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A Family of Ovals with Few Collineations

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A recently discovered [1] family of ovals in $PG(2, q)$, $q = 2^e$, e odd, is shown to have a cyclic collineation group of order $2e$.

1. Introduction

An oval of $PG(2, q)$ is a set of $q + 1$ points no three of which are collinear; a hyperoval of $PG(2, q)$, q even, is a set of $q + 2$ points not three of which are collinear. For any oval Ω of $PG(2, q)$, q even, there is a unique point n, called the nucleus of Ω, such that $\Omega \cup \{n\} = \Omega^*$ is a hyperoval. For a survey on ovals and hyperovals we refer to [2, pp. 45, 207 and 278–285]. Recently, S. E. Payne discovered a new family of ovals which we now describe.

Let $F = GF(q)$, $q = 2^e$, e odd. Define $\delta: F \to F$ by

$$\delta: x \mapsto x^{1/6} + x^{1/2} + x^{5/6}, \qquad \text{for all } x \in F. \tag{1}$$

In [1] it was shown that

$$\Omega(\delta) = \{(0, 1, 0)\} \cup \{(1, c, c^\delta): c \in F\} \tag{2}$$

is an oval in $PG(2, q)$ with nucleus $(0, 0, 1)$, and that $\Omega(\delta)$ is new provided $e \geqslant 5$. It is clear that δ commutes with each automorphism of F, so that

$$\sigma: (x, y, z) \mapsto (x^2, y^2, z^2) \tag{3}$$

generates a group of order e of collineations of $PG(2, q)$ leaving invariant the oval $\Omega(\delta)$. A simple computation shows that

$$(x^{-1})^\delta = x^\delta / x. \tag{4}$$

From this it follows that

$$\theta: (x, y, z) \mapsto (y, x, z) \tag{5}$$

is a projectivity of $PG(2, q)$ that fixes the points $(1, 1, 1)$ and $(0, 0, 1)$, interchanges $(1, 0, 0)$ and $(0, 1, 0)$, and interchanges the points $(1, c, c^\delta)$ and $(1, c^{-1}, (c^{-1})^\delta)$, $c \neq 0$. Hence θ leaves invariant the hyperoval

$$\Omega^* = \Omega(\delta) \cup \{(0, 0, 1)\}. \tag{6}$$

The goal of this essay is to show that for $e \geqslant 5$ the cyclic group G of order $2e$ generated by σ and θ is the full group G^* of collineations of $PG(2, q)$ leaving invariant the hyperoval Ω^*. For $e \geqslant 7$, Bezout's theorem (cf. [3, p. 44]) yields a fairly efficient proof. The case $q = 32$ is more stubborn, even requiring the assistance of a computer.

The G-orbits on Ω^* are $\{(0, 0, 1)\}$, $\{(1, 1, 1)\}$, $\{(1, 0, 0), (0, 1, 0)\}$, and sets of size $2d$, where $1 < d$ and d divides e. One reason for interest in this result is the many pairwise non-isomorphic generalized quadrangles (GQ) that arise from Ω^*. As explained in [1], there are the following cases:

(I) Ω^* yields one GQ of order $(q - 1, q + 1)$.

(II) For each G-orbit of Ω^* there arises a distinct GQ of order (q, q), with none of these isomorphic to the dual of any of them.

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(III) For each G-orbit on the unordered pairs of distinct points of Ω^* there arises a distinct GQ of order $(q+1, q-1)$, none of which is the dual of that one given in (I).

2. An Algebraic Curve of Degree Six

For each $x \in F$ a routine calculation shows that

$$(x^\delta)^6 = x(x^\delta)^4 + x^5 + x. \tag{7}$$

Taking square roots and putting

$$f_x(T) = T^3 + x^{1/2}T^2 + x^{5/2} + x^{1/2}, \tag{8}$$

we have

$$f_x(T) = (T - x^\delta)(T^2 + (x^{1/6} + x^{5/6})T + x^\delta(x^{1/6} + x^{5/6})). \tag{9}$$

(2.1) *If* $1 \neq x \in F$, *then* x^δ *is the unique root in* F *of* $f_x(T) = 0$. *If* $x = 1$, *then* $1^\delta = 1$ *is a root, but so is* $T = 0$.

Proof. For $x = 0$, $f_x(T) = T^3 = 0$ has only the root $T = 0^\delta = 0$. For $x = 1$, $f_x(T) = T^3 + T^2 = 0$ has the root $T = 1^\delta = 1$ and also the *extraneous* root $T = 0$. Recall that the elements of F are partitioned into two sets: $C_0 = \{x^2 + x : x \in F\}$ and its other additive coset C_1, where since e is odd $C_1 = C_0 + 1$. Then $T^2 + aT + b$ is irreducible over F iff $b/a^2 \in C_1$ (cf. [2]). So for $0 \neq x \neq 1$, $T^2 + (x^{1/6} + x^{5/6})T + x^\delta(x^{1/6} + x^{5/6})$ is irreducible over F iff $x^\delta(x^{1/6} + x^{5/6})/(x^{1/6} + x^{5/6})^2 = 1 + x^{1/3}/(1 + x^{1/3})^2 \in C_1$, which is the case since $1 \in C_1$ and $x^{1/3}/(1 + x^{1/3})^2 = A^2 + A \in C_0$, with $A = (1 + x^{1/3})^{-1}$. □

As a corollary one may prove that if $\gamma = \delta^{-1}$ and $0 \neq y \in F$, then y^γ is the unique root in F of $T^5 + T(1 + y^4) + y^6 = 0$.

