JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 51, 705–723 (1975)

On the Determination of a Hill's Equation From Its Spectrum*

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Submitted by J. P. LaSalle

I. INTRODUCTION

A Hill's equation is an equation of the form

$$\left.\begin{aligned} y'' + [\lambda - q(z)]y &= 0 \\ q(z + \pi) &= q(z), \end{aligned}\right\} \tag{1}$$

where $q(z)$ is assumed to be integrable over $[0, \pi]$. Without loss of generality, it is customary to assume that

$$\int_0^\pi q(z)\, dz = 0.$$

The discriminant of (1) is defined by

$$\Delta(\lambda) = y_1(\pi) + y_2'(\pi),$$

where y_1 and y_2 are solutions of (1) satisfying

$$y_1(0) = y_2'(0) = 1$$

and

$$y_1'(0) = y_2(0) = 0.$$

Pertinent information about the analytic structure of the discriminant can be found in Magnus and Winkler [1]. $\Delta(\lambda)$ is an entire function of order $1/2$ and $\Delta(\lambda) - 2$ has infinitely many zeros with no finite limit point. To each zero there corresponds a solution of (1) satisfying

$$y(\pi) = y(0) \qquad \text{and} \qquad y'(\pi) = y'(0).$$

* Taken from a dissertation submitted to the Faculty of the Polytechnic Institute of New York in partial fulfillment of the requirements for the degree of Doctor of Philosophy (Mathematics).

This condition, combined with (1), defines a self-adjoint boundary value problem. It has only real eigenvalues which are the zeros of $\Delta(\lambda) - 2$. They are denoted by λ_i ($i = 0, 1, 2,...$) and are arranged so that

$$\lambda_0 < \lambda_1 \leqslant \lambda_2 < \lambda_3 \leqslant \lambda_4 < \cdots.$$

Similarly, the eigenvalues corresponding to the boundary condition

$$y(\pi) = -y(0) \qquad \text{and} \qquad y'(\pi) = -y'(0)$$

are the zeros of $\Delta(\lambda) + 2$ which are denoted by λ_i' ($i = 1, 2,...$) and are arranged so that

$$\lambda_1' \leqslant \lambda_2' < \lambda_3' \leqslant \lambda_4' < \cdots.$$

The two sequences are interlaced so that

$$\lambda_0 < \lambda_1' \leqslant \lambda_2' < \lambda_1 \leqslant \lambda_2 < \lambda_3' \leqslant \lambda_4' < \cdots.$$

The following intervals are now formed:

$$(-\infty, \lambda_0], (\lambda_0, \lambda_1'), [\lambda_1', \lambda_2'], (\lambda_2', \lambda_1), [\lambda_1, \lambda_2],... .$$

In the intervals of type $(-\infty, \lambda_0]$ and $[\lambda_{2n-1}, \lambda_{2n}]$ we have $\Delta \geqslant 2$. In those of type $(\lambda_{2n}, \lambda_{2n+1}')$ and $(\lambda_{2n}', \lambda_{2n-1})$ we have $|\Delta| < 2$. In intervals of type $[\lambda_{2n-1}', \lambda_{2n}']$ we have $\Delta \leqslant -2$.

For values of λ such that $|\Delta| > 2$, (1) has no solution which is bounded for all real z. When $|\Delta| = 2$, there exists at least one bounded solution. When $|\Delta| < 2$, all solutions of (1) are bounded for all real z.

Therefore, the intervals for which $|\Delta| < 2$ are called stability intervals, while the remaining intervals are called instability intervals. All instability intervals are finite except for $(-\infty, \lambda_0]$.

The following result will be proved.

THEOREM. *If $q(z)$ is real and integrable, and if precisely n finite instability intervals fail to vanish, then $q(z)$ must satisfy a differential equation of the form*

$$q^{(2n)} + H(q, q',..., q^{(2n-2)}) = 0, \quad a.e., \tag{2}$$

where H is a polynomial of maximal degree $n + 2$.

Borg [2], Hochstadt [3], and Ungar [4] proved this theorem for the case $n = 0$, i.e., when all finite instability intervals vanish, and found that

$$q(z) = 0, \quad \text{a.e.} \tag{3}$$

For the case $n = 1$, Hochstadt [5] showed that $q(z)$ is an elliptic function that satisfies

$$q'' = 3q^2 + Aq + B, \quad \text{a.e.}, \tag{4}$$

where A and B are constants. Equations (3) and (4) are equivalent to (2) for the cases $n = 0$ and 1, respectively. In particular, for the case $n = 2$, the explicit expression of (2) is

$$q^{(4)} = 10qq'' + Aq'' + 5(q')^2 - 10q^3 + Bq^2 + Cq + D, \quad \text{a.e.,} \tag{5}$$

where A, B, C, and D are constants.

Erdelyi [6] investigated a Hill's equation where $q(z)$ is a Lame function and discovered situations where all but a finite number of finite instability intervals vanish. Equation (4) provides a converse to some of his results.

Lax [7] later showed that if $q(z)$ is a periodic solution of

$$cq + K_n(q) = 0, \tag{6}$$

where c is a constant and K_n is an nth order Korteweg–deVries operator, then the Hill's equation (1) has only a finite number of instability intervals. Equations (3), (4), and (5) are equivalent to (6) when n is 0, 1, and 2, respectively, and hence provide a converse to Lax's result for these cases.

Hochstadt [5], also proved that $q(z)$ is infinitely differentiable a.e. when n finite instability intervals fail to vanish. We assume this result throughout this paper.

II. A Necessary Lemma

As in [5], y_2 is represented by the series

$$y_2(z) = \sum_0^\infty w_k(z), \tag{7a}$$

where

$$w_0(z) = (\sin \lambda^{1/2} z)/\lambda^{1/2} \tag{7b}$$

and

$$w_{k+1}(z) = (1/\lambda^{1/2}) \int_0^z \sin \lambda^{1/2}(z - \zeta)\, q(\zeta)\, w_k(\zeta)\, d\zeta. \tag{7c}$$

We let $Q(z) = \int_0^z q(\zeta)\, d\zeta$ and write $w_k(z)$ in the form

$$w_k(z) = f_k(z) \sin \lambda^{1/2} z + g_k(z) \cos \lambda^{1/2} z. \tag{8}$$

We now prove the following.

Lemma. *Let $q(z)$ be infinitely differentiable. Then*

$$f_k(z) = \sum_{l=0}^{M} [(P_l^k(z))/(\lambda^{k+l-1/2})] + O[1/(\lambda^{M+k+1/2})] \qquad (k = 1, 2, \ldots) \tag{9}$$

and

$$g_k(z) = \sum_{l=0}^{M} [(R_l^k(z))/(\lambda^{k+l})] + O[1/(\lambda^{M+k+1})] \qquad (k = 1, 2, \ldots), \tag{10}$$

where P_0^k and R_0^k are polynomials in Q and q plus integral terms, P_l^k is a polynomial in $Q, q, q', \ldots, q^{(2l-2)}$ plus integral terms, and R_l^k is a polynomial in $Q, q, \ldots, q^{(2l-1)}$ plus integral terms. Furthermore, the maximal degree of the nonintegral terms in P_l^k and R_l^k is k.

