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Fourier transformation of Sato's hyperfunctions

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Abstract

A new generalized function space in which all Gelfand–Shilov classes S'^{0}_{α} $(\alpha > 1)$ of analytic functionals are embedded is introduced. This space of *ultrafunctionals* does not possess a natural nontrivial topology and cannot be obtained via duality from any test function space. A canonical isomorphism between the spaces of hyperfunctions and ultrafunctionals on $\mathbb{R}^k$ is constructed that extends the Fourier transformation of Roumieu-type ultradistributions and is naturally interpreted as the Fourier transformation of hyperfunctions. The notion of carrier cone that replaces the notion of support of a generalized function for ultrafunctionals is proposed. A Paley–Wiener–Schwartz-type theorem describing the Laplace transformation of ultrafunctionals carried by proper convex closed cones is obtained and the connection between the Laplace and Fourier transformations is established.

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1. Introduction

It is well known that Sato's hyperfunctions cannot be interpreted as continuous linear functionals on any test function space. For this reason, the standard definition of the Fourier transformation of generalized functions is inapplicable to hyperfunctions. This

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difficulty does not appear in the framework of Fourier hyperfunctions [11] that grow at infinity no faster than any linear exponential. Kawai [5] has established that the space of Fourier hyperfunctions on $\mathbb{R}^k$ is naturally identified with the continuous dual of a suitable test function space (actually coinciding with the Gelfand–Shilov space $S_1^1(\mathbb{R}^k)$) and is taken to itself by the Fourier transformation. However, the question is still open whether it is possible to construct the Fourier transformation of general hyperfunctions with no growth restrictions imposed. The aim of this paper is to fill this gap.

The proposed construction naturally arises from the consideration of analytic functionals defined on Gelfand–Shilov test function spaces $S_\alpha^0(\mathbb{R}^k)$ with $\alpha > 1$. According to [2] the Fourier transformation induces a topological isomorphism between $S_\alpha^0(\mathbb{R}^k)$ and the space $S_0^\alpha(\mathbb{R}^k)$, whose continuous dual $S_0^{\prime\alpha}(\mathbb{R}^k)$ is exactly the space of Roumieu's ultradistributions [10] of class $\{k^{\alpha k}\}$. The space $\mathcal{B}(\mathbb{R}^k)$ of hyperfunctions on $\mathbb{R}^k$ can be thought of as the "limiting case" of the spaces $S_0^{\prime\alpha}(\mathbb{R}^k)$ as $\alpha \downarrow 1$. Therefore, we can try to define the Fourier transform $\mathcal{U}(\mathbb{R}^k)$ of the space $\mathcal{B}(\mathbb{R}^k)$ by passing to the limit $\alpha \downarrow 1$ in the definition of the spaces $S_\alpha^{\prime 0}(\mathbb{R}^k)$.

Unfortunately, we cannot just set $\mathcal{U}(\mathbb{R}^k) = S_1^{\prime 0}(\mathbb{R}^k)$ because the space $S_1^0(\mathbb{R}^k)$ is trivial [2]. The way of overcoming this difficulty is suggested by the results of the papers [13,14] concerning the localization of analytic functionals belonging to $S_\alpha^{\prime 0}(\mathbb{R}^k)$. In these works, the notion of *carrier cone* that replaces the notion of support of a generalized function for analytic functionals was proposed (the standard definition of support does not work because of the lack of test functions with compact support). The definition of carrier cones is based on introducing, for every closed cone K, a suitable test function space $S_\alpha^0(K)$ in which $S_\alpha^0(\mathbb{R}^k)$ is densely embedded (the precise definition will be given in Section 2); a functional $u \in S_\alpha^{\prime 0}(\mathbb{R}^k)$ is said to be carried by a closed cone K if u has a continuous extension to $S_\alpha^0(K)$. As shown in [13], every functional in $S_\alpha^{\prime 0}(\mathbb{R}^k)$ has a uniquely defined minimal carrier cone. The definition of the spaces associated with cones is naturally extended to the case $\alpha = 1$ and it turns out that the spaces $S_1^0(K)$ over *proper* [2] closed cones are nontrivial. The space $\mathcal{U}(\mathbb{R}^k)$ is obtained by "gluing together" the generalized function spaces $S_1^{\prime 0}(K)$ associated with proper closed cones $K \subset \mathbb{R}^k$ (this procedure will be given a precise meaning in Section 3).

The properties of the elements of $\mathcal{U}(\mathbb{R}^k)$, which will be named *ultrafunctionals*, are quite similar to those of analytic functionals in $S_\alpha^{\prime 0}(\mathbb{R}^k)$. In particular, the definition of carrier cones is extended to the case of the space $\mathcal{U}(\mathbb{R}^k)$ and it turns out that every ultrafunctional has a uniquely defined minimal carrier cone. For a closed proper cone K, the space $\mathcal{U}(K)$ consisting of ultrafunctionals carried by K coincides with $S_1^{\prime 0}(K)$. The spaces $S_\alpha^{\prime 0}(K)$ are naturally embedded in $\mathcal{U}(K)$ for any closed cone K. If $K, K_1, \ldots, K_n$ are closed cones in $\mathbb{R}^k$ such that $K = \bigcup_{j=1}^n K_j$, then every ultrafunctional $u \in \mathcal{U}(K)$ is representable in the form

$$u = \sum_{j=1}^{n} u_j, \quad u_j \in \mathcal{U}(K_j). \tag{1.1}$$

[2] A cone U in $\mathbb{R}^k$ will be called proper if $\bar{U} \setminus \{0\}$ is contained in an open half-space of $\mathbb{R}^k$ (the bar denotes closure). For convex closed cones, this definition is equivalent to the usual one according to which a cone is called proper if it contains no straight lines.

Every exponential decreasing in an open half-space containing a convex proper closed cone K belongs to the space $S_1^0(K)$. This allows us to define the Laplace transform $\mathcal{L}_K u$ of every ultrafunctional u carried by K. We prove an elegant Paley–Wiener–Schwartz-type theorem asserting that the Laplace transformation $\mathcal{L}_K$ induces a topological isomorphism between $\mathcal{U}(K)$ and the space of all functions analytic in the tubular domain $\mathbb{R}^k + iV$, where V is the interior of the dual cone of K. The Fourier transform $\mathcal{F}u$ of an ultrafunctional u carried by a convex proper closed cone K is by definition the boundary value in $\mathcal{B}(\mathbb{R}^k)$ of the Laplace transform of u. For a general $u \in \mathcal{U}(\mathbb{R}^k)$, we take a decomposition of form (1.1), where all K_j are convex proper closed cones, and set $\mathcal{F}u = \sum_{j=1}^n \mathcal{F}u_j$. The hyperfunction $\mathcal{F}u$ so defined does not depend on the chosen decomposition. We prove that the operator $\mathcal{F}$ maps $\mathcal{U}(\mathbb{R}^k)$ isomorphically onto $\mathcal{B}(\mathbb{R}^k)$ and that its restriction to $S_\alpha^{\prime 0}(\mathbb{R}^k)$ coincides with the ordinary Fourier transformation determined via duality by the Fourier transformation of test functions.

The paper is organized as follows. In Section 2, we give a brief exposition of the results of the works [13,14] concerning the spaces $S_\alpha^{\prime 0}(\mathbb{R}^k)$ with $\alpha > 1$ and obtain a useful representation of $S_\alpha^{\prime 0}(\mathbb{R}^k)$ in terms of the spaces associated with proper closed cones. In Section 3, we introduce the spaces $S_1^0(K)$ and $\mathcal{U}(K)$ and give the precise formulations of the main results. In the same section, we prove the compatibility of the operator $\mathcal{F}$ with the Fourier transformation of ultradistributions. Section 4 is devoted to a detailed study of the spaces $S_1^0(K)$ over proper closed cones and to the proof of the above-mentioned PWS-type theorem. In Section 5, the results concerning carrier cones (the existence of a unique minimal carrier cone of an ultrafunctional and the existence of decompositions of form (1.1)) are established. In Section 6, the bijectivity of the Fourier operator $\mathcal{F}$ is proved. In Section 7, we indicate some possible further developments of these results. The proofs of some algebraic statements of Section 5 are given in Appendices A and B.

2. Localization of analytic functionals on Gelfand–Shilov spaces

The space $S_\alpha^\beta(\mathbb{R}^k)$ is by definition [2] the union (inductive limit) with respect to $A, B > 0$ of the Banach spaces consisting of smooth functions on $\mathbb{R}^k$ with the finite norm

$$\sup_{x\in\mathbb{R}^k,\, \lambda,\, \mu} \frac{|x^\lambda \partial^\mu f(x)|}{A^{|\lambda|} B^{|\mu|} |\lambda|^{\alpha|\lambda|} |\mu|^{\beta|\mu|}}, \tag{2.1}$$

where λ and μ run over all multi-indices and the standard multi-index notation is used. The spaces S_α^β are nontrivial if $\alpha + \beta > 1$ or if $\alpha, \beta > 0$ and $\alpha + \beta = 1$. For $\alpha = 0$, the spaces S_α^β consist of functions of compact support. If $0 \leqslant \beta < 1$, then S_α^β consists of (the restrictions to $\mathbb{R}^k$ of) entire analytic functions and an alternative description of these spaces in terms of complex variables is possible [2]. Namely, an analytic function

f on $\mathbb{C}^k$ belongs to the class S^{β}_{α} if and only if

$$|f(z)| \leqslant C\, \exp(-|x/A|^{1/\alpha} + |By|^{1/(1-\beta)}), \quad z = x + iy \in \mathbb{C}^k$$

for some $A, B > 0$ depending on f. For definiteness, we assume the norm $|\cdot|$ on $\mathbb{R}^k$ to be uniform, i.e., $|x| = \sup_{1\leqslant j\leqslant k} |x_j|$. As shown in [2], the Fourier transformation isomorphically maps the space S^{β}_{α} onto S^{α}_{β}. The Fourier transformation of generalized functions on S^{β}_{α} is defined in a standard way, as the dual mapping of the Fourier transformation of test functions, and maps S'^{β}_{α} onto S'^{α}_{β}.

In what follows, we confine our discussion to the case $\beta = 0$ which is of primary interest to us, but in fact only the condition $\beta < 1$ guaranteeing the analyticity of test functions is necessary for the constructions described in the rest of this section. We say that a cone W is a conic neighborhood of a cone U if W has an open projection [3] and contains U.

Definition 2.1. Let $\alpha > 1$ and U be a nonempty cone in $\mathbb{R}^k$. The Banach space $S^{0,B}_{\alpha,A}(U)$ consists of entire analytic functions on $\mathbb{C}^k$ with the finite norm

$$\|f\|^{\alpha}_{U,A,B} = \sup_{z=x+iy\in\mathbb{C}^k} |f(z)| \exp(|x/A|^{1/\alpha} - \delta_U(Bx) - |By|),$$

where $\delta_U(x) = \inf_{x'\in U} |x - x'|$ is the distance from x to U. The space $S^0_{\alpha}(U)$ is defined by the relation

$$S^0_{\alpha}(U) = \bigcup_{A,B>0,\ W\supset U} S^{0,B}_{\alpha,A}(W),$$

where W runs over all conic neighborhoods of U and the union is endowed with the inductive limit topology.

According to the above, for $U = \mathbb{R}^k$, this definition is equivalent to the initial definition of $S^0_{\alpha}(\mathbb{R}^k)$. From now on and throughout the paper, all cones in question will be supposed nonempty. As a rule, the word 'nonempty' will be omitted. In the rest of this section, we assume that the nontriviality condition $\alpha > 1$ is satisfied. If $U' \subset U$, then the space $S^0_{\alpha}(U)$ is continuously embedded into $S^0_{\alpha}(U')$. If $W \subset \mathbb{R}^k$ is a cone with open projection, then Definition 2.1 gives

$$S^0_{\alpha}(W) = \bigcup_{A,B>0} S^{0,B}_{\alpha,A}(W). \tag{2.2}$$

[3] The projection $\operatorname{Pr} W$ of a cone $W \subset \mathbb{R}^k$ is by definition the canonical image of W in the sphere $\mathbb{S}_{k-1} = (\mathbb{R}^k \setminus \{0\})/\mathbb{R}_+$; the projection of W is meant to be open in the topology of this sphere. Note that the degenerate cone $\{0\}$ is a cone with an open (empty) projection.

