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Construction of the Rudvalis Group of Order 145,926,144,000*

J. H. CONWAY

Department of Pure Mathematics and Mathematical Statistics, Cambridge, England

AND

D. B. WALES

Department of Mathematics, California Institute of Technology, Pasadena, California 91109

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Recently, Arunas Rudvalis [1] provided evidence for the existence of a new simple group R of order $145{,}926{,}144{,}000 = 2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$. He describes the group as a rank 3 permutation group on 4060 letters, in which the stabilizer of a point is the (nonsimple) Ree group $F = {}^2F_4(2)$, which has orbits of sizes 1,1755,2304, corresponding to subgroups of F of orders 20480 and 15600. The first of these is the centralizer of an involution of F, while the second has the known subgroup $L = \mathrm{PSL}_2(25)$ of F' (see Ref. [4]) as a subgroup of index 2.

Since one of the involutions in R has no fixed point and so just 2030 2-cycles, an argument of Griess and Schur [2] shows that R has a proper double cover $2R$. Rudvalis and Frame gave evidence for supposing that this group had a 28-dimensional complex representation not splitting over F. (Feit and Lyons have proved this under further assumptions on R.) We suppose that this unitary representation of $2R$ exists while constructing the group, but then conclude independently of this supposition that the construction defines a group $2R$ whose central quotient is R.

It turns out that in the 28-dimensional representation of $2R$ the 4060 letters become 4060 quadruplets of four vectors v, iv, $-v$, $-iv$. The stabilizer of any quadruplet is a (not proper) double cover $2F$ of F, and the stabilizer

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of an individual vector is the subgroup F' of index 2 in F, the so-called Tits simple group. From known properties of the 27-dimensional representation of F we can infer that $2F$ acts on these quadruplets with orbits of lengths 1,2304,1755, the inner product of a vector from the orbit of length 1 with a vector from the 1755 orbit being 0, while its inner product with one from the 2304 orbit is $15i^n$, on a scale in which vector-norms are 60.

Under the subgroup L these orbits split into orbits of lengths

$$1 + (1 + 78 + 300 + 300 + 325 + 325 + 975) + (975 + 390 + 390).$$

Transitivity is established by finding an involution, preserving inner products and, therefore, the graph, which fuses the orbits of F. Since in fact this involution normalizes the subgroup L, it is only necessary to verify that it takes one representative of each orbit of L into another orbit of L. This was established with the aid of computers at McGill University and California Institute of Technology. We would like to thank John MacKay (at McGill) and Chris Landauer (at C.I.T.) for their very generous assistance with these calculations.

We now examine the 28-dimensional representation in more detail, quoting certain facts about the representations of F'. (In general, properties of F' and aut(L) which are easily deduced from their character tables will be given without proof.)

Since Griess [3] has shown that F' has trivial multiplier, F' appears as a subgroup of $2R$. The smallest irreducible representations of F' have degrees 1, 26, 27, and there is no irreducible representation of degree 28. If a 26-dimensional irreducible representation of F' appears in the 28-dimensional representation, there is a two-dimensional fixed space of F' which would have 4060 images under $2R$, and some *two* of these images would be fixed by the subgroup L. However, the 26-dimensional representations remain irreducible when restricted to L, so that the appropriate representation of F' can only be the sum of an irreducible 27-dimensional representation with the trivial one.

Since the group does not split when restricted to F, the group $2F$ and its representation must have the following properties. There is a subgroup $Z_2 \times F'$ of index 2, whose central element is represented by -1, and each element outside this subgroup has square outside F'. The 27-dimensional representation of $2F$ is obtained by taking a 27-dimensional representation of F and multiplying elements not in F' by $\pm i$ and those in F' by ± 1. The 28-dimensional representation of $2F$ is obtained by adding a one-dimensional representation in which elements outside $Z_2 \times F'$ take values $\pm i$. (The sign is found from the condition that the determinant be 1.) The 27-dimensional representation now restricts on L to the sum of a particular 26-dimensional representation with the trivial representation.

Our strategy for constructing the group is as follows. We first explicitly construct the 26-dimensional representation of the subgroup L, which will be generated by certain monomial matrices α, $-\beta^2$, γ (the relevance of the element β appears later). Additional elements β, δ, ϵ which extend L to $2R$ are then found in the normalizer of a Sylow 5-subgroup P of L.

