finite velocity.

The discontinuity of the flow field at these singularities can be circumvented by permitting small 'leaks' allowing fluid to enter or leave the cavity. The boundary geometry of the leaks, which ensues on the basis of this assumption, is shown in Figure 1 below.

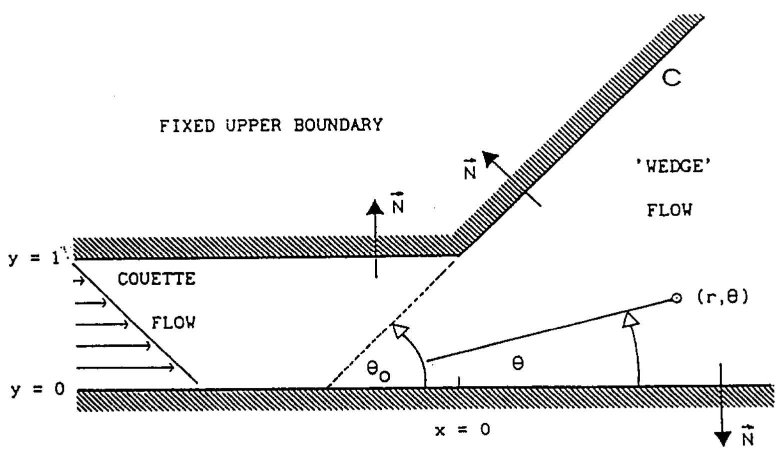

Figure 1. Problem geometry and boundary conditions.

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BIE solutions of Stokes flows in the neighbourhood of re-entrant corners have already been found by Kelmanson [2] using an extension of the singularity subtraction technique introduced by Symm [3] in connection with harmonic problems. However, in [2] the original integral equations contained a vorticity gradient, which is non-integrable at the re-entrant corner. Our approach, like that of Xu [4], differs somewhat in that it is based upon integral equations containing boundary pressure and shear stress, which are both integrable, though singular, at the corner. Furthermore, it makes use of ***singularity incorporation*** near the re-entrant corner, extending the harmonic applications of Hansen [5] and Kelmanson [6] to the present biharmonic problem.

2. FORMULATION

We seek to determine the stationary, incompressible Stokes flow in the region shown in Figure 1. The non-dimensional, biharmonic stream function $\psi = \psi(x, y)$ may readily be sought via integral equation techniques, provided we can write the boundary conditions in a suitable form. There are no-slip conditions on the solid boundaries, the upper of which (C in Figure 1) is stationary, the lower of which slides with unit speed in the positive x direction. Boundary conditions for ψ on the solid walls and far back into the channel are easily found but, in the wedge, its asymptotic form is a little more complicated. Using the polar coordinate system of Figure 1, we find that the asymptotic form of $\psi = \psi(r, \vartheta)$ is :

$$\psi = r\,\frac{\vartheta \sin\vartheta_0\, \sin(\vartheta-\vartheta_0) - \vartheta_0\,(\vartheta-\vartheta_0)\sin\vartheta}{\vartheta_0{}^2 - \sin^2\vartheta_0} + \frac{(\vartheta-\vartheta_0)\cos\vartheta_0 - \sin(\vartheta-\vartheta_0)\cos\vartheta}{2\,(\vartheta_0\cos\vartheta_0 - \sin\vartheta_0)}, \tag{1}$$

where ϑ_0 is the fixed wedge angle shown in Figure 1. Near the re-entrant corner, a consideration of singular nature of the flow must be undertaken, as in [2]. A prolonged analysis, based upon the far-field asymptotic behaviour of the flow in both the wedge and channel regions, and the singular nature of the flow in the re-entrant region, leads to the following coupled integral equations in the unknown values of the boundary pressures p and wall stresses $\tau \equiv -\Delta\,\psi$:

$$\int_C \left\{ p(s)\,\frac{\partial G}{\partial s} - \tau(s)\,\frac{\partial G}{\partial N} \right\} ds = y_0 - \frac{1}{2}, \ (x_0, y_0) \in C, \tag{2}$$

$$\frac{1}{2}\,\Delta_0\,\psi(x_0, y_0) + \int_C \left\{ p(s)\,\frac{\partial \Delta_0 G}{\partial s} - \tau(s)\,\frac{\partial \Delta_0 G}{\partial N} \right\} ds = 0, \ (x_0, y_0) \in C, \tag{3}$$

where G is a Greens function, specifically chosen to annihilate certain integral contributions on the sliding plate, given by

$$G(x-x_0, y, y_0) = \frac{1}{16\pi}\{(x-x_0)^2 + (y-y_0)^2\}\ln\frac{(x-x_0)^2 + (y+y_0)^2}{(x-x_0)^2 + (y-y_0)^2} - \frac{y\,y_0}{4\pi}, \tag{4}$$

and $\Delta_0 \equiv \frac{\partial^2}{\partial x_0{}^2} + \frac{\partial^2}{\partial y_0{}^2}$. In Equations (3) and (4), $\frac{\partial}{\partial N}$ refers to differentiation with respect to the vector $\vec{N}$ of Figure 1. The details of this rather lengthy derivation may be found in Hansen and Kelmanson [7]. On solving Equations (3) and (4) for the boundary stress and pressure distributions, the stream function may be subsequently found using

$$\psi(x_0, y_0) = y_0 - \frac{1}{2} - \int_C \left\{ p(s)\,\frac{\partial G}{\partial s} - \tau(s)\,\frac{\partial G}{\partial N} \right\} ds, \ (x_0, y_0) \in \text{interior}. \tag{5}$$