Writing homogeneous co-ordinates for the points of $\Omega(\delta)$, we find

$$\left.\begin{aligned}
\Omega(\delta) &= \{(x, y, z) : x = z = 0 \neq y \quad \text{or } x \neq 0 \\
&\text{and} \\
&(y/x)^\delta = z/x\} = \{(0,1,0), (1,1,1)\} \cup \{(x, y, z) : 0 \neq x \neq y \\
&\text{and} \\
&(z/x)^6 + (y/x)(z/x)^4 + (y/x)^5 + y/x = 0\} \\
&= \{(0,1,0), (1,1,1)\} \cup \{(x, y, z) : 0 \neq x \neq y \\
&\text{and} \\
&z^6 = xy(x + y + z)^4\}.
\end{aligned}\right\} \tag{10}$$

Putting $x = 0$ in $z^6 = xy(x + y + z)^4$ allows only the point $(0, 1, 0)$; and $x = y \neq 0$ does allow $(1, 1, 1)$, but it also allows the extraneous point $(1, 1, 0)$. Define the algebraic curve Γ in $PG(2, q)$ by

$$\Gamma = \{(x, y, z) \in PG(2, q) : z^6 = xy(x + y + z)^4\}. \tag{11}$$

Then the preceding paragraph proves the following:

(2.2) $\Gamma = \Omega(\delta) \cup \{(1, 1, 0)\}$.

Our goal is to show that for $e \geqslant 5$, $G^* = G$. Since $\sigma \in G^*$, it suffices to determine all linear (i.e. projective) collineations of $PG(2, q)$ leaving Ω^* invariant—and for $e > 32$ we are already able to do this!

(2.3) *Let α be a linear collineation in G^*. Then*:
(*i*) *if* $q > 32$, $\alpha \in \{id, \theta\}$;
(*ii*) *if* $q = 32$ *and* $(0, 0, 1)^\alpha = (0, 0, 1)$, then $\alpha \in \{\text{id}, \theta\}$;
(*iii*) *if* $q = 32$ *and* $(1, 1, 0)^\alpha = (1, 1, 0)$, *then* $\alpha \in \{id, \theta\}$.

PROOF. Clearly, α transforms the algebraic curve Γ into an algebraic curve Γ', also of the sixth degree. The point $(1, 1, 0)$ is a 4-tuple point of Γ, and moreover is the only singular point of Γ. Hence $(1, 1, 0)^\alpha$ is the unique singular point of Γ'. Suppose $\Gamma \neq \Gamma'$. Since Γ and Γ' are irreducible, by Bezout's theorem (cf. [3, p. 44]) we have $|\Gamma \cap \Gamma'| \leqslant 36$. As $\Omega(\delta) \cap (\Omega(\delta))^\alpha \subset \Gamma \cap \Gamma'$, we have $q \leqslant |\Gamma \cap \Gamma'|$. Hence if $q > 32$, $\Gamma = \Gamma'$ and $(1, 1, 0)^\alpha = (1, 1, 0)$.

Now suppose $q = 32$ and $(0, 0, 1)^\alpha = (0, 0, 1)$. The tangents of Γ at the simple points of Γ (i.e. the points of $\Omega(\delta)$) concur at $(0, 0, 1)$. Consequently, the tangents of Γ' at the points of $(\Omega(\delta))^\alpha = \Omega(\delta)$ concur at $(0, 0, 1)$. Therefore the tangents of Γ and Γ' at the points of $\Omega(\delta)$ coincide. This means that, if $\Gamma \neq \Gamma'$ and considering the intersection multiplicities of points of $\Gamma \cap \Gamma'$, the points of $\Omega(\delta)$ account for at least $2(q + 1) = 66$ common points, contradicting Bezout's theorem. Hence also in this case we have $\Gamma = \Gamma'$ and $(1, 1, 0)^\alpha = (1, 1, 0)$.

Next, suppose that $q = 32$ and $(1, 1, 0)^\alpha = (1, 1, 0)$, so the 4-tuple points of Γ and Γ' coincide. Assume $\Gamma \neq \Gamma'$. Then $(1, 1, 0)$ accounts for at least $4 \cdot 4 = 16$ common points of Γ and Γ'. Since $32 + 16 > 36$, we again have a contradiction, by Bezout's theorem.