The proof is accomplished by induction. By repeated integrations by parts, we get, for $k = 1$:

$$w_1(z) = \sin \lambda^{1/2} z \left\{ \sum_{n=0}^{N} \frac{(-1)^n [q^{(2n)}(z) + q^{(2n)}(0)]}{2^{2n+2} \lambda^{n+3/2}} + O\left[\frac{1}{\lambda^{N+5/2}}\right] \right\}$$

$$+ \cos \lambda^{1/2} z \left\{ \frac{-Q(z)}{2\lambda} + \sum_{n=1}^{N} \frac{(-1)^{n-1} [q^{(2n-1)}(z) - q^{(2n-1)}(0)]}{2^{2n+1} \lambda^{n+1}} + O\left[\frac{1}{\lambda^{N+2}}\right] \right\},$$

which can be put in the form of (9) and (10). (For details see Appendix I.)

We now substitute series (9) and (10) into (7c) and perform repeated integrations by parts. This yields

$$\begin{aligned} w_{k+1} = \sin \lambda^{1/2} z & \left\{ \frac{1}{2\lambda^{1/2}} \int_0^z q(\zeta) \sum_{l=0}^{M} \frac{R_l^k(\zeta)}{k+l} \, d\zeta \right. \\ & + \sum_{n=0}^{N} \frac{(-1)^n \left[q(z) \sum_{l=0}^{M} \frac{P_l^k(z)}{\lambda^{k+l-1/2}} + q(z) \sum_{l=0}^{M} \frac{P_l^k(0)}{\lambda^{k+l-1/2}} \right]^{(2n)}}{2^{2n+2} \lambda^{n+1}} \\ & + \sum_{n=1}^{N} \frac{(-1)^{n-1} \left[q(z) \sum_{l=0}^{M} \frac{R_l^k(z)}{\lambda^{k+l}} + q(z) \sum_{l=0}^{M} \frac{R_l^k(0)}{\lambda^{k+l}} \right]^{(2n-1)}}{2^{2n+1} \lambda^{n+1/2}} \\ & \left. + O\left(\frac{1}{\lambda^{M+k}}\right) \right\} \\ & + \cos \lambda^{1/2} z \left\{ \frac{-1}{2\lambda^{1/2}} \int_0^z q(\zeta) \sum_{l=0}^{M} \frac{P_l^k(\zeta)}{\lambda^{k+l-1/2}} \, d\zeta \right. \\ & + \sum_{n=0}^{N} \frac{(-1)^n \left[q(z) \sum_{l=0}^{M} \frac{R_l^k(z)}{\lambda^{k+l}} - q(z) \sum_{l=0}^{M} \frac{R_l^k(0)}{\lambda^{k+l}} \right]^{(2n)}}{2^{2n+2} \lambda^{n+1}} \\ & + \sum_{n=1}^{N} \frac{(-1)^{n-1} \left[q(z) \sum_{l=0}^{M} \frac{P_l^k(z)}{\lambda^{k+l-1/2}} - q(z) \sum_{l=0}^{M} \frac{P_l^k(0)}{\lambda^{k+l-1/2}} \right]^{(2n-1)}}{2^{2n+1} \lambda^{n+1/2}} \\ & \left. + O\left(\frac{1}{\lambda^{M+k}}\right) \right\} + O\left(\frac{1}{\lambda^{N+3/2}}\right). \end{aligned}$$

(For details, see Appendix I.) Hence,

$$f_{k+1}(z) = \sum_{m=0}^{M} \frac{P_m^{k+1}(z)}{\lambda^{k+m+1/2}} + O\left(\frac{1}{\lambda^{k+M+3/2}}\right)$$

and

$$g_{k+1}(z) = \sum_{m=0} \frac{R_m^{k+1}(z)}{\lambda^{k+m+1}} + O\left(\frac{1}{\lambda^{M+k+1}}\right),$$

where

$$\begin{aligned} P_m^{k+1}(z) = {} & \tfrac{1}{2} \int_0^z q(\zeta)\, R_m{}^k(\zeta)\, d\zeta \\ & + \sum_{n+l=m} \frac{(-1)^n [P_l{}^k(z)\, q(z) + P_l{}^k(0)\, q(z)]^{(2n)}}{2^{2n+2}} \\ & + \sum_{\substack{l+n=m \\ n\neq 0}} \frac{(-1)^{n-1} [R_l{}^k(z)\, q(z) + R_l{}^k(0)\, q(z)]^{(2n-1)}}{2^{2n+1}} \end{aligned} \tag{11}$$

and

$$\begin{aligned} R_m^{k+1}(z) = {} & -\tfrac{1}{2} \int_0^z q(\zeta)\, P_m{}^k(\zeta)\, d\zeta \\ & + \sum_{l+n=m} \frac{(-1)^n [R_l{}^k(z)\, q(z) - R_l{}^k(0)\, q(z)]^{(2n)}}{2^{2n+2}} \\ & + \sum_{l+n=m+1} \frac{(-1)^{n-1} [P_l{}^k(z)\, q(z) - P_l{}^k(0)\, q(z)]^{(2n-1)}}{2^{2n+1}}. \end{aligned} \tag{12}$$

A multiplication by q increases the maximal degree of each term in (11) and (12) by 1. Furthermore, for fixed $m > 0$, the highest order terms in the nonintegral parts of P_m^{k+1} and R_m^{k+1} are $q^{(2m-2)}$ and $q^{(2m-1)}$, respectively. For $m = 0$, the nonintegral parts of P_0^{k+1} and R_0^{k+1} are polynomials in q and Q.

Therefore, P_m^{k+1} consists of a polynomial of maximal degree $k+1$ in $Q, q, q', \ldots, q^{(2m-2)}$ plus integral terms, and R_m^{k+1} consists of a polynomial of maximal degree $k+1$ in $Q, q, q', \ldots, q^{(2m-1)}$ plus integral terms.

The proof of the Lemma is now complete.

III. Proof of Theorem

The proof of the theorem is accomplished by investigating the related problem

$$u'' + [\lambda - q(z + \tau)]u = 0; \ \tau \text{ real, arbitrary} \tag{13}$$

and by assuming the result [5] that $q(z)$ is infinitely differentiable a.e. when n finite instability intervals fail to vanish.

