High-Girth Graphs Avoiding a Minor are Nearly Bipartite

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Let $H$ be a fixed graph. We show that any $H$-minor free graph $G$ of high enough girth has circular chromatic number arbitrarily close to two. Equivalently, each such graph $G$ admits a homomorphism to a large odd circuit. In particular, graphs of high girth and of bounded genus, or of bounded tree width, are “nearly bipartite” in this sense. For example, any planar graph of girth at least 16 admits a homomorphism to a pentagon. We also obtain tight bounds on the girth of $G$ in a few specific cases of small forbidden minors $H$.

Key Words: homomorphism; circular chromatic number; star chromatic number; girth; genus; forbidden minor; nearly bipartite graph.

1. INTRODUCTION

A homomorphism of a graph $G$ to a graph $H$ is a mapping $f: V(G) \rightarrow V(H)$ such that $uv \in E(G)$ implies $f(u) f(v) \in E(H)$. If $G$ admits a homomorphism to $H$ we write $G \rightarrow H$. The girth of a graph $G$ is the minimum length...
of a circuit in $G$. The odd girth of a graph $G$ is the minimum length of an odd circuit in $G$. (The odd girth of a bipartite graph is infinite.) Both girth and odd girth can be viewed as measures of “sparsity” of a graph. A minor of $G$ is any graph $H$ obtained from $G$ by contracting and deleting edges. A graph is $H$-minor free if it has no minor isomorphic to $H$. The circular chromatic number (sometimes called the “star chromatic number”) of a graph $G$, denoted $\chi_c(G)$, is the infimum of the set of real numbers $r$ with the following property: there exists a mapping of $V(G)$ to the set of unit-length open arcs of a circle with circumference $r$ which maps adjacent vertices to disjoint arcs. It turns out that the same definition with “circle” replaced by “interval” (and “circumference” by “length”) yields the usual chromatic number $\chi(G)$, that the infimum in this definition can be achieved (and hence is the minimum), and that $\chi(G) = \lceil \chi_c(G) \rceil$; cf. the excellent survey article of X. Zhu [25].

We investigate the connections between the properties of embeddability and sparsity of a graph and its (circular) chromatic number. The first result of this flavour is probably due to Grötzsch [9], who proved that every planar graph with no triangle can be 3-coloured. Some years later, Kronk and White [13] showed that every toroidal graph of girth at least six is 3-colourable, while Cook [5] proved that if $G$ is a graph of genus $\gamma$ and the girth of $G$ is at least $\max\{9, 6 + 2 \log_2 \gamma\}$, then $G$ is 3-colourable. Thomassen [21] improved the bound for toroidal graphs by showing that every toroidal graph of girth at least five can be 3-coloured. Let $C_\ell$ denote the $\ell$-circuit with vertices $1, 2, \ldots, \ell$. A 3-colouring of a graph $G$ is precisely a homomorphism to $C_3$. Thus we are led to ask analogous questions about other odd circuits $C_\ell$. (As we shall see, these are often best expressed in terms of the circular chromatic number.)

**Question 1.1.** Does there exist an integer $g_0$ such that each planar graph with girth at least $g_0$ admits a homomorphism to $C_5$?

It is not difficult to find planar graphs of girth seven which do not admit a homomorphism to $C_5$. In fact, one can find such graphs with very low degrees. In Fig. 1 we give one such graph, due to Albertson and Moore [1], which has maximum degree three. In any event, if such an integer $g_0$ exists, then $g_0 \geq 8$. Jaeger’s “circular flow conjecture” [11] would imply that one can take $g_0 = 8$. It will follow from our results that the question does have an affirmative answer, and that one can take $g_0 = 16$. (X. Zhu has recently shown that $g_0 = 15$ will also work.)

We shall also consider graphs embedded on other surfaces, and graphs avoiding a fixed minor. For this purpose we define the following two parameters:
Definition 1.2. Let $\gamma \geq 0$, $\ell \geq 3$ be integers, $\ell$ odd, and let $H$ be a fixed graph.

We define $g_\gamma(\ell)$ to be the infimum of all integers $g$ such that every graph $G$ of genus at most $\gamma$ and girth at least $g$ is homomorphic to $C_\ell$.

We also define $g_\gamma^H(\ell)$ to be the infimum of all integers $g$ such that every $H$-minor free graph $G$ of girth at least $g$ is homomorphic to $C_\ell$.