3. NUMERICAL SOLUTION

The boundary C was broken into sections C_j, $j = 1, \ldots, 6$. On C_1, which extended along $y = 1$ from $x = -\infty$ to $x = -4$, the asymptotic Couette behaviour of $p(x, 1)$ and $\tau(x, 1)$ was enforced. C_2, on $y = 1$, from $x = -4$ to $x = -0.4$, was further subdivided into 40 uniform elements, over each of which both p and τ were assumed to be piecewise constant functions. On C_3, on

$y = 1$, from $x = -0.4$ to $x = 0$, the singular forms of p and τ were incorporated into the integral equations via series expansions with unknown coefficients, i.e., singularity incorporation was employed on the boundary elements nearest the re-entrant corner. The unknowns' behaviours on C_4, C_5 and C_6 essentially mirrored that enforced over C_3, C_2 and C_1, respectively, except that these sections were on $\vartheta = \vartheta_0$, and that the pressure and wall stress on C_6 were obtained by suitable differentiation of Equation (1). For a full discussion of the above parameter choices, the reader is again referred to Hansen and Kelmanson [7]. The subsequently discretized versions of Equations (3) and (4) provided linear equations in the boundary unknowns and series coefficients, solution of which yielded the stream function via a discretized form of Equation (5). We merely mention here that much analysis and computational effort was required to perform the integrations to obtain the coefficients in these linear equations, particularly in those cases where the collocation points (x_0, y_0) lay on the boundary C.

4. RESULTS

Results are presented for $\vartheta_0 = 90°$. Figure 2 shows streamlines on a scale in which each border graticule represents 5 channel widths (5-CW). The dotted streamlines are those from Taylor's [1] scraper problem, in which there is no gap. What is interesting is that the presence of the channel (i.e., leak) clearly affects the flow at distances of order O(100-CW) from the leak; this is certainly greater than one might intuitively expect. In Figure 3, each border graticule represents a distance of 50-CW, and we can see that the perturbed flow of the present work has asymptotically approached the theoretical zero-gap flow at distances of order O(1000-CW) from the leak. This we expect, and indeed it is a check on the accuracy of the results produced by the integral equation method.

Our final observation, which we could not have predicted, is illustrated in Figure 4, in which we provide a comparison between the asymptotic solution of Equation (1) with the full numerical solution generated by our method. Unexpectedly, the asymptotic solution, which is far easier to obtain, is a remarkably accurate representation of the flow even at distances of order O(1-CW) away from the leak.

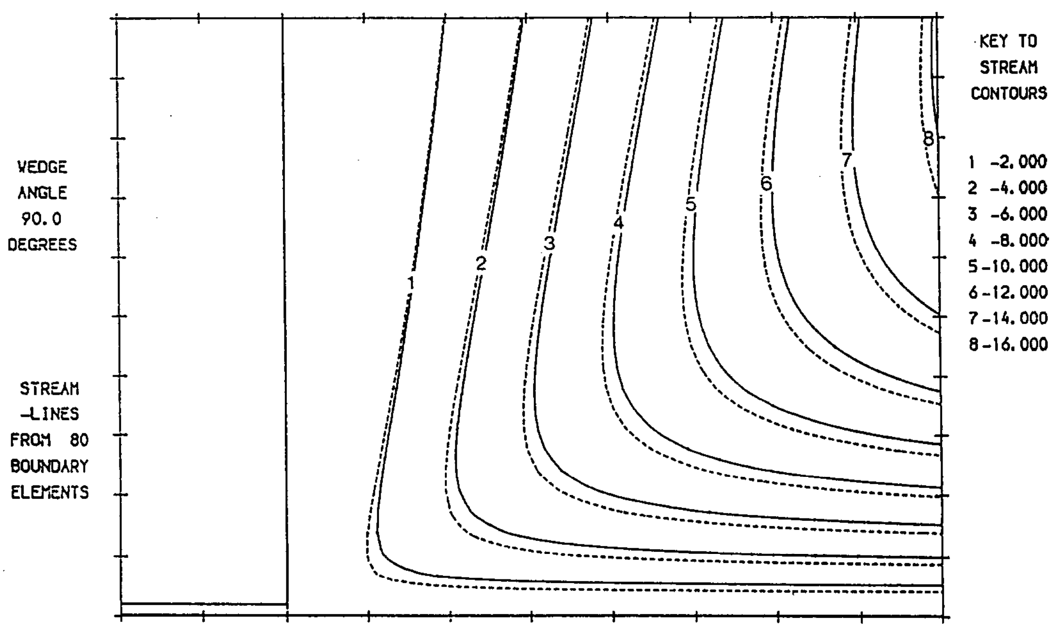

Figure 2. Streamline comparisons between present solution (——) and Taylor (1962) solution (– – –) for $\vartheta_0 = 90°$. Each border graticule is 5 channel widths.