At this point we know that each of the hypotheses of (2.3) leads to $\Gamma = \Gamma'$ and $(1, 1, 0)^\alpha = (1, 1, 0)$, which we now take as our hypothesis.

Since the tangents of Γ at the simple points of Γ concur at $(0, 0, 1)$, we have $(0, 0, 1)^\alpha = (0, 0, 1)$. Let L be a line through $(0, 0, 1)$. If $L \cap \Omega(\delta) = I$, then the intersection multiplicity of L and Γ at I is exactly 6 iff L is $x = 0$ or $y = 0$, in which case I is $(0, 1, 0)$ or $(1, 0, 0)$. Hence, with $[a, b, c]$ denoting the line with equation $aX + bY + cZ = 0$, we have all of the following: $(\Omega(\delta))^\alpha = \Omega(\delta)$; $(1, 1, 0)^\alpha = (1, 1, 0)$; $(0, 0, 1)^\alpha = (0, 0, 1)$; $[1, 1, 0]^\alpha = [1, 1, 0]$; $(1, 1, 1)^\alpha = \{\Omega(\delta) \cap [1, 1, 0]\}^\alpha = (\Omega(\delta))^\alpha \cap [1, 1, 0]^\alpha = \Omega(\delta) \cap [1, 1, 0] = (1, 1, 1)$; $\{(0, 1, 0), (1, 0, 0)\}^\alpha = \{(0, 1, 0), (1, 0, 0)\}$; $[0, 0, 1]^\alpha = [0, 0, 1]$. Since α is linear, it is now easy to check that $\alpha \in \{id, \theta\}$. □

The preceding result has as an immediate consequence that $G = G^*$ if $q > 32$ or if $q = 32$ and $(1, 1, 0)^\alpha = (1, 1, 0)$ for each linear α in G^*. Hence the remainder of the paper is devoted to showing that when $q = 32$, $(1, 1, 0)^\alpha = (1, 1, 0)$ for each linear α in G^*. Before proceeding, however, we make one interesting (but probably useless) observation. The map τ: $(x, y, z) \mapsto (z, y, x)$ maps $\Omega(\delta) \cup \{(0, 0, 1)\}$ to $\Omega(\delta^*) \cup \{(0, 0, 1)\}$, where $(x^\delta)^{-1} = (x/x^\delta)^{\delta^*}$, for all $x \neq 0$, and $0^{\delta^*} = 0$. Using (4), for all $x \neq 0$ it follows that $(1/x^\delta)^{\delta^*} = (x^{-1}/(x^\delta/x))^{\delta^*} = (x^{-1}/(x^{-1})^\delta)^{\delta^*} = ((x^{-1})^\delta)^{-1} = x/x^\delta$. Hence δ^*: $x/x^\delta \mapsto 1/x^\delta$ $(x \neq 0)$, implying that δ^* is a product of disjoint transpositions.

3. The Special Role of (1, 1, 0)

For any $c \in F - \{0, 1\}$, the line joining the points $(1, c, c^\delta)$ and $(1, c^{-1}, (c^{-1})^\delta)$ of $\Omega(\delta)$ contains the point $(1, 1, 0)$.

(3.1) *If the distinct lines* p_1p_2 and q_1q_2, $p_1, p_2, q_1, q_2 \in \Omega(\delta)$, *contain the point* $(1, 1, 0)$, *then the line joining the diagonal points of the complete quadrangle* $p_1p_2q_1q_2$ *is always the line* $[1, 1, 0]$.

PROOF. This is a straightforward computation using (4). □

(3.2) *Let* p_1p_2, q_1q_2, r_1r_2, *with* $p_1, p_2, q_1, q_2, r_1, r_2 \in \Omega(\delta)$, *be distinct lines containing the point* $(1, 1, 0)$. *Then* $p_1, p_2, q_1, q_2, r_1, r_2$ *belong to an irreducible conic C. Moreover, the line* $[1, 1, 0]$ *is the tangent from* $(1, 1, 0)$ *to C.*

PROOF. Let a, b, c, d be four points of an irreducible conic C. Then the line joining the diagonal points of the complete quadrangle $abcd$ is tangent to C. Let C be the irreducible conic through the points p_1, p_2, q_1, q_2, r_1. Then, by (3.1), $[1, 1, 0]$ is the line joining the diagonal points of $p_1p_2q_1q_2$ and is the tangent to C through $(1, 1, 0)$. Hence the line L through $(1, 1, 0)$ and r_1 is a secant to C and passes through a second point r of C. It follows that $[1, 1, 0]$ is the line joining the diagonal points of $q_1q_2r_1r$ and of $q_1q_2r_1r_2$, forcing $r = r_2$. □

(3.3) *Let* $(1, 1, 0)$ *be on the line* p_1p_2, *with* p_1, p_2 *points of* $\Omega(\delta)$. *Then the line joining the diagonal points of* p_1p_2rs, *with* $r(0, 0, 1)$ *and* $s(1, 1, 1)$, *is external to* $\Omega(\delta)$. *In this way there arise all* $q/2$ *lines through* $(1, 1, 0)$ *having no point in common with* $\Omega(\delta)$.

