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On fast algorithms for the evaluation of Legendre coefficients

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ABSTRACT

In this paper, we present formulas betwixt Legender and Chebyshev expansion coefficients for a piecewise smooth (or Dini–Lipschitz) function, analyze the error bounds for the Piessens' algorithm and present a new algorithm with $O(N \log N)$ operations for computation of the first $N + 1$ coefficients of the Legendre expansion. Finally, we show the identity to the formulas given by Iserles [5] for analytic functions in a neighborhood of $[-1, 1]$.

1. Introduction

Legendre expansions are widely used in approximation theory, numerical integration, solutions of partial differential equations, analysis of pseudospectral methods, special function theory, etc. Given a function $f \in L^2[-1, 1]$, it takes $O(N^2)$ operations to compute the first $N + 1$ terms of the Legendre expansion

$$f(x) = \sum_{m=0}^{\infty} a_m P_m(x) \tag{1.1}$$

by the standard discrete expansion [1]

$$a_m = \left(m + \frac{1}{2}\right) \int_{-1}^{1} f(x)P_m(x)dx \approx \left(m + \frac{1}{2}\right) \sum_{n=0}^{N} w_n f(x_n)P_m(x_n), \quad m = 0, 1, \ldots, N, \tag{1.2}$$

where $\{x_n\}_{n=0}^{N}$ is the Gauss-point set (the roots of $P_{N+1}(x) = 0$) and w_n the corresponding weights in the Gauss–Legendre quadrature [2].

Piessens [3] presented another efficient algorithm with order $O(N^2)$ to approximate the first $N + 1$ Legender coefficients by using the Chebyshev expansion:

$$f(x) = {\sum_{m=0}^{\infty}}' c_m T_m(x), \qquad c_m = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_m(x)}{\sqrt{1 - x^2}} dx, \quad m \in \mathbb{Z}_+, \tag{1.3}$$

where $\sum'$ denotes a sum whose first term is halved. Then a_m can be approximated [3] by

$$a_m \approx \left(m + \frac{1}{2}\right) {\sum_{k=0}^{N}}' c_k I_{m,k}, \quad m = 0, 1, \ldots, N$$

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which yields a fast algorithm [3]

$$\bar{a}_m = \left(m + \frac{1}{2}\right) \sum_{j=0}^{N}{}'' I_{m,j} \widetilde{c}_j, \quad m = 0, 1, \ldots, N. \tag{1.4}$$

Here $I_{m,k} = \int_{-1}^{1} P_m(x) T_k(x) dx$ can be calculated by recursion in $O(N^2)$ operations

$$I_{m,k} = \begin{cases} \dfrac{2}{1-k^2}, & m = 0, k \text{ is even} \\ \dfrac{2^{2m}(m!)^2}{(2m+1)!}, & k = m \geq 1 \\ \dfrac{[(m+2j-3)(m+2j-2) - m(m+1)](m+2j)}{[(m+2j+1)(m+2j) - m(m+1)](m+2j-2)} I_{m,m+2j-2}, & m \geq 1, \ k = m+2j, \ j \geq 1 \\ 0, & \text{otherwise}, \end{cases} \tag{1.5}$$