The following statement is an immediate consequence of Definition 2.1, formula (2.2), and the associativity property of inductive limit topologies.

Lemma 2.2. *Let U be a cone in $\mathbb{R}^k$. Then*

$$S^0_\alpha(U) = \bigcup_{W \supset U} S^0_\alpha(W), \tag{2.3}$$

where the union is taken over all conic neighborhoods of U and is endowed with the inductive limit topology.

A closed cone K is called a carrier cone of a functional $u \in S'^0_\alpha(\mathbb{R}^k)$ if u can be extended continuously to the space $S^0_\alpha(K)$. The following three basic theorems were established in [13,14].

Theorem 2.3. *The space $S^0_\alpha(\mathbb{R}^k)$ is dense in $S^0_\alpha(U)$ for any cone $U \subset \mathbb{R}^k$.*

Theorem 2.4. *If both K_1 and K_2 are carrier cones of $u \in S'^0_\alpha(\mathbb{R}^k)$, then so is $K_1 \cap K_2$.*

Theorem 2.5. *Let K_1 and K_2 be closed cones in $\mathbb{R}^k$ and $u \in S'^0_\alpha(\mathbb{R}^k)$ be carried by $K_1 \cup K_2$. Then there are $u_{1,2} \in S'^0_\alpha(\mathbb{R}^k)$ carried by $K_{1,2}$ and such that $u = u_1 + u_2$.*

Theorem 2.3 shows that the space of the functionals carried by a closed cone K is naturally identified with the space $S'^0_\alpha(K)$. It follows from Theorem 2.3 and Lemma 2.2 that a functional $u \in S'^0_\alpha(\mathbb{R}^k)$ is carried by a closed cone K if and only if u has a continuous extension to the space $S^0_\alpha(W)$ for every conic neighborhood W of K. Theorem 2.4 implies that the intersection of an arbitrary family $\{K_\omega\}_{\omega\in\Omega}$ of carrier cones of a functional $u \in S'^0_\alpha(\mathbb{R}^k)$ is again a carrier cone of u. Indeed, let W be a conic neighborhood of $K = \bigcap_{\omega\in\Omega} K_\omega$. Then by standard compactness arguments, there is a finite family $\omega_1, \ldots, \omega_n \in \Omega$ such that $\tilde{K} = \bigcap_{j=1}^n K_{\omega_j} \subset W$. By Theorem 2.4, $\tilde{K}$ is a carrier cone of u and, therefore, u has a continuous extension to $S^0_\alpha(W)$. Hence K is a carrier cone of u. In particular, every functional $u \in S'^0_\alpha(\mathbb{R}^k)$ has a uniquely defined minimal carrier cone—the intersection of all carrier cones of u.

Remark 2.6. In [13,14], only open and closed cones were considered. The space $S^0_\alpha(W)$ associated with an open cone W was defined by formula (2.2). For a closed cone K, the space $S^0_\alpha(K)$ was defined as the right-hand side of (2.3), where the union is taken over all open cones W such that $K \setminus \{0\} \subset W$. Definition 2.1 covers both these cases. Using cones with open projection instead of open cones allows treating the degenerate cone $\{0\}$ on the same footing as nondegenerate closed cones. Theorem 2.3 was actually proved in [14] only for open and closed U. This implies that Theorem 2.3 holds for cones with open projection and Lemma 2.2 ensures that it is valid for arbitrary U.

Let K, K' be closed cones in $\mathbb{R}^k$ such that $K' \subset K$. We denote by $\rho^{\alpha}_{K',K}$ the natural mapping from $S'^0_{\alpha}(K')$ to $S'^0_{\alpha}(K)$ (if $u \in S'^0_{\alpha}(K')$ then $\rho^{\alpha}_{K',K}u$ is the restriction of u to $S^0_{\alpha}(K)$). It follows from Theorems 2.3–2.5 that

(a) The mappings $\rho^{\alpha}_{K',K}$ are injective for any $K' \subset K$.
(b) If $u \in S'^0_{\alpha}(K_1 \cup K_2)$, then there are $u_{1,2} \in S'^0_{\alpha}(K_{1,2})$ such that $u = \rho^{\alpha}_{K_1, K_1\cup K_2}u_1 + \rho^{\alpha}_{K_2, K_1\cup K_2}u_2$.
(c) If $u_{1,2} \in S'^0_{\alpha}(K_{1,2})$, $K_{1,2} \subset K$, and $\rho^{\alpha}_{K_1,K}u_1 = \rho^{\alpha}_{K_2,K}u_2$, then there is a $u \in S'^0_{\alpha}(K_1 \cap K_2)$ such that $u_1 = \rho^{\alpha}_{K_1\cap K_2, K_1}u$ and $u_2 = \rho^{\alpha}_{K_1\cap K_2, K_2}u$.

Remark 2.7. Starting from the spaces $S'^0_{\alpha}(K)$, one can construct a flabby sheaf $\mathfrak{F}_{\alpha}$ on the sphere $\mathbb{S}_{k-1} = (\mathbb{R}^k \setminus \{0\})/\mathbb{R}_+$. For $Q \subset \mathbb{S}_{k-1}$, let $C(Q)$ denote the cone in $\mathbb{R}^k$ containing the origin and such that $\Pr C(Q) = Q$. For an open set $O \subset \mathbb{S}_{k-1}$, set $\mathfrak{F}_{\alpha}(O) = S'^0_{\alpha}(C(\bar{O}))/S'^0_{\alpha}(C(\partial O))$, where ∂O is the boundary of O and the bar stands for closure in $\mathbb{S}_{k-1}$. Proceeding as in Section 9.2 of the book [4], where hyperfunctions are constructed from analytic functionals, and using properties (a)–(c) reformulated in terms of closed subsets of $\mathbb{S}_{k-1}$, one can define the restriction mappings $\mathfrak{F}_{\alpha}(O_1) \to \mathfrak{F}_{\alpha}(O_2)$ for $O_1 \supset O_2$ and prove that $\mathfrak{F}_{\alpha}$ is indeed a flabby sheaf. Note however that $\mathfrak{F}_{\alpha}(\mathbb{S}_{k-1}) = S'^0_1(\mathbb{R}^k)/S'^0_1(\{0\})$. Thus, passing from the spaces $S'^0_{\alpha}(K)$ to the sheaf $\mathfrak{F}_{\alpha}$ leads to the loss of information concerning the functionals carried by the origin. Moreover, since $S'^0_{\alpha}(\{0\})$ is dense in every space $S'^0_{\alpha}(K)$, all information about the topology of these spaces is also lost.

Lemma 2.8. *Let M be a closed cone in $\mathbb{R}^k$ and P be a set of closed subcones of M such that $K_1 \cap K_2 \in P$ for any $K_1, K_2 \in P$. Suppose there is a finite subset P' of P such that $M = \bigcup_{K\in P'} K$. Then the space $S'^0_{\alpha}(M)$ is canonically isomorphic as a topological vector space to $\varinjlim_{K\in P} S'^0_{\alpha}(K)$ (the set P is meant to be naturally ordered by inclusion).*

The inductive limit in Lemma 2.8 is taken, in general, over a partially ordered but not directed set of indices. The definitions of the inductive system and inductive limit, which are usually formulated for the case of a directed set of indices, are immediately extended to this more general case. Moreover, the usual inductive limit universality property remains valid in this more general case. Precise formulations concerning such generalized inductive systems will be given in the end of this section.

Proof of Lemma 2.8. For $K \in P$, we denote by ρ^{α}_K and ι_K the canonical mapping from $S'^0_{\alpha}(K)$ to $\varinjlim_{K\in P} S'^0_{\alpha}(K)$ and the canonical embedding of $S'^0_{\alpha}(K)$ into $\oplus_{K\in P} S'^0_{\alpha}(K)$, respectively. If $K, K' \in P$ and $K' \subset K$, then we have $\rho^{\alpha}_{K',M} = \rho^{\alpha}_{K,M}\rho^{\alpha}_{K',K}$ and by the inductive limit universality property, there is a unique continuous mapping $l\colon \varinjlim_{K\in P} S'^0_{\alpha}(K) \to S'^0_{\alpha}(M)$ such that $\rho^{\alpha}_{K,M} = l\rho^{\alpha}_K$ for any $K \in P$. It follows from property (b) that l is surjective because M can be represented as a union of finitely many cones belonging to P. We now prove the injectivity of l. Let N be the subspace of

$\oplus_{K\in P} S'^0_\alpha(K)$ spanned by all vectors of the form $\imath_{K'}u - \imath_K \rho^\alpha_{K',K} u$, where $K, K' \in P$, $K' \subset K$, and $u \in S'^0_\alpha(K')$. The space $\lim\limits_{\longrightarrow K\in P} S'^0_\alpha(K)$ is by definition the quotient space $\oplus_{K\in P} S'^0_\alpha(K)/N$. It suffices to show that for any $K_1,\ldots,K_n \in P$ and every $u_1 \in S'^0_\alpha(K_1),\ldots,u_n \in S'^0_\alpha(K_n)$, the relation $\rho^\alpha_{K_1,M}u_1+\cdots+\rho^\alpha_{K_n,M}u_n = 0$ implies that $\imath_{K_1}u_1+\cdots+\imath_{K_n}u_n$ belongs to N. The proof is by induction on n. If $n=1$ and $\rho^\alpha_{K_1,M}u_1 = 0$, then by (a) we have $u_1 = 0$. We now assume $n>1$ and prove the statement supposing it holds for $n-1$. Let $K = K_1\cup\cdots\cup K_{n-1}$, $K' = K\cap K_n$, and $u = \rho^\alpha_{K_1,K}u_1+\cdots+\rho^\alpha_{K_{n-1},K}u_{n-1}$. Then we have $\rho^\alpha_{K_n,M}u_n = -\rho^\alpha_{K,M}u$ and by property (c), there is a $u' \in S'^0_\alpha(K')$ such that $u_n = \rho^\alpha_{K',K_n}u'$ and $u = -\rho^\alpha_{K',K}u'$. Let $K'_j = K_j\cap K_n$, $j=1,\ldots,n-1$. Since P is closed under finite intersections, we have $K'_j \in P$. By property (b), there are $u'_1 \in S'^0_\alpha(K'_1),\ldots,u'_{n-1}\in S'^0_\alpha(K'_{n-1})$ such that $u' = \rho^\alpha_{K'_1,K'}u'_1+\cdots+\rho^\alpha_{K'_{n-1},K'}u'_{n-1}$. We therefore obtain

$$u_n = \rho^\alpha_{K'_1,K_n}u'_1+\cdots+\rho^\alpha_{K'_{n-1},K_n}u'_{n-1}. \tag{2.4}$$

Set $v_j = u_j + \rho^\alpha_{K'_j,K_j}u'_j$, $j=1,\ldots,n-1$, and $v = \imath_{K_1}v_1+\cdots+\imath_{K_{n-1}}v_{n-1}$. By the induction hypothesis, we have $v\in N$ because

$$\rho^\alpha_{K_1,M}v_1+\cdots+\rho^\alpha_{K_{n-1},M}v_{n-1} = \rho^\alpha_{K_1,M}u_1+\cdots+\rho^\alpha_{K_n,M}u_n = 0.$$

Further, we have

$$\begin{aligned}\imath_{K_1}u_1+\cdots+\imath_{K_n}u_n &= v + [\imath_{K_n}u_n - \imath_{K'_1}u'_1 - \cdots - \imath_{K'_{n-1}}u'_{n-1}]\\ &\quad + (\imath_{K'_1}u'_1 - \imath_{K_1}\rho^\alpha_{K'_1,K_1}u'_1)\\ &\quad + \cdots + (\imath_{K'_{n-1}}u'_{n-1} - \imath_{K_{n-1}}\rho^\alpha_{K'_{n-1},K_{n-1}}u'_{n-1}).\end{aligned}$$