When restricted to this group P of order 25, the 28-dimensional representation consists of all the 24-nontrivial linear representations together with a fixed space of dimension 4. Since each nontrivial representation appears just once, it defines a unique one-dimensional subspace, and we compute an orthogonal basis for the space whose first 24 vectors generate these 24 spaces.

The details are as follows. It is best to consider along with L a group four times the size, which is related to it in the same way that $2F$ is related to F'. This group has a 26-dimensional representation obtained from an irreducible representation of $\mathrm{PSL}_2(25)$ by multiplying all matrices outside PSL(25) by $\pm i$, and those inside by ± 1. Since this representation is induced from a linear character of a subgroup of index 26, its action is monomial in terms of a suitable base of 26 vectors $v_x (x \in \mathbb{F}_{25} \cup \{\infty\})$, namely,

$$\alpha\colon v_x \to v_{x+1}, \qquad \beta\colon \begin{matrix} v_x \to i\omega v_{\theta x} (x \neq \infty), \\ v_\infty \to i\bar{\omega} v_\infty , \end{matrix} \qquad \gamma\colon \begin{matrix} v_x \to \omega^n v_{x^{-1}} (x = \theta^n), \\ v_0 \leftrightarrow v_\infty , \end{matrix}$$

where the typical element of $\mathbb{F}_{25}$ can be written as $a + b\sqrt{3}$ or, if nonzero, as θ^n ($\theta = 1 + \sqrt{3}$), and $\omega = e^{2\pi i/3}$. The elements α, $-\beta^2$, γ generate $\mathrm{PSL}_2(25)$ and can be checked to yield the character values restricted from a 27-dimensional representation of F', so that we can identify L with this $\mathrm{PSL}_2(25)$.

The vector

$$q_0 = \tfrac{1}{5} \sum_{a,b} e^{2\pi a i/5} v_{a+b\sqrt{3}}$$

is an eigenvector of eigenvalue $e^{2\pi i/5}$ for each of the elements $x \to x + 1$ and $x \to x + \theta$ of L, which generate a Sylow 5-subgroup P. If we, therefore, define q_n to be the image of q_0 under β^n ($n = 0, 1,..., 23$), we obtain 24 vectors which will serve as the first 24 vectors of our second basis. We can complete them to an orthonormal basis for the 28-space by adjoining four more vectors:

$$r = \tfrac{1}{5} \sum v_{a+b\sqrt{3}}, \qquad s = -v_\infty , t, \qquad \text{and} \qquad u.$$

We shall refer to this as the *second basis*, the *first basis* being v_x, v_∞, v_+, v_-, where we define $4v_+ = -t\sqrt{10} - u\sqrt{6}$, $4v_- = -t\sqrt{6} + u\sqrt{10}$.

Now the abstract group $\mathrm{PSL}_2(25)$ is contained to index 2 in just three distinct groups obtained by adjoining outer automorphisms, namely:

$\mathrm{PGL}_2(25)$, which can be obtained by adjoining the involution $x \to \theta/x$,
$\mathrm{P\Sigma L}_2(25)$, which can be obtained by adjoining the involution $x \to x^5$,
and a third group L^* obtainable by adjoining the map taking $x \to \theta x^5$.

Since all involutions of the abstract group F are already in F', we see that F contains L^*, and so that $2F$ contains a group $2L^*$ in which elements outside the subgroup $Z_2 \times L$ have squares outside L.

The 26-dimensional representation of L extends to one of $\mathrm{P\Sigma L}_2(25)$ by adjoining the map interchanging q_n with $-q_{5n}$ and r with s, and combining this with β we obtain a representation of $2L^*$, namely, that obtained by adjoining to L the element:

$$\delta\beta\colon q_n \to -i\omega q_{5n+1}\,, \qquad r \to i\bar{\omega} s, \qquad s \to i\omega r.$$

Since this is one of just two faithful representations of $2L^*$ which restrict properly to L, the other being obtained by changing signs of elements outside $Z_2 \times L$, we can suppose without loss of generality that there is an element $\delta\beta$ in $2F$ which has this action on the 26-space. Moreover, since it is outside $Z_2 \times L$ we can suppose it takes t to $\pm it$, and we can make it take u to $\pm iu$. If it is to be an element of $2R$, we must have the same sign in each case, since its determinant must then be 1. It is important for our construction to show that the ambiguous sign here is $+$ without supposing the existence of the Rudvalis group, and we shall later describe a computation which proves this. Modulo this ambiguity, we know the element $\delta\beta$ on the entire 28-space.