$\widetilde{c}_m$ $(m = 0, 1, \ldots, N)$ can be evaluated by FFT in $O(N \log N)$ operations [4]:

```
function b = coefficients(f, n)                          % (n + 1) coefficients for f
a0 = −1; b0 = 1                                          % approximate interval of f
x = (a0 + b0)/2 + (b0 − a0)/2 ∗ cos(pi ∗ (0 : n)′/n);   % Chebyshev points of the second kind
fx = feval(f, x)/(2 ∗ n);                                % f evaluated at these points
g = fft(fx([1 : N0 + 1N0 : −1 : 2]));                    % FFT
c̃= [2 ∗ g(1); g(2 : N0) + g(2 ∗ N0 : −1 : N0 + 2); 2 ∗ g(N0 + 1)];   % Chebyshev coefficients
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and $\sum''$ denotes a sum whose first and last terms are halved.

More recently, Iserles [5] proposed a new improvement by using the Cauchy theorem and Taylor expansions of the Legendre and hypergeometric functions Φ_m and ${}_2F_1$, respectively, for $f(x)$ analytic on and inside the Bernstein ellipse $\mathcal{E}_\rho$ with foci ± 1 and major and minor semiaxis lengths summing to $\rho (> 1)$. The derivation is rather convoluted but the outcome is a surprisingly simple numerical algorithm [5]. The total error bound on the algorithm remains open there [5].

In this paper, we will focus on fast algorithms for the evaluation of Legendre expansion for non-analytic functions of finite regularity and their error analysis. We will show that the Legendre coefficient a_m for piecewise smooth (or Dini–Lipschitz) $f(x)$ on $[-1, 1]$ can be represented by the Chebyshev coefficients as

$$a_0 = \frac{1}{2}c_0 + \frac{1}{2}\sum_{n=1}^{\infty} I_{0,2n} c_{2n}, \qquad a_m = \left(m + \frac{1}{2}\right) \sum_{n=0}^{\infty} I_{m,m+2n} c_{m+2n}, \quad m = 1, 2, \ldots, \tag{1.6}$$

from which we can construct an efficient algorithm to compute a_m

$$\begin{aligned} &\widetilde{a}_0 = \frac{1}{2}c_0 + \frac{1}{2}\sum_{n=1}^{N_0} I_{0,2n} \widetilde{c}_{2n}, \qquad \widetilde{a}_m = \left(m + \frac{1}{2}\right) \sum_{n=0}^{N_0} I_{m,m+2n} \widetilde{c}_{m+2n}, \quad m = 1, \ldots, N-1, \\ &\widetilde{a}_N = \left(N + \frac{1}{2}\right) \sum_{n=0}^{N_0-1} I_{N,N+2n} \widetilde{c}_{N+2n} + \frac{1}{2}\left(N + \frac{1}{2}\right) I_{N,N+2N_0} \widetilde{c}_{N+2N_0}, \end{aligned} \tag{1.7}$$