By definition of the space N, the terms in the round brackets belong to N and in view of (2.4) the term in the square brackets also belongs to N. Therefore, the expression in the left-hand side belongs to N and the injectivity of l is proved. It remains to show that l^{-1} is continuous. Suppose at first that the set P is finite. Since $S'^0_\alpha(K)$ are Fréchet spaces [14], $\oplus_{K\in P} S'^0_\alpha(K)$ is also a Fréchet space. By the above, N coincides with the kernel of the continuous mapping $\{u_K\}_{K\in P} \to \sum_{K\in P}\rho^\alpha_{K,M}u_K$. Therefore, N is a closed subspace of $\oplus_{K\in P} S'^0_\alpha(K)$ and $\lim\limits_{\longrightarrow K\in P} S'^0_\alpha(K)$ is a Fréchet space. The continuity of l^{-1} now follows from the open mapping theorem. If P is arbitrary, then choose a finite set P' such that $M = \bigcup_{K\in P'} K$. We can assume that P' is closed under intersections of its elements (otherwise we can add to P' all cones that are intersections of elements of P'). Let l' and m be the canonical mappings from $\lim\limits_{\longrightarrow K\in P'} S'^0_\alpha(K)$ to $S'^0_\alpha(M)$ and from $\lim\limits_{\longrightarrow K\in P'} S'^0_\alpha(K)$ to $\lim\limits_{\longrightarrow K\in P} S'^0_\alpha(K)$ respectively. Then we have $l' = lm$. By the above, l' is a topological isomorphism and, therefore, $l^{-1} = ml'^{-1}$ is continuous. The lemma is proved. $\square$

In particular, the conditions of Lemma 2.8 are satisfied if P is equal to the set $\mathcal{P}(M)$ of all nonempty closed proper subcones of M. We thus have the canonical isomorphism

$$S'^0_\alpha(M) \simeq \varinjlim_{K\in\mathcal{P}(M)} S'^0_\alpha(K). \tag{2.5}$$

We end this section by reformulating some standard definitions and facts related to inductive limits for the case of partially ordered, but not necessarily directed sets of indices. By an inductive system $\mathcal{X}$ of (locally convex topological) vector spaces indexed by a partially ordered set A, we mean the following data: [4]

(1) a family $\{\mathcal{X}(\alpha)\}_{\alpha\in A}$ of (locally convex topological) vector spaces;
(2) a family of (continuous) linear mappings $\rho^{\mathcal{X}}_{\alpha\alpha'}: \mathcal{X}(\alpha) \to \mathcal{X}(\alpha')$ defined for $\alpha \leqslant \alpha'$ and satisfying the conditions
    (i) $\rho^{\mathcal{X}}_{\alpha\alpha}$ is the identity mapping for any $\alpha \in A$;
    (ii) $\rho^{\mathcal{X}}_{\alpha\alpha''} = \rho^{\mathcal{X}}_{\alpha'\alpha''}\rho^{\mathcal{X}}_{\alpha\alpha'}$ for $\alpha \leqslant \alpha' \leqslant \alpha''$.

In other words, $\mathcal{X}$ is a covariant functor from the small category A to the category of (locally convex topological) vector spaces. Let $\iota^{\mathcal{X}}_\alpha$ denote the canonical embedding of $\mathcal{X}(\alpha)$ in $\oplus_{\alpha'\in A}\mathcal{X}(\alpha')$. The inductive limit $\varinjlim_{\alpha\in A} \mathcal{X}(\alpha)$ (or simply $\varinjlim \mathcal{X}$) is by definition the quotient space $[\oplus_{\alpha\in A}\mathcal{X}(\alpha)]/N^{\mathcal{X}}$, where $N^{\mathcal{X}}$ is the subspace of $\oplus_{\alpha\in A}\mathcal{X}(\alpha)$ spanned by all elements of the form $\iota^{\mathcal{X}}_\alpha x - \iota^{\mathcal{X}}_{\alpha'}\rho^{\mathcal{X}}_{\alpha\alpha'}x$, $x \in \mathcal{X}(\alpha)$. The canonical mapping $\rho^{\mathcal{X}}_\alpha: \mathcal{X}(\alpha) \to \varinjlim \mathcal{X}$ is defined by the relation $\rho^{\mathcal{X}}_\alpha = j^{\mathcal{X}}\iota^{\mathcal{X}}_\alpha$, where $j^{\mathcal{X}}$ is the canonical surjection of $\oplus_{\alpha\in A}\mathcal{X}(\alpha)$ onto $\varinjlim \mathcal{X}$. As usual, we have the following inductive limit universality property:

> Let E be a (locally convex topological) vector space and h_α be(continuous) linear mappings from $\mathcal{X}(\alpha)$ to E such that $h_{\alpha'}\rho^{\mathcal{X}}_{\alpha\alpha'} = h_\alpha$ for any $\alpha \leqslant \alpha'$. Then there is a unique (continuous) linear mapping $h: \varinjlim \mathcal{X} \to E$ such that $h_\alpha = h\rho^{\mathcal{X}}_\alpha$ for any $\alpha \in A$.

Remark 2.9. Although the above definitions are quite standard, the properties of such generalized inductive systems may be very different from those of inductive systems indexed by directed sets. For example, canonical mappings $\rho^{\mathcal{X}}_\alpha$ may be not injective even if all connecting morphisms $\rho^{\mathcal{X}}_{\alpha\alpha'}$ are injective. Indeed, let E be a vector space and A be the four-element set $\{\alpha, \beta, \gamma, \delta\}$ with the order defined by the relations $\alpha \leqslant \gamma$, $\alpha \leqslant \delta$, $\beta \leqslant \gamma$, and $\beta \leqslant \delta$. We define the inductive system $\mathcal{X}$ setting $\mathcal{X}(\alpha) = \mathcal{X}(\beta) =$

[4] In the rest of this section and in Section 5, where abstract inductive systems are discussed, the Greek letter α is systematically used to denote an element of a partially ordered set A and has nothing in common with the index of Gelfand–Shilov spaces.

$\mathcal{X}(\gamma) = \mathcal{X}(\delta) = E$ and $\rho^{\mathcal{X}}_{\alpha\gamma} = -\rho^{\mathcal{X}}_{\alpha\delta} = \rho^{\mathcal{X}}_{\beta\gamma} = \rho^{\mathcal{X}}_{\beta\delta} = \mathrm{id}_E$, where id_E is the identity mapping. Fix $x \in E$ and set $z_1 = \iota^{\mathcal{X}}_\alpha x - \iota^{\mathcal{X}}_\gamma x$, $z_2 = \iota^{\mathcal{X}}_\alpha x + \iota^{\mathcal{X}}_\delta x$, $z_3 = \iota^{\mathcal{X}}_\beta x - \iota^{\mathcal{X}}_\delta x$, and $z_4 = \iota^{\mathcal{X}}_\gamma x - \iota^{\mathcal{X}}_\beta x$. Obviously $z_1, \ldots, z_4 \in N^{\mathcal{X}}$ and, therefore, $\iota^{\mathcal{X}}_\alpha x = (z_1+z_2+z_3+z_4)/2$ belongs to $N^{\mathcal{X}}$. This means that $\rho^{\mathcal{X}}_\alpha x = 0$ for any $x \in E$.

3. Basic definitions and formulations of main results

We now extend the constructions of the preceding section to the case $\alpha = 1$ which is of primary interest to us. By analogy with Definition 2.1, we introduce suitable test function spaces associated with cones in $\mathbb{R}^k$.

Definition 3.1. Let U be a cone in $\mathbb{R}^k$. The Banach space $S^{0,B}_{1,A}(U)$ consists of entire analytic functions on $\mathbb{C}^k$ with the finite norm

$$\|f\|_{U,A,B} = \sup_{z=x+iy\in\mathbb{C}^k} |f(z)| \exp(|x/A| - \delta_U(Bx) - |By|),$$

where $\delta_U(x) = \inf_{x'\in U} |x - x'|$ is the distance from x to U. The space $S^0_1(U)$ is defined by the relation

$$S^0_1(U) = \bigcup_{A,B>0,\, W\supset U} S^{0,B}_{1,A}(W),$$

where W runs over all conic neighborhoods of U and the union is endowed with the inductive limit topology.

For $U = \mathbb{R}^k$, Definition 3.1 is equivalent to the standard definition of $S^0_1(\mathbb{R}^k)$ given in [2]. Therefore, the space $S^0_1(U)$ is trivial for $U = \mathbb{R}^k$. A sufficient condition for the nontriviality of $S^0_1(U)$ will be given in Lemma 3.3. Representation (2.2) for the spaces $S^0_\alpha(W)$ associated with cones with open projection and Lemma 2.2 obviously remain valid for $\alpha = 1$ (in fact, one can show that formula (2.2) holds for $\alpha = 1$ even without the assumption that W has an open projection, but we shall not prove this fact here). As shown in [14], $S^0_\alpha(U)$ with $\alpha > 1$ are DFS-spaces (we recall that DFS-spaces are, by definition, the inductive limits of injective compact sequences of locally convex spaces). In particular, they (and their duals) are reflexive, complete, and Montel spaces [6]. The following lemma shows that the spaces $S^0_1(U)$ enjoy the same nice topological properties.

Lemma 3.2. *The space $S^0_1(U)$ is DFS for any cone $U \subset \mathbb{R}^k$.*

Proof. It suffices to show that the inclusion mapping $S^{0,B}_{1,A}(U) \to S^{0,B'}_{1,A'}(U)$ is compact for any $A' > A$, $B' > B$ and every cone $U \subset \mathbb{R}^k$. Let $\{f_m\}_{m\in\mathbb{N}}$ be a sequence of functions belonging to the unit ball of the space $S^{0,B}_{1,A}(U)$. By Montel's theorem, this sequence contains a subsequence $\{f_{m_n}\}$ which converges uniformly on compact sets in $\mathbb{C}^k$ to an entire analytic function f. To prove the statement, it suffices to show that the sequence $\{f_{m_n}\}$ converges to f in $S^{0,B'}_{1,A'}(U)$. Set $\varrho_{U,A,B}(x+iy) = -|x/A| + B\delta_U(x) + B|y|$, $x, y \in \mathbb{R}^k$. Since $\|f_{m_n}\|_{U,A,B} \leqslant 1$, we have $|f_{m_n}(z)| \leqslant e^{\varrho_{U,A,B}(z)}$. Passing to the limit $n \to \infty$ in this inequality, we conclude that $f \in S^{0,B}_{1,A}(U)$ and $\|f\|_{U,A,B} \leqslant 1$. Further, for any $R > 0$, we have

$$\begin{aligned}\|f - f_{m_n}\|_{U,A',B'} &\leqslant e^{R/A'} \sup_{|z|\leqslant R} |f(z) - f_{m_n}(z)| \\ &\qquad + \|f - f_{m_n}\|_{U,A,B} \sup_{|z|>R} e^{\varrho_{U,A,B}(z) - \varrho_{U,A',B'}(z)} \\ &\leqslant e^{R/A'} \sup_{|z|\leqslant R} |f(z) - f_{m_n}(z)| + 2e^{-LR},\end{aligned}$$

where $L = \min((A'-A)/A'A, B'-B)$. Choose $R(\varepsilon)$ and $n(\varepsilon)$ such that $2e^{-LR(\varepsilon)} < \varepsilon/2$ and $e^{R(\varepsilon)/A'} \sup_{|z|\leqslant R(\varepsilon)} |f(z) - f_{m_n}(z)| < \varepsilon/2$ for any $n \geqslant n(\varepsilon)$. Then $\|f - f_{m_n}\|_{U,A',B'} < \varepsilon$ for any $n \geqslant n(\varepsilon)$. The lemma is proved. $\square$

Lemma 3.3. *Let U be a cone in $\mathbb{R}^k$. If U is a proper cone, then the space $S^0_1(U)$ is nontrivial. If U contains a straight line, then $S^0_1(U)$ is trivial.*

Proof. Let U be a proper cone and l be a linear functional on $\mathbb{R}^k$ such that $\bar{U} \setminus \{0\} \subset \{x \in \mathbb{R}^k \mid l(x) > 0\}$. Then $S^0_1(U)$ contains the function $f(z) = e^{-l(z)}$ and, therefore, is nontrivial. Now let U contain a straight line and $f \in S^0_1(U)$. Let W be a conic neighborhood of U such that $f \in S^{0,B}_{1,A}(W)$ for some $A, B > 0$ and $\tilde{W}$ be the union of all straight lines contained in W. Clearly, $\tilde{W}$ is a cone with a nonempty interior. For $x \in \tilde{W} \setminus \{0\}$ and $\tau \in \mathbb{C}$, we set $g(\tau) = f(\tau x)$. It easily follows from Definition 3.1 that $g \in S^{0,B|x|}_{1,A/|x|}(\mathbb{R})$ and hence $g \equiv 0$. Therefore, $f(x) = g(1) = 0$, i.e., f vanishes on $\tilde{W}$. By the uniqueness theorem, we conclude that f is identically zero. The lemma is proved. $\square$

Lemma 3.3 suggests that we can try to define the desired "nontrivialization" $\mathcal{U}(\mathbb{R}^k)$ of the space $S'^0_1(\mathbb{R}^k)$ (and, more generally, of the space $S'^0_1(M)$ over an arbitrary closed cone M) as the right-hand side of (2.5) with $\alpha = 1$. We then arrive at the following definition.