PROOF. First suppose $p_1p_2 = [0, 0, 1]$, say $p_1(1, 0, 0)$ and $p_2(0, 1, 0)$. Then the line joining the diagonal points of p_1p_2rs is $[1, 1, 1]$, and it will have a point of $\Omega(\delta)$ iff there is some $c \in F$ with $c^\delta = 1 + c$. Clearly, $c = 1$ is not a solution, but (2.1) implies $0 = f_c(c^\delta) = (1 + c)^3$, an impossibility. So, suppose $p_1(1, c, c^\delta)$ and $p_2(1, c^{-1}, (c^{-1})^\delta)$, $0 \neq c \neq 1$. The line joining the diagonal points of p_1p_2rs is $[1, 1, (1 + c)/(1 + c + c^\delta)]$. This line meets $\Omega(\delta)$ iff there is some $x \in F - \{0, 1\}$ with $x^\delta/(1 + x) = 1 + c^\delta/(1 + c)$. Use $t^\delta/(1 + t) = t^{1/6}/(1 + t^{1/3})$ for $t \neq 1$, and put $a = x^{1/6}$, $b = c^{1/6}$ to rewrite this last equation as $a^2(1 + b + b^2) + a(1 + b^2) + 1 + b + b^2 = 0$. For a given b, there is no solution for a iff $(1 + b + b^2)^2/(1 + b^2)^2 \in C_1$ iff $(1 + b + b^2)/(1 + b^2) \in C_1$ iff $1 + (1 + b + b^2)/(1 + b^2) = b/(1 + b)^2 \in C_0$, which is easily seen to hold, completing the proof of the first statement. For the second statement it suffices to show that for $x, y \in F - \{0, 1\}$, $(1 + x)/(1 + x + x^\delta) = (1 + y)/(1 + y + y^\delta)$ iff $x = y$ or $x = y^{-1}$. And this is easily seen to be the case using steps similar to those just above. □

The results of this section show that the point $(1, 1, 0)$ plays a rather remarkable role for the oval $\Omega(\delta)$. To complete the determination of all collineations of $PG(2, q)$, leaving Ω^* invariant in case $q = 32$, we show that $(1, 1, 0)$ is unique in this respect.

4. THE CASE $q = 32$

Let $F = GF(32)$, and let w be a primitive root of F satisfying $w^5 = 1 + w^2$. The effect of the permutation δ is given in (5.1).

Let α be a projective (i.e. linear) collineation of $PG(2, 32)$ leaving the hyperoval Ω^* invariant and *moving* the point $(1, 1, 0)$. We must find a contradiction. The proof is arranged into a number of cases according to the form of the co-ordinates for the point $(1, 1, 0)^\alpha$.

CASE 1: $(1, 1, 0)^\alpha = (1, b, c) = u$, *with* $b \neq 0 \neq c$

Let e_1, e_2, e_3 be the points of Ω^* with co-ordinates $e_1(1, 0, 0)$, $e_2(0, 1, 0)$, $e_3(0, 0, 1)$. Let $e_iu \cap \Omega^* = \{e_i, f_i\}$, $i = 1, 2, 3$, with co-ordinates $f_1(1, x, x^\delta)$, $f_2(1, y, y^\delta)$, $f_3(1, z, z^\delta)$, $xyz \neq 0$.

Writing out the condition that e_i, u and f_i are collinear for $i = 1, 2, 3$ yields $b = z$, $c = y^\delta$, and $c/b = x^\delta/x$. Hence

$$(x^{-1})^\delta = x^\delta/x = y^\delta/z. \tag{12}$$

(a) $e_3^\alpha \notin e_i u$, $i = 1, 2, 3$

Here $e_1, e_2, e_3, f_1, f_2, f_3$ are on an irreducible conic C, by (3.2). Since C contains e_1, e_2, e_3 it must have an equation of the form

$$C\colon XY + mYZ + nZX = 0. \tag{13}$$

But since C also contains f_1, f_2, f_3, we also have

$$\left.\begin{aligned} x + mx^{\delta+1} + nx^\delta &= 0 \\ y + my^{\delta+1} + ny^\delta &= 0 \\ z + mz^{\delta+1} + nz^\delta &= 0 \end{aligned}\right\} \Rightarrow \begin{vmatrix} 1 & x^\delta & x^{\delta-1} \\ 1 & y^\delta & y^{\delta-1} \\ 1 & z^\delta & z^{\delta-1} \end{vmatrix} = 0.$$

Now add the first row to each of the other two and expand the determinant by the first column to obtain

$$x^\delta y^\delta(x + y)z + x^\delta z^\delta(x + z)y + y^\delta z^\delta(y + z)x = 0. \tag{14}$$

Now use (12) in (14) to solve for x and rewrite (12):

$$x = z(y^{\delta+1} + z^{\delta+1})/(zy^\delta + yz^\delta), \tag{15}$$

$$x^\delta = y^\delta(y^{\delta+1} + z^{\delta+1})/(zy^\delta + yz^\delta). \tag{16}$$

Keep in mind that x, y, z are distinct and non-zero, so that both $xy^\delta + yz^\delta \neq 0$ and $y^{\delta+1} + z^{\delta+1} \neq 0$.