Note that $g_0(5)$ is the smallest value of $g_0$ which suffices for a positive answer in Question 1.1.

We shall show that both of the above parameters are finite, and we shall give some explicit bounds.

Let $\mathcal{C}_\ell$ denote the class of all graphs $G$ such that $G \rightarrow C_\ell$. It is easy to see that

$$\mathcal{C}_3 \supseteq \mathcal{C}_5 \supseteq \mathcal{C}_7 \supseteq \cdots$$

and that $\bigcap \mathcal{C}_\ell$ is the class of all bipartite graphs.

Thus we may view a graph in $\mathcal{C}_\ell$ ($\ell \gg 1$, $\ell$ odd) as being “nearly bipartite”, with $\ell$ being a measure of the “bipartiteness” of the graph. This notion can be nicely formulated in terms of circular chromatic numbers.
Proposition 1.3 [25]. For any graph $G$ and integer $k$, the following are equivalent:

- $\chi_c(G) \leq 2 + \frac{1}{k}$
- $G \to C_{2k+1}$

Thus a graph is “nearly bipartite” if its circular chromatic number is close to two.

We study classes of graphs for which high girth implies low circular chromatic number. Formally, we say that a class $\mathcal{G}$ of graphs is **girth-bipartite** if for any $\epsilon > 0$ there exists an integer $g(\epsilon)$ such that every graph in $\mathcal{G}$ having girth at least $g(\epsilon)$ satisfies $\chi_c(G) \leq 2 + \epsilon$. (We use the term “girth-bipartite” to distinguish from other variants of the intuitive notion of “near bipartiteness”.)

The main result of this paper is the following:

**Theorem 1.4.** For every graph $H$, the class of $H$-minor free graphs is girth-bipartite.

In particular, no minor-closed class of graphs whose members have unbounded girth can have its circular chromatic numbers bounded away from two.

The condition that the graphs be $H$-minor free cannot be omitted, since there exist graphs with arbitrarily high girth and chromatic number [2].

We remark that Theorem 1.4 applies to any class of graphs of bounded genus. In Theorem 3.2 we give a specific upper bound on the circular chromatic number of graphs with a given girth and genus. The bound $g_0 = 16$ mentioned above for planar graphs is a special case of Theorem 3.2.

The special case of planar graphs was first proved to be girth-bipartite by Nešetřil and Zhu [15, Corollary 2]. In fact, they proved a stronger result, by replacing both the girth and the planarity condition with weaker hypotheses.

Let $k \geq 3$ be an integer. Consider the following properties of a graph $G$:

- **P1.** $G$ has girth at least $k$;
- **P2.** $G$ admits a homomorphism to some graph of girth at least $k$;
- **P3.** $G$ has odd girth at least $k$.

It is easy to see that we have:

**Proposition 1.5.** For any graph $G$ and any integer $k \geq 3$, property P1 implies property P2, and property P2 implies property P3.

On the other hand, P2 does not imply P1, since $C_4$ satisfies $P_2$ for any $k$, but has girth four. Similarly, P3 does not imply P2, since the Grötzsch
graph has odd girth five, but it is easily seen not to admit a homomorphism to any graph of girth five or more. (Any two non-adjacent vertices of the Grötzsch graph are joined by a path of length three, and so identifying them creates a triangle.)

Hence, we may give two other definitions analogous to “girth-bipartite”, by replacing P1 by P2 (or P3) in the definition: We say that a class of graphs $\mathcal{G}$ is hom-girth-bipartite (respectively odd-girth-bipartite) if for some function $g(\varepsilon)$ and all $\varepsilon > 0$, every graph in $\mathcal{G}$ satisfying P2 (respectively P3) with $k \geq g(\varepsilon)$ satisfies $\chi_2(G) \leq 2 + \varepsilon$.