where N_0 is a fixed positive integer depending on $f(x)$ and independent of N for a given tolerance $\delta > 0$, $\widetilde{c}_m$ $(m = 0, 1, \ldots, N + 2N_0)$ can be evaluated by FFT in $O((N + 2N_0) \log(N + 2N_0))$ operations, and $I_{m,j}$ can be obtained by (1.5) in $O(NN_0)$ operations. Therefore, the total cost is $O(N \log N)$. In Section 2, we will derive the error bounds for Piessens' algorithm (1.4) and the new algorithm (1.7). In the final remark, we will show that (1.6) is identity to that given in [5] for analytic functions in a neighborhood of $[-1, 1]$.

2. Main results

Lemma 2.1. *Suppose $f(x) \in L^2[-1, 1]$ is piecewise smooth (or Dini–Lipschitz) on $[-1, 1]$, then*

$$a_0 = \frac{1}{2}c_0 + \frac{1}{2}\sum_{n=1}^{\infty} I_{0,2n} c_{2n}, \qquad a_m = \left(m + \frac{1}{2}\right) \sum_{n=0}^{\infty} I_{m,m+2n} c_{m+2n}, \quad m = 1, 2, \ldots.$$

Proof. Recall that a function $h(x)$ is called piecewise continuous on $[-1, 1]$ provided (i) h is continuous on $[-1, 1]$ except at finitely many points $y_1, \ldots, y_k$; and (ii) at each of the points $y_1, \ldots, y_k$, the left-hand and right-hand limits of h exist. h is called piecewise smooth if h and h' are both piecewise continuous on $[-1, 1]$ [6, p. 32]. Since $f(x)$ is piecewise smooth on $[-1, 1]$, then $f(\cos(\theta))$ is piecewise smooth on $[-\pi, \pi]$ and

$$f(\cos(\theta)) = {\sum_{n=0}^{\infty}}' c_n \cos(n\theta), \quad \cos(\theta) \neq y_j,\ j = 1, \ldots, k$$

(see [6, p. 35]), where c_n is the same as that in (1.3). Consequently we have

$$f(x) = {\sum_{n=0}^{\infty}}' c_n T_n(x), \quad x \neq y_j,\ j = 1, \ldots, k. \tag{2.1}$$

Thus, from (2.1) and Theorem 1.27 [7, p. 22] it follows

$$\begin{aligned} a_m &= \left(m + \frac{1}{2}\right) \int_{-1}^{1} f(x) P_m(x) dx = \left(m + \frac{1}{2}\right) \int_{-1}^{1} {\sum_{n=0}^{\infty}}' c_n T_n(x) P_m(x) dx \\ &= \left(m + \frac{1}{2}\right) {\sum_{n=0}^{\infty}}' c_n \int_{-1}^{1} T_n(x) P_m(x) dx, \quad m \in \mathbb{Z}_+, \end{aligned}$$

which directly leads to (1.6).

In the case that f is Dini–Lipschitz, from the Dini–Lipschitz Theorem [8, p. 129], the Chebyshev expansion uniformly converges to $f(x)$. By the above same proof, (1.6) is also satisfied. □

Lemma 2.2. *For the Legendre expansion (1.1) and Chebyshev expansion (1.3), the coefficients satisfy*

$$|a_n| \leq \frac{2\sqrt{n}M}{\rho^{n-1}(\rho^2 - 1)} \quad (\text{see [9]}), \qquad |c_n| \leq \frac{2M}{\rho^n} \quad (\text{see [10]}),\ n = 1, 2, \ldots \tag{2.2}$$

if f is analytic with $|f(z)| \leq M$ in the region bounded by the ellipse $\mathcal{E}_\rho$ with foci ± 1 and major and minor semiaxis lengths summing to $\rho > 1$; or