Let $u_2(z)$ denote the solution of (13) which satisfies

$$u_2(0) = 0 \qquad \text{and} \qquad u_2'(0) = 1.$$

In [5], Hochstadt showed that (13) when subject to $u(0) = u(\pi)$, has eigenvalues $\mu_n(\tau)$, where $\mu_i(\tau)$ lies in the ith finite instability interval of (1). Therefore, all but n zeros of $u_2(\pi)$ and $y_2(\pi)$ coincide when precisely n finite instability intervals fail to vanish. Furthermore, as functions of λ, $y_2(\pi)$ and $u_2(\pi)$ are entire functions of order 1/2. Hence,

$$u_2(\pi) = f(\tau)\, y_2(\pi) \prod_{i=1}^{n} \left[\frac{\lambda - \mu_i(\tau)}{\lambda - \mu_i(0)}\right],$$

where $f(\tau)$ is a constant which may depend on τ.

From the lemma it follows that, for real λ

$$y_2(\pi) = (\sin \lambda^{1/2}\pi/\lambda^{1/2}) + O(1/\lambda),$$

and similarly for $u_2(\pi)$. This implies that $f(\tau) = 1$, and

$$u_2(\pi) \prod_{i=1}^{n} [\lambda - \mu_i(0)] = y_2(\pi) \prod_{i=1}^{n} [\lambda - \mu_i(\tau)].$$

This is rewritten as

$$u_2(\pi) \sum_{i=0}^{n} \sigma_j(0)\lambda^{n-j} = y_2(\pi) \sum_{j=0}^{n} \sigma_j(\tau)\lambda^{n-j}, \tag{14}$$

where $\sigma_0(\tau) = 1$ and

$$\sigma_j(\tau) = (-1)^j \sum_{i_1<i_2<\cdots<i_j} \mu_{i_1}(\tau)\, \mu_{i_2}(\tau) \cdots \mu_{i_j}(\tau) \qquad \text{for} \quad j = 1, 2, \ldots, n.$$

From the lemma it follows that

$$y_2(z) = \sum_{k=0}^{\infty} [f_k(z) \sin \lambda^{1/2}z + g_k(z) \cos \lambda^{1/2}z]$$

$$= \frac{\sin \lambda^{1/2}z}{\lambda^{1/2}} \left[1 + \sum_{k=1}^{N} \sum_{l=0}^{M} \frac{P_l^k(z)}{\lambda^{k+l-1}} + O\left(\frac{1}{\lambda^{M+N}}\right)\right]$$

$$+ \cos \lambda^{1/2}z \left[\sum_{k=1}^{N} \sum_{l=0}^{M} \frac{R_l^k(z)}{\lambda^{k+l}} + O\left(\frac{1}{\lambda^{M+N+1}}\right)\right]$$

$$= \frac{\sin \lambda^{1/2}z}{\lambda^{1/2}} \left[1 + \sum_{n=0}^{N} \sum_{\substack{k+l-1=n \\ k=1,2,\ldots,n+1 \\ l=0,1,2,\ldots,n}} \frac{P_l^k(z)}{\lambda^{k+l-1}} + O\left(\frac{1}{\lambda^{N+1}}\right)\right]$$

$$+ \cos \lambda^{1/2}z \left[\sum_{n=1}^{N} \sum_{\substack{k+l=n \\ k=1,2,\ldots,n \\ l=0,1,\ldots,n-1}} \frac{R_l^k(z)}{\lambda^{k+l}} + O\left(\frac{1}{\lambda^{N+1}}\right)\right].$$

Hence,

$$y_2(z) = \frac{\sin \lambda^{1/2}z}{\lambda^{1/2}} \left[\sum_{n=0}^{N} \frac{P_n(z)}{\lambda^n} + O\left(\frac{1}{\lambda^{N+1}}\right)\right]$$

$$+ \cos \lambda^{1/2}z \left[\sum_{n=1}^{N} \frac{R_n(z)}{\lambda^n} + O\left(\frac{1}{\lambda^{N+1}}\right)\right], \tag{15}$$

where

$$P_0(z) = 1, \qquad P_n(z) = \sum_{k+l-1=n} P_l^k(z) \qquad \text{and}$$

$$R_n(z) = \sum_{k+l=n} R_l^k(z) \qquad \text{for} \quad n = 1, 2, \ldots, N.$$

For fixed $n \geqslant 1$,

$$P_n(z) = P_n^1(z) + \sum_{\substack{k+l-1=n \\ k=2,3,\ldots,n+1 \\ l=0,1,\ldots,n-1}} P_l^k(z)$$

$$= [(-1)^{n-1}/2^{2n}][q^{(2n-2)}(z) + q^{(2n-2)}(0)] + \sum_{\substack{k+l-1=n \\ k=2,3,\ldots,n+1 \\ l=0,1,\ldots,n-1}} P_l^k(z)$$

$$= [(-1)^{n-1}/2^{2n}]\, q^{(2n-2)}(z) + P_n{}^*(z). \tag{16}$$

Clearly, $P_n{}^*(z)$ is a polynomial of maximal degree $n + 1$ in $Q, q, q', \ldots, q^{(2n-4)}$ plus integral terms.

$u_2(z)$ can also be represented by a series of the form

$$u_2(z) = \sum_0^\infty v_k(z),$$

where

$$v_0(z) = \frac{\sin \lambda^{1/2} z}{\lambda^{1/2}}$$

and

$$v_{k+1}(z) = \frac{1}{\lambda^{1/2}} \int_0^z \sin \lambda^{1/2}(z - \zeta)\, q(\zeta + \tau)\, v_k(\zeta)\, d\zeta.$$

Following the same steps as in deriving (15) we get

$$u_2(\pi) = \frac{\sin \lambda^{1/2}\pi}{\lambda^{1/2}} \left[\sum_{n=0}^{N} \frac{S_n(\tau)}{\lambda^n} + O\left(\frac{1}{\lambda^{N+1}}\right)\right]$$
$$+ \cos \lambda^{1/2}\pi \left[\sum_{n=0}^{N} (T_n/\lambda^n) + O\left(\frac{1}{\lambda^{N+1}}\right)\right]. \tag{17}$$

From the analog of (11), using the periodicity of q, it follows that $S_n(\tau)$ differs from the nonintegral terms in $P_n(\tau)$ by a constant. In fact $P_n(\pi) = S_n(0)$. Similarly, (12) can be used to show that T_n is precisely the constant $R_n(\pi)$.