Returning to the result of Nešetril and Zhu [15, Corollary 2], we first note that when $H$ is planar, then all $H$-minor free graphs have bounded tree width [16]. Their theorem can be stated as follows:

**Theorem 1.6.** Any class of graphs with bounded tree width is hom-girth-bipartite.

In particular, any planar graph is hom-girth-bipartite. This result suggests that our Theorem 1.4 or Theorem 3.2 might generalize to properties P2 or P3. However, there exist projective planar graphs of chromatic number four and arbitrary odd girth [23]; thus property P3 is not guaranteed even for all classes of graphs of bounded genus. On the other hand, Zhang has recently shown [24] that the class of planar graphs is odd-girth-bipartite. (It is further proved in [6] that for any orientable surface $\Sigma$ there exists an integer $r = r(\Sigma)$ such that the set of locally bipartite graphs embeddable on $\Sigma$ with representativity at least $r$ is odd-girth-bipartite.) Thus we are mainly left with the following open question:

**Question 1.7.** Is it true that for every graph $H$, the class of $H$-minor free graphs is hom-girth-bipartite?

We do not know the answer even in the special case of graphs of bounded genus.

The concepts “girth-bipartite” and “odd-girth-bipartite” are easily dualized in the matroid sense. A circulation in a directed graph is a real-valued function on its arcs satisfying the usual flow-conservation law. The *circular flow number* of a graph $G = (V, E)$ is defined by

$$\phi_c(G) = \inf \{r \in \mathbb{R} : G \text{ has an orientation } \vec{G} \text{ and}
\text{a circulation } f : E(\vec{G}) \to [1, r - 1]\}.$$  

For finite 2-edge connected graphs, the infimum is attained at a rational number since for such graphs [8]

$$\phi_c(G) = \min_{\sigma} \max_{X \subseteq V} \frac{|\delta^-(X)|}{|\delta^+(X)|}$$
where the minimum is over all strong orientations of $G$. (Here $\delta^+(X)$

denotes the set of arcs in $G$ with tails in $X$ and heads in $V - X$, and

$\delta(X) = \delta^+(X) + \delta^+(V - X)$.) As in the case of chromatic numbers, the usual flow number of a graph $G$ [8] is given by $\lceil \phi_c(G) \rceil$. The cogirth (respectively odd-cogirth) of $G$ is the minimum size of a (odd-size) non-empty edge cut of $G$. By planar duality, our answer to Question 1.1 is equivalent to the statement: any planar graph of cogirth at least 16 has circular flow number at most $\frac{5}{2}$.

A class of graphs $\mathcal{G}$ is cogirth-even (odd-cogirth-even) if for some function $\lambda(\varepsilon)$ and all $\varepsilon > 0$, every graph in $\mathcal{G}$ with cogirth (odd-cogirth) at least $\lambda(\varepsilon)$ satisfies $\phi_c(G) \leq 2 + \varepsilon$. In general, flow numbers behave better than chromatic numbers. For example, all bridgeless graphs $G$ satisfy $\phi_c(G) \leq 6$, and all graphs of cogirth at least four have $2 \leq \phi_c(G) \leq 4$ [18, p. 297]. Jaeger has proposed the following "circular flow conjecture" [11]:

**Conjecture 1.8.** The class of all graphs is cogirth-even.

In fact, he proposes that $\lambda(1/k) = 4k$ suffices for integers $k \geq 1$. By planar duality, this would imply that $g_0(5) = \lambda(\frac{1}{2}) = 8$ suffices in Question 1.1. However, it is possible that a stronger statement holds.

**Question 1.9.** Is the class of all graphs odd-cogirth-even?

Zhang [24] has shown that any class of graphs of bounded genus is odd-cogirth-even, but little else is known about this problem.

2. PROOF OF THEOREM 1.4

Our results rely on the simple observation that deleting a long path whose internal vertices have degree two does not change the circular chromatic number of the graph. Specifically, we have:

**Lemma 2.1.** Let $G$ be a graph, and $\ell$ an odd integer. Suppose $P = u_1, u_2, \ldots, u_p$ is a path in $G$, with $p \geq \ell$, and let $U$ denote the set of internal vertices of $P$, i.e., $U = \{u_2, u_3, \ldots, u_{p-1}\}$. If all vertices of $U$ have degree two in $G$, then $G \rightarrow C_\ell$ if and only if $(G - U) \rightarrow C_\ell$.