Definition 3.4. Let M be a closed cone in $\mathbb{R}^k$. The space $\mathcal{U}(M)$ is defined to be the inductive limit $\varinjlim_{K\in\mathcal{P}(M)} S'^0_1(K)$, where $\mathcal{P}(M)$ is the set of all nonempty proper closed cones contained in M. The elements of $\mathcal{U}(\mathbb{R}^k)$ are called ultrafunctionals. A closed cone K is said to be a carrier cone of an ultrafunctional u if the latter belongs to the image of the canonical mapping from $\mathcal{U}(K)$ to $\mathcal{U}(\mathbb{R}^k)$.

In this definition, the set $\mathcal{P}(M)$ is meant to be ordered by inclusion and the inductive limit is taken with respect to the natural morphisms $\rho_{K',K}: S_1'^0(K') \to S_1'^0(K)$ which are defined for $K' \subset K$ and map the functionals belonging to $S_1'^0(K')$ into their restrictions to the space $S_1^0(K)$. The canonical mappings from $\mathcal{U}(K')$ to $\mathcal{U}(K)$ will be denoted by $\rho^{\mathcal{U}}_{K',K}$. Note that if K is a proper closed cone, then $\mathcal{U}(K)$ is canonically isomorphic to $S_1'^0(K)$.

We shall see that $\mathcal{U}(\mathbb{R}^k)$ is Fourier-isomorphic to the space $\mathcal{B}(\mathbb{R}^k)$ which is known to have no natural topology. Therefore, the following result is by no means surprising.

Lemma 3.5. *Let M be a closed cone in $\mathbb{R}^k$ containing a straight line. Then the inductive limit topology on $\mathcal{U}(M)$ is trivial* (*i.e.*, *$\mathcal{U}(M)$ and $\emptyset$ are the only open sets*).

Proof. It suffices to prove that any continuous linear functional l on $\mathcal{U}(M)$ is equal to zero. For $K \in \mathcal{P}(M)$, we denote by ρ_K the canonical mapping from $S_1'^0(K)$ to $\mathcal{U}(M)$. The continuity of l means that the functional $l\rho_K$ is continuous on $S_1'^0(K)$ for any $K \in \mathcal{P}(M)$. By Lemma 3.2, the space $S_1^0(K)$ is reflexive for any cone K. Hence for any $K \in \mathcal{P}(M)$, there is a function $f_K \in S_1^0(K)$ such that $l\rho_K u = u(f_K)$ for every $u \in S_1'^0(K)$. If $K' \subset K$, then we have

$$u(f_{K'}) = l\rho_{K'} u = l\rho_K \rho_{K',K} u = (\rho_{K',K} u)(f_K) = u(f_K), \quad u \in S_1'^0(K')$$

and, consequently, $f_{K'} = f_K$. Choosing K' equal to the degenerate cone $\{0\}$, we see that $f_K = f_{\{0\}}$ does not depend on $K \in \mathcal{P}(M)$ and, therefore, belongs to the space $L = \bigcap_{K \in \mathcal{P}(M)} S_1^0(K)$. Let $K_1, \ldots, K_n \in \mathcal{P}(M)$ be such that $M = K_1 \cup \cdots \cup K_n$. Since $\delta_{K_1 \cup \cdots \cup K_n}(x) = \min(\delta_{K_1}(x), \ldots, \delta_{K_n}(x))$ for any $x \in \mathbb{R}^k$, it follows from Definition 3.1 that $S_1^0(M) = S_1^0(K_1 \cup \cdots \cup K_n) = S_1^0(K_1) \cap \cdots \cap S_1^0(K_n)$. Hence $L \subset S_1^0(M)$ is trivial by Lemma 3.3 and $f_K = 0$ for any $K \in \mathcal{P}(M)$. This means that $l\rho_K = 0$ for every $K \in \mathcal{P}(M)$ and, therefore, $l = 0$. The lemma is proved. $\square$

Thus, there is, in general, no natural way to define a reasonable topology on $\mathcal{U}(K)$. Because of this, we do not endow these spaces with any topology and consider them only from algebraic point of view. One of the main results of this paper is that the ultrafunctionals have the same localization properties as the analytic functionals belonging to $S_\alpha'^0(\mathbb{R}^k)$ with $\alpha > 1$. More precisely, the following analog of Theorems 2.3–2.5 are valid.

Theorem 3.6. *The natural mapping $\rho^{\mathcal{U}}_{K',K}: \mathcal{U}(K') \to \mathcal{U}(K)$ is injective for any closed cones K and K' such that $K' \subset K$.*

Theorem 3.7. *Let $\{K_\omega\}_{\omega\in\Omega}$ be an arbitrary family of carrier cones of an ultrafunctional u. Then $\bigcap_{\omega\in\Omega} K_\omega$ is also a carrier cone of u.*

Theorem 3.8. *Let K_1 and K_2 be closed cones in $\mathbb{R}^k$ and an ultrafunctional u be carried by $K_1 \cup K_2$. Then there are $u_{1,2} \in \mathcal{U}(\mathbb{R}^k)$ carried by $K_{1,2}$ such that $u = u_1 + u_2$.*

These theorems will be proved in Section 5.

Remark 3.9. The spaces $\mathcal{U}(K)$ determine a flabby sheaf on the sphere $\mathbb{S}_{k-1}$ in the same way as the spaces $S'^0_\alpha(K)$ (see Remark 2.7).

For $u \in S'^0_1(K)$, one can in a standard way define the operators of partial differentiation and multiplication by an entire function g of infra-exponential type (i.e., satisfying the bound $|g(z)| \leqslant C_\varepsilon e^{\varepsilon|z|}$ for every $\varepsilon > 0$):

$$\partial u/\partial x_j(f) = -u(\partial f/\partial x_j), \quad (gu)(f) = u(gf), \quad f \in S^0_1(K), \quad j = 1, \ldots, k.$$

These operations are obviously compatible with the connecting morphisms $\rho_{K',K}$ and, therefore, can be lifted to the spaces $\mathcal{U}(K)$ over arbitrary closed cones. Let $\alpha > 1$. The natural mappings from $S'^0_\alpha(K)$ to $S'^0_1(K)$ taking functionals in $S'^0_\alpha(K)$ to their restrictions to $S^0_1(K)$ are compatible with the connecting morphisms $\rho^\alpha_{K',K}$ and $\rho_{K',K}$ and in view of (2.5) determine a mapping from $S'^0_\alpha(K)$ to $\mathcal{U}(K)$ for any closed cone K. Below we shall see that these mappings are injective, i.e., the space $S'^0_\alpha(K)$ can be regarded as a subspace of $\mathcal{U}(K)$.

We now describe the construction of the Fourier transformation of hyperfunctions. As a first step, we consider the Laplace transformation of analytic functionals on the spaces $S^0_1(K)$ over convex proper closed cones. In the rest of this section, we identify $S'^0_1(K)$ with $\mathcal{U}(K)$ for $K \in \mathcal{P}(\mathbb{R}^k)$. For brevity, the natural embeddings $\rho^{\mathcal{U}}_{K,\mathbb{R}^k}:\mathcal{U}(K) \to \mathcal{U}(\mathbb{R}^k)$ and $\rho^\alpha_{K,\mathbb{R}^k}: S'^0_\alpha(K) \to S'^0_\alpha(\mathbb{R}^k)$ will be denoted by σ_K and σ^α_K respectively. Let $\langle\cdot,\cdot\rangle$ be a symmetric nondegenerate bilinear form on $\mathbb{R}^k$. Given a cone $U \subset \mathbb{R}^k$, we denote by U^* its dual cone $\{x \in \mathbb{R}^k \mid \langle x, \eta\rangle \geqslant 0 \text{ for any } \eta \in U\}$. Note that U^* is always closed and convex. A cone U is proper if and only if U^* has a nonempty interior. If V is an open cone, then the function $e^{i\langle\cdot,\zeta\rangle}$ belongs to $S^0_1(V^*)$ for every $\zeta \in T^V \stackrel{\text{def}}{=} \mathbb{R}^k + iV$. Given an open set $O \subset \mathbb{R}^k$, we denote by $\mathcal{A}(O)$ the space of functions analytic in an open set $T^O \subset \mathbb{C}^k$. The space $\mathcal{A}(O)$ is endowed with the topology of uniform convergence on compact subsets of T^O.

Theorem 3.10. *Let K be a convex proper closed cone in $\mathbb{R}^k$ and $V = \operatorname{int} K^*$. For any $u \in \mathcal{U}(K)$, the function $\zeta \to u(e^{i\langle\cdot,\zeta\rangle})$ is analytic in T^V. The linear mapping $\mathcal{L}_K:\mathcal{U}(K) \to \mathcal{A}(V)$ taking $u \in \mathcal{U}(K)$ to this function is a topological isomorphism.*

The proof of this theorem will be given in Section 4. The function $\mathcal{L}_V u$ is called the Laplace transform of u. By definition, we have

$$(\mathcal{L}_K u)(\zeta) = u(e^{i\langle\cdot,\zeta\rangle}), \quad \zeta \in \mathbb{R}^k + i \operatorname{int} K^*. \tag{3.1}$$

For an open cone $V \subset \mathbb{R}^k$, we denote by $\mathbf{b}_V$ the linear mapping taking functions in $\mathcal{A}(V)$ to their boundary values in the space of hyperfunctions $\mathcal{B}(\mathbb{R}^k)$. Let $K, K' \subset \mathbb{R}^k$ be proper convex closed cones, $V = \operatorname{int} K^*$, and $V' = \operatorname{int} K'^*$. If $K' \subset K$, then $\mathcal{L}_K \rho_{K',K} u$ is the restriction of $\mathcal{L}_{K'} u$ to T^V for any $u \in S_1'^0(K')$. This implies that $\mathbf{b}_{V'} L_{K'} = \mathbf{b}_V \mathcal{L}_K \rho^{\mathcal{U}}_{K',K}$, and by the inductive limit universality property,[5] there is a unique mapping $\mathcal{F}: \mathcal{U}(\mathbb{R}^k) \to \mathcal{B}(\mathbb{R}^k)$ such that

$$\mathcal{F}\sigma_{V^*} = \mathbf{b}_V \mathcal{L}_{V^*} \tag{3.2}$$

for any open convex cone $V \subset \mathbb{R}^k$.