$$|a_n| \leq \frac{\sqrt{\pi} V_p}{\sqrt{2(n - p - 1)} \left(n - \frac{1}{2}\right) \left(n - \frac{3}{2}\right) \cdots \left(n - \frac{2p-1}{2}\right)} \quad (\text{see [11]}), \tag{2.3a}$$

$$|c_n| \leq \frac{2V_p}{\pi n(n-1)\cdots(n-p)} \quad (\text{see [12]}) \tag{2.3b}$$

if $f, \ldots, f^{(p-1)}$ are absolutely continuous on $[-1, 1]$ and $f^{(p)}$ has bounded variation V_p for some $p \geq 1$.

Theorem 2.1. *The error bound for the Piessens' algorithm (1.4) for evaluating a_m can be given as*

$$|\bar{a}_m - a_m| \leq \begin{cases} \dfrac{(2m+1) I_{m,m} V_p}{p\pi N(N-1)\cdots(N-p+1)} + \dfrac{V_p}{(p-1)\left(N - \frac{1}{2}\right)\left(N - \frac{3}{2}\right)\cdots\left(N - \frac{2p-3}{2}\right)} \sqrt{\dfrac{\pi}{2(N-p)}}, \\ \quad V_p < \infty \\ \dfrac{(4m+2) I_{m,m} M}{(\rho - 1)\rho^N} + \dfrac{2M[(N+1)(\rho - 1) + 1]}{\sqrt{N+1}\rho^{N-1}(\rho - 1)^3(\rho + 1)}, \quad f(x) \text{ analytic in } \mathcal{E}_\rho \end{cases} \tag{2.4}$$

for $m = 0, 1, 2, \ldots, N$.

Proof. From the definitions of $I_{m,k}$s, it follows for $m \geq 0$ that

$$|a_m| \leq \left(m + \frac{1}{2}\right) \sum_{n=0}^{\infty} |c_n I_{m,n}| = \left(m + \frac{1}{2}\right) |c_m I_{m,m}| + \left(m + \frac{1}{2}\right) \sum_{j=1}^{\infty} |c_{m+2j} I_{m,m+2j}|,$$

and for $j \geq 0$ that

$$|I_{m,m+2j+2}| = \frac{|(m+2j-3)(m+2j-2) - m(m+1)|(m+2j)}{[(m+2j+1)(m+2j) - m(m+1)](m+2j-2)} |I_{m,m+2j-2}| \leq \frac{|2j-1|}{2j+1} |I_{m,m+2j}|,$$

which implies

$$|I_{m,m+2j}| \leq \frac{1}{2j-1} |I_{m,m}|, \quad j = 0, 1, 2, \ldots.$$

Moreover, from (1.4)–(1.6), the error bound for evaluation of a_m by (1.4) can be given as

$$|\bar{a}_m - a_m| \leq \left(m+\frac{1}{2}\right) I_{m,m} \left(\sum_{j=0}^{N-1}{}' |\widetilde{c}_j - c_j| + \frac{1}{2}|\widetilde{c}_N - c_N|\right) + \sum_{j=N+1}^{\infty} |a_j|, \quad m = 0, 1, 2, \ldots, N. \tag{2.5}$$

Recalling (2.13.1.11) in [2] (also see [13, p. 96])

$$\widetilde{c}_j - c_j = \sum_{\ell=1}^{\infty} (c_{2\ell n-j} + c_{2\ell n+j}), \quad j = 0, 1, \ldots, n, \tag{2.6}$$

which, together with (2.5) and (2.6), gives

$$|\bar{a}_m - a_m| \leq \left(m+\frac{1}{2}\right) I_{m,m} \sum_{j=N+1}^{\infty} |c_j| + \sum_{j=N+1}^{\infty} |a_j|. \tag{2.7}$$

Furthermore, note that $\sum_{j=N+1}^{\infty} |c_j| \leq \frac{2V_p}{p\pi N(N-1)\cdots(N-p+1)}$ [14, Theorem 2.1] for $V_p < \infty$ and