We next try to find an element ϵ which extends L to the Tits simple group F'. From the character table of F' we see that an element of order 5 centralizes a group of order 50, so that there is an involution ϵ (of trace -5 in the 27-dimensional representation of F') which normalizes P and centralizes a subgroup of order 5. Now the normalizer of P in F' is a split extension of P by a group $4A_4$ having a central cyclic subgroup of order 4 with quotient A_4. A cyclic subgroup of order 12 (generated by $-\beta^2$) is already in L, and so the action of ϵ is restricted by the condition that it generate $4A_4$ with this. This leaves essentially only one possibility for the action on P, knowing which we can assert that ϵ must permute the vectors q_n as follows, possibly with monomial factors:

$$(q_0q_{17})(q_1q_{13})(q_3q_4)(q_5q_{12})(q_6q_{23})(q_7q_{19})(q_9q_{10})$$
$$(q_{11}q_{18})(q_{15}q_{16})(q_{21}q_{22})(q_2)(q_8)(q_{14})(q_{20}).$$

When this permutation takes q_j to q_k, we shall understand that ϵ takes q_j to E_jq_k, so that we must have $E_jE_k = 1$. In fact we shall show that all the E_j

are -1. Since P is self-centralizing in F, elements of $2F$ which permute the spaces $\langle q_j \rangle$ in the same way can differ only in sign and must be equal if they are both in F'. We obtain in this way the relations

$$R_1\colon \beta^6\epsilon = \epsilon\beta^6, \qquad R_2\colon (\epsilon\beta^8)^3 = -\beta^6, \qquad R_3\colon \epsilon^{\delta\beta} = \beta^8\epsilon\beta^4.$$

Now R_1 implies $E_{n+6} = E_n$, using which the condition on the trace of ϵ implies $E_2 = -1$, and we obtain the equations:

$$E_1{}^2 = E_2{}^2 = E_3E_4 = E_0E_5 = 1, \qquad \text{from} \quad \epsilon^2 = 1,$$
$$E_0E_1E_3 = E_2E_4E_5 = -1, \qquad \text{from } R_2,$$

and

$$E_1 = E_2, \qquad E_0 = E_3, E_4 = E_5, \qquad \text{from } R_3.$$

These equations together imply that $E_1 = E_2 = -1$, $E_0 = E_3 = E_4 = E_5 = \pm 1$. We show that the sign here is $-$, without loss of generality, as follows. Up to this point, the only irrational complex numbers we have mentioned are i and ω, so we are still free to replace the number i by its algebraic conjugate $-i$. This has the effect of changing the sign of the vectors q_{2n+1} and so of the numbers E_0, E_3, E_4, E_5. It also changes the sign of the operation $\delta\beta$. This amounts to choosing a particular one of the two algebraically conjugate 27-dimensional representations of F'.

Now we must compute ϵ on the four space spanned by r, s, t, and u. In this space, the group $4A_4$ has a representation which is the sum of an irreducible representation on the space $\langle r, s, t\rangle$ with the trivial representation on $\langle u \rangle$, since it is a subgroup of F'. Now we can see that the central subgroup of order 4 (generated by $-\beta^6$) acts trivially on the 3-space, and so we have in essence a representation of the group A_4, which extends by $\delta\beta$ to give a representation of a group $2S_4$, in which elements outside the subgroup $Z_2 \times A_4$ have squares outside A_4. There is a unique irreducible representation of A_4, in which the group permutes four vectors with zero sum, say t_1, t_2, t_3, t_4, in 3 dimensions.

We can let

$$t_1 = 2ar + 2bs - ct,$$

when we find

$$t_2 = 2a\bar{\omega}r + 2b\omega s - ct,$$
$$t_3 = 2a\omega r + 2b\bar{\omega}s - ct,$$

on applying β^8, and so $t_4 = 3ct$.