Theorem 3.11. *The operator $\mathcal{F}$ maps $\mathcal{U}(\mathbb{R}^k)$ isomorphically onto $\mathcal{B}(\mathbb{R}^k)$.*

This theorem will be proved in Section 6. The operator $\mathcal{F}$ is naturally interpreted as the inverse Fourier transformation of hyperfunctions. Indeed, for any $j = 1, \dots, k$ and $u \in \mathcal{U}(\mathbb{R}^k)$, we obviously have the standard relations

$$\mathcal{F}\left[\frac{\partial u}{\partial x_j}\right](\xi) = -i\xi_j[\mathcal{F}u](\xi), \quad \mathcal{F}\left[x_j u\right](\xi) = -i\frac{\partial[\mathcal{F}u]}{\partial \xi_j}.$$

Moreover, the restriction of $\mathcal{F}^{-1}$ to ultradistributions of the class $S_0'^\alpha(\mathbb{R}^k)$ coincides with the ordinary Fourier transformation determined via duality by the Fourier transformation of test functions. More precisely, let $\alpha > 1$ and the Fourier transformation $\hat{f}$ of a test function $f \in S_0^\alpha(\mathbb{R}^k)$ be defined by the relation $\hat{f}(x) = \int f(\xi) e^{\langle x,\xi\rangle}\, \mathrm{d}\xi$. As mentioned in Section 2, the mapping $f \to \hat{f}$ is a topological isomorphism from $S_0^\alpha(\mathbb{R}^k)$ onto $S_\alpha^0(\mathbb{R}^k)$. Let $\mathcal{F}^\alpha$ denote its dual mapping acting on generalized functions. Then we have

$$\mathcal{F}e^\alpha = i^\alpha \mathcal{F}^\alpha, \tag{3.3}$$

where $e^\alpha: S_\alpha'^0(\mathbb{R}^k) \to \mathcal{U}(\mathbb{R}^k)$ and $i^\alpha: S_0'^\alpha(\mathbb{R}^k) \to \mathcal{B}(\mathbb{R}^k)$ are canonical mappings (see [7] for the construction of the natural embedding of ultradistributions into the space of hyperfunctions). To prove (3.3), we recall some results concerning the Laplace transformation of analytic functionals belonging to the spaces $S_\alpha'^0$ with $\alpha > 1$. For an

[5] Note that in the definition of $\mathcal{U}(\mathbb{R}^k)$, it suffices to take the inductive limit over all proper *convex* closed cones in $\mathbb{R}^k$ because the convex hull of a proper closed cone is again a proper closed cone.

open cone V, we denote by $\mathcal{A}^\alpha(V)$ the Fréchet space consisting of functions analytic in T^V and having the finite norms

$$|||\mathbf{v}|||_{V',\varepsilon,R} = \sup_{\zeta\in T^{V'},\, |\zeta|\leqslant R} |\mathbf{v}(\zeta)| \exp[-\varepsilon|\eta|^{-1/(\alpha-1)}], \quad \eta = \operatorname{Im}\zeta$$

for any $\varepsilon, R > 0$ and every compact subcone V' of V. The following result has been established in [14].

Theorem 3.12. *Let $\alpha > 1$, K be a convex proper closed cone in $\mathbb{R}^k$, and $V = \operatorname{int} K^*$. For any $u \in S'^0_\alpha(K)$, the function $T^V \ni \zeta \to u(e^{i\langle\cdot,\zeta\rangle})$ belongs to $\mathcal{A}^\alpha(V)$. The linear mapping $\mathcal{L}^\alpha_K: S'^0_\alpha(K) \to \mathcal{A}^\alpha(V)$ taking $u \in S'^0_\alpha(K)$ to this function is a topological isomorphism. The function $(\mathcal{L}^\alpha_K u)(\cdot + i\eta)$ tends to $\mathcal{F}^\alpha\sigma^\alpha_K u$ in the topology of $S'^\alpha_0(\mathbb{R}^k)$ as $\eta \to 0$ inside a fixed compact subcone V' of V.*

This theorem implies the existence, for every open convex cone V, of the continuous boundary value operator $\mathbf{b}^\alpha_V: \mathcal{A}^\alpha(V) \to S'^\alpha_0(\mathbb{R}^k)$ satisfying the relation

$$\mathcal{F}^\alpha\sigma^\alpha_{V*} = \mathbf{b}^\alpha_V\mathcal{L}^\alpha_{V*}. \tag{3.4}$$

Let j^α_V be the inclusion of $\mathcal{A}^\alpha(V)$ into $\mathcal{A}(V)$ and e^α_K be the canonical mapping from $S'^0_\alpha(K)$ to $\mathcal{U}(K)$ (in particular, $e^\alpha_{\mathbb{R}^k} = e^\alpha$). By definition of the mappings e^α_K, $\mathcal{L}^\alpha_K$, and $\mathcal{L}_K$, we have the relations $j^\alpha_V\mathcal{L}^\alpha_{V*} = \mathcal{L}_{V*}e^\alpha_{V*}$ and $\sigma_K e^\alpha_K = e^\alpha\sigma^\alpha_K$ for any open convex cone V and every closed cone K. Theorem 11.5 of [7] ensures that for an open convex cone V, the boundary values of functions in $\mathcal{A}^\alpha(V)$ in the sense of ultradistributions coincide with those in the sense of hyperfunctions. This means that $i^\alpha\mathbf{b}^\alpha_V = \mathbf{b}_V j^\alpha_V$. It follows from these relations and formulas (3.2) and (3.4) that

$$i^\alpha\mathcal{F}^\alpha\sigma^\alpha_{V*} = i^\alpha\mathbf{b}^\alpha_V\mathcal{L}^\alpha_{V*} = \mathbf{b}_V j^\alpha_V\mathcal{L}^\alpha_{V*} = \mathbf{b}_V\mathcal{L}_{V*}e^\alpha_{V*} = \mathcal{F}\sigma_{V*}e^\alpha_{V*} = \mathcal{F}e^\alpha\sigma^\alpha_{V*}$$

for any open convex cone V. Relation (3.3) now follows from the inductive limit universality property.

Lemma 3.13. *The canonical mapping $e^\alpha_K: S'^0_\alpha(K) \to \mathcal{U}(K)$ is injective for any closed cone $K \subset \mathbb{R}^k$.*

Proof. For $K = \mathbb{R}^k$, the statement follows from (3.3) because the Fourier operator $\mathcal{F}^\alpha$ is an isomorphism and the canonical mapping $i^\alpha: S'^\alpha_0(\mathbb{R}^k) \to \mathcal{B}(\mathbb{R}^k)$ is injective by Theorem 7.5 of [7]. The injectivity of e^α_K for an arbitrary K now follows from the relation $\sigma_K e^\alpha_K = e^\alpha\sigma^\alpha_K$ and the injectivity of the natural mapping $\sigma^\alpha_K: S'^0_\alpha(K) \to S'^0_\alpha(\mathbb{R}^k)$ ensured by Theorem 2.3. The lemma is proved. $\square$

We end this section by establishing the connection between the analytic wave front set (singular spectrum) of hyperfunctions and of their Fourier transforms.

Lemma 3.14. *Let an ultrafunctional u be carried by a closed convex proper cone $K \subset \mathbb{R}^k$ and let $f = \mathcal{F}u$. Then the analytic wave front set $WF_A(f)$ of the hyperfunction f satisfies the relation*

$$WF_A(f) \subset \mathbb{R}^k \times (K \setminus \{0\}).$$

Proof. Theorem 9.3.3 of [4] implies that $WF_A(\mathbf{b}_V \mathbf{v}) \subset \mathbb{R}^k \times (V^* \setminus \{0\})$ for any connected open cone V and every $\mathbf{v} \in \mathcal{A}(V)$. Hence the assertion of the lemma follows because by definition of the Fourier operator $\mathcal{F}$, we have $f = \mathbf{b}_{\mathrm{int}\, K^*} \mathcal{L}_K \sigma_K u$. $\square$

Lemma 3.14 strengthens analogous results for tempered distributions and ultradistributions given by Lemma 8.4.17 of [4] and Lemma 2 of [15], respectively.

4. Spaces $S_1^0(K)$ over proper cones

In this section, we show that the properties (a)–(c) listed in Section 2 hold also for the spaces $S_1^{\prime 0}(K)$ and the mappings $\rho_{K', K}$ provided that all involved cones are proper. The verification of these properties constitutes the "functional analytic" part of the proof of Theorems 3.6–3.8. In the end of this section, we prove Theorem 3.10 describing the Laplace transformation of ultrafunctionals carried by proper convex closed cones.

As above, let $\langle \cdot, \cdot \rangle$ be a symmetric nondegenerate bilinear form on $\mathbb{R}^k$. For any $x, y \in \mathbb{R}^k$, we have $|\langle x, y\rangle| \leqslant a|x||y|$, where

$$a = \sup_{|x|,\, |y| \leqslant 1} |\langle x, y\rangle|. \tag{4.1}$$

Lemma 4.1. *Let $A, B > 0$, U be a cone in $\mathbb{R}^k$, and W be a conic neighborhood of U. Suppose $\eta \in \mathbb{R}^k$ is such that $|\eta| < 1/Aa$, where a is given by (4.1). Then the function $e^{\langle \cdot, \eta\rangle} f$ belongs to $S_1^0(U)$ for any $f \in S_{1,A}^{0,B}(W)$ and the mapping $f \to e^{\langle \cdot, \eta\rangle} f$ from $S_{1,A}^{0,B}(W)$ to $S_1^0(U)$ is continuous.*

Proof. Let $f \in S_{1,A}^{0,B}(W)$ and $\eta \in \mathbb{R}^k$ be such that $|\eta| < 1/Aa$. Then

$$|e^{\langle z, \eta\rangle} f(z)| \leqslant \|f\|_{U,A,B} \exp[-(1/A - a|\eta|)|x| + B\delta_U(x) + B|y|], \quad z = x + iy.$$

Therefore, $e^{\langle \cdot, \eta\rangle} f \in S_{1,A'}^{0,B}(W)$, where $A' = A/(1 - Aa|\eta|)$, and the mapping $f \to e^{\langle \cdot, \eta\rangle} f$ from $S_{1,A}^{0,B}(W)$ to $S_{1,A'}^{0,B}(W)$ is continuous. It remains to note that the space $S_{1,A'}^{0,B}(W)$ is continuously embedded into $S_1^0(U)$. $\square$

Corollary 4.2. *Let $A, B > 0$ and U be a cone in $\mathbb{R}^k$. Then for every $f \in S_1^0(U)$, there is an $\varepsilon > 0$ such that $fe^{\langle\cdot,\eta\rangle} \in S_1^0(U)$ for any $\eta \in \mathbb{R}^k$ with $|\eta| < \varepsilon$.*

Let U be a cone in $\mathbb{R}^k$, $\alpha \geqslant 1$, and $\eta \in \operatorname{int} U^*$. We denote by $M^\eta_{\alpha, U}$ the continuous mapping $f \to fe^{-\langle\cdot,\eta\rangle}$ from $S_\alpha^0(U)$ to $S_1^0(U)$.

Lemma 4.3. *Let U, U' be nonempty proper cones in $\mathbb{R}^k$ such that $U' \subset U$. Then the space $S_1^0(U)$ is dense in the space $S_1^0(U')$.*

Proof. Fix $\alpha > 1$ and let $f \in S_1^0(U')$. By Corollary 4.2, there is an $\eta \in \operatorname{int} U^*$ such that $fe^{\langle\cdot,\eta\rangle} \in S_1^0(U')$. This means that f belongs to the image $\operatorname{Im} M^\eta_{\alpha, U'}$ of the mapping $M^\eta_{\alpha, U'}$. It follows from Theorem 2.3 that $S_\alpha^0(U)$ is dense in $S_\alpha^0(U')$. Since the image of the closure of a set under a continuous mapping is contained in the closure of its image, we have the inclusions $\operatorname{Im} M^\eta_{\alpha, U'} \subset \overline{\operatorname{Im} M^\eta_{\alpha, U}} \subset \overline{S_1^0(U)}$, where the bar stands for closure in $S_1^0(U')$. This implies that $f \in \overline{S_1^0(U)}$. Since $f \in S_1^0(U')$ is arbitrary, the lemma is proved. □

Corollary 4.4. *Let K, K' be closed proper cones in $\mathbb{R}^k$ such that $K' \subset K$. Then the natural mapping $\rho_{K', K}: S_1'^0(K') \to S_1'^0(K)$ is injective.*

Corollary 4.5. *Let U be a cone in $\mathbb{R}^k$ and U' be a proper cone containing a conic neighborhood of U. A functional $u \in S_1'^0(U')$ has a continuous extension to $S_1^0(U)$ if and only if u can be extended to every space $S_1^0(W)$, where W is a conic neighborhood of U contained in U'.*