Since $(t^2)^\delta = (t^\delta)^2$ for all $t \in F$, it follows that a triple (x, y, z) satisfies (15) and (16) (and hence (12)) iff (x^2, y^2, z^2) does. It is also easy to check that (x, y, z) satisfies (15) and (16) (and hence (12)) iff (y^{-1}, x^{-1}, z^{-1}) does. Hence a given triple (x, y, z) (of distinct non-zero elements of F) belongs to an 'orbit' of 'ten' triples corresponding to the automorphisms in G such that all satisfy or all fail to satisfy (15) and (16). In Table 5 (Section 5) we list all pairs (x, y) for which there is a z $(= xy^\delta/x^\delta)$ satisfying (15) and (16). Using the elements of G as indicated just above we can restrict our attention to the pairs: (w^0, w^3), (w, w^{21}), (w^3, w^{13}), (w^5, w), (w^7, w^4), (w^{11}, w^9).

Consider the equation of the line D joining the diagonal points of the complete quadrangle $e_1e_2f_1f_2$, with $u = e_1f_1 \cap e_2f_2 = (1, b, c) = (1, z, y^\delta)$. Also $e_1f_2 \cap e_2f_1 = (y^\delta, yx^\delta, y^\delta x^\delta)$. This leads to an equation for D:

$$X(y/y^\delta + x/x^\delta) + Y(1/x^\delta + 1/y^\delta) + Z(x/x^{2\delta} + y/y^{2\delta}) = 0. \tag{17}$$

Since D is the image of the line $[1, 1, 0]$ under the action of α, we have that $D \cap \Omega^*$ must not be empty. This eliminates classes 4 and 5, leaving four possibilities from Table 3 yet to be considered. In each case we pick some line L through $u = (1, b, c)$ with $L \neq D$, e_1f_1, e_2f_2, e_3f_3 and with $|L \cap \Omega^*| = 2$; say $L \cap \Omega^* = \{e_4, f_4\}$. Then the line D' joining the diagonal points of $e_1f_1e_4f_4$ must coincide with D. Below, we have indicated our choice of L for each of the four cases. A quick comparison with Table 3 will show that in each of these cases $D' \neq D$.

TABLE 1.

(i, j)	L	$L \cap \Omega^* = \{e_4, f_4\}$	D'
(0, 3)	$X + w^{10}Y + Z = 0$	$(1, w^{19}, w^3)$, $(1, w, w^{19})$	$X + w^{21}Y + w^3Z = 0$
(1, 21)	$X + w^2Y + w^5Z = 0$	$(1, w^{20}, w^2)$, $(1, 1, 1)$	$X + w^{22}Y + w^6Z = 0$
(3, 13)	$X + w^5Y + w^{15}Z = 0$	$(1, w^{12}, w^{15})$, $(1, w^2, w^7)$	$w^6X + w^2Y + Z = 0$
(11, 9)	$X + w^2Y + w^6Z = 0$	$(1, w^5, w^{16})$, $(1, w^3, w^{27})$	$X + Y + wZ = 0$

This completes a proof that case 1(a) cannot occur.

(*b*) $e_3^\alpha \in e_1 u$

Let p be the diagonal point $p = e_2 f_3 \cap e_3 f_2 = (1, y, z^\delta)$. Then the line pu: $X(z^{\delta+1} + y^{\delta+1}) + Y(y^\delta + z^\delta) + Z(z + y) = 0$ must coincide with the line $e_3^\alpha u = e_1 u$: $y^\delta Y + zZ = 0$. Hence $z^{\delta+1} = y^{\delta+1}$. The pair (y, z) satisfies $y^{\delta+1} = z^{\delta+1}$ iff the pair (y^2, z^2) also satisfies $(y^2)^{\delta+1} = (z^2)^{\delta+1}$. It turns out that every pair (y, z) satisfying $y^{\delta+1} = z^{\delta+1}$ lies in the 'σ-orbit' of one of the pairs (w^{21}, w^{23}), (w^{23}, w^{21}). In both cases $x = w^7$ (by (15)). In each case we again choose a line $L \neq e_1 f_1, e_2 f_2, e_3 f_3$ through $u = (1, z, y^\delta)$ meeting Ω^* in two points e_4, f_4. The line D' joining the diagonal points of $e_3 e_4 f_3 f_4$ must be the line $D = e_1 u$: $y^\delta Y + zZ = 0$. For $(x, y, z) = (w^7, w^{21}, w^{23})$ put $e_4(1, w^{11}, w^{13})$ and $f_4(1, 1, 1)$. For $(x, y, z) = (w^7, w^{23}, w^{21})$, put $e_4(1, w^{10}, w)$ and $f_4(1, 1, 1)$. In both cases $u \in e_4 f_4$ and $e_4 f_4 \notin \{e_1 f_1, e_2 f_2, e_3 f_3\}$.

