Completely Positive Linear Maps on Complex Matrices

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ABSTRACT

A linear map Φ from $\mathfrak{M}_n$ to $\mathfrak{M}_m$ is completely positive iff it admits an expression $\Phi(A)=\Sigma_i V_i^* A V_i$ where V_i are $n\times m$ matrices.

In this paper, we describe the tractable structure of completely positive linear maps between complex matrix algebras. The objective is (pursuing the work of Stinespring [8], Størmer [9], and Arveson [1,2]) to establish that completely positive linear maps, rather than positive linear maps, are the natural generalization of positive linear functionals. The results presented here are 'finite' and 'concrete' in essence. The reader may consult [1, Chapter 1] for general abstract information about the infinite-dimensional case.

Our main theorems reveal that the class of completely positive linear maps is the positive cone of the class of hermitian-preserving maps endowed with a natural ordering. Thus, a thorough structure theory follows immediately (Theorem 5). Finally (Theorem 7), we show that positive linear maps have the same effect as completely positive linear maps on 2×2 symmetric matrices.

For a complex matrix A, A^* denotes the transpose of the complex conjugate of A. We say a square matrix A is *symmetric* iff A equals its transpose, A is *hermitian* iff $A=A^*$, A is *positive* (or *positive semi-definite* for exactness) iff A is hermitian and its eigenvalues are nonnegative. We denote by $\mathfrak{M}_n$ the collection of $n\times n$ complex matrices. The Kronecker delta δ_{jk} equals 1 if $j=k$, and 0 if $j\neq k$; hence $I=(\delta_{jk})\in\mathfrak{M}_n$ is the identity $n\times n$ matrix. $E_{jk}\in\mathfrak{M}_n$ is the $n\times n$ matrix with 1 at the j,k component and zeros elsewhere. $\mathfrak{M}_n(\mathfrak{M}_m)=\mathfrak{M}_m\otimes\mathfrak{M}_n$ is the collection of all $n\times n$ block matrices with $m\times m$ matrices as entries; each element of $\mathfrak{M}_n(\mathfrak{M}_m)$ can also be regarded as an $nm\times nm$ matrix with numerical entries.

A linear map $\Phi:\mathfrak{M}_n\to\mathfrak{M}_m$ is *positive* (resp. *hermitian-preserving*) iff $\Phi(A)$ is positive (resp. hermitian) for all positive (resp. hermitian) A in $\mathfrak{M}_n$. We define $\Phi\otimes 1_p:\mathfrak{M}_p(\mathfrak{M}_n)\to\mathfrak{M}_p(\mathfrak{M}_m)$ by $\Phi\otimes 1_p((A_{jk})_{1\leqslant j,k\leqslant p})=(\Phi(A_{jk}))_{1\leqslant j,k\leqslant p}$. Φ is *completely positive* iff $\Phi\otimes 1_p$ is positive for all positive integers p. The reader is referred to [4] for the discrimination in a precise way between completely positive linear maps and positive linear maps.

For each $n\times m$ matrix V, it is evident that the map: $\mathfrak{M}_n\to\mathfrak{M}_m$ with $A\to V^*AV$ is completely positive. In the following, we show that the combinations of maps of the above form constitute all completely positive linear maps.

THEOREM 1. *Let $\Phi:\mathfrak{M}_n\to\mathfrak{M}_m$. Then Φ is completely positive iff Φ is of the form $\Phi(A)=\Sigma_i V_i^*AV$ for all A in $\mathfrak{M}_n$ where V_i are $n\times m$ matrices.*

Proof. The 'if' part is obvious. We proceed to prove the converse. Each $1\times nm$ matrix v can be regarded as a $1\times n$ block matrix $(x_1,\dots,x_n)$ with $1\times m$ matrices x_j as entries; hence we associate with it the $n\times m$ matrix V which has x_j as the j-th row. A simple computation leads to

$$(V^*E_{jk}V)_{1<j,k<n}=(x_j^*x_k)_{1<j,k<n}=v^*v.$$

Now suppose $\Phi:\mathfrak{M}_n\to\mathfrak{M}_m$ is completely positive. As $(E_{jk})_{1\leqslant j,k\leqslant n}$ is positive, so $(\Phi(E_{jk}))_{1\leqslant j,k\leqslant n}\in\mathfrak{M}_n(\mathfrak{M}_m)$ is positive; thus there exist vectors v_i^* (regarded as $nm\times 1$ matrices) such that $(\Phi(E_{jk}))_{jk}=\Sigma_i v_i^*v_i$. Let V_i be the $n\times m$ matrices associated with v_i. Then by the preceding result, $(\Phi(E_{jk}))_{jk}=\Sigma_i(V_i^*E_{jk}V_i)_{jk}$. Therefore, we conclude that $\Phi(A)=\Sigma_i V_i^*AV_i$ for all A. ■