Proof. Only the sufficiency part of the statement needs proving. If $W \subset U'$ is a conic neighborhood of U, we denote by u_W the extension of u to $S_1^0(W)$. By Lemma 4.3, the functionals u_W are uniquely defined and are compatible with the inclusion mappings (i.e., if $U \subset W \subset W' \subset U'$, then $u_{W'}$ is the restriction of u_W to $S_1^0(W')$). As mentioned in Section 3, Lemma 2.2 remains valid for $\alpha = 0$. Moreover, the union in (2.3) obviously can be taken only over conic neighborhoods of U contained in U'. The functionals u_W therefore determine a functional $\tilde{u} \in S_1'^0(U)$ such that u_W are restrictions of $\tilde{u}$ to $S_1^0(W)$. Since u_W are extensions of u, we conclude that $\tilde{u}$ is an extension of u and the corollary is proved. □

Lemma 4.6. *Let K_1 and K_2 be nonempty proper closed cones in $\mathbb{R}^k$ such that $K_1 \cup K_2$ is a proper cone. Then for every $f \in S_1^0(K_1 \cap K_2)$, there are $f_{1,2} \in S_1^0(K_{1,2})$ such that $f = f_1 + f_2$.*

Proof. Let $A, B > 0$ be such that $f \in S^{0,B}_{1,A}(K_1 \cap K_2)$. Fix $\alpha > 1$ and choose $\eta \in \mathbb{R}^k$ such that $|\eta| \leqslant 1/Aa$ and $\eta \in \operatorname{int}(K_1 \cup K_2)^*$. Then the function $g = fe^{\langle\cdot,\eta\rangle}$ belongs to

$S^0_1(K_1 \cap K_2)$ and, consequently, to $S^0_\alpha(K_1 \cap K_2)$. As shown in [13] (see also Lemma 1 of [12]), there are $g_{1,2} \in S^0_\alpha(K_{1,2})$ such that $g = g_1 + g_2$. Set $f_{1,2} = g_{1,2}e^{-\langle \cdot,\eta\rangle}$. Then $f_{1,2} \in S^0_1(K_{1,2})$ and $f = f_1 + f_2$. The lemma is proved. □

Lemma 4.7. *Let K_1 and K_2 be closed cones in $\mathbb{R}^k$. Then for every $u \in S'^0_1(K_1 \cup K_2)$, one can find $u_{1,2} \in S'^0_1(K_{1,2})$ such that $u = \rho_{K_1,\, K_1\cup K_2}u_1 + \rho_{K_2,\, K_1\cup K_2}u_2$.*

Proof. Let $l\colon S^0_1(K_1 \cup K_2) \to S^0_1(K_1) \oplus S^0_1(K_2)$ and $m\colon S^0_1(K_1) \oplus S^0_1(K_2) \to S^0_1(K_1 \cap K_2)$ be the continuous linear mappings taking f to (f, f) and (f_1, f_2) to $f_1 - f_2$, respectively. The mapping l has a closed image because by Definition 3.1, we have $S^0_1(K_1) \cap S^0_1(K_2) = S^0_1(K_1 \cup K_2)$ and, therefore, $\mathrm{Im}\, l = \mathrm{Ker}\, m$. In view of Lemma 3.2 this implies that the space $\mathrm{Im}\, l$ is DFS.[6] By the open mapping theorem, the linear functional $(f, f) \to u(f)$ is continuous on $\mathrm{Im}\, l$ and by the Hahn–Banach theorem, there exists a continuous extension v of this functional to the whole of $S^0_1(K_1) \oplus S^0_1(K_2)$. Let u_1 and u_2 be the restrictions of v to $S^0_1(K_1)$ and $S^0_1(K_2)$, respectively. Then for any $f \in S^0_1(K_1 \cup K_2)$, we have $u(f) = v(f, f) = u_1(f) + u_2(f)$. This means that $u = \rho_{K_1,\, K_1\cup K_2}u_1 + \rho_{K_2,\, K_1\cup K_2}u_2$ and the lemma is proved. □

Lemma 4.8. *Let K_1 and K_2 be proper closed cones in $\mathbb{R}^k$ such that $K_1 \cup K_2$ is a proper cone. Let $u_{1,2} \in S'^0_1(K_{1,2})$ be such that $\rho_{K_1,\, K_1\cup K_2}u_1 = \rho_{K_2,\, K_1\cup K_2}u_2$. Then there is a $u \in S'^0_1(K_1 \cap K_2)$ such that $u_1 = \rho_{K_1\cap K_2,\, K_1}\, u$ and $u_2 = \rho_{K_1\cap K_2,\, K_2}\, u$.*

Proof. Let the mappings l and m be as in the proof of Lemma 4.7. By Lemma 4.6, the mapping m is surjective and by the open mapping theorem, $S^0_1(K_1 \cap K_2)$ is topologically isomorphic to the quotient space $(S^0_1(K_1) \oplus S^0_1(K_2))/\mathrm{Ker}\, m$. Let v be the continuous linear functional on $S^0_1(K_1) \oplus S^0_1(K_2)$ defined by the relation $v(f_1, f_2) = u_1(f_1) - u_2(f_2)$. The condition $\rho_{K_1,\, K_1\cup K_2}u_1 = \rho_{K_2,\, K_1\cup K_2}u_2$ means that $u_1(f) = u_2(f)$ for every $f \in S^0_1(K_1 \cup K_2)$. We therefore have $\mathrm{Ker}\, v \supset \mathrm{Im}\, l$. Since $\mathrm{Ker}\, m = \mathrm{Im}\, l$ (see the proof of Lemma 4.7), this inclusion implies the existence of a functional $u \in S'^0_1(K_1 \cap K_2)$ such that $v = um$. If $f_{1,2} \in S^0_1(K_{1,2})$, then we have $u(f_1) = v(f_1, 0) = u_1(f_1)$ and $u(f_2) = v(0, -f_2) = u_2(f_2)$. The lemma is proved. □

Corollary 4.9. *Let $\{K_\omega\}_{\omega\in\Omega}$ be a family of closed cones in $\mathbb{R}^k$, $K \subset \mathbb{R}^k$ be a proper closed cone such that $K_\omega \subset K$ for every $\omega \in \Omega$, and $\tilde{K} = \bigcap_{\omega\in\Omega} K_\omega$. Let $\{u_\omega\}_{\omega\in\Omega}$ be a family of functionals such that $u_\omega \in S'^0_1(K_\omega)$ and $\rho_{K_\omega,\, K}\, u_\omega = \rho_{K_{\omega'},\, K}\, u_{\omega'}$ for every $\omega, \omega' \in \Omega$. Then there is a $u \in S'^0_1(\tilde{K})$ such that $u_\omega = \rho_{\tilde{K},\, K_\omega} u$ for every $\omega \in \Omega$.*

Proof. If Ω is finite, then the statement follows by induction from Lemmas 4.8 and 4.3. Now let Ω be arbitrary and $K' \supset K$ be a proper closed cone containing a conic

[6] Recall that the direct sum of a finite family of DFS spaces and a closed subspace of a DFS space are again DFS spaces, see [6].

neighborhood of $\tilde{K}$. Clearly, the functional $v = \rho_{K_\omega, K'}(u_\omega)$ does not depend on the choice of $\omega \in \Omega$. Let $W \subset K'$ be a conic neighborhood of $\tilde{K}$. By standard compactness arguments, there is a finite family $\omega_1, \ldots, \omega_n \in \Omega$ such that $M = \bigcap_{j=1}^n K_{\omega_j} \subset W$. Since this corollary holds for finite Ω, we conclude that v has a continuous extension to $S_1^0(M)$ and, therefore, to $S_1^0(W)$. Corollary 4.5 now ensures that v has a continuous extension u to $S_1^0(\tilde{K})$. By Lemma 4.3, $\rho_{\tilde{K}, K_\omega} u$ coincides with u_ω for any $\omega \in \Omega$ because both functionals have the same restriction to $S_1^0(K')$. The corollary is proved. □

To prove Theorem 3.10, we need the following lemma.

Lemma 4.10. *Let V be a convex open cone in $\mathbb{R}^k$ and $K = V^*$. Suppose a mapping $V \ni y \to u^y \in S_1'^0(K)$ is such that $e^{-\langle \cdot, \eta' \rangle} u^\eta = u^{\eta+\eta'}$ for any $\eta, \eta' \in V$. Then there is a unique $u \in S_1'^0(K)$ such that $u^\eta = e^{-\langle \cdot, \eta \rangle} u$ for any $\eta \in V$.*

Proof. Let $A, B > 0$, W be a conic neighborhood of K, and $\eta \in \mathbb{R}^k$ be such that $|\eta| \leqslant 1/Aa$, where a is defined by (4.1). We denote by $L^\eta_{W,A,B}$ the mapping $f \to e^{\langle \cdot, \eta \rangle} f$ from $S^{0,B}_{1,A}(W)$ to $S_1^0(K)$. Lemma 4.1 shows that this mapping is well defined and continuous. Let $\eta \in V$ be such that $|\eta| \leqslant 1/Aa$. We define the continuous functional $u_{W,A,B}$ on $S^{0,B}_{1,A}(W)$ by the relation

$$u_{W,A,B}(f) = u^\eta(L^\eta_{W,A,B} f), \quad f \in S^{0,B}_{1,A}(W).$$

Although η enters in the expression in the right-hand side, $u_{W,A,B}$ actually does not depend on the choice of η. Indeed, let $\eta' \in V$ be such that $|\eta'| \leqslant 1/Aa$. Set $\eta'' = t\eta'$, where $0 < t < 1$. Since V is open, $\eta - \eta'' \in V$ for t sufficiently small, and we have

$$\begin{aligned} u^\eta(L^\eta_{W,A,B} f) &= u^{\eta''+(\eta-\eta'')}(e^{\langle \cdot, \eta \rangle} f) = u^{\eta''}(e^{\langle \cdot, \eta'' \rangle} f) \\ &= u^{\eta''+(\eta'-\eta'')}(e^{\langle \cdot, \eta' \rangle} f) = u^{\eta'}(L^{\eta'}_{W,A,B} f) \end{aligned}$$

for any $f \in S^{0,B}_{1,A}(W)$. Let $A' > A$, $B' > B$, and $W' \subset W$. If $\eta \in V$ satisfies the bound $|\eta| \leqslant 1/A'a$, then $L^\eta_{W,A,B}$ is the restriction of $L^\eta_{W',A',B'}$ to $S^{0,B}_{1,A}(W)$ and we have

$$u_{W,A,B}(f) = u^\eta(L^\eta_{W,A,B} f) = u^\eta(L^\eta_{W',A',B'} f) = u_{W',A',B'}(f), \quad f \in S^{0,B}_{1,A}(W).$$

Thus, the functionals $u_{W,A,B}$ are compatible with the embeddings $S^{0,B}_{1,A}(W) \to S^{0,B'}_{1,A'}(W')$ and, therefore, determine a functional $u \in S_1'^0(K)$. Let $\eta, \eta' \in V$ be such that $|\eta'| \leqslant 1/Aa$ and $\eta - \eta' \in V$. Fix $f \in S_1^0(K)$ and choose $A, B > 0$ and a conic neighborhood W of K such that the function $e^{-\langle \cdot, \eta \rangle} f$ belongs to $S^{0,B}_{1,A}(W)$. We then obtain

$$(e^{-\langle \cdot, \eta \rangle} u)(f) = u_{W,A,B}(e^{-\langle \cdot, \eta \rangle} f) = u^{\eta'}(e^{-\langle \cdot, \eta-\eta' \rangle} f) = u^\eta(f).$$

This means that $e^{-\langle\cdot,\eta\rangle}u = u^\eta$. It remains to prove the uniqueness of u. Suppose $u' \in S_1'^0(K)$ is such that $e^{-\langle\cdot,\eta\rangle}u' = u^\eta$ for any $\eta \in V$. Then $v = u' - u$ satisfies the relation $e^{-\langle\cdot,\eta\rangle}v = 0$ for any $\eta \in V$. Let $f \in S_1^0(K)$. By Corollary 4.2, there is an $\eta \in V$ such that $e^{\langle\cdot,\eta\rangle}f \in S_1^0(K)$. We therefore have $v(f) = (e^{-\langle\cdot,\eta\rangle}v)(e^{\langle\cdot,\eta\rangle}f) = 0$. Thus, $v = 0$ and the lemma is proved. $\square$