**Proof.** Clearly, if $G \rightarrow C_\ell$, then $(G - U) \rightarrow C_\ell$. Conversely, suppose that $f$ is a homomorphism of $G - U$ to $C_\ell$. Since the vertices of $U$ have degree two, we will be able to extend the homomorphism $f$ to all of $G$. Consider the vertices $f(u_1)$ and $f(u_p)$ of $C_\ell$. They separate $C_\ell$ into two paths whose lengths have different parities and are both less than or equal to $\ell$. (If $f(u_1) = f(u_p)$, then the even path has length zero and the odd path has
Thus at least one of these paths has the same parity as $P$ and its length does not exceed the length of $P$. Therefore, there is a homomorphism of $P$ to $C_\ell$ taking $u_1$ to $f(u_1)$ and $u_p$ to $f(u_p)$, allowing us to define an extension of $f$ to all of $G$. To illustrate the idea we give the details for $\ell = p = 5$: by symmetry, we may assume that $f(u_1) = 1$ and $f(u_3) \in \{1, 2, 3\}$.

- when $f(u_3) = 1$ we set $f(u_2) = 2$, $f(u_4) = 1$, $f(u_5) = 2$;
- when $f(u_3) = 2$ we set $f(u_2) = 5$, $f(u_4) = 4$, $f(u_5) = 3$; and
- when $f(u_3) = 3$ we set $f(u_2) = 2$, $f(u_4) = 3$, $f(u_5) = 2$.

A graph $G$ is $p$-path degenerate if there is a sequence $G = G_0, G_1, \ldots, G_t$ of 2-connected subgraphs of $G$ such that $G_t$ is bipartite, and each $G_i$ ($i > 0$) is obtained from $G_{i-1}$ by deleting the internal vertices of a path of length at least $p$, all of degree two. By repeatedly applying Lemma 2.1 and observing that bipartite graphs have circular chromatic number 2, we obtain the following:

**Corollary 2.2.** A $p$-path degenerate graph $G$ admits a homomorphism to any odd circuit of length at most $p + 1$. In other words, a $p$-path degenerate graph $G$ has

$$\chi_c(G) \leq 2 + \left\lfloor \frac{p}{2} \right\rfloor.$$  

Next, we show that every graph avoiding a fixed minor and of “high enough” girth is $p$-path degenerate. For this, we rely on an observation of Thomassen [20, p. 115].

**Lemma 2.3.** For any graph $H$ there exists an integer $k$ such that every $H$-minor free graph with minimum degree at least three has girth at most $k$.

**Definition 2.4.** Let $k_H$ denote the least integer $k$ satisfying Lemma 2.3.

**Lemma 2.5.** Let $H$ be a graph and $p$ a positive integer. Every $H$-minor free graph with girth at least $(p - 1)k_H + 1$ is $p$-path degenerate.

**Proof.** Let $G$ be an $H$-minor free graph of girth at least $(p - 1)k_H + 1$. We may assume that $G$ is 2-connected, since a graph with all of its 2-connected blocks $p$-path degenerate is $p$-path degenerate as well. If $G$ is a circuit, then it has length at least $(p - 1)k_H + 1 \geq p + 1$ and thus is $p$-path degenerate. Henceforth we assume that $G$ is a 2-connected graph different from an edge or a circuit; therefore, there is a unique graph $G'$ with minimum degree at least three, which is homeomorphic to $G$. In other words,
$G$ can be obtained from $G'$ by replacing each edge $e \in E(G')$ with a path $P(e)$ of length at least one. As $G'$ has no $H$-minor, Lemma 2.3 implies that $G'$ must have a circuit $C'$ of length at most $k_H$. There is a circuit $C$ in $G$ which corresponds to $C'$. Since the length of $C$ is at least $(p - 1) k_H + 1$, we conclude that $C$ contains a path $P(e_0)$ of length at least $p$, for some $e_0 \in E(C')$. Since the internal vertices of $P(e_0)$ have degree two in $G$, their deletion leaves a smaller $H$-minor free graph with girth at least $(p - 1) k_H + 1$. Thus $G$ is $p$-path degenerate by induction.

Proof of Theorem 1.4. Every $H$-minor free graph $G$ of girth at least $(p - 1) k_H + 1$ is $p$-path degenerate, according to Lemma 2.5, and hence has a homomorphism to any odd circuit of length $\ell \leq p + 1$, according to Corollary 2.2. Thus $\chi_c(G) \leq 2 + \lfloor p/2 \rfloor$. An upper bound for $\chi_c(G)$ is obtained by choosing $p$ as large as possible in Lemma 2.5:

**Corollary 2.6.** For any $H$-minor free graph $G$ of girth $g$

$$G \rightarrow C, \quad \text{for any odd integer } \ell \leq \frac{g - 1}{k_H} + 2.$$  

Determining $k_H$ is difficult in general; however, in the following we provide an upper bound when $H = K_r$.