$$\begin{aligned}\sum_{j=N+1}^{\infty} |a_j| &\leq V_p \sqrt{\frac{\pi}{2(N-p)}} \sum_{j=N+1}^{\infty} \frac{1}{\left(j-\frac{1}{2}\right)\left(j-\frac{3}{2}\right)\cdots\left(j-\frac{2p-1}{2}\right)} \\ &= V_p \sqrt{\frac{\pi}{2(N-p)}} \frac{1}{p-1} \sum_{j=N+1}^{\infty} \left[\frac{1}{\left(j-\frac{3}{2}\right)\cdots\left(j-\frac{2p-1}{2}\right)} - \frac{1}{\left(j-\frac{1}{2}\right)\cdots\left(j-\frac{2p-3}{2}\right)}\right] \\ &= \frac{V_p}{(p-1)\left(N-\frac{1}{2}\right)\left(N-\frac{3}{2}\right)\cdots\left(N-\frac{2p-3}{2}\right)} \sqrt{\frac{\pi}{2(N-p)}}.\end{aligned}$$

These together lead to (2.4) for $V_p < \infty$.

Similarly, by (2.2) together with Theorem 2.1 [14] and Corollary 2.1 [9], it obtains (2.4) for $f(x)$ analytic in $\mathcal{E}_\rho$. □

Theorem 2.2. *The error bound for the algorithm (1.7) for evaluating a_m can be given as*

$$|\widetilde{a}_m - a_m| \leq \begin{cases} \dfrac{(4m+2)I_{m,m}V_p}{p\pi(N_0+2m)(N_0+2m-1)\cdots(N_0+2m-p+1)}, & V_p < \infty \\ \dfrac{(4m+2)I_{m,m}M}{(\rho-1)\rho^{m+2N_0}}, & f(x) \text{ analytic in } \mathcal{E}_\rho \end{cases} \tag{2.8}$$

for $m = 0, 1, 2, \ldots, N$.

Proof. In a similar way, the error for computation of each a_m by $\widetilde{a}_m$ can be estimated from (1.6) and (1.7) for $m = 0, 1, \ldots, N$ by

$$\begin{aligned}|\widetilde{a}_m - a_m| &\leq \left(m+\frac{1}{2}\right) I_{m,m} \left(\sum_{n=0}^{m+2N_0-1}{}' |\widetilde{c}_n - c_n| + \left|\frac{1}{2}\widetilde{c}_{N+2N_0} - c_{N+2N_0}\right| + \sum_{n=N_0+1}^{\infty} |c_{m+2n}|\right) \\ &\leq \left(m+\frac{1}{2}\right) I_{m,m} \left(\sum_{n=m+2N_0+1}^{\infty} |c_n| + \sum_{n=N_0+1}^{\infty} |c_{m+2n}|\right) \quad \text{by (2.6)} \\ &\leq (2m+1)I_{m,m} \sum_{n=m+2N_0+1}^{\infty} |c_n|,\end{aligned}$$

which together with $\sum_{j=N+1}^{\infty} |c_j| \leq \frac{2V_p}{p\pi N(N-1)\cdots(N-p+1)}$ for $V_p < \infty$ and $\sum_{j=N+1}^{\infty} |c_j| \leq \frac{2M}{(\rho-1)\rho^N}$ [14, Theorem 2.1] yields the desired result. □

Remark 1. From the definition of $I_{m,m}$s, we see that $\frac{1}{2}I_{0,0} = 1$ and $(m+\frac{1}{2})I_{m,m} \leq \sqrt{m}$ for $m \geq 1$.

Remark 2. Both algorithms (1.2) and (1.5) can achieve higher accuracy as N increases. However, these methods cannot improve the accuracy of the relative errors at the last few coefficients as reported in [3]. But from (2.8), it shows that the relative errors by (1.7) at the last few coefficients can achieve higher accuracy as N_0 increases.

Example 1. We illustrate the convergence rates of (1.7) and compare with that evaluated by the discrete expansion (1.2) and (1.5) with $N = 100$ (Tables 1 and 2).

Table 1
Approximation of coefficients of the Legendre expansions by (1.7).

$f(x)$	a_k	$N_0 = 2$	$N_0 = 10$	$N_0 = 10^2$	$N_0 = 10^3$	Exact value