In the first case $u = (1, z, y^\delta) = (1, w^{23}, w^{22})$ and one other diagonal point is $(1, w^{11}, w^{24})$; and e_1 on D' says that

$$0 = \begin{vmatrix} 1 & w^{23} & w^{22} \\ 1 & w^{11} & w^{24} \\ 1 & 0 & 0 \end{vmatrix},$$

which is not true.

In the second case $u = (1, z, y^\delta) = (1, w^{21}, w^{20})$ and one other diagonal point is $(1, w^{10}, w^{30})$. Here e_1 on D' says

$$0 = \begin{vmatrix} 1 & w^{21} & w^{20} \\ 1 & w^{10} & w^{30} \\ 1 & 0 & 0 \end{vmatrix},$$

which is impossible. Hence case 1(b) does not occur.

(*c*) $e_3^\alpha \in e_2 f_2$

Applying the automorphism θ shows that this case is equivalent to the previous case (b).

(*d*) $e_3 \in e_3 f_3$

The diagonal points of $e_1 e_2 f_1 f_2$ must be on the line $e_3 f_3$, so the points $(0, 0, 1)$, $n = (1, z, y^\delta)$ and $(y^\delta, yx^\delta, x^\delta y^\delta)$ are all collinear. Writing out this condition we find $z = yx^\delta / y^\delta$. By (12) we have $z = xy^\delta / x^\delta$, and so $y^{2\delta}/y = x^{2\delta}/x$. With the help of Table 1 (and using $x \neq y$) we see that $x = y^{-1}$ and $z = 1$. Using the automorphisms $\sigma, \sigma^2, \sigma^3, \sigma^4$, we only need to consider the cases $x = w, w^3, w^5, w^7, w^{11}, w^{15}$. The procedure is now similar to that in the preceding cases. Choose a line L through u meeting Ω^* in two points e_4, f_4, so that $L \neq e_1 f_1, e_2 f_2, e_3 f_3$. Then the line joining the diagonal points of the complete quadrangle $e_1 f_1 e_4 f_4$ must contain $(0, 0, 1)$. For each choice of x we make an appropriate choice of e_4, f_4 and one of the other diagonal points d. (For (i, j, k) under e_4, f_4, d, read (w^i, w^j, w^k); * denotes the corresponding co-ordinate to be zero.)

TABLE 2.

x	e_4	f_4	d
w	(0, 2, 7)	(0, 12, 15)	(*, 0, 5)
w^3	(0, 2, 7)	(0, 20, 2)	(23, 0, 5)
w^5	(0, 2, 7)	(0, 19, 3)	(30, 0, 5)
w^7	(0, 2, 7)	(0, 18, 8)	(19, 0, 5)
w^{11}	(0, 2, 7)	(0, 13, 21)	(22, 0, 5)
w^{15}	(0, 2, 7)	(0, 8, 28)	(30, 0, 5)

In each case it is easy to check that the line ud does not contain e_3, so that Case 1 is completely eliminated.

CASE 2: $(1, 1, 0)^\alpha = (1, b, 0) = u$

Since $(1, 1, 0)^\alpha$ is different from $(1, 1, 0)$ and $u \notin \Omega^*$, we have $1 \neq b \neq 0$. Put $L_1 = e_1e_2 = e_1u$. Let $L_2 = ue_0$ with $e_0(1, 1, 1)$ and suppose $f_0(1, y, y^\delta)$ is in $L_2 \cap \Omega^*$, $0 \neq y \neq 1$. Finally, let $L_3 = ue_3$ and suppose $f_3(1, x, x^\delta)$ is in $L_3 \cap \Omega^*$, $0 \neq x \neq 1$. Therefore L_1, L_2, L_3 are three distinct lines through u each meeting Ω^* in two points. It follows readily that

$$b = x = (y + y^\delta)/(1 + y^\delta). \tag{18}$$

Considering the automorphisms σ, σ^2, σ^3, σ^4, we can restrict our attention to the values of y given in Table 3.

TABLE 3

$i: y = w^i$	1	3	5	7	11	15
$j: b = x = w^j$	22	12	15	28	2	20

(*a*) $e_3^\alpha \notin L_1, L_2, L_3$

Then e_0, f_0, f_3 are on the conic $XY + mYZ + nXZ = 0$; so

$$0 = \begin{vmatrix} 1 & 1 & 1 \\ y & y^{\delta+1} & y^\delta \\ x & x^{\delta+1} & x^\delta \end{vmatrix}, \quad \text{i.e. } X^{\delta+1}(y^\delta + y) + X^\delta(y^{\delta+1} + y) + X(y^\delta + y^{\delta+1}) = 0. \tag{19}$$

One may check that the pairs (x, y) from Table 3 never satisfy the equation in (19).