Proof of Theorem 3.10. As in the preceding section, we identify $\mathcal{U}(K)$ with $S_1'^0(K)$. Fix $\alpha > 1$. For $u \in S_1'^0(K)$ and $\eta \in V$, we define the functional $v^\eta \in S_\alpha'^0(K)$ by the relation $v^\eta(f) = u(e^{-\langle\cdot,\eta\rangle}f)$, $f \in S_\alpha^0(K)$. We then have

$$(\mathcal{L}_K u)(\zeta + i\eta) = v^\eta(e^{i\langle\cdot,\zeta\rangle}) = (\mathcal{L}_K^\alpha v^\eta)(\zeta), \quad \zeta \in T^V \tag{4.2}$$

and in view of Theorem 3.12 the function $(\mathcal{L}_K u)(\cdot + i\eta)$ is analytic in T^V. Since $\eta \in V$ is arbitrary, this implies that $\mathcal{L}_K u$ is analytic in T^V. If $\mathcal{L}_K u = 0$, then by (4.2) we have $\mathcal{L}_K^\alpha v^\eta = 0$ for any $\eta \in V$. This implies that $v^\eta = 0$ for any $\eta \in V$ because the Laplace transformation $\mathcal{L}_K^\alpha$ is injective by Theorem 3.12. Denoting by u^η the restriction of v^η to $S_1^0(K)$ and applying the uniqueness part of Lemma 4.10, we conclude that $u = 0$. Thus, the operator $\mathcal{L}_K$ is injective. The mapping $u \to v^\eta$ from $S_1'^0(K)$ to $S_\alpha'^0(K)$ is continuous for any $\eta \in V$ being the dual mapping of the continuous mapping $f \to e^{-\langle\cdot,\eta\rangle}f$. It therefore follows from (4.2) and Theorem 3.12 that the mapping $u \to (\mathcal{L}_K u)(\cdot + i\eta)$ is continuous as a mapping from $S_1'^0(K)$ to $\mathcal{A}^\alpha(V)$ and, consequently, as a mapping from $S_1'^0(K)$ to $\mathcal{A}(V)$. This implies the continuity of $\mathcal{L}_K$ because for every compact set $K \subset T^V$, one can find an $\eta \in V$ such that $K - \eta \subset T^V$. We now prove the surjectivity of $\mathcal{L}_K$. Let $\mathbf{v} \in \mathcal{A}(V)$. Clearly, $\mathbf{v}(\cdot + i\eta) \in \mathcal{A}^\alpha(V)$ for any $\eta \in V$. We set $v^\eta = (\mathcal{L}_K^\alpha)^{-1}\mathbf{v}(\cdot + i\eta)$ and denote by u^η the restrictions of v^η to $S_1^0(K)$. For every $\eta \in V$ and $\zeta \in T^V$, we have

$$(\mathcal{L}_K u^\eta)(\zeta) = u^\eta(e^{i\langle\cdot,\zeta\rangle}) = v^\eta(e^{i\langle\cdot,\zeta\rangle}) = (\mathcal{L}_K^\alpha v^\eta)(\zeta) = \mathbf{v}(\zeta + i\eta).$$

Hence it follows that

$$\mathcal{L}_K(e^{-\langle\cdot,\eta'\rangle}u^\eta) = (\mathcal{L}_K u^\eta)(\cdot + i\eta') = \mathbf{v}(\cdot + i(\eta + \eta')) = \mathcal{L}_K u^{\eta+\eta'}, \quad \eta, \eta' \in V$$

and in view of the injectivity of $\mathcal{L}_K$ we have $e^{-\langle\cdot,\eta'\rangle}u^\eta = u^{\eta+\eta'}$. By Lemma 4.10, there is a $u \in S_1'^0(K)$ such that $u^\eta = e^{-\langle\cdot,\eta\rangle}u$ for any $\eta \in V$. Fix $\zeta = \xi + i\eta \in T^V$ and choose $\eta' \in V$ such that $\eta - \eta' \in V$. Then we have

$$(\mathcal{L}_K u)(\zeta) = (e^{-\langle\cdot,\eta'\rangle}u)(e^{i\langle\cdot,\xi+i(\eta-\eta')\rangle}) = (\mathcal{L}_K u^{\eta'})(\xi + i(\eta - \eta')) = \mathbf{v}(\zeta).$$

Thus, $\mathcal{L}_K u = \mathbf{v}$ and, consequently, $\mathcal{L}_K$ is a continuous one-to-one mapping. Since both $S_1'^0(K)$ and $\mathcal{A}(V)$ are Fréchet spaces, the continuity of the inverse operator $\mathcal{L}_K^{-1}$ is ensured by the open mapping theorem. Theorem 3.10 is proved. $\square$

5. Localizable inductive systems

The results of the preceding section show that the localization properties described by Theorems 3.6–3.8 hold for ultrafunctionals carried by proper closed cones. To prove these theorems in their full volume, we have to show that the properties of the inductive system S formed by the spaces $S_1'^0(K)$ over proper closed cones are inherited by the inductive system $\mathcal{U}$ formed by the spaces $\mathcal{U}(K)$ over arbitrary closed cones. We shall obtain the desired localization properties of $\mathcal{U}$ as a consequence of a more general algebraic construction formulated in terms of (*pre*)*localizable inductive systems* introduced by Definition 5.3 below. In contrast to Section 4, all considerations in this section are purely algebraic.

Recall that a partially ordered set A is called a lattice if every two-element subset $\{\alpha_1, \alpha_2\}$ of the set A has a supremum $\alpha_1 \vee \alpha_2$ and an infimum $\alpha_1 \wedge \alpha_2$. A lattice A is called distributive if $\alpha_1 \wedge (\alpha_2 \vee \alpha_3) = (\alpha_1 \wedge \alpha_2) \vee (\alpha_1 \wedge \alpha_3)$ for any $\alpha_1, \alpha_2, \alpha_3 \in A$.

Definition 5.1. Let A be a partially ordered set. We say that A is a quasi-lattice if every two-element subset of A has an infimum and every bounded above two-element subset of A has a supremum. We say that a quasi-lattice A is distributive if $\alpha_1 \wedge (\alpha_2 \vee \alpha_3) = (\alpha_1 \wedge \alpha_2) \vee (\alpha_1 \wedge \alpha_3)$ for every bounded above pair $\alpha_2, \alpha_3 \in A$ and every $\alpha_1 \in A$.

Clearly, every (distributive) lattice is a (distributive) quasi-lattice. If A is a distributive lattice, then we have

$$\begin{aligned}(\alpha_1 \vee \alpha_2) \wedge (\alpha_1 \vee \alpha_3) &= ((\alpha_1 \vee \alpha_2) \wedge \alpha_1) \vee ((\alpha_1 \vee \alpha_2) \wedge \alpha_3) \\ &= \alpha_1 \vee ((\alpha_1 \wedge \alpha_3) \vee (\alpha_2 \wedge \alpha_3)) = \alpha_1 \vee (\alpha_2 \wedge \alpha_3)\end{aligned}$$

for any $\alpha_1, \alpha_2, \alpha_3 \in A$. By induction, it follows that

$$\inf_{\omega \in \Omega} (\alpha \vee \alpha_\omega) = \alpha \vee \inf_{\omega \in \Omega} \alpha_\omega \tag{5.1}$$

for any $\alpha \in A$ and every nonempty finite family $\{\alpha_\omega\}_{\omega \in \Omega}$ of elements of A.

Definition 5.2. We call a lattice A infinitely distributive if every nonempty subset of A has an infimum and condition (5.1) is satisfied for an arbitrary (not necessarily finite) family $\{\alpha_\omega\}_{\omega \in \Omega}$ of elements of A.

Note that a distributive lattice may be not infinitely distributive even if all its subsets have an infimum (see, e.g., [3, Section II.4, Exercises 17 and 18]).

We call a nondecreasing mapping λ from a quasi-lattice A into a quasi-lattice B a morphism of quasi-lattices if $\lambda(\alpha_1 \wedge \alpha_2) = \lambda(\alpha_1) \wedge \lambda(\alpha_2)$ for any $\alpha_1, \alpha_2 \in A$ and $\lambda(\alpha_1 \vee \alpha_2) = \lambda(\alpha_1) \vee \lambda(\alpha_2)$ for every bounded above pair $\alpha_1, \alpha_2 \in A$.

In the rest of this section, we study abstract inductive systems of vector spaces indexed by (quasi-)lattices and systematically use the corresponding notation introduced in the end of Section 2.

Definition 5.3. An inductive system $\mathcal{X}$ of vector spaces over a quasi-lattice A is called to be prelocalizable if the following conditions are satisfied:

(I) The mappings $\rho^{\mathcal{X}}_{\alpha\alpha'}$ are injective for any $\alpha, \alpha' \in A$, $\alpha \leqslant \alpha'$.
(II) If a pair $\alpha_1, \alpha_2 \in A$ is bounded above and $x \in \mathcal{X}(\alpha_1 \vee \alpha_2)$, then there are $x_{1,2} \in \mathcal{X}(\alpha_{1,2})$ such that $x = \rho^{\mathcal{X}}_{\alpha_1,\, \alpha_1\vee\alpha_2}(x_1) + \rho^{\mathcal{X}}_{\alpha_2,\, \alpha_1\vee\alpha_2}(x_2)$.
(III) If a pair $\alpha_1, \alpha_2 \in A$ is bounded above by an element $\alpha \in A$, $x_{1,2} \in \mathcal{X}(\alpha_{1,2})$, and $\rho^{\mathcal{X}}_{\alpha_1,\, \alpha}(x_1) = \rho^{\mathcal{X}}_{\alpha_2,\, \alpha}(x_2)$, then there is an $x \in \mathcal{X}(\alpha_1 \wedge \alpha_2)$ such that $x_1 = \rho^{\mathcal{X}}_{\alpha_1\wedge\alpha_2,\, \alpha_1}(x)$ and $x_2 = \rho^{\mathcal{X}}_{\alpha_1\wedge\alpha_2,\, \alpha_2}(x)$.

We say that the inductive system $\mathcal{X}$ is localizable if every nonempty subset of A has an infimum and instead of (III) the following stronger condition is satisfied:

(III′) Let $\{\alpha_\omega\}_{\omega\in\Omega}$ be a nonempty family of elements of A bounded above by an $\alpha \in A$, and let a family $\{x_\omega\}_{\omega\in\Omega}$ be such that $x_\omega \in \mathcal{X}(\alpha_\omega)$ and $\rho^{\mathcal{X}}_{\alpha_\omega,\, \alpha}(x_\omega) = \rho^{\mathcal{X}}_{\alpha_{\omega'},\, \alpha}(x_{\omega'})$ for any $\omega, \omega' \in \Omega$. Then there is an $x \in \mathcal{X}(\tilde{\alpha})$ ($\tilde{\alpha} = \inf_{\omega\in\Omega} \alpha_\omega$) such that $x_\omega = \rho^{\mathcal{X}}_{\tilde{\alpha},\, \alpha_\omega}(x)$ for any $\omega \in \Omega$.

Let M be a closed cone in $\mathbb{R}^k$. The (ordered by inclusion) set $\mathcal{P}(M)$ of all proper closed cones contained in M is a distributive quasi-lattice, while the set $\mathcal{K}(M)$ of all closed cones contained in M is an infinitely distributive lattice. As shown by the properties (a)–(c) listed in Section 2, the inductive system over $\mathcal{K}(\mathbb{R}^k)$ formed by the spaces $S'^0_\alpha(K)$ ($\alpha > 1$) is prelocalizable (in fact, it is even localizable, see the paragraph following the formulation of Theorem 2.5).

A subset I of a quasi-lattice A will be called $\wedge$-closed if $\alpha_1 \wedge \alpha_2 \in I$ for any $\alpha_1, \alpha_2 \in I$. If I is a finite subset of a quasi-lattice A, then one can find a finite $\wedge$-closed set $I' \subset A$ containing I (for instance, the set consisting of infima of all subsets of I can be taken as I').