Mader [14] (see [22, p. 333] for an English language version) proved that:

**Lemma 2.7.** For each positive integer $t$ there exists an integer $d(t)$ such that every simple graph with minimum degree at least $d(t)$ contains $K_t$ as a minor.

Later, Kostochka [12] showed that any simple graph with average degree at least $2t$ has a $K_{\eta}$-minor where $\eta \geq 0.064t/\sqrt{\log_2 t}$. This implies that $d(t) = 66t/\sqrt{\log_2 t}$ suffices in Lemma 2.7, when $t \geq 3$. Finally, Thomason [19] proved that the constant 66 may be lowered to 5.36 if $t$ is large.

**Lemma 2.8.** If $H = K_t$ and $t \geq 3$, then $k_H < 4c_1 t \sqrt{\log_2 t}$, where $c_1 = 66$.

**Proof.** Consider a graph $G$ with minimum degree at least three and girth at least $4c_1 t \sqrt{\log_2 t}$. By Thomassen’s proof of Lemma 2.3, $G$ contains as a minor a simple subgraph $F$ whose minimum degree $\delta(F)$ is at least $c_1 t \sqrt{\log_2 t}$. Hence by the remark following Lemma 2.7, there exists a $K_r$-minor in $G$.  


For any simple graph $H$ of order $t$, the above lemma yields an upper bound for $k_H$ depending only on $t$ and the girth of $G$. Thus, by substituting this bound in the formula in Corollary 2.6, we obtain the following bound on $g_H(\ell)$:

**Corollary 2.9.** Let $H$ be a fixed graph of order $t$, and let $\ell > 1$ be an odd integer. Then,

$$g_H(\ell) \leq 264(\ell - 2) t \sqrt{\log_2 t} + 1.$$

Finding better upper bounds on $g_H(\ell)$ for specific forbidden minors $H$ is the topic of Section 4.

It is worth noting that a result analogous to Lemma 2.7 is true [3] for $K_t$ contained as a topological minor, i.e., a subgraph which is a subdivision of $K_t$. All results in this paper may be stated in this setting, including our main theorem:

**Theorem 2.10.** For every graph $H$, the class of graphs which do not contain $H$ as a topological minor is girth-bipartite.

### 3. GRAPHS OF BOUNDED GENUS

We refer the reader to [22] for definitions of genus and embeddings. Planar graphs have genus zero. All graphs of genus at most $\gamma$ are $H$-minor free for any graph $H$ of genus greater than $\gamma$. Thus any class of graphs of bounded genus is girth-bipartite by Theorem 1.4. In this section we specialize the bounds of the previous section to graphs of bounded genus.

**Definition 3.1.** Let $k_\gamma$ denote the least integer such that all graphs with genus at most $\gamma$ with minimum degree at least three have girth at most $k_\gamma$.

For example, it is well known that $k_0 = 5$ and $k_1 = 7$. By exactly the same argument as in the proof of Lemma 2.5, we see that any graph of genus at most $\gamma$ and girth at least $(p - 1)k_\gamma + 1$ is $p$-path degenerate. Thus any graph of genus at most $\gamma$ satisfies the inequalities of Corollary 2.6 with $k_H$ replaced by $k_\gamma$. The goal of this section is to derive the following upper bound on $k_\gamma$.

**Theorem 3.2.** For any $\gamma \geq 0$, we have

$$k_\gamma \leq 4 + \lceil 2 \log_2 (\gamma + 3/2) \rceil.$$  

(1)
Corollary 3.3. For any graph $G$ of girth $g$ and genus at most $\gamma$,
- $G \to C_{\ell}$ for any odd integer $\ell \leq 2 + \frac{1}{\ell - 1} \left( 2 + \frac{\ell}{2} \log_2 (\frac{\ell}{2}) \right)$; and
- $g, (\ell) \leq (\ell - 2)(4 + 2 \log_2 (\gamma + 3/2)) + 1$.

In particular, any planar graph with girth at least 16 has a homomorphism to a pentagon, answering Question 1.1.