(*b*) $e_3^\alpha \in L_1$

The diagonal point $(x + y, x + y, yx^\delta + xy^\delta + x^\delta + y^\delta)$ of $e_0f_0e_3f_3$ must be on L_1, implying

$$xy^\delta + yx^\delta + x^\delta + y^\delta = 0. \tag{20}$$

From (18) and (20) it follows that $y = (x + x^\delta)/(x^\delta + 1)$, but a check with the values of (x, y) from Table 3 shows this is impossible.

(*c*) $e_3^\alpha \in L_2$

The diagonal point $(0, x, x^\delta)$ of $e_1e_2e_3f_3$ must be incident with L_2. From this it follows that

$$(x^{-1})^\delta + x^\delta = 1. \tag{21}$$

But (21) is satisfied for x iff it is satisfied for x^{-1} and x^2. Hence it suffices to check (21) for the cases $x = w$, $x = w^3$, $x = w^5$. For each of these three values of x, (21) fails to hold.

(*d*) $e_3^\alpha \in L_3$

The diagonal point $(1, y^\delta, y^\delta)$ of $e_1e_2e_0f_0$ must be on L_3. This implies $x = y^\delta$. With (18) this forces $y = y^{2\delta} = y^{1/3} + y + y^{5/3}$, so $y = y^5$. Hence $y = 0$ or $y = 1$, both values being excluded from Table 3.

CASE 3: $(1, 1, 0)^\alpha = (1, 0, c) = u, c \neq 1$

With $e_1(1, 0, 0)$, $e_2(0, 1, 0)$, $e_3(0, 0, 1)$, $e_0(1, 1, 1)$, $f_0(1, x, x^\delta)$, $f_2(1, y, y^\delta)$, put $L_1 = e_1e_3$, $L_2 = e_0f_0$, $L_3 = e_2f_2$ as three distinct lines through $u = (1, 0, c)$. From $ue_0 = uf_0$ and $ue_2 = uf_2$ we find

$$c = (x + x^\delta)/(x + 1) = y^\delta. \tag{22}$$

Again we need only consider the following values of x:

$$\left.\begin{array}{lcccccc} i: x = w^i & 1 & 3 & 5 & 7 & 11 & 15 \\ j: y^\delta = w^j & 15 & 20 & 22 & 2 & 28 & 12 \\ k: y = w^k & 12 & 23 & 21 & 20 & 8 & 14 \end{array}\right\} \qquad (23)$$

(*a*) $e_3^\alpha \notin L_1, L_2, L_3$

Here e_0, f_0, f_2 are on the conic $XY + mYZ + nZX = 0$, so

$$0 = \begin{vmatrix} 1 & 1 & 1 \\ y & y^{\delta+1} & y^\delta \\ x & x^{\delta+1} & x^\delta \end{vmatrix} = y^\delta(x + x^{\delta+1}) + y(x^{\delta+1} + x^\delta) + y^{\delta+1}(x + x^\delta).$$

Then checking the six values of (x, y) from (23) shows that case (a) cannot occur.

(*b*) $e_3^\alpha \in L_1$

The diagonal point $(1 + y^\delta, y^\delta + y + x^\delta(y + 1), y^\delta x^\delta + x^\delta)$ of $e_0 f_0 e_2 f_2$ must be incident with L_1. This forces $x^\delta = (y^\delta + y)/(y + 1)$, which is not permitted by (23).

(*c*) $e_3^\alpha \in L_2$

The diagonal point $(1, y, 0)$ of $e_1 e_3 e_2 f_2$ must be incident with L_2. This forces $(y^{-1})^\delta + y^\delta = 1$. This is the equation of (21), which was shown to be impossible in case 2(c).

(*d*) $e_3^\alpha \in L_3$

The diagonal point $(1, x, x)$ of $e_1 e_3 e_0 f_0$ must be on L_3. This forces $c = x$, and from (22) it follows that $x^2 = x^\delta$. We check the values of x permitted by (23):

$$\left.\begin{array}{lcccccc} i: x = w^i & 1 & 3 & 5 & 7 & 11 & 15 \\ j: x^\delta = w^j & 19 & 27 & 16 & 6 & 13 & 9 \\ k: x^2 = w^k & 2 & 6 & 10 & 14 & 22 & 30 \end{array}\right\} \qquad (24)$$

This completes a proof that case 3 cannot arise.

CASE 4: $(1, 1, 0)^\alpha = (1, 0, 1) = u$

Let L_1, L_2, L_3 be distinct lines through u with $L_1 = e_1 e_3$, $L_2 = e_0 e_2$, $L_3 = f_1 f_2$; $e_1(1, 0, 0)$, $e_2(0, 1, 0)$, $e_3(0, 0, 1)$, $e_0(1, 1, 1)$, $f_1(1, x, x^\delta)$, $f_2(1, y, y^\delta)$; $x, y \notin \{0, 1\}$, $x \neq y$. From $u \in f_1 f_2$ we have $xy^\delta + yx^\delta + x + y = 0$. For example, we may choose

$$x = w^{30} \text{ and } y = w^{18} \qquad (\text{so } x^\delta = w^{18}, y^\delta = w^8). \qquad (25)$$

(*a*) $e_3^\alpha \notin L_1, L_2, L_3$

Then e_0, f_1, f_2 must be on the conic $XY + mYZ + nZX = 0$. But this is not satisfied for $x = w^{30}$ and $y = w^{18}$.