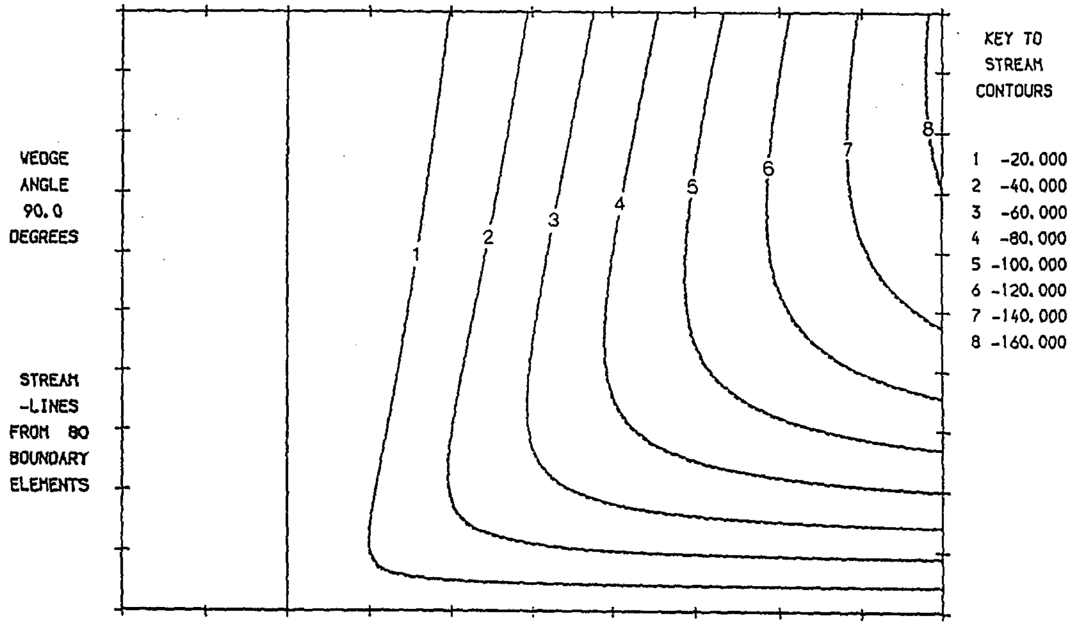

Figure 3. As Figure 2; each border graticule is 50 channel widths.

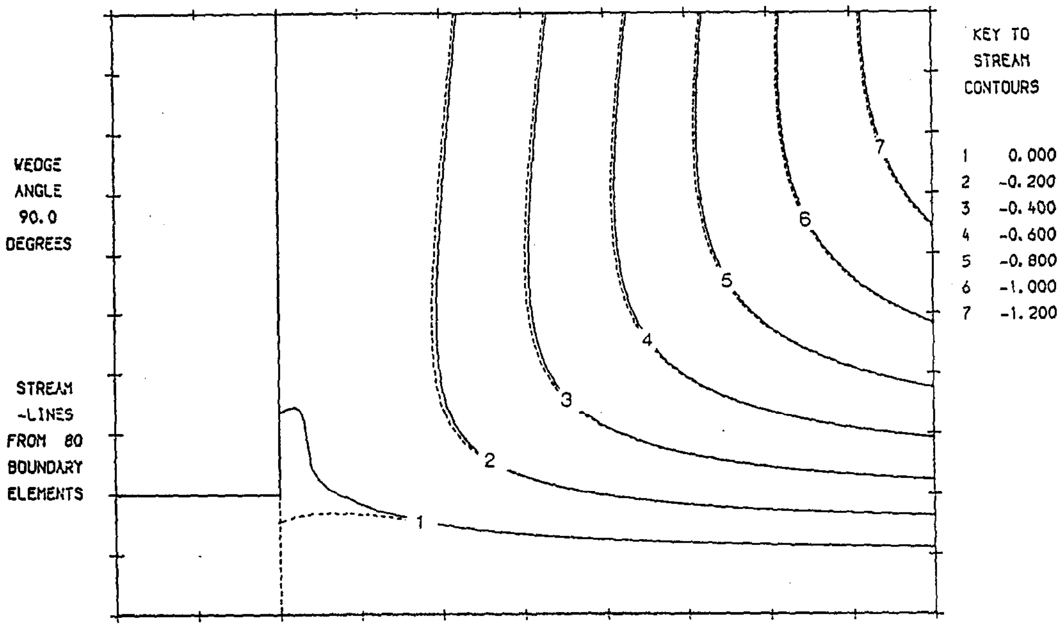

Figure 4. Streamline comparisons between present solution (——) and asymptotic solution (– – –) of Equation (1), for $\vartheta_0 = 90°$.

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