The trigonometric terms in (17) are expressed as exponentials giving

$$u_2(\pi) = \frac{e^{i\lambda^{1/2}\pi}}{2} \left[\sum_{k=0}^{n+1} \frac{-iS_k(\tau)}{\lambda^{k+1/2}} + \sum_{k=1}^{n+1} \frac{T_k}{\lambda^k} + O\left(\frac{1}{\lambda^{n+2}}\right)\right]$$
$$+ \frac{e^{-i\lambda^{1/2}\pi}}{2} \left[\sum_{k=0}^{n+1} \frac{iS_k(\tau)}{\lambda^{k+1/2}} + \sum_{k=1}^{n+1} \frac{T_k}{\lambda^k} + O\left(\frac{1}{\lambda^{n+2}}\right)\right].$$

$y_2(\pi)$ has the same representation with $\tau = 0$. Equation (14) now becomes

$$\Bigg\{ \frac{e^{i\lambda^{1/2}\pi}}{2} \left[\sum_{k=0}^{n+1} \frac{-iS_k(\tau)}{\lambda^{k+1/2}} + \sum_{k=1}^{n+1} \frac{T_k}{\lambda^k} + O\left(\frac{1}{\lambda^{n+2}}\right)\right]$$
$$+ \frac{e^{-i\lambda^{1/2}\pi}}{2} \left[\sum_{k=0}^{n+1} \frac{iS_k(\tau)}{\lambda^{k+1/2}} + \sum_{k=1}^{n+1} \frac{T_k}{\lambda^k} + O\left(\frac{1}{\lambda^{n+2}}\right)\right] \Bigg\} \cdot \sum_{j=0}^{n} \sigma_j(0)\lambda^{n-j}$$
$$= \Bigg\{ \frac{e^{i\lambda^{1/2}\pi}}{2} \left[\sum_{k=0}^{n+1} \frac{-iS_k(0)}{\lambda^{k+1/2}} + \sum_{k=1}^{n+1} \frac{T_k}{\lambda^k} + O\left(\frac{1}{\lambda^{n+2}}\right)\right]$$
$$+ \frac{e^{-i\lambda^{1/2}\pi}}{2} \left[\sum_{k=0}^{n+1} \frac{iS_k(0)}{\lambda^{k+1/2}} + \sum_{k=1}^{n+1} \frac{T_k}{\lambda^k} + O\left(\frac{1}{\lambda^{n+2}}\right)\right] \Bigg\} \cdot \sum_{j=0}^{n} \sigma_j(\tau)\lambda^{n-j}. \tag{18}$$

We note that for $\mathrm{Im}(\lambda^{1/2}) > 0$, $e^{i\lambda^{1/2}\pi} \to 0$ as $\mathrm{Im}(\lambda^{1/2}) \to \infty$, while for $\mathrm{Im}(\lambda^{1/2}) < 0$, $e^{-i\lambda^{1/2}\pi} \to 0$ as $\mathrm{Im}(\lambda^{1/2}) \to \infty$. Therefore, the coefficients of $e^{i\lambda^{1/2}\pi}$ and $e^{-i\lambda^{1/2}\pi}$ from both sides of (18) must be independently equal. Hence

$$\left[\sum_{k=0}^{n+1} \frac{-iS_k(\tau)}{\lambda^{k+1/2}} + \sum_{k=1}^{n+1} \frac{T_k}{\lambda^k} + O\left(\frac{1}{\lambda^{n+2}}\right)\right] \sum_{j=0}^{n} \sigma_j(0)\lambda^{n-j}$$
$$= \left[\sum_{k=0}^{n+1} \frac{-iS_k(0)}{\lambda^{k+1/2}} + \sum_{k=1}^{n+1} \frac{T_k}{\lambda^k} + O\left(\frac{1}{\lambda^{n+2}}\right)\right] \sum_{j=0}^{n} \sigma_j(\tau)\lambda^{n-j}. \qquad (19)$$

This, in turn, implies that all coefficients of powers of $\lambda^{-1/2}$ in (19) vanish. In particular,

$$\sum_{j+k=l} [\sigma_j(0)\, S_k(\tau) - \sigma_j(\tau)\, S_k(0)] = 0 \qquad \text{for} \quad l = 1, 2,..., n.$$

This is rewritten as

$$\sigma_l(\tau) = \sigma_l(0) + \sum_{j=0}^{l-1} [\sigma_j(0)\, S_{l-j}(\tau) - S_{l-j}(0)\, \sigma_j(\tau)] \qquad \text{for} \quad l = 1, 2,..., n. \quad (20)$$

We solve (20) recursively for $\sigma_1(\tau),..., \sigma_{l-1}(\tau)$ in terms of $S_0(\tau),..., S_{l-1}(\tau)$ and conclude that $\sigma_l(\tau)$ is a linear function of $S_0(\tau),..., S_l(\tau)$. That is, $\sigma_l(\tau)$ is a polynomial in $q, q',..., q^{(2l-2)}$ of maximal degree $l + 1$, for $l = 1, 2,..., n$.

Matching coefficients of $\lambda^{n-3/2}$ in (19) gives

$$\sum_{\substack{j+k=n+1\\k\neq 0}} \sigma_j(0)\, S_k(\tau) = \sum_{\substack{j+k=n+1\\k\neq 0}} S_k(0)\, \sigma_j(\tau),$$

which is rewritten as

$$\sum_{j=0}^{n} [\sigma_j(0)\, S_{n+1-j}(\tau) - S_{n+1-j}(0)\, \sigma_j(\tau)] = 0. \qquad (21)$$

From (16), we see that $S_{n+1}(\tau) = [(-1)^n/2^{2n+2}]\, q^{(2n)}(\tau) + S^*_{n+1}(\tau)$, where $S^*_{n+1}(\tau)$ is a polynomial of maximal degree $n + 2$ in $q, q',..., q^{(2n-2)}$.

We conclude that (21) is of the form

$$q^{(2n)} + H(q, q',..., q^{(2n-2)}) = 0$$

and the theorem is proven.

We now derive explicit expressions of these equations. The lengthy process of repeated integrations by parts gives the following convergent series:

$$u_2(\pi) = \frac{\sin \lambda^{1/2}\pi}{\lambda^{1/2}} \Big\{ 1 + [q(\tau)/2\lambda] + [3q^2(\tau) - q''(\tau)]/8\lambda^2$$
$$+ [q^{(4)}(\tau) - 10q''(\tau)\, q(\tau) - 5[q'(\tau)]^2 + 10q^3(\tau)]/32\lambda^3 + O(1/\lambda^4) \Big\}$$
$$+ \cos \lambda^{1/2}\pi \Big\{ (-1/8\lambda^2) \int_0^\pi q^2(z + \tau)\, dz + O(1/\lambda^3) \Big\}. \tag{22}$$

(A simpler formal procedure for deriving this series is found in Appendix II.)

When all finite instability intervals vanish, (14) becomes $u_2(\pi) = y_2(\pi)$.

Matching coefficients of $\lambda^{-3/2}$ yields

$$q(\tau) = q(0), \quad \text{a.e.}$$

from which (3) immediately follows.