Now multiplying a by ω^n and b by ω^{-n} simply rotates t_1, t_2, t_3, so that, since we know $\delta\beta$ takes t_4 to $\pm it_4$, we can suppose without loss of generality that it takes t_1 to $\pm it_3$, t_2 to $\pm it_2$, and t_3 to $\pm it_1$, with the same ambiguous sign, since the vectors have zero sum. This proves that $a = \pm b$, when by scaling the t_i we can suppose $a = \pm b = 1$, whence $|c| = 1$ since the norms of the t_i must be equal. Since we can still scale t by any complex number of norm 1, we can further suppose that $c = 1$.

Now since $\epsilon \in F'$ it must permute the four *vectors* t_i (not just their four spaces), in one of the three ways $(t_1t_4)(t_2t_3)$, $(t_2t_4)(t_1t_3)$, $(t_3t_4)(t_1t_2)$, and the equation $\epsilon^{\delta\beta} = \beta^8\epsilon\beta^4$ shows that we can have only the first possibility, so that ϵ is completely determined to within the ambiguity of sign. If the sign is $+$, we have

$$\epsilon: r \to \tfrac{2}{3}(r + s + t) - r, \quad s \to \tfrac{2}{3}(r + s + t) - s, \quad t \to \tfrac{2}{3}(r + s + t) - t,$$

and otherwise we have the expressions obtained from these by everywhere changing the sign of s.

We were unable to find a simple argument within F which distinguished these cases, so we computed the first 18 images of a certain vector under $\gamma\epsilon$, finding that if the sign is $-$ this element has too large an order to be in F', so that the sign must be $+$, and the elements $\delta\beta$ and ϵ are completely determined on the 28-space. It is important to realize that at no time in the computation of these elements have we presupposed the existence of the Rudvalis group, so that we now know that the elements

$$\alpha, -\beta^2, \gamma, \delta\beta, \qquad \text{and } \epsilon$$

do indeed generate a 28-dimensional representation of $2F$, and that, if in fact the Rudvalis group does exist, these elements belong to the group $2R$.

Suppose now that the group $2R$ exists, we show that β must be an element of it. Since $2304 \neq 1755$, the 2304 orbit must be self-paired, so that there is some element interchanging a vector from the 1 orbit with one from the 2304 orbit. Since the joint stabilizer of these two vectors is L, any such element must induce an (obviously outer) automorphism of L, which, since we already have the automorphism $\delta\beta$, can be chosen to be that induced by β. Again, there are only two possibilities for the representation of the extended group, differing only in the sign outside $Z_2 \times L$, so that we can suppose the new element acts exactly like β on the 26-dimensional space.

Examining now the action of β, $\delta\beta$, ϵ on the vectors q_j, we see that they generate a group $2S_5$ modulo the element $-\beta^6$, so that on the remaining four space we must have a representation of $2S_5$ extending our previous representation of $2S_4$, elements outside the subgroup $Z_2 \times A_5$ having squares outside A_5. There must now be a set of five vectors with zero sum which are permuted by the A_5, and we can take these to be

$$u_1 = \tfrac{1}{3}t_1\sqrt{15} - du, \qquad u_2 = \tfrac{1}{3}t_2\sqrt{15} - du, \qquad u_3 = \tfrac{1}{3}t_3\sqrt{15} - du,$$
$$u_4 = \tfrac{1}{3}t_4\sqrt{15} - du, \qquad \text{and} \qquad u_5 = 4\,du.$$

Since these must all have the same norm we must have $|d| = 1$, and can suppose $d = 1$ by scaling u, and we can complete the determination of β by observing that its action must be:

$$u_1 \to iu_3, \qquad u_2 \to iu_1, \qquad u_3 \to iu_2, \qquad u_4 \to iu_5, \qquad u_5 \to iu_4.$$

(We know the action on r, s, and β may not fix $\langle u \rangle$.)

TABLE 1

The Operations $\alpha, \beta, \gamma, \delta, \epsilon$

The first base consists of vectors $v_x (x \in \mathbb{F}_{25})$, v_∞, v_+, v_-. We have

$$4v_+ = -t\sqrt{10} - u\sqrt{6},\ 4v_- = -t\sqrt{6} + u\sqrt{10}, \text{ so that } \langle x_+, v_- \rangle = \langle t, u \rangle.$$