Lemma 5.4. *Let A be a distributive lattice, $\alpha \in A$, and $\mathcal{X}$ be a prelocalizable inductive system over $\mathcal{X}$. Suppose I is a $\wedge$-closed subset of A such that $\alpha' \leqslant \alpha$ for any $\alpha' \in I$ and $\alpha = \alpha_1 \vee \cdots \vee \alpha_n$ for some $\alpha_1, \ldots, \alpha_n \in I$. Then the space $\mathcal{X}(\alpha)$ is canonically isomorphic to $\varinjlim_{\alpha' \in I} \mathcal{X}(\alpha')$.*

The proof of this lemma is completely analogous to the algebraic part of the proof of Lemma 2.8 and is omitted. The following result is an immediate consequence of Corollary 4.4, Lemma 4.7, and Corollary 4.9.

Lemma 5.5. *The inductive system S over $\mathcal{P}(\mathbb{R}^k)$ formed by the spaces $S'^0_1(K)$ is localizable.*

Theorems 3.6–3.8 can be reformulated in terms of localizable inductive systems as follows.

Theorem 5.6. *The inductive system $\mathcal{U}$ over $\mathcal{K}(\mathbb{R}^k)$ formed by the spaces $\mathcal{U}(K)$ is localizable.*

Let $\mathcal{X}$ be an inductive system over a partially ordered set A. For every $I \subset A$, we define the inductive system $\mathcal{X}^I$ over I setting $\mathcal{X}^I(\alpha) = \mathcal{X}(\alpha)$ and $\rho^{\mathcal{X}^I}_{\alpha\alpha'} = \rho^{\mathcal{X}}_{\alpha\alpha'}$ for $\alpha, \alpha' \in I$, $\alpha \leqslant \alpha'$ (i.e., $\mathcal{X}^I$ is the "restriction" of $\mathcal{X}$ to I). If $I \subset J \subset A$, then there are canonical mappings $\tau^{\mathcal{X}}_{I,J}\colon \varinjlim \mathcal{X}^I \to \varinjlim \mathcal{X}^J$ satisfying the relation $\tau^{\mathcal{X}}_{I,J}\rho^{\mathcal{X}^I}_{\alpha} = \rho^{\mathcal{X}^J}_{\alpha}$ for any $\alpha \in I$. Let λ be a nondecreasing mapping from A to a partially ordered set B. With every $\beta \in B$ we associate the set $A_\beta = \{\alpha \in A \mid \lambda(\alpha) \leqslant \beta\}$ and define the inductive system $\lambda(\mathcal{X})$ over B setting $\lambda(\mathcal{X})(\beta) = \varinjlim \mathcal{X}^{A_\beta}$ and $\rho^{\lambda(\mathcal{X})}_{\beta\beta'} = \tau^{\mathcal{X}}_{A_\beta, A_{\beta'}}$ for $\beta, \beta' \in B$, $\beta \leqslant \beta'$. $\square$

The inclusion mapping $\theta\colon \mathcal{P}(\mathbb{R}^k) \to \mathcal{K}(\mathbb{R}^k)$ is clearly a morphism of quasi-lattices. By definition of the inductive system $\mathcal{U}$, we have $\mathcal{U} = \theta(S)$ and, therefore, Theorem 5.6 follows from the following more general statement.

Theorem 5.7. *Let A be a distributive quasi-lattice, B be a distributive lattice and $\lambda\colon A \to B$ be an injective quasi-lattice morphism such that every element $\beta \in B$ is representable in the form $\beta = \lambda(\alpha_1) \vee \cdots \vee \lambda(\alpha_n)$, where $\alpha_1, \ldots, \alpha_n \in A$. If $\mathcal{X}$ is a prelocalizable inductive system of vector spaces over A, then $\lambda(\mathcal{X})$ is a prelocalizable inductive system of vector spaces over B. If $\mathcal{X}$ is a localizable inductive system over A and the lattice B is infinitely distributive, then the inductive system $\lambda(\mathcal{X})$ is localizable.*

The proof of Theorem 5.7 is given in Appendix A.

Remark 5.8. Under the conditions of Theorem 5.7, for every quasi-lattice morphism φ from A to a distributive lattice L, there is a unique lattice morphism $\psi\colon B \to L$ such that $\varphi = \psi\lambda$. This means that B is the *free distributive lattice* over the partially ordered set A (see [3, Section I.5, Definition 2]).

Let $K_1, \ldots, K_n$ be convex closed proper cones in $\mathbb{R}^k$ such that $\bigcup_{i=1}^n K_i = \mathbb{R}^k$ and let I be the set consisting of all intersections of the cones $K_1, \ldots, K_n$. It follows from Theorems 3.10 and 5.6 and Lemma 5.4 that $\mathcal{U}(\mathbb{R}^k)$ is canonically isomorphic to the space $\varinjlim_{K \in I} \mathcal{A}(\operatorname{int} K^*)$. In the next section, we shall establish the bijectivity of the Fourier transformation by proving that for some choice of the cones K_i, the latter space is isomorphic to $\mathcal{B}(\mathbb{R}^k)$. To this end, we shall need to pass from the above inductive limit representation of $\mathcal{U}(\mathbb{R}^k)$ to another representation similar to that given by Martineau's edge of the wedge theorem for hyperfunctions. We conclude this section by describing the corresponding procedure in terms of abstract inductive systems.

Recall [3] that a partially ordered set A is called a lower semilattice if every two-element subset $\{\alpha_1, \alpha_2\}$ of the set A has an infimum $\alpha_1 \wedge \alpha_2$. Recall also that a subset I of a partially ordered set A is called cofinal in A if every element of A is majorized by an element of I.

Lemma 5.9. *Let T be a set, A be a lower semilattice, $\mathcal{X}$ be an inductive system over A, and λ be a mapping from T to A such that $\lambda(T)$ is cofinal in A. Let $\mathcal{N}$ be the* *subspace of* $\oplus_{\tau\in T}\mathcal{X}(\lambda(\tau))$ *spanned by all vectors of the form*

$$j_\tau\rho^{\mathcal{X}}_{\lambda(\tau)\wedge\lambda(\tau'),\,\lambda(\tau)}x - j_{\tau'}\rho^{\mathcal{X}}_{\lambda(\tau)\wedge\lambda(\tau'),\,\lambda(\tau')}x, \tag{5.2}$$

where $\tau,\tau'\in T$, $x\in\mathcal{X}(\lambda(\tau)\wedge\lambda(\tau'))$, *and* j_τ *is the canonical embedding of* $\mathcal{X}(\lambda(\tau))$ *into* $\oplus_{\tau\in T}\mathcal{X}(\lambda(\tau))$. *Then we have a natural isomorphism*

$$\oplus_{\tau\in T}\mathcal{X}(\lambda(\tau))/\mathcal{N}\simeq\varinjlim\mathcal{X}.$$

The proof of Lemma 5.9 is given in Appendix B.

Corollary 5.10. *Let A be a finite lower semilattice and* $\mathcal{X}$ *be an inductive system over A. Let* $\alpha_1,\ldots,\alpha_n$ *be a family of elements of A containing all maximal elements of A and* $\mathcal{N}$ *be the subspace of* $\oplus_{i=1}^n\mathcal{X}(\alpha_i)$ *consisting of the vectors* $(x_1,\ldots,x_n)$ *whose components are representable in the form*

$$x_i=\sum_{l=1}^n j_l\rho^{\mathcal{X}}_{\alpha_i\wedge\alpha_l,\,\alpha_l}x_{il},\quad i=1,\ldots,n, \tag{5.3}$$

where $x_{il}\in\mathcal{X}(\alpha_i\wedge\alpha_l)$, $x_{il}=-x_{li}$, *and* j_i *is the canonical embedding of* $\mathcal{X}(\alpha_i)$ *into* $\oplus_{i=1}^n\mathcal{X}(\alpha_i)$. *Then we have a natural isomorphism*

$$\oplus_{i=1}^n\mathcal{X}(\alpha_i)/\mathcal{N}\simeq\varinjlim\mathcal{X}.$$

Proof. Obviously, a subset I of a finite partially ordered set A is cofinal in A if and only if it contains all maximal elements of A, and in view of Lemma 5.9 it suffices to show that $\mathcal{N}$ coincides with the subspace $\mathcal{N}'$ of $\oplus_{i=1}^n\mathcal{X}(\alpha_i)$ spanned by the vectors of the form $j_i\rho^{\mathcal{X}}_{\alpha_i\wedge\alpha_l,\,\alpha_i}y - j_l\rho^{\mathcal{X}}_{\alpha_i\wedge\alpha_l,\,\alpha_l}y$ with $y\in\mathcal{X}(\alpha_i\wedge\alpha_l)$. Let $x\in\mathcal{N}'$. Then we have

$$x=\sum_{i,l=1}^n(j_i\rho^{\mathcal{X}}_{\alpha_i\wedge\alpha_l,\,\alpha_i}y_{il}-j_l\rho^{\mathcal{X}}_{\alpha_i\wedge\alpha_l,\,\alpha_l}y_{il})=\sum_{i=1}^n j_i\sum_{l=1}^n\rho^{\mathcal{X}}_{\alpha_i\wedge\alpha_l,\,\alpha_i}(y_{il}-y_{li}),$$

where $y_{il}\in\mathcal{X}(\alpha_i\wedge\alpha_l)$. Setting $x_{il}=y_{il}-y_{li}$, we see that the components of x have form (5.3) and, therefore, $x\in\mathcal{N}$. Conversely, let x be the element of $\mathcal{N}$ whose components have form (5.3). Then in view of the antisymmetry of x_{il} we have

$$x=\sum_{i=1}^n j_i\sum_{l=1}^n\rho^{\mathcal{X}}_{\alpha_i\wedge\alpha_l,\,\alpha_i}x_{il}=\frac{1}{2}\sum_{i,l=1}^n(j_i\rho^{\mathcal{X}}_{\alpha_i\wedge\alpha_l,\,\alpha_i}x_{il}-j_l\rho^{\mathcal{X}}_{\alpha_i\wedge\alpha_l,\,\alpha_l}x_{il})$$

and, therefore, $x\in\mathcal{N}'$. Thus, $\mathcal{N}=\mathcal{N}'$ and the corollary is proved. $\square$

6. Bijectivity of Fourier transformation

In this section, we give the proof of Theorem 3.11.

We first consider the one-dimensional case, when the spaces of hyperfunctions and ultrafunctionals have very simple structure and the bijectivity of $\mathcal{F}$ can be derived immediately from Theorem 3.10 without any reference to algebraic constructions of the preceding section. Let $H(V)$ denote the space of functions holomorphic in an open set $V \subset \mathbb{C}$. According to Sato's definition, hyperfunctions on an open set $O \subset \mathbb{R}$ are the elements of the quotient space $H(V \setminus O)/H(V)$, where V is an open set in $\mathbb{C}$ containing O as a relatively closed subset and $H(V)$ is assumed to be embedded in $H(V \setminus O)$ via the restriction mapping. It is important that all such quotient spaces are naturally isomorphic to each other and, therefore, this definition actually does not depend on the choice of V (see, e.g., Section 2 of [8] or Section 3.1 of [9]). In particular, we can set $\mathcal{B}(\mathbb{R}) = H(\mathbb{C} \setminus \mathbb{R})/H(\mathbb{C})$. For $\mathbf{v} \in H(\mathbb{C} \setminus \mathbb{R})$, we denote by $[\mathbf{v}]$ the corresponding element of $\mathcal{B}(\mathbb{R})$. Let the operators $j_\pm : H(\mathbb{C}_\pm) \to H(\mathbb{C} \setminus \mathbb{R})$ be defined by the relations

$$(j_\pm \mathbf{v}_\pm)(\zeta) = \begin{cases} \mathbf{v}_\pm(\zeta) & \text{for } \zeta \in \mathbb{C}_\pm \\ 0 & \text{for } \zeta \in \mathbb{C}_\mp \end{cases}, \quad \mathbf{v}_\pm \in H(\mathbb{C}_\pm).$$

The boundary value operators $\mathbf{b}_{\mathbb{R}_\pm} : \mathcal{A}(\mathbb{R}_\pm) \to \mathcal{B}(\mathbb{R})$ and $\mathbf{b}_{\mathbb{R}} : \mathcal{A}(\mathbb{R}) \to \mathcal{B}(\mathbb{R})$ are given