For $n \geqslant 1$ when we substitute (22) into (14) and match the coefficients of $\lambda^{n-3/2}$ we get

$$\sigma_1(\tau) - \sigma_1(0) = (1/2)[q(\tau) - q(0)], \quad \text{a.e.} \tag{23}$$

The coefficients of $\lambda^{n-5/2}$ yield

$$\sigma_2(0) + [q(\tau)/2]\, \sigma_1(0) + (3/8)\, q^2(\tau) - [q''(\tau)/8]$$
$$= \sigma_2(\tau) + [q(0)/2]\, \sigma_1(\tau) + (3/8)\, q^2(0) - [q''(0)/8], \quad \text{a.e.}$$

A substitution of (23) into the above expression gives

$$\sigma_2(\tau) = [-q''(\tau)/8] + (3/8)\, q^2(\tau) - \{[q(0)/4] - [\sigma_1(0)/2]\}\, q(\tau)$$
$$- [q^2(0)/8] + [q''(0)/8] - [q(0)\, \sigma_1(0)/2] + \sigma_2(0), \quad \text{a.e.} \tag{24}$$

When all finite instability intervals vanish, the only nonzero symmetric function in (14) is $\sigma_0(\tau)$. When precisely one finite instability interval fails to vanish, $\sigma_0(\tau)$ and $\sigma_1(\tau)$ are the only nonzero symmetric functions in (14).

Hence, when all finite instability intervals vanish, the explicit expression of (21) is (23) with $\sigma_1(\tau) = 0$.

Similarly, when precisely one finite instability interval fails to vanish, the explicit expression of (21) is (24) with $\sigma_2(\tau) = 0$ and (4) then follows.

To obtain (5) we need only substitute (23) and (24) into the explicit expression of (21) when precisely two finite instability intervals fail to vanish. This yields an equation of the form

$$q^{(4)}(\tau) - 10q(\tau)\, q''(\tau) + Aq''(\tau) - 5[q'(\tau)]^2 + 10q^3(\tau) + Bq^2(\tau)$$
$$+ Cq(\tau) + D = 0, \quad \text{a.e.,}$$

where

$$\begin{aligned}
A &= 2q(0) - 4\sigma_1(0)\\
B &= 12\sigma_1(0) - 6\\
C &= 16\sigma_2(0) + 2q(0)A\\
D &= -10q^3(0) + 8[q(0)\,q''(0) + \sigma_1(0)\,q^2(0) - 2\sigma_2(0)\,q(0)]\\
&\quad + 5[q'(0)]^2 - q^{(4)}(0).
\end{aligned}$$

Appendix I

In this section we derive the expressions for $w_1(z)$ and $w_{\lambda+1}(z)$ which were stated in the lemma.

For $F \in C^N[0, z]$, the integration by parts formula yields

$$\begin{aligned}
&\int_0^z \sin 2\lambda^{1/2}\zeta F(\zeta)\,d\zeta\\
&\quad = \sum_{n=0}^{N} (-1)^{n+1}\left[\frac{\cos 2\lambda^{1/2}zF^{(2n)}(z) - F^{(2n)}(0)}{2^{2n+1}\lambda^{n+1/2}}\right]\\
&\quad\quad + \sum_{n=1}^{N} \frac{(-1)^{n-1}\sin 2\lambda^{1/2}zF^{(2n-1)}(z)}{2^{2n}\lambda^n} + O\left(\frac{1}{\lambda^{N+1}}\right),
\end{aligned}\tag{25}$$

and

$$\begin{aligned}
&\int_0^z \cos 2\lambda^{1/2}\zeta F(\zeta)\,d\zeta\\
&\quad = \sum_{n=0}^{N} \frac{(-1)^n \sin 2\lambda^{1/2}zF^{(2n)}(z)}{2^{2n+1}\lambda^{n+1/2}}\\
&\quad\quad + \sum_{n=1}^{N} (-1)^{n-1}\left[\frac{\cos 2\lambda^{1/2}zF^{(2n-1)}(z) - F^{(2n-1)}(0)}{2^{2n}\lambda^n}\right] + O\left(\frac{1}{\lambda^{N+1}}\right).
\end{aligned}\tag{26}$$

Upon inserting (7b) into (7c) we get

$$\begin{aligned}
w_1(z) &= \frac{1}{\lambda}\int_0^z \sin\lambda^{1/2}(z-\zeta)\sin\lambda^{1/2}\zeta q(\zeta)\,d\zeta\\
&= \frac{1}{\lambda}\int_0^z [\sin\lambda^{1/2}z\cos\lambda^{1/2}\zeta - \cos\lambda^{1/2}z\sin\lambda^{1/2}\zeta]\sin\lambda^{1/2}\zeta q(\zeta)\,d\zeta\\
&= \frac{\sin\lambda^{1/2}z}{2\lambda}\int_0^z \sin 2\lambda^{1/2}\zeta q(\zeta)\,d\zeta\\
&\quad - \frac{\cos\lambda^{1/2}z}{2\lambda}\int_0^z [1 - \cos 2\lambda^{1/2}\zeta]\,q(\zeta)\,d\zeta.
\end{aligned}$$

Using (25) and (26) this becomes

$$w_1(z) = \frac{\sin \lambda^{1/2}z}{2\lambda}\left\{\sum_{n=0}^{N}(-1)^{n-1}\left[\frac{\cos 2\lambda^{1/2}z q^{(2n)}(z) - q^{(2n)}(0)}{2^{2n+1}\lambda^{n+1/2}}\right]\right.$$

$$\left. + \sum_{n=1}^{N}\frac{(-1)^{n-1}\sin 2\lambda^{1/2}z q^{(2n-1)}(z)}{2^{2n}\lambda^{n}} + O\left(\frac{1}{\lambda^{N+1}}\right)\right\}$$

$$+ \frac{\cos \lambda^{1/2}z}{2\lambda}\left\{-\int_0^z q(\zeta)\,d\zeta + \sum_{n=0}^{N}\frac{(-1)^n \sin 2\lambda^{1/2}z q^{(2n)}(z)}{2^{2n+1}\lambda^{n+1/2}}\right.$$

$$\left. + \sum_{n=1}^{N}(-1)^{n-1}\left[\frac{\cos 2\lambda^{1/2}z q^{(2n-1)}(z) - q^{(2n-1)}(0)}{2^{2n}\lambda^{n}}\right] + O\left(\frac{1}{\lambda^{N+1}}\right)\right\}.$$

$$= \sum_{n=0}^{N}(-1)^n\frac{[\sin 2\lambda^{1/2}z \cos \lambda^{1/2}z - \cos 2\lambda^{1/2}z \sin \lambda^{1/2}z]\, q^{(2n)}(z)}{2^{2n+2}\lambda^{n+3/2}}$$

$$+ \sum_{n=0}^{N}\frac{(-1)^n \sin \lambda^{1/2}z q^{(2n)}(0)}{2^{2n+2}\lambda^{n+3/2}}$$

$$+ \sum_{n=1}^{N}\frac{(-1)^{n-1}[\cos 2\lambda^{1/2}z \cos \lambda^{1/2}z + \sin 2\lambda^{1/2}z \sin \lambda^{1/2}z]\, q^{(2n-1)}(z)}{2^{2n+1}\lambda^{n+1}}$$

$$- \sum_{n=1}^{N}\frac{(-1)^{n-1}\cos \lambda^{1/2}z q^{(2n-1)}(0)}{2^{2n+1}\lambda^{n+1}}$$

$$- \frac{\cos \lambda^{1/2}z}{2\lambda}\int_0^z q(\zeta)\,d\zeta + O(1/\lambda^{N+2}).$$

By factoring $\sin \lambda^{1/2}z$ and $\cos \lambda^{1/2}z$ out of their respective terms, the desired result for $w_1(z)$ is obtained.