We shall display the coordinates of vectors with respect to this base in the arrangement

$$\begin{array}{ccccc} \theta^7 & \theta^{10} & \theta^9 & \theta^{14} & \theta^{23} \\ \theta^{20} & \theta^{13} & \theta^{15} & \theta^5 & \theta^{16} \\ \theta^6 & \theta^{12} & 0 & 1 & \theta^{18} \\ \theta^4 & \theta^{17} & \theta^3 & \theta & \theta^8 \\ \theta^{11} & \theta^2 & \theta^{21} & \theta^{22} & \theta^{19} \\ + & & \infty & & - \end{array}, \text{ equivalently } \begin{array}{ccccc} -2-2\sqrt{3} & -1-2\sqrt{3} & -2\sqrt{3} & 1-2\sqrt{3} & 2-2\sqrt{3} \\ -2-\sqrt{3} & -1-\sqrt{3} & -\sqrt{3} & 1-\sqrt{3} & 2-\sqrt{3} \\ -2 & -1 & 0 & 1 & 2 \\ -2+\sqrt{3} & -1+\sqrt{3} & \sqrt{3} & 1+\sqrt{3} & 2+\sqrt{3} \\ -2+2\sqrt{3} & -1+2\sqrt{3} & 2\sqrt{3} & 1+2\sqrt{3} & 2+2\sqrt{3} \\ + & & \infty & & - \end{array}$$

The relation between the two bases on the 26-space orthogonal to $\langle t, u \rangle$ is given on $p3$. In the above notation, with $\xi = e^{2\pi i/5}$, the vector q_0 is shown:

$$q_0 = \frac{1}{5}\begin{pmatrix} \xi^3 & \xi^4 & \xi^0 & \xi^1 & \xi^2 \\ \xi^3 & \xi^4 & \xi^0 & \xi^1 & \xi^2 \\ \xi^3 & \xi^4 & \xi^0 & \xi^1 & \xi^2 \\ \xi^3 & \xi^4 & \xi^0 & \xi^1 & \xi^2 \\ \xi^3 & \xi^4 & \xi^0 & \xi^1 & \xi^2 \\ & 0 & 0 & 0 & \end{pmatrix}$$

With respect to the first base we have:

$$\begin{array}{lll} \alpha\colon\ v_x \to v_{x+1} & \beta\colon\ v_x \to i\omega v_{\theta x} & \gamma\colon\ v_x \to \omega^n v_{1/x}\ (x = \theta^n) \\ \quad v_\infty \to v_\infty & \quad v_\infty \to i\bar{\omega} v_\infty & \quad v_\infty \to v_0 \\ \quad v_+ \to v_+ & \quad v_+ \to iv_+ & \quad v_+ \to v_+ \\ \quad v_- \to v_- & \quad v_- \to -iv_- & \quad v_- \to v_- \end{array}$$

With respect to the second base (only the eigenvalues of α are shown):

	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}	q_{11}	q_{12}	q_{13}	q_{14}	q_{15}	q_{16}	q_{17}	q_{18}	q_{19}	q_{20}	q_{21}	q_{22}	q_{23}	r	s	t	u
α:	ξ^4	ξ^3	ξ^4	ξ^0	ξ^2	ξ^3	ξ^3	ξ^1	ξ^3	ξ^0	ξ^4	ξ^1	ξ^1	ξ^2	ξ^1	ξ^0	ξ^3	ξ^2	ξ^2	ξ^4	ξ^2	ξ^0	ξ^1	ξ^4	ξ^0	ξ^0	ξ^0	ξ^0
β:	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}	q_{11}	q_{12}	q_{13}	q_{14}	q_{15}	q_{16}	q_{17}	q_{18}	q_{19}	q_{20}	q_{21}	q_{22}	q_{23}	q_0	$i\omega r$	$i\bar{\omega} s$	it'	iu'
$-\delta$:	q_0	q_5	q_{10}	q_{15}	q_{20}	q_1	q_6	q_{11}	q_{16}	q_{21}	q_2	q_7	q_{12}	q_{17}	q_{22}	q_3	q_8	q_{13}	q_{18}	q_{23}	q_4	q_9	q_{14}	q_{19}	$-s$	$-r$	$-t'$	$-u'$
$-\epsilon$:	q_{17}	q_{13}	q_2	q_4	q_3	q_{12}	q_{23}	q_{19}	q_8	q_{10}	q_9	q_{18}	q_5	q_1	q_{14}	q_{16}	q_{15}	q_0	q_{11}	q_7	q_{20}	q_{22}	q_{21}	q_6	$-r^*$	$-s^*$	$-t^*$	$-u$

Here $\xi = e^{2\pi i/5}$, $4t' = t + u\sqrt{15}$, $4u' = t\sqrt{15} - u$,

$$r^* + r = s^* + s = t^* + t = \tfrac{2}{3}(r + s + t).$$

We have now completely determined the actions of certain elements α, β, γ, δ, ϵ on the space, and for the reader's convenience we describe these elements in Table I. It remains only to verify that these transformations with i generate a group $4R$ of the required properties, and here we use the fact that α, $-\beta^2$, γ, ϵ are known to generate the group F' (since L is known to be a maximal subgroup of F'—see Ref. [4]).

