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The Schur Indices of the Reflection Group $\mathcal{I}_4$

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It was shown in 1971 by M. Benard [1] that the irreducible complex characters of the crystallographic reflection groups $\mathscr{E}_6$, $\mathscr{E}_7$, and $\mathscr{E}_8$ all have Schur index 1 over the rational field Q. The same result for other crystallographic reflection groups had been proved in earlier papers by A. Young [11] in 1930, W. Specht [9] in 1932, and T. Kondo [6] in 1965.

As for the noncrystallographic groups, it can be seen that each Schur index is 1 for the dihedral groups $\mathscr{H}_2^n$ by formulas of Yamada [10] (1968); also a particularly simple proof appears in Plotkin [7] (1972). It is easily seen (e.g., by the methods of the present paper) that the group $\mathcal{I}_3$ of all symmetries of the icosahedron has all Schur indices 1, so the only irreducible group remaining is the group $\mathcal{I}_4$ of symmetries of the regular 120-hedroid in Euclidean four-space. We shall extend the results above to the case of $\mathcal{I}_4$, thereby providing the final step in the proof of the following theorem.

THEOREM. *If $\mathscr{G}$ is a finite group generated by reflections and χ is an irreducible complex character of $\mathscr{G}$, then the Schur index $m_Q(\chi)$ of χ over the rational field Q is equal to* 1, *with the exception of the character of degree* 48 *of* $\mathcal{I}_4$, *whose Schur index is* 2.

Since the theorem has been proved by means of a case-by-case analysis, a general proof covering all cases simultaneously would be of considerable interest.

All notation involving $\mathcal{I}_4$ and its characters will be that used in [5].

All characters of $\mathcal{I}_4$ are real valued. Since $\chi_{11}, \chi_{12}, \chi_{13}, \chi_{14}, \chi_{27}, \chi_{28}$ are of odd degree (and real), they all have Schur index 1 by theorem of A. Speiser (see [3, p. 165]). Each of the characters of even degree has Schur index either 1 or 2 by a theorem of Brauer and Hasse (again see [3, p. 165]). Thus it will suffice to show that $m_Q(\chi)$ is odd for each χ of even degree. The most important tool will be the following theorem of Schur (see [4, 11.4]; also see [8, Theorem 2]).

THEOREM (Schur). *Suppose F is a subfield of the complex field C and χ is an irreducible C-representation of a finite group $\mathscr{G}$. If χ is a constituent of the character η of an F-representation of $\mathscr{G}$ then* $m_F(\chi) \mid (\chi, \eta)$.

Two reductions are possible. Given an irreducible character χ of $\mathscr{I}_4$, suppose a character η of a Q-representation has been found, as in Schur's theorem, for which (χ, η) is odd. Multiplying by the alternating character χ_2, we obtain a possibly different character $\chi_2\chi$, and $\chi_2\eta$ is also afforded by a Q-representation. Since

$$(\chi_2\chi, \chi_2\eta) = (\chi, \eta)$$

is odd, we see that $\chi_2\chi$ also has Schur index 1. Inspection of the character table shows then that we need not consider χ_i for $i = 3, 5, 18, 20$, and 32 (since $\chi_3 = \chi_2\chi_4$, etc.).

Next set $F = Q(5^{1/2})$, and define $\phi \in \mathrm{Gal}(F : Q)$ by means of $\phi(5^{1/2}) = -5^{1/2}$. Suppose η is the character of a Q-representation T and (χ, η) is odd. Then $T^\phi = T$ has character $\eta^\phi = \eta$, and

$$(\chi^\phi, \eta) = (\chi^\phi, \eta^\phi) = (\chi, \eta),$$

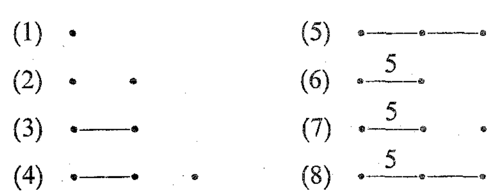
so $m_Q(\chi^\phi) = 1$. Since

$$\alpha = \cos \pi/5 = (1 + 5^{1/2})/4 \qquad \text{and} \qquad \beta = \cos 2\pi/5 = (-1 + 5^{1/2})/4,$$

the effect of ϕ on the entries of the character table is to interchange α and $-\beta$. Thus we also need not consider χ_i for $i = 6, 7, 17, 24, 26$, and 30 (since $\chi_6 = \chi_4{}^\phi$, etc.).

The remaining characters of even degree are χ_i, $i = 4, 8, 9, 10, 15, 16, 19, 21, 22, 23, 25, 29, 31, 33$, and 34. For each we shall exhibit a character η of a Q-representation for which (χ_i, η) is odd, $i \neq 34$.