(*b*) $e_3^\alpha \in L_1$

The diagonal point $(y^\delta + 1, y^\delta + y + yx^\delta + x^\delta, y^\delta x^\delta + x^\delta)$ of $e_0 e_2 f_1 f_2$ must be incident with L_1, forcing $y^\delta + y + (y + 1)x^\delta = 0$. But this does not hold for $x = w^{30}$ and $y = w^{18}$.

(*c*) $e_3^\alpha \in L_2$

The diagonal point $(1, y, x^\delta y/x)$ of $e_1 e_3 f_1 f_2$ must be on L_2. This leads to $x = x^\delta y$, which does not hold for $x = w^{30}$ and $y = w^{18}$.

(*d*) $e_3^\alpha \in L_3$

The diagonal point $(0, 1, 1)$ of $e_1 e_3 e_0 e_2$ must be on L_3, forcing $x^\delta + x + 1 = 0$. Again this is not satisfied for $x = w^{30}$.

Case 5: $(1, 1, 0)^{\alpha} = (0, 1, c) = u$

Then $(1, 1, 0)^{\alpha\theta} = (1, 0, c)$, which is excluded by case 4.

5. Helpful Tables

These tables were compiled with computer assistance.

(5.1) *For* $F = GF(32)$ *with primitive root* w *satisfying* $w^5 = 1 + w^2$, *the action of the ovoidal permutation* δ: $x \mapsto x^{1/6} + x^{1/2} + x^{5/6}$ *is given by the following table*:

Table 4.
$(w^i)^{\delta} = w^j$

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
j	0	19	7	27	14	16	23	6	28	4	1	13	15	21	12	9

i	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
j	25	29	8	3	2	22	26	20	30	17	11	10	24	5	18

(5.2) *All pairs* (i, j) *are given for which there exists* $z = w^k$, *with* $(x, y, z) = (w^i, w^j, w^k)$ *satisfying both* (15) *and* (16):

Table 5.

Class	(i, j)	$(2i, 2j)$	$(4i, 4j)$	$(8i, 8j)$	$(16i, 16j)$
1	(0, 3)	(0, 6)	(0, 12)	(0, 24)	(0, 17)
2	(1, 21)	(2, 11)	(4, 22)	(8, 13)	(16, 26)
3	(3, 13)	(6, 26)	(12, 21)	(24, 11)	(17, 22)
4	(5, 1)	(10, 2)	(20, 4)	(9, 8)	(18, 16)
5	(7, 4)	(14, 8)	(28, 16)	(25, 1)	(19, 2)
6	(11, 9)	(22, 18)	(13, 5)	(26, 10)	(21, 20)

Class	$(-j, -i)$	$(-2j, -2i)$	$(-4j, -4i)$	$(-8j, -8i)$	$(-16j, -16i)$
1	(28, 0)	(25, 0)	(19, 0)	(7, 0)	(14, 0)
2	(10, 30)	(20, 29)	(9, 27)	(18, 23)	(5, 15)
3	(18, 28)	(5, 25)	(10, 19)	(20, 7)	(9, 14)
4	(30, 26)	(29, 21)	(22, 11)	(23, 22)	(15, 13)
5	(27, 24)	(23, 17)	(15, 3)	(30, 6)	(19, 12)
6	(22, 20)	(13, 9)	(26, 18)	(21, 5)	(11, 10)

(5.3) *For one* (i, j) *from each class in Table* 5 *the corresponding* $D \cap \Omega^*$ *is computed.* (*Here* $x = w^i$, $y = w^j$, $z = w^k$ *so that* (x, y, z) *satisfies* (15) *and* (16).)

Table 6.

(i, j)	k	D	$D \cap \Omega^*$
(0, 3)	27	$w^{12}X + Y + w^9Z = 0$	$(1, w^{13}, w^{21})$, $(1, w^{25}, w^{17})$
(1, 21)	4	$w^5X + Y + Z = 0$	$(1, w^6, w^{23})$, $(1, w^{15}, w^9)$
(3, 13)	28	$w^{16}X + Y + w^{18}Z = 0$	$(1, w^6, w^{23})$, $(1, w^{23}, w^{20})$
(5, 1)	8	$w^{25}X + Y + w^{18}Z = 0$	Empty
(7, 4)	15	$w^3X + Y + w^{12}Z = 0$	Empty
(11, 9)	2	$w^{17}X + Y + w^{22}Z = 0$	$(1, w, w^{19})$, $(1, w^{26}, w^{11})$

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