e^x	a_0	1.17520248187784	1.17520119364380	1.17520119364380	1.17520119364380	1.17520119364380
	a_{100}	0.00000000000000	0.00000000000000	0.00000000000000	0.00000000000000	0.00000000000000
$\frac{1+x}{4+x^2}$	a_0	0.23182165921697	0.23182380450040	0.23182380450040	0.23182380450040	0.23182380450040
	a_{100}	0.00000000000000	0.00000000000000	0.00000000000000	0.00000000000000	0.00000000000000
e^{-1/x^2}	a_0	0.08855032961820	0.08907377800247	0.08907385589078	0.08907385589078	0.08907385589078
	a_{100}	0.00000000037222	0.00000000032330	0.00000000032424	0.00000000032424	0.00000000032424
$\|x\|^3$	a_0	0.24979747886432	0.24999996006293	0.25000000009753	0.25000000000004	0.25000000000000
	a_{100}	0.00000154312226	0.00000117911069	0.00000094424591	0.00000094223996	0.00000094223929

Table 2
Approximation of coefficients of the Legendre expansions by (1.2) and (1.5) respectively.

$f(x)$	a_k	By (1.2)	By (1.5)	Exact value
e^x	a_0	1.17520119364380	1.17520119364380	1.17520119364380
	a_{100}	0.00000000000000	0.00000000000000	0.00000000000000
$\frac{1+x}{4+x^2}$	a_0	0.23182380450040	0.23182380450040	0.23182380450040
	a_{100}	0.00000000000001	0.00000000000000	0.00000000000000
e^{-1/x^2}	a_0	0.08907385589078	0.08907385589078	0.08907385589078
	a_{100}	0.00000000091831	0.00000000016170	0.00000000032424
$\|x\|^3$	a_0	0.25000000765123	0.25000000812140	0.25000000000000
	a_{100}	0.00000183956737	0.00000068850364	0.00000094223929

3. Final remarks

Iserles [5] proposed fast algorithms for analytic function $f(x)$ on and inside the Bernstein ellipse $\mathcal{E}_\rho$. By using the Cauchy theorem and Taylor expansions of the Legendre and hypergeometric functions Φ_m and ${}_2F_1$, the Legendre coefficient a_m can be represented by [5] in the form of

$$a_m = \frac{2^{2m}(m!)^2}{2(2m)!\rho^m}\sum_{j=0}^{\infty}\int_{-\pi}^{\pi}(1-\rho^{-2}e^{2i\theta})f\left(\frac{1}{2}(\rho^{-1}e^{i\theta}+\rho e^{-i\theta})\right){}_2F_1\left(m+1,\frac{1}{2};m+\frac{3}{2};\rho^{-2}e^{2i\theta}\right)e^{im\theta}d\theta.$$

Letting $\rho \to 1$ it follows [5]

$$a_m = \frac{1}{2\pi}\sum_{j=0}^{\infty}\widetilde{g}_{m,j}\int_{-\pi}^{\pi}(1-e^{2i\theta})f(\cos(\theta))e^{i(m+2j)\theta}d\theta = \frac{1}{2}\sum_{j=0}^{\infty}\widetilde{g}_{m,j}(c_{m+2j}-c_{m+2j+2}), \tag{3.1}$$

where $\widetilde{g}_{m,j} = \frac{2^{2m}(m!)^2(m+1)_j(\frac{1}{2})_j}{(2m)!j!(m+\frac{3}{2})_j}$ can be computed by

$$\widetilde{g}_{0,0} = 1, \qquad \widetilde{g}_{m,0} = \frac{m}{m-\frac{1}{2}}\widetilde{g}_{m-1,0}, \qquad \widetilde{g}_{m,j} = \frac{(m+j)\left(j-\frac{1}{2}\right)}{j\left(m+j+\frac{1}{2}\right)}\widetilde{g}_{m,j-1}, \quad m=1,2,\ldots,\ j=1,2,\ldots,$$