We note here that the dependence on genus arises only through Euler's characteristic formula. Thus a similar formula holds regarding the unoriented genus of $G$: if $G$ embeds on $S_\gamma$, the sphere with $\gamma'$ crosscaps, then the conclusion of Corollary 3.2 holds upon substituting $2\gamma'$ for $\gamma$.

A $(g, k)$-cage is a graph with minimum degree at least $k$ and girth at least $g$, and with the fewest possible number of vertices. There is a well-known lower bound (see [2]) for the number $n(g, k)$ of vertices in a $(g, k)$-cage. In particular,

$$n(g, 3) \geq \begin{cases} 2 \cdot 2^{g/2} - 2 & \text{if } g \text{ is even} \\ 3 \cdot 2^{(g+1)/2} & \text{if } g \text{ is odd} \end{cases} \geq 2 \cdot 2^{g/2} - 2.$$

Lemma 3.4. If $G$ is a graph having minimum degree at least three, girth $g \geq 6$ and genus $\gamma$, then

$$\gamma \geq 1 + \left\lceil \frac{g - 6}{4g} n(g, 3) \right\rceil.$$

Proof. Let $G$ be a graph with $n$ vertices, $e$ edges, girth $g$, genus $\gamma$ and minimum degree at least three. If $g = 6$, then $G$ is not planar and we are done, so we may assume $g \geq 7$. Suppose $G$ has $r$ regions when embedded to $S_\gamma$. We follow Cook's argument in [4] which uses Euler's formula and the inequalities $ge \leq 3n \leq 2e$ to derive

$$n \leq \frac{4g(g - 1)}{g - 6}.$$

On the other hand, we have, by the definition of a cage, $n \geq n(g, 3)$. The two inequalities imply (3).

Proof of Theorem 3.2. Let $G$ be a graph having minimum degree at least three and genus $\gamma$. Let $h(\gamma)$ be the function expressed in the right hand side of (1). Our goal is to show that the girth $g$ of $G$ is at most $h(\gamma)$. We have $h(0) = 5$, $h(1) = 7$ which are well-known upper bounds for planar and toroidal graphs. We may assume that $G$ has genus $\gamma \geq 2$ and apply Lemma 3.4. From (2) and (3) we have

$$\gamma \geq 1 + \frac{1}{2} \left( 1 - \frac{6}{g} \right) (2^{g/2} - 1).$$
If \( 7 \leq g \leq 11 \), then we verify the theorem numerically by checking that \( g \leq h(\gamma) \) when the right hand side of (4) is substituted for \( \gamma \). If \( g \geq 12 \), then we have from (4)

\[
4 \left( \gamma + 3 \right) \geq 4 + (2\pi^2 + 2\pi^2 - 2) - \frac{12}{g} (2\pi^2 - 1) + 6 > 2\pi^2
\]

and the result follows by taking logarithms.

The upper bound (1) compares quite well to the best upper bound to \( k_\gamma \) that can be possibly derived from (3) and the first inequality in (2). In fact, the two bounds are equal for \( \gamma \leq 7 \), they differ by at most one when \( \gamma \leq 11 \), and they never differ by more than two. The exact value of \( k_\gamma \) appears to be difficult to determine.

The best upper bound to \( g_\gamma(\ell) \) in terms of the girth and genus of \( G \) also seems hard to determine, even for planar graphs.

### 4. Bounds for Specific Forbidden Minors II

The general bounds given in the previous sections can be improved when specific minors are forbidden. For instance, it follows from the work of Hell and Zhu [10] that every \( K_4 \)-minor free graph \( G \) of girth at least \( 2\left(3k - 1\right)/2 \) is homomorphic to \( C_{2k-1} \). Thus \( g_{K_4}(\ell) \leq (3\ell + 1)/2 \).

It was observed by Gerards [7] that subdivisions of \( K_4 \) play an important role in homomorphisms to odd circuits. An odd \( K_4 \) and an odd \( K^2_3 \) are subdivisions of \( K_4 \) and \( K^2_3 \). (\( K^2_3 \) is a \( K_3 \) with all edges doubled and each vertex replaced with a path of length at least zero) shown in Fig. 2. (The word “odd” inside a face means that the length of that face is odd.)

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**FIG. 2.** (a) Odd \( K_4 \), (b) Odd \( K^2_3 \).
In fact, Gerards proved [7] the following result:

**Theorem 4.1.** Let $G$ be a nonbipartite graph. If $G$ contains neither an odd $K_4$ nor an odd $K_2^3$, then $G$ admits a homomorphism to its shortest odd circuit.