We know that it is possible for F' to permute two distinct sets of 2304 and 1755 vector-quadruplets, and that one vector from one of the 2304 quadruplets can be chosen to be

$$\tfrac{1}{2}\sqrt{15}\cdot u_4 = \tfrac{1}{2}(15t - \sqrt{15}\cdot u).$$

Now by studying the permutation characters of the subgroups of indices 2304 and 1755 in F', we can show that under the subgroup L the orbits of sizes 1,2304,1755 split into orbits

$$1 + (1 + 78 + 300 + 300 + 325 + 325 + 975) + (975 + 390 + 390).$$

(An elegant way of presenting the calculations involved here was shown to us by Professor Marshall Hall, Jr.) A quadruple from the first 390 orbit is invariant under a subgroup of order 20 in L, and a vector from it is invariant under a group of order 20 in $\langle L, i\rangle$, generated by α and $\pm i\beta^6$. The vector can therefore be chosen to be

$$\sqrt{3}\cdot\sum_{a,b}(\pm i)^b\cdot v_{a+2b\sqrt{3}}, \qquad (a = 0, 1, 2, 3, 4,\ b = 0, 1, 2, 3)$$

since this vector lies in the unique one space pointwise fixed by this group. The case with $+i$ appears in Table II, since that with $-i$ was shown by computation to have more than 1755 images under F.

We now have vectors which under the group $2F$ yields sets of 1, 2304, and 1755 quadruplets, and we know the way in which these orbits split under the subgroup L. John McKay and Chris Landauer programmed computers for us which applied α, $-\beta^2$, γ, ϵ to these vectors, and found the vectors listed in Table II, which are representatives of the orbits under L, just one vector being given from each quadruplet. Since β normalizes L it is only necessary to verify that it takes one vector from each of these orbits into a vector from some orbit, and this is very easy to check since the operations involved are all monomial.

With this we have verified the existence of a rank 3 group with parameters 1, 1755, 2304 in which the stabilizer of a point contains F. Modulo the center, this group acts on a graph Γ whose nodes are the quadruplets, two nodes being joined if and only if vectors from the corresponding quadruplets are not orthogonal. It follows from the subsequent remarks that the Rudvalis group is the full automorphism group of this graph.

TABLE 2

Representatives of the Orbits of L

These vectors are arranged as explained in Table 1. We use the numbers

$P = 1 + \sqrt{3}$ $\quad Q = 1 - \sqrt{3}$ $\quad R = \sqrt{3}$ $\quad S = \sqrt{6}$ $\quad T = \sqrt{10}$

$= 1 + i\bar{\omega} - i\omega$ $\quad = 1 + i\omega - i\bar{\omega}$ $\quad = i\bar{\omega} - i\omega$

$p = P(1+i)/2$ $\quad q = Q(1+i)/2$ $\quad s = S(1+i)/2$ $\quad t = T(1+i)/2$

$= -i\omega - \bar{\omega}$ $\quad = -i\bar{\omega} - \omega$

0	0	0	0	0	0	0	0	0	0	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1
	$3/2T$	0	$-5/2S$			$3/2T$	0	$+5/2S$			T	-5	0	
		Length 1					Length 1					Length 78		

p	$\bar{\omega}q$	ωq	p	ωq	ωq	ωp	p	ωq	p
$\bar{\omega}q$	q	q	$\bar{\omega}p$	$\bar{\omega}p$	ωp	$\bar{\omega}p$	$\bar{\omega}p$	q	q
p	$\bar{\omega}p$	ωp	$\bar{\omega}p$	p	ωq	q	$\bar{\omega}q$	q	ωq
$\bar{\omega}p$	$\bar{\omega}p$	q	q	$\bar{\omega}q$	q	q	$\bar{\omega}p$	$\bar{\omega}p$	ωp
ωq	p	ωq	$\bar{\omega}q$	p	p	ωq	p	ωp	ωq
	t	$\bar{\omega}q$	$-is$			t	ωp	is	
		Length 300					Length 300		