Each linear map $\Phi:\mathfrak{M}_n\to\mathfrak{M}_m$ is determined by its values on E_{jk}. Hence Φ is completely determined by the single element $(\Phi(E_{jk}))_{1\leqslant j,k\leqslant n}\in\mathfrak{M}_n(\mathfrak{M}_m)$. The proof of Theorem 1 also provides another characterization of completely positive linear maps:

THEOREM 2. *Let Φ be a linear map from $\mathfrak{M}_n$ to $\mathfrak{M}_m$. Then Φ is completely positive iff $(\Phi(E_{jk}))_{1\leqslant j,k\leqslant n}$ is positive.*

REMARK 3. For a linear map $\Phi:\mathfrak{M}_n\to\mathfrak{M}_m$, it is obvious that Φ is hermitian-preserving iff $(\Phi(E_{jk}))_{jk}$ is hermitian. Endowed with the natural ordering induced by $\mathfrak{M}_n(\mathfrak{M}_m)$, the class of hermitian-preserving maps is a partially ordered vector space over the reals, while the class of completely positive linear maps is just the positive cone.

Referring again to the proof of Theorem 1, we deduce another pertinent fact (cf., [7, p. 134, Theorem 2.1] and [5, p. 259, Theorem 2]): $\Phi:\mathfrak{M}_n\to\mathfrak{M}_m$ is hermitian-preserving iff Φ admits an expression $\Phi(A)=\Sigma\epsilon_i V_i^* A V_i$ where $\epsilon_i=\pm 1$, V_i are $n\times m$ matrices. Since there are no such elegant expressions for positive linear maps, we may be convinced that completely positive linear maps, rather than positive linear maps, deserve the adjective 'positive'.

Remark 4. In the proof of Theorem 1, the expression $(\Phi(E_{jk}))_{jk} = \Sigma_i v_i^* v_i$ is not unique, hence $\{V_i\}$ is not uniquely determined. For some improvement, we may require $\{v_i^*\}$ to be linearly independent, then $\{V_i\}$ must be linearly independent too.

This additional condition on $\{V_i\}_i^\ell$ ensures that $\Phi(A)=\Sigma_i^\ell V_i^* A V_i$ is a 'canonical' expression for Φ, in the following precise sense: Let $\{W_p\}_p^{\ell'}$ be a class of $n\times m$ matrices, then Φ has the expression $\Phi(A)=\Sigma_p^{\ell'} W_p^* A W_p$ iff there exists an isometric $\ell'\times\ell$ matrix $(\mu_{pi})_{pi}$, such that $W_p=\Sigma_i \mu_{pi} V_i$ for all p. Moreover, if $\{W_p\}_p^{\ell'}$ is also a linearly independent set, then $\ell'=\ell$, and $(\mu_{pi})_{pi}$ is unitary.

Proof. The 'if' part follows by direct computation. We proceed to prove the 'only if' part. Denote by w_p, the display of W_p as a $1\times nm$ matrix. As in the proof of Theorem 1, $\Sigma_p w_p^* w_p = (\Phi(E_{jk}))_{jk} = \Sigma_i v_i^* v_i$. Thus w_p^* belongs to $sp\{v_i^*\}_i$, the linear span of $\{v_i^*\}_i$; namely, there exists $(\mu_{pi})_{pi}$ such that $w_p^* = \Sigma_i \overline{\mu_{pi}}\, v_i^*$. It follows that $W_p=\Sigma_i \mu_{pi} V_i$.