Clearly, an odd $K_2^3$ contains $K_2^3$ as a minor. On the other hand, $K_2^3$-minor free graphs may contain an odd $K_4$-minor and, hence, constitute a non-trivial class for homomorphisms to odd circuits.

By dualizing the characterization of Seymour [17] of $K_{2,3}$-minor free graphs, we may state the following:

**Theorem 4.2.** Let $G$ be a 2-connected $K_2^3$-minor free graph. Then either $G$ does not contain a $K_4$-minor or $G$ is a subdivision of $K_4$.

By the above result and Gerards' theorem, we expect that odd $K_4$'s are the obstructions for homomorphisms to odd circuits. This expectation is confirmed by the next lemma. We denote by $x_1, x_2, x_3, x_4$ the vertices of degree three in a subdivision of $K_4$, and we call arms the paths $[x_i, x_j]$ joining different pairs $x_i, x_j$, $1 \leq i, j \leq 4$.

**Lemma 4.3.** Let $G$ be an odd $K_4$ of girth $3k$. If all arms have length $k$, then $G$ is not homomorphic to $C_{2k+1}$.

**Proof.** Let $V(C_{2k+1}) = \{0, 1, ..., 2k\}$. Suppose there exists a homomorphism $f$ of $G$ to $C_{2k+1}$. We may assume without loss of generality that $x_1$ receives colour 0, i.e., $f(x_1) = 0$. Now, we distinguish two cases: either both the vertices $x_2$ and $x_3$ receive an even colour or they both receive an odd colour. In the first case, since the lengths of the paths between $x_1$ and $x_2$, and $x_1$ and $x_3$, respectively, are odd and equal to $k$, it follows that the $f(x_2)$ and $f(x_3)$ must be at least $k+1$. This implies that the circuit $[x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_1]$ cannot be mapped to $C_{2k+1}$.

Similarly, if the colours of $x_2$ and $x_3$ are both odd, they must have values smaller than or equal to $k$, which again implies that the circuit $[x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_1]$ cannot be mapped to $C_{2k+1}$.

Now we show that any odd $K_4$, regardless of the length of its arms, but of girth at least $3k$, can be mapped to $C_{2k-1}$. To prove this, we introduce the operation of *folding*, i.e., identification of two nonadjacent vertices having a common neighbour.

**Lemma 4.4.** Let $G$ be an odd $K_4$ of girth $\geq 3k$. Then $G$ is homomorphic to $C_{2k-1}$.
Proof. It is enough to prove the statement when $G$ has girth $3k$. First of all observe that at least one of the arms of $K_4$ has length at most $k$, since otherwise the girth of $G$ would be greater than $3k$. We may assume without loss of generality that the arm $[x_1, x_3]$ has length $l \leq k$.

If $l$ is even, then we fold the arm $l/2$ times; if $l$ is odd, then we fold it $(l-1)/2$ times, thus obtaining an arm of length 1. We then fold this remaining edge with the first edge of the arm $[x_1, x_4]$.

In each case, the resulting graph has girth at least $2k - 1$ and contains neither an odd $K_4$ nor an odd $K_3^2$. Hence, by Theorem 4.1, it is isomorphic to its shortest odd circuit, which has length at least $2k - 1$, and hence also to $C_{2k-1}$.

It follows easily from Theorems 4.2, 4.1 and Lemma 4.4 that:

**Theorem 4.5.** Let $\ell > 1$ be an odd integer. Then,

$$g_{K_3^2}(\ell) \leq \frac{3(\ell + 1)}{2}.$$  

Note that if $F$ is a minor of $H$, then the bound for $F$-minor free graphs is at most the bound obtained for $H$-minor free graphs. It follows that if $H$ is a minor of $K_4$ or of $K_3^2$, and $G$ is an $H$-minor free graph, then $G$ is isomorphic to $C_{2k+1}$, provided the girth of $G$ is at least $3(k + 1)$.

Observe also that by Gerards’ theorem we know that if $H$ is a minor of both $K_4$ and $K_3^2$ and $G$ is $H$-minor free, then $G$ is isomorphic to $C_{2k+1}$ provided that the girth of $G$ is at least $2k + 1$.

**REFERENCES**

1. M. Albertson and E. Moore, personal communication.