which together leads to a fast algorithm [5]

$$\begin{aligned}\widehat{a}_m &= \frac{1}{2}\sum_{j=0}^{N_0-1}\widetilde{g}_{m,j}(\widetilde{c}_{m+2j}-\widetilde{c}_{m+2j+2}), \quad m=0,1,\ldots,N-1\\ \widehat{a}_N &= \frac{1}{2}\sum_{j=0}^{N_0-2}\widetilde{g}_{N,j}(\widetilde{c}_{N+2j}-\widetilde{c}_{N+2j+2})+\frac{1}{2}\widetilde{g}_{N,j}\left(\widetilde{c}_{N+2N_0-2}-\frac{1}{2}\widetilde{c}_{N+2N_0}\right).\end{aligned} \tag{3.2}$$

In the following, we will show that (3.1) and (1.6) are identity, that is,

$$a_0 = \frac{1}{2}c_0 + \frac{1}{2}\sum_{n=1}^{\infty}I_{0,2n}c_{2n} = \frac{1}{2}\sum_{j=0}^{\infty}\widetilde{g}_{0,j}(c_{2j}-c_{2j+2})$$

$$a_m = \left(m+\frac{1}{2}\right)\sum_{n=0}^{\infty}I_{m,m+2n}c_{m+2n} = \frac{1}{2}\sum_{j=0}^{\infty}\widetilde{g}_{m,j}(c_{m+2j}-c_{m+2j+2}), \quad m \geq 1.$$

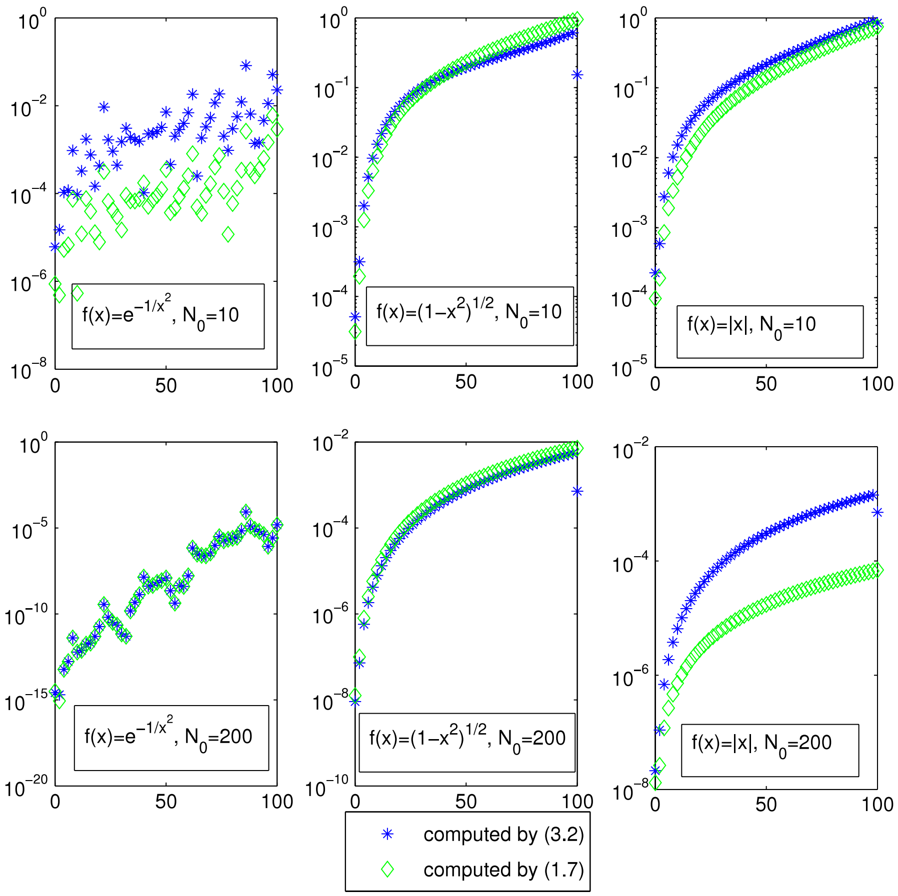

Fig. 1. The relative errors computed by (3.2) and (1.7) respectively: $m = 0 : 2 : 100$.

Notice that (3.1) can be rewritten as

$$a_m = \frac{1}{2}\widetilde{g}_{m,0}c_m + \frac{1}{2}\sum_{j=1}^{\infty}(\widetilde{g}_{m,j} - \widetilde{g}_{m,j-1})c_{m+2j} = \frac{1}{2}\widetilde{g}_{m,0}c_m - \frac{1}{2}\sum_{j=1}^{\infty}\frac{\frac{1}{2}m+j}{j\left(m+j+\frac{1}{2}\right)}\widetilde{g}_{m,j-1}c_{m+2j}. \tag{3.3}$$

Comparing the first coefficient in (3.3) with that in (1.6), it is obvious by the definitions of $I_{m,m}$ and $\widetilde{g}_{m,0}$ that

$$\left(m+\frac{1}{2}\right)I_{m,m} = \frac{1}{2}\widetilde{g}_{m,0}, \quad m = 0, 1, 2, \ldots.$$

Hence, it is only necessary to prove that for each fixed m

$$\left(m+\frac{1}{2}\right)I_{m,m+2j} = -\frac{1}{2}\frac{\frac{1}{2}m+j}{j\left(m+j+\frac{1}{2}\right)}\widetilde{g}_{m,j-1}, \qquad j = 1, 2, \ldots, \tag{3.4}$$

which can be proved by induction on j. Suppose (3.4) is true for $k = j$, then for $k = j+1$, by (1.6) it yields

$$\begin{aligned}
&\left(m+\frac{1}{2}\right)I_{m,m+2j+2} \\
&= \left(m+\frac{1}{2}\right)I_{m,m+2j}\frac{[(m+2j-1)(m+2j)-m(m+1)](m+2j+2)}{[(m+2j+3)(m+2j+2)-m(m+1)](m+2j)} \\
&= -\frac{1}{2}\frac{\frac{1}{2}m+j}{j\left(m+j+\frac{1}{2}\right)}\widetilde{g}_{m,j-1}\frac{[(m+2j-1)(m+2j)-m(m+1)](m+2j+2)}{[(m+2j+3)(m+2j+2)-m(m+1)](m+2j)} \\
&= -\frac{1}{2}\frac{\frac{1}{2}m+j+1}{(j+1)\left(m+j+1+\frac{1}{2}\right)}\cdot\frac{(m+j)\left(j-\frac{1}{2}\right)}{j\left(m+j+\frac{1}{2}\right)}\widetilde{g}_{m,j-1}
\end{aligned}$$