Since $\{v_i^*\}_i$ is a linearly independent set, $\{v_i^* v_j\}_{ij}$ is also a linearly independent set. (In fact, $\{v_i^* v_j\}_{ij}$ is a basis of the linear transformation space on $sp\{v_i^*\}_i$.) From

$$\Sigma_i v_i^* v_i = \Sigma_p w_p^* w_p = \Sigma_{pij} \overline{\mu_{pi}}\, \mu_{pj} v_i^* v_j,$$

we obtain $\Sigma_p \overline{\mu_{pi}}\, \mu_{pj} = \delta_{ij}$. Hence $(\mu_{pi})_{pi}$ is an isometry. In case that $\{W_p\}_p$ is also a linearly independent set, from $sp\{v_i^*\}_i^\ell = sp\{w_p^*\}_p^{\ell'}$, we derive that $\ell=\ell'$ and $(\mu_{pi})_{pi}$ is unitary. ■

For each fixed positive K in $\mathfrak{M}_m$, we write $\mathbf{CP}[\mathfrak{M}_n,\mathfrak{M}_m;K] = \{\Phi:\mathfrak{M}_n\to\mathfrak{M}_m \mid \Phi \text{ is completely positive and } \Phi(I)=K\}$. It is evident that $\mathbf{CP}[\mathfrak{M}_n,\mathfrak{M}_m;K]$ is a convex set, hence it is the convex hull of its extreme points. The following theorem gives a thorough description of the structure of completely positive linear maps.

Theorem 5. *Let $\Phi:\mathfrak{M}_n\to\mathfrak{M}_m$. Then Φ is extreme in $\mathbf{CP}[\mathfrak{M}_n,\mathfrak{M}_m;K]$ iff Φ admits an expression $\Phi(A)=\Sigma_i V_i^* A V_i$ for all A in $\mathfrak{M}_n$, where V_i are $n\times m$ matrices, $\Sigma_i V_i^* V_i = K$, and $\{V_i^* V_j\}_{ij}$ is a linearly independent set.*

Proof. *'The only if part'*

Assume Φ is extreme in $\mathbf{CP}[\mathfrak{M}_n, \mathfrak{M}_m; K]$. Express Φ in canonical form (Remark 4) $\Phi(A) = \Sigma V_i^* A V_i$ with $\{V_i\}$ linearly independent. Now suppose $\Sigma_{ij} \lambda_{ij} V_i^* V_j = 0$, we wish to prove that $(\lambda_{ij})_{ij} = 0$.

We may assume that $(\lambda_{ij})_{ij}$ is a hermitian matrix. (In fact, from $\Sigma \lambda_{ij} V_i^* V_j = 0$ we infer that $\Sigma(\lambda_{ij} \pm \overline{\lambda_{ji}}) V_i^* V_j = 0$. Then, if we prove $(\lambda_{ij} \pm \overline{\lambda_{ji}})_{ij} = 0$, that will yield $(\lambda_{ij})_{ij} = 0$.) By a scalar multiplication, we may further assume $-I \leqslant (\lambda_{ij})_{ij} \leqslant I$.

Define $\Psi_\pm : \mathfrak{M}_n \to \mathfrak{M}_m$ by $\Psi_\pm(A) = \Sigma V_i^* A V_i \pm \Sigma \lambda_{ij} V_i^* A V_j$. Hence $\Psi_\pm(I) = \Sigma V_i^* V_i = K$. Let $I + (\lambda_{ij})_{ij} = (\alpha_{ij})_{ij}^* (\alpha_{ij})_{ij}$ and $W_i = \Sigma_j \alpha_{ij} V_j$. By direct computation, $\Psi_+(A) = \Sigma W_i^* A W_i$; thus Ψ_+ is completely positive. In the same manner, Ψ_- is also completely positive. From $\Phi = \frac{1}{2}(\Psi_+ + \Psi_-)$ and the extremeness of Φ, we obtain $\Phi = \Psi_+$. By Remark 4, $(\alpha_{ij})_{ij}$ is an isometry. Therefore, $I + (\lambda_{ij})_{ij} = I$, i.e., $(\lambda_{ij})_{ij} = 0$ as required. ∎