By writing (7c) in the form

$$w_{k+1}(z) = \frac{1}{\lambda^{1/2}}\int_0^z [\sin \lambda^{1/2}z \cos \lambda^{1/2}\zeta - \cos \lambda^{1/2}z \sin \lambda^{1/2}\zeta]\, q(\zeta)\, w_k(\zeta)\, d\zeta$$

and using (8), we get

$$w_{k+1}(z) = \frac{\sin \lambda^{1/2} z}{\lambda^{1/2}} \int_0^z \cos \lambda^{1/2}\zeta[\sin \lambda^{1/2}\zeta f_k(\zeta)\, q(\zeta) + \cos \lambda^{1/2}\zeta g_k(\zeta)\, q(\zeta)]\, d\zeta$$

$$- \frac{\cos \lambda^{1/2} z}{\lambda^{1/2}} \int_0^z \sin \lambda^{1/2}\zeta[\sin \lambda^{1/2}\zeta f_k(\zeta)\, q(\zeta) + \cos \lambda^{1/2}\zeta g_k(\zeta)\, q(\zeta)]\, d\zeta.$$

$$= \frac{\sin \lambda^{1/2} z}{\lambda^{1/2}} \int_0^z \frac{\sin 2\lambda^{1/2}\zeta}{2}\, q(\zeta)\, f_k(\zeta)\, d\zeta$$

$$+ \frac{\sin \lambda^{1/2} z}{\lambda^{1/2}} \int_0^z \cos^2 \lambda^{1/2}\zeta q(\zeta)\, g_k(\zeta)\, d\zeta$$

$$- \frac{\cos \lambda^{1/2} z}{\lambda^{1/2}} \int_0^z \sin^2 \lambda^{1/2}\zeta q(\zeta)\, f_k(\zeta)\, d\zeta$$

$$- \frac{\cos \lambda^{1/2} z}{\lambda^{1/2}} \int_0^z \frac{\sin 2\lambda^{1/2}\zeta}{2}\, q(\zeta)\, f_k(\zeta)\, d\zeta.$$

Let

$$F_k(z) = q(z)\, f_k(z), \qquad G_k(z) = q(z)\, g_k(z)$$

and note that

$$\cos^2 \lambda^{1/2}\zeta = (1/2) + (\cos 2\lambda^{1/2}\zeta/2), \qquad \sin^2 \lambda^{1/2}\zeta = (1/2) - (\cos 2\lambda^{1/2}\zeta/2).$$

Then

$$w_{k+1}(z) = \frac{\sin \lambda^{1/2} z}{2\lambda^{1/2}} \int_0^z G_k(\zeta)\, d\zeta + \frac{\sin \lambda^{1/2} z}{2\lambda^{1/2}} \int_0^z \sin 2\lambda^{1/2}\zeta F_k(\zeta)\, d\zeta$$

$$+ \frac{\sin \lambda^{1/2} z}{2\lambda^{1/2}} \int_0^z \cos 2\lambda^{1/2}\zeta G_k(\zeta)\, d\zeta - \frac{\cos \lambda^{1/2} z}{2\lambda^{1/2}} \int_0^z F_k(\zeta)\, d\zeta$$

$$- \frac{\cos \lambda^{1/2} z}{2\lambda^{1/2}} \int_0^z \sin 2\lambda^{1/2}\zeta G_k(\zeta)\, d\zeta + \frac{\cos \lambda^{1/2} z}{2\lambda^{1/2}} \int_0^z \cos 2\lambda^{1/2}\zeta F_k(\zeta)\, d\zeta. \tag{27}$$

Now integrate the appropriate terms in (27) by parts [using (25) and (26)]; then

$$w_{k+1}(z) = \frac{\sin \lambda^{1/2}z}{2\lambda^{1/2}} \int_0^z G_k(\zeta)\, d\zeta - \frac{\cos \lambda^{1/2}z}{2\lambda^{1/2}} \int_0^z F_k(\zeta)\, d\zeta$$

$$+ \frac{\sin \lambda^{1/2}z}{2\lambda^{1/2}} \left\{ \sum_{n=0}^{N} \frac{(-1)^{n+1}[\cos \lambda^{1/2}z F_k^{(2n)}(z) - F_k^{(2n)}(0)]}{2^{2n+1}\lambda^{n+1/2}} \right.$$

$$\left. + \sum_{n=1}^{N} \frac{(-1)^{n-1} \sin 2\lambda^{1/2}z F_k^{(2n-1)}(z)}{2^{2n}\lambda^{n}} \right\}$$

$$+ \frac{\cos \lambda^{1/2}z}{2\lambda^{1/2}} \left\{ \sum_{n=0}^{N} \frac{(-1)^{n} \sin 2\lambda^{1/2}z F_k^{(2n)}(z)}{2^{2n-1}\lambda^{n-1/2}} \right.$$

$$\left. + \sum_{n=1}^{N} \frac{(-1)^{n-1}[\cos 2\lambda^{1/2}z F_k^{(2n-1)}(z) - F_k^{(2n-1)}(0)]}{2^{2n}\lambda^{n}} \right\}$$

$$+ \frac{\sin \lambda^{1/2}z}{2\lambda^{1/2}} \left\{ \sum_{n=0}^{N} \frac{(-1)^{n} \sin 2\lambda^{1/2}z G_k^{(2n)}(z)}{2^{2n-1}\lambda^{n-1/2}} \right.$$

$$\left. + \sum_{n=1}^{N} \frac{(-1)^{n-1}[\cos 2\lambda^{1/2}z G_k^{(2n-1)}(z) - G_k^{(2n-1)}(0)]}{2^{2n}\lambda^{n}} \right\}$$

$$- \frac{\cos \lambda^{1/2}z}{2\lambda^{1/2}} \left\{ \sum_{n=0}^{N} \frac{(-1)^{n+1}[\cos 2\lambda^{1/2}z G_k^{(2n)}(z) - G_k^{(2n)}(0)]}{2^{2n+1}\lambda^{n+1/2}} \right.$$

$$\left. + \sum_{n=1}^{N} \frac{(-1)^{n-1} \sin 2\lambda^{1/2}z G_k^{(2n-1)}(z)}{2^{2n}\lambda^{n}} \right\} + O\left(\frac{1}{\lambda^{N+3/2}}\right).$$