A subgroup $\mathscr{H}$ of $\mathscr{I}_4$ is called *parabolic* if it is generated by a subset of a set $\{S_1, S_2, S_3, S_4\}$ of fundamental reflections. Thus a nontrivial proper parabolic subgroup is a Coxeter group with one of the Coxeter graphs



and hence is of type $\mathcal{A}_1$, $\mathcal{A}_1 \times \mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_2 \times \mathcal{A}_1$, $\mathcal{A}_3$, $\mathcal{H}_2^5$, $\mathcal{H}_2^5 \times \mathcal{A}_1$, or $\mathcal{I}_3$ (see [2, Chap. 5]).

If $1_{\mathcal{H}}$ is the principal character of a subgroup $\mathcal{H}$ of $\mathcal{I}_4$, then the induced character $1_{\mathcal{H}}{}^*$ of $\mathcal{I}_4$ is clearly the character of a rational representation of $\mathcal{I}_4$, so $1_{\mathcal{H}}{}^*$ is a candidate for the role of η in Schur's theorem. By the Frobenius Reciprocity Theorem we have, for each character χ of $\mathcal{I}_4$,

$$(\chi, 1_{\mathcal{H}}{}^*) = (\chi \mid \mathcal{H}, 1_{\mathcal{H}}) = |H|^{-1} \sum \{\chi(\psi) : \psi \in \mathcal{H}\}.$$

We shall compute this formula explicitly for parabolic subgroups of each type.

The simplest case is, of course, $\mathcal{H} = \mathcal{A}_1 = \{1, \psi\}$ for any $\psi \in K_{27}$, since K_{27} is the class of reflections in $\mathcal{I}_4$. Thus

$$(\chi, 1_{\mathcal{H}}{}^*) = (1/2)[\chi(1) + \chi(K_{27})].$$

Let us carry out the computation for another type, say $\mathcal{H} = \mathcal{H}_2^5$. Then $\mathcal{H}$ has four conjugacy classes: $C_1 = \{1\}$, the class $C_2 \subseteq K_{27}$ of five reflections, and classes C_3 and C_4 each having two elements of order 5 (rotations through angles $\pm 2\pi/5$ and $\pm 4\pi/5$). The elements of C_3 are products of two reflections having angle $\pi/5$ between their reflecting planes. Thus for an appropriate basis we have the representing matrix

$$\begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} \begin{bmatrix} \cos 3\pi/5 & \sin 3\pi/5 & & 0 \\ \sin 3\pi/5 & -\cos 3\pi/5 & & \\ & 0 & 1 & 0 \\ & & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\cos 3\pi/5 & -\sin 3\pi/5 & & 0 \\ \sin 3\pi/5 & -\cos 3\pi/5 & & \\ & 0 & 1 & 0 \\ & & 0 & 1 \end{bmatrix}$$

having trace $2 - 2\cos 3\pi/5 = 2 + 2\beta = 2\alpha + 1$. Since χ_3 is a character afforded by a faithful representation of $\mathcal{I}_4$ as a group of transformations of $\mathcal{R}^4$, and the only class of elements of order 5 at which χ_3 takes the value $2\alpha + 1$ is K_6, we may assume that $C_3 \subseteq K_6$, and similarly $C_4 \subseteq K_8$. As a result, when $\mathcal{H} = \mathcal{H}_2^5$ we have

$$(\chi, 1_{\mathcal{H}}{}^*) = (1/10)[\chi(1) + 2\chi(K_6) + 2\chi(K_8) + 5\chi(K_{27})].$$

The computations are omitted for parabolic subgroups of other types; we simply list the resulting formulas:

$$\mathcal{H} = \mathcal{A}_1; \quad (\chi, 1_{\mathcal{H}}{}^*) = (1/2)[\chi(1) + \chi(K_{27})], \tag{1}$$

$$\mathcal{H} = \mathcal{A}_1 \times \mathcal{A}_1; \quad (\chi, 1_{\mathcal{H}}{}^*) = (1/4)[\chi(1) + \chi(K_3) + 2\chi(K_{27})], \tag{2}$$

$$\mathcal{H} = \mathcal{A}_2; \quad (\chi, 1_{\mathcal{H}}{}^*) = (1/6)[\chi(1) + 2\chi(K_4) + 3\chi(K_{27})], \tag{3}$$

$$\mathcal{H} = \mathcal{A}_2 \times \mathcal{A}_1; \quad (\chi, 1_{\mathcal{H}}{}^*) = (1/12)[\chi(1) + 3\chi(K_3) + 2\chi(K_4) + 4\chi(K_{27}) + 2\chi(K_{29})], \tag{4}$$