0	ωR	R	$\bar{\omega}R$	0	$-iR$	$-iR$	$-iR$	$-iR$	$-iR$	1	$\bar{\omega}$	Q	1	ω
$\bar{\omega}R$	0	R	0	ωR	$-R$	$-R$	$-R$	$-R$	$-R$	1	1	Q	ω	$\bar{\omega}$
iR	iR	iR	iR	iR	0	0	0	0	0	P	P	-2	P	P
ωR	0	R	0	$\bar{\omega}R$	R	R	R	R	R	$\bar{\omega}$	ω	Q	1	1
0	$\bar{\omega}R$	R	ωR	0	iR	iR	iR	iR	iR	ω	1	Q	$\bar{\omega}$	1
	0	iR	$-S$			0	0	0			$-1/2T$	-2	$+1/2S$	
		Length 325					Length 390					Length 975		

ωR	0	iR	0	$\bar{\omega}R$	$-iR$	$-R$	0	R	iR	$\bar{\omega}$	1	P	ω	1
0	ωR	iR	$\bar{\omega}R$	0	$-iR$	$-R$	0	R	iR	ω	$\bar{\omega}$	P	1	1
R	R	iR	R	R	$-iR$	$-R$	0	R	iR	Q	Q	-2	Q	Q
0	ωR	iR	ωR	0	$-iR$	$-R$	0	R	iR	1	1	P	$\bar{\omega}$	ω
ωR	0	iR	0	ωR	$-iR$	$-R$	0	R	iR	1	ω	P	1	$\bar{\omega}$
	0	iR	$+S$			0	0	0			$-1/2T$	-2	$-1/2S$	
		Length 325					Length 390					Length 975		

To the group $\langle \alpha, \beta, \gamma, \delta, \epsilon \rangle$ we can add if necessary the scalar multiplication by i to obtain a group $4R$ which still fixes the system of 4060 quadruplets. We now verify that this group is the largest group of unitary automorphisms of the quadruplets, having the expected order $4 \times 145,926,144,000$. If the order were strictly larger, there would be a strictly larger group than $4F$ fixing a quadruplet, and a group strictly larger than $4L^*$ fixing a pair of quadruplets with nonzero inner products. The orbit of length 78 is determined by its inner products with these, and so would be mapped to itself. Now $4L^*$ has 26 imprimitivity sets of size 3 on this orbit, two nonorthogonal quadruplets being in the same set only if there is no quadruplet orthogonal to both. $4H$ must, therefore, have the same imprimitivity sets, and induce a strictly larger doubly transitive group on them (since any automorphism which fixes each of the imprimitivity sets, one of them pointwise, can be shown to be the identity by an elementary graphical argument). The doubly transitive groups on 26 letters are known, and neither of the candidates (A_{26} and S_{26}) has an imprimitive representation on 78 letters.

Since calculation reveals that a node of Γ is in the 78 orbit corresponding to two nodes A and B only if it is joined to A and B and to just 400 of the 780 nodes joined to neither of A and B, this proof extends to show that the Rudvalis group R is the full automorphism group of the graph Γ. Also, we can show that R has just one conjugacy class of subgroups F, and, hence, has no outer automorphism.

To see that R is simple, observe that R acts primitively on Γ, and, since there is no characteristically simple group of order 4060, a proper normal subgroup must have index 2 and intersect F in F'. A subgroup of index 2 would permute 4060 pairs of opposite vectors chosen from the quadruplets—but this is impossible since F' acts transitively on the 4×1755 vectors from the 1755 quadruplets.

A Sylow 2-group S of $4R$ permutes four quadruplets and acts on their four space as a group Q with $i \notin Q'$. Gaschutz's theorem implies $i \notin (4R)' = \langle \alpha, \beta, \gamma, \delta, \epsilon \rangle$, which must be a *double* cover $2R$ of R. We have shown the existence and uniqueness of a rank 3 extension R of F with parameters 1, 1755, 2304, and also that a double cover $2R$ of R has a 28-dimensional representation. (Uniqueness of the rank 3 graph can be deduced from consideration of various orbit-sizes.)

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