or

$$w_{k+1}(z) = \frac{\sin \lambda^{1/2} z}{2\lambda^{1/2}} \int_0^z G_k(\zeta)\, d\zeta - \frac{\cos \lambda^{1/2} z}{2\lambda^{1/2}} \int_0^z F_k(\zeta)\, d\zeta$$

$$+ \sum_{n=0}^{N} \frac{(-1)^n [\sin 2\lambda^{1/2} z \cos \lambda^{1/2} z - \cos 2\lambda^{1/2} z \sin \lambda^{1/2} z]\, F_k^{(2n)}(z)}{2^{2n+2} \lambda^{n+1}}$$

$$+ \sum_{n=0}^{N} \frac{(-1)^n \sin \lambda^{1/2} z F_k^{(2n)}(0)}{2^{2n+2} \lambda^{n+1}}$$

$$+ \sum_{n=1}^{N} \frac{(-1)^{n-1} [\cos 2\lambda^{1/2} z \cos \lambda^{1/2} z + \sin 2\lambda^{1/2} z \sin \lambda^{1/2} z]\, F_k^{(2n-1)}(z)}{2^{2n+1} \lambda^{n+1/2}}$$

$$- \sum_{n=1}^{N} \frac{(-1)^{n-1} [\cos \lambda^{1/2} z]\, F_k^{(2n-1)}(0)}{2^{2n+1} \lambda^{n+1/2}}$$

$$+ \sum_{n=0}^{N} \frac{(-1)^n [\cos 2\lambda^{1/2} z \cos \lambda^{1/2} z + \sin 2\lambda^{1/2} z \sin \lambda^{1/2} z]\, G_k^{(2n)}(z)}{2^{2n+2} \lambda^{n+1}}$$

$$- \sum_{n=0}^{N} \frac{(-1)^n \cos \lambda^{1/2} z G_k^{(2n)}(0)}{2^{2n+2} \lambda^{n+1}}$$

$$+ \sum_{n=1}^{N} \frac{(-1)^{n-1} [\cos 2\lambda^{1/2} z \sin \lambda^{1/2} z - \sin 2\lambda^{1/2} z \cos \lambda^{1/2} z]\, G_k^{(2n-1)}(z)}{2^{2n+1} \lambda^{n+1/2}}$$

$$- \sum_{n=1}^{N} \frac{(-1)^{n-1} \sin \lambda^{1/2} z G_k^{(2n-1)}(0)}{2^{2n+1} \lambda^{n+1/2}} + O\left(\frac{1}{\lambda^{N+3/2}}\right).$$

Therefore,

$$w_{k+1}(z) = \sin \lambda^{1/2} z \left\{ \frac{1}{2\lambda^{1/2}} \int_0^z G_k(\zeta)\, d\zeta + \sum_{n=0}^{N} (-1)^n \left[\frac{F_k^{(2n)}(z) + F_k^{(2n)}(0)}{2^{2n+2} \lambda^{n+1}} \right] \right.$$

$$\left. + \sum_{n=1}^{N} (-1)^n \left[\frac{G_k^{(2n-1)}(z) + G_k^{(2n-1)}(0)}{2^{2n+1} \lambda^{n+1/2}} \right] \right\}$$

$$+ \cos \lambda^{1/2} z \left\{ \frac{-1}{2\lambda^{1/2}} \int_0^z F_k(\zeta)\, d\zeta \right.$$

$$+ \sum_{n=1}^{N} (-1)^{n-1} \left[\frac{F_k^{(2n-1)}(z) - F_k^{(2n-1)}(0)}{2^{2n+1} \lambda^{n+1/2}} \right]$$

$$\left. + \sum_{n=0}^{N} (-1)^n \left[\frac{G_k^{(2n)}(z) - G_k^{(2n)}(0)}{2^{2n+2} \lambda^{n+1}} \right] \right\} + O\left(\frac{1}{\lambda^{N+3/2}}\right) \tag{28}$$

and we need only substitute the equivalent series for F_k and G_k from (9) and (10), respectively, into the above expression.

Appendix II

Due to the complexity of calculating series (22) by repeated integrations by parts, we offer a simpler formal procedure for deriving this series in an asymptotic sense.

In Coddington and Levinson [7] it is shown that the equation

$$y'' + [\lambda^2 - q(t)]y = 0 \tag{29}$$

has asymptotic solutions of the form

$$y^+(t) = e^{i\lambda t}\left[\sum_{n=0}^{N} \frac{f_n(t)}{\lambda^n} + O\left(\frac{1}{\lambda^{N+1}}\right)\right]$$

and

$$y^-(t) = e^{-i\lambda t}\left[\sum_{n=0}^{N} \frac{g_n(t)}{\lambda^n} + O\left(\frac{1}{\lambda^{N+1}}\right)\right].$$

Let $y_2(t) = ay^+(t) + by^-(t)$, where $y_2(0) = 0$ and $y_2'(0) = 1$. A direct substitution of $y_2(t)$ into (29) yields

$$ae^{i\lambda t}\left\{\sum_{n=0}^{N}\left[\frac{-\lambda^2 f_n}{\lambda^n} + \frac{2i\lambda f_n'}{\lambda^n} + \frac{f_n''}{\lambda^n} + \frac{\lambda^2 f_n}{\lambda^n} - \frac{qf_n}{\lambda^n}\right] + O\left(\frac{1}{\lambda^{N+1}}\right)\right\}$$

$$+ be^{-i\lambda t}\left\{\sum_{n=0}^{N}\left[\frac{-\lambda^2 g_n}{\lambda^n} - \frac{2i\lambda g_n'}{\lambda^n} + \frac{g_n''}{\lambda^n} + \frac{\lambda^2 g_n}{\lambda^n} - \frac{qg_n}{\lambda^n}\right] + O\left(\frac{1}{\lambda^{N+1}}\right)\right\} = 0.$$

Hence $a[2if'_{n+1} + f''_n - qf_n]\,e^{i\lambda t} + b[-2ig'_{n+1} + g''_n - qg_n]\,e^{-i\lambda t} = 0$, from which we conclude that

$$f_{n+1}(t) = \frac{1}{2i}\int_0^t [qf_n - f''_n]\,dt$$

and

$$g_{n+1}(t) = \frac{1}{2i}\int_0^t [-qg_n + g''_n]\,dt, \qquad \text{for} \quad n \geqslant 1. \tag{30}$$