$$\times\left\{\frac{j(j+1)\left(m+j+\frac{1}{2}\right)\left(m+j+\frac{3}{2}\right)}{\left(\frac{1}{2}m+j+1\right)(m+j)\left(j-\frac{1}{2}\right)}\frac{\left(\frac{1}{2}m+j\right)[(m+2j-1)(m+2j)-m(m+1)](m+2j+2)}{j\left(m+j+\frac{1}{2}\right)[(m+2j+3)(m+2j+1)-m(m+1)](m+2j)}\right\}$$
$$=-\frac{1}{2}\frac{\frac{1}{2}m+j+1}{j\left(m+j+\frac{3}{2}\right)}\widetilde{g}_{m,j}$$

where we used Maple 11 to verify the last factor in the right-hand side of the third equality is equal to 1.

Therefore, the fast algorithm (3.2) can also be applied to piecewise smooth (or Dini–Lipschitz) functions. Comparing (3.3) with (1.7) (N_0-1 instead of N_0 in (1.7)), we see that for fixed $m=0,1,\ldots,N$ in (3.2) and (1.7)

$$(3.2)=(1.7)-\frac{1}{2}\widetilde{g}_{m,N0}\widetilde{c}_{m+2N_0+2}.$$

Example 2. We use $f(x)=e^{-1/x^2}$, $\sqrt{1-x^2}$, $|x|$ to compare (3.2) with (1.7) to show the efficiency of these two algorithms (Fig. 1).

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References

[1] J.S. Hesthaven, S. Gottlieb, D. Gottlieb, Spectral Methods for Time-Dependent Problems, Cambridge University Press, 2007.
[2] P.J. Davis, P. Rabinowitz, Methods of Numerical Integration, second ed., Academic Press, 1984.
[3] R. Piessens, Computation of Legendre series coefficients, Algorithm 473, Commun. ACM 17 (1974) 25.
[4] L.N. Trefethen, Is Gauss quadrature better than Clenshaw–Curtis? SIAM Rev. 50 (2008) 67–87.
[5] A. Iserles, A fast (and simple) algorithm for the computation of Legendre coefficients, DAMTP Tech. Rep. 2010/NA01.
[6] G.B. Folland, Fourier Analysis and its Applications, Thomson Learning, 1992.
[7] W. Rudin, Real and Complex Analysis, McGraw-Hill Company, Inc., 1974.
[8] E.W. Cheney, Introduction to Approximation Theory, McGraw-Hill, New York, 1966.
[9] S. Xiang, On error bounds for orthogonal polynomial expansions and Gauss-type quadrature, SIAM J. Numer. Anal. 50 (2012) 1240–1263.
[10] S. Bernstein, Sur l'order de la meilleure approximation des fonctions continues par des polynomes de degré donné, Mem. Acad. Roy. Belg. (2) 4 (1912) 1–103.
[11] H. Wang, S. Xiang, On the convergence rates of Legendre approximation, Math. Comp. 81 (2012) 861–877.
[12] L.N. Trefethen, Approximation Theory and Approximation in Practice, University of Oxford, 2011. http://www2.maths.ox.ac.uk/chebfun/ATAP/.
[13] J.P. Boyd, Chebyshev and Fourier Spectral Methods, Dover Publications, New York, 2000.
[14] S. Xiang, X. Chen, H. Wang, Error bounds for approximation in Chebyshev points, Numer. Math. 116 (2010) 463–491.