Proof. *'The if part'.*

Assume $\Phi(A) = \Sigma V_i^* A V_i$, $\Sigma V_i^* V_i = K$ and $\{V_i^* V_j\}_{ij}$ is a linearly independent set. (Consequently, $\{V_i\}_i$ is a linearly independent set.) Now suppose $\Phi = \frac{1}{2}(\Psi_1 + \Psi_2)$ with $\Psi_1(A) = \Sigma W_p^* A W_p$, $\Psi_2(A) = \Sigma Z_q^* A Z_q$, and $\Sigma W_p^* W_p = \Sigma Z_q^* Z_q = K$. Since $\Psi(A) = \frac{1}{2}\Sigma W_p^* A W_p + \frac{1}{2}\Sigma Z_q^* A Z_q$, W_p and Z_q can be expressed in terms of V_i (Remark 4). Let $W_p = \Sigma_i \mu_{pi} V_i$ for each p. Then $\Sigma V_i^* V_i = \Sigma W_p^* W_p = \Sigma_{pij} \overline{\mu_{pi}} \mu_{pj} V_i^* V_j$, so $\Sigma_p \overline{\mu_{pi}} \mu_{pj} = \delta_{ij}$; i.e., $(\mu_{pi})_{pi}$ is an isometry. From Remark 4 again, we conclude that $\Phi = \Psi_1$; therefore Φ is extreme in $\mathbf{CP}[\mathfrak{M}_n, \mathfrak{M}_m; K]$. ∎

REMARK 6. Suppose $\Phi : \mathfrak{M}_n \to \mathfrak{M}_m$ is completely positive. From Remark 4, we can write Φ in the form $\Phi(A) = \Sigma_i^\ell V_i^* A V_i$ where $\{V_i\}_i^\ell$ is a class of linearly independent $n \times m$ matrices; hence $\ell \leqslant nm$. In case Φ is extreme in $\mathbf{CP}[\mathfrak{M}_n, \mathfrak{M}_m; K]$, ℓ can be reduced to $\leqslant m$. In fact, $\{V_i^* V_j\}_{ij}$ is a linearly independent set only if the cardinal number of $\{V_i^* V_j\}_{ij} \leqslant \dim \mathfrak{M}_m$, hence only if $\ell^2 \leqslant m^2$, i.e., $\ell \leqslant m$.

In particular, if $m = 1$, we obtain the well-known fact that Φ is a 'pure state' (an extreme identity-preserving positive functional) on $\mathfrak{M}_n$ iff Φ is a 'vector state' (i.e., there exists a unit vector v such that $\Phi(A) = v^* A v$ for all A).

The general structure of positive linear maps is very complicated (see [9, Chapter 8]). We will treat it for 2×2 matrices only. Following is an 'almost global' property of positive linear maps.

THEOREM 7. *If* $\Phi : \mathfrak{M}_2 \to \mathfrak{M}_m$ *is positive, then there exist* $2 \times m$ *matrices* V_i *such that* $\Phi(A) = \Sigma_i V_i^* A V_i$ *for all* 2×2 *symmetric* A.

Proof. We will associate each linear map with a matrix-coefficient quadratic form. First, we call attention to a known result (see [3, Theorem 2] for an elegant proof; the statement also appeared in an earlier paper [6, Appendix III]): Let F be an $\mathfrak{M}_m$-coefficient quadratic form $F(s,t)=B_1s^2+B_2st+B_3t^2$ with real indeterminates s, t. If $F(s,t)$ is positive for all s, t, then there exist $k\times m$ matrices C, D, such that $F(s,t)=(Cs+Dt)^*(Cs+Dt)$. (k is a certain integer.)

Now suppose Φ is positive, then $\Phi\left(\begin{bmatrix} s^2 & st \\ st & t^2 \end{bmatrix}\right)$ is positive for all real s, t; i.e., $F(s,t)=\Phi(E_{11})s^2+\Phi(E_{12}+E_{21})st+\Phi(E_{22})t^2$ is a positive $\mathfrak{M}_m$-coefficient quadratic form. From the preceding paragraph, there exist matrices C, D such that $\Phi(E_{11})=C^*C$, $\Phi(E_{12}+E_{21})=C^*D+D^*C$, and $\Phi(E_{22})=D^*D$. Define $\Psi:\mathfrak{M}_2\to\mathfrak{M}_m$ by

$$\left(\Psi(E_{jk})\right)_{jk}=\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix};$$

then Ψ is completely positive from Theorem 2. Since Φ agrees with Ψ on every symmetric matrix, we obtain the desired expression from Theorem 1. ■

We remark that Theorem 7 is not valid for higher order matrices. This is just because the quoted result for an $\mathfrak{M}_m$-coefficient quadratic form cannot be generalized to the case of more than 2 real indeterminates, as will be shown in a forthcoming paper.

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