$$\mathcal{H} = \mathcal{A}_3; \quad (\chi, 1_{\mathcal{H}}{}^*) = (1/24)[\chi(1) + 3\chi(K_3) + 8\chi(K_4) + 6\chi(K_{27}) + 6\chi(K_{28})], \tag{5}$$

$$\mathcal{H} = \mathcal{H}_2{}^5; \quad (\chi, 1_{\mathcal{H}}{}^*) = (1/10)[\chi(1) + 2\chi(K_6) + 2\chi(K_8) + 5\chi(K_{27})], \tag{6}$$

$$\mathcal{H} = \mathcal{H}_2{}^5 \times \mathcal{A}_1; \quad (\chi, 1_{\mathcal{H}}{}^*) = (1/20)[\chi(1) + 5\chi(K_3) + 2\chi(K_6) + 2\chi(K_8) + 6\chi(K_{27}) + 2\chi(K_{31}) + 2\chi(K_{34})], \tag{7}$$

$$\mathcal{H} = \mathcal{S}_3; \quad (\chi, 1_{\mathcal{H}}{}^*) = (1/120)[\chi(1) + 15\chi(K_3) + 20\chi(K_4) + 12\chi(K_6) + 12\chi(K_8) + \chi(K_{26}) + 15\chi(K_{27}) + 20\chi(K_{30}) + 12\chi(K_{32}) + 12\chi(K_{33})] \tag{8}.$$

Applying the above formulas we find, for example, that for $\mathcal{H} = \mathcal{A}_1$ we have

$$(\chi_4, 1_{\mathcal{H}}{}^*) = (1/2)[4 - 2] = 1,$$

and hence $m_Q(\chi_4) = 1$. In fact, for each remaining character χ_i of even order, with the exception of χ_{34}, there is a parabolic subgroup $\mathcal{H}$ for which $(\chi_i, 1_{\mathcal{H}}{}^*) = 1$. They are listed in the following table:

Character	χ_4	χ_8	χ_9	χ_{15}	χ_{16}	χ_{19}	χ_{21}
Subgroup	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_2 \times \mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_2 \times \mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_2$

Character	χ_{22}	χ_{23}	χ_{25}	χ_{29}	χ_{31}	χ_{33}
Subgroup	$\mathcal{H}_2{}^5$	$\mathcal{A}_3$	$\mathcal{A}_3$	$\mathcal{H}_2{}^5 \times \mathcal{A}_1$	$\mathcal{S}_3$	$\mathcal{A}_3$

There is a subgroup $\mathcal{H} = (K \times L)/\pm 1$ of order 48, where $K = \langle p_5, k \rangle \leqslant I$ and $L \leqslant I$ is the 8-element quaternion group. The faithful characters of degree 2 on K and L give an irreducible character η of $\mathcal{H}$, with $m_Q(\eta) = 2$ (see [1, p. 98, 6(b)]). Thus $m_Q(\chi_{34}) = 2$ by Lemma 2.2 of [1], since $(\chi_{34} \mid \mathcal{H}, \eta)$ is odd and χ_{34} and η are rational valued.

References

1. M. Benard, On the Schur indices of characters of the exceptional Weyl groups, *Ann. of Math.* **94** (1971), 89–107.
2. C. Benson and L. Grove, "Finite Reflection Groups," Bogden & Quigley, Tarrytown-on-Hudson, N. Y., 1971.
3. R. Brauer, Representations of finite groups, *in* "Lectures on Modern Mathematics" (T. L. Saaty, Ed.), Wiley, New York, 1963.
4 W. Feit, "Characters of Finite Groups," Benjamin, Menlo Park, Ca., 1967.

5. L. Grove, The characters of the hecatonicosahedroidal group, *J. Reine Angew. Math.*, to appear.
6. T. Kondo, The characters of the Weyl group of type F_4, *J. Fac. Sci. Univ. Tokyo* 1 (1965), 145–153.
7. N. Plotkin, Representations of Finite Groups in Quadratic Fields, Ph.D. Dissertation, Syracuse University, Syracuse, N.Y., 1972.
8. I. Reiner, The Schur index in the theory of group representations, *Michigan Math. J.* **8** (1961), 39–47.
9. W. Specht, Eine Verallgemeinerung der symmetrischen Gruppe, *Schr. Math. Sem. Berlin* **1** (1932), 1–32.
10. T. Yamada, On the group algebras of metacyclic groups over algebraic number fields, *J. Fac. Sci. Univ. Tokyo* **15** (1968), 179–199.
11. A. Young, Quantitative substitutional analysis, IV, V, *Proc. London Math. Soc.* (2) **31** (1930), 253–288.