By taking $f_0(t) = g_0(t) = 1$, it immediately follows that

$$g_n(t) = (-1)^n f_n(t) \qquad \text{for} \quad n = 0, 1, 2, \ldots .$$

From $y_2(0) = 0$ we get $b = -a$ and hence

$$y_2(t) = a \left\{ e^{i\lambda t} \left[\sum_{n=0}^{N} \frac{f_n(t)}{\lambda^n} + O\left(\frac{1}{\lambda^{N+1}}\right) \right] \right.$$

$$\left. - e^{-i\lambda t} \left[\sum_{n=0}^{N} \frac{(-1)^n f_n(t)}{\lambda^n} + O\left(\frac{1}{\lambda^{N+1}}\right) \right] \right\}$$

$$y_2'(t) = a(i\lambda) \left\{ e^{i\lambda t} \left[\sum_{n=0}^{N} \frac{f_n(t)}{\lambda^n} + O\left(\frac{1}{\lambda^{N+1}}\right) \right] \right.$$

$$\left. + e^{-i\lambda t} \left[\sum_{n=0}^{N} \frac{(-1)^n f_n(t)}{\lambda^n} + O\left(\frac{1}{\lambda^{N+1}}\right) \right] \right\}$$

$$+ a \left\{ e^{i\lambda t} \left[\sum_{n=0}^{N} \frac{f_n'(t)}{\lambda^n} + O\left(\frac{1}{\lambda^{N+1}}\right) \right] \right.$$

$$\left. - e^{-i\lambda t} \left[\sum_{n=0}^{N} \frac{(-1)^n f_n'(t)}{\lambda^n} + O\left(\frac{1}{\lambda^{N+1}}\right) \right] \right\}.$$

From $y_2'(0) = 1$, it follows that

$$a = \left[1 \middle/ \left(2i\lambda + \sum_{n\,\mathrm{odd}}^{N} \frac{f_n'(0)}{\lambda^n} + O\left(\frac{1}{\lambda^N}\right) \right) \right]$$

Therefore,

$$y_2(t) \sim \frac{\left(\dfrac{e^{i\lambda t} - e^{-i\lambda t}}{2i}\right) \sum\limits_{n\,\mathrm{even}} \dfrac{2if_n(t)}{\lambda^n} + \left(\dfrac{e^{i\lambda t} + e^{-i\lambda t}}{2}\right) \sum\limits_{n\,\mathrm{odd}} \dfrac{2f_n(t)}{\lambda^n}}{2i\lambda + \sum\limits_{n\,\mathrm{odd}} \dfrac{2f_n'(0)}{\lambda^n}}$$

or

$$y_2(t) \sim \frac{\sin \lambda t \sum\limits_{n\,\mathrm{even}} [if_n(t)/\lambda^n] + \cos \lambda t \sum\limits_{n\,\mathrm{odd}} [f_n(t)/\lambda^n]}{i\lambda + \sum\limits_{n\,\mathrm{odd}} [f_n'(0)/\lambda^n]}. \tag{31}$$

Let $T = \sum_{n\,\mathrm{odd}} [f_n'(0)/\lambda^n]$. Then

$$\left\{ 1 \middle/ \left[i\lambda + \sum_{n\,\mathrm{odd}} (f_n'(0)/\lambda^n) \right] \right\} = 1/(i\lambda + T) = (1/i\lambda) \left[1 \middle/ \left(1 - \frac{iT}{\lambda} \right) \right]$$

$$\sim -i \sum_{n=0}^{N} [(iT)^n/\lambda^{n+1}] + O(1/\lambda^{N+2})$$

After expanding this series sufficiently and substituting the results into (31), we get

$$\begin{aligned} y_2(\pi) = {} & \sin\lambda\pi[(1/\lambda) + \{[f_2(\pi) + if_1'(0)]/\lambda^3\} \\ & + \{[f_4(\pi) + if_3'(0) - [f_1'(0)]^2 + if_2(\pi) f_1'(0)]/\lambda^5\} \\ & + \{[f_6(\pi) + if_5'(0) - 2f_1'(0) f_3'(0) - i[f_1'(0)]^3 + if_2(\pi) f_3'(0) \\ & \quad - f_2(\pi)[f_1'(0)]^2]/\lambda^7\} + O(1/\lambda^9)] \\ & + \cos\lambda\pi[(-if_1(\pi)/\lambda^2) - \{i[f_3(\pi) + if_1(\pi) f_1'(0)]/\lambda^4\} + O(1/\lambda^6)]. \quad (32) \end{aligned}$$

The first few results of repeated applications of (30) are

$$f_0(t) = 1$$

$$f_1(t) = (1/2i)\, Q(t)$$

$$f_2(t) = -(1/4)\{[Q^2(t)/2] - q(t) + q(0)\}$$

$$f_3(t) = -(1/8i)\left[[Q^3(t)/6] - \int_0^t q^2(\tau)\, d\tau + q(0) Q(t) - Q(t)\, q(t) + q'(t) - q'(0)\right]$$

$$\begin{aligned} f_4(t) = {} & (1/16)[[Q^4(t)/24] - Q(t)\, q^2(t) + [q(0)\, Q^2(t)/2] + (5/2)\, q^2(t) \\ & - (3/2)\, q^2(0) - q(0)\, q(t) - q'(0)\, Q(t) - [q(t)\, Q^2(t)/2] \\ & + Q(t)\, q'(t) - q''(t) + q''(0)]. \end{aligned}$$

Due to the length of $f_5(t)$ and $f_6(t)$, we simply state that

$$f_5'(0) = (-1/32i)[-2q^3 + 5(q')^2 + 6qq'' - q'''']_{t=0}$$

and $f_6(\pi) = 0$.

These results are now substituted into (32) and we finally get

$$\begin{aligned} y_2(\pi) = {} & \sin\lambda\pi[(1/\lambda) + [q(0)/2\lambda^3] + \{[q^2(0) - q''(0) + 2q^2(0)]/8\lambda^5\} \\ & + \{[-5[q'(0)]^2 - 10q(0)\, q''(0) + 10q^3(0) + q^{(4)}(0)]/32\lambda^7\} + O(1/\lambda^9)] \\ & + \cos\lambda\pi\left[(-1/8\lambda^4)\int_0^\pi q^2(\tau)\, d\tau + O(1/\lambda^6)\right]. \end{aligned}$$

Acknowledgment

The author thanks his advisor, Harry Hochstadt, for his generous advice and guidance.

References

1. W. Magnus and S. Winkler, "Hill's Equation," Interscience, New York, 1966.
2. G. Borg, Eine Umkehrung der Sturm–Liouvilleschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte, *Acta Math.* **78** (1946), 1–96.
3. H. Hochstadt, Function theoretic properties of the discriminant of Hill's equation, *Math. Z.* **82** (1963), 237–242.
4. P. Ungar, Stable Hill equations, *Comm. Pure Appl. Math.* **14** (1961), 707–710.
5. H. Hochstadt, On the determination of a Hill's equation from its spectrum, *Arch. Rational Mech. Anal.* **19** (1965), 353–362.
6. A. Erdélyi, "Higher Transcendental Functions, Vol. III," p. 63 ff., McGraw–Hill, New York, 1955.
7. P. Lax, Periodic solutions of the KdV equation, *in* "Lectures in Applied Math.," Vol. 15, AMS, Providence, Rhode Island, 1974.
8. E. A. Coddington and N. Levinson, "Theory of Ordinary Differential Equations," pp. 174–184, McGraw–Hill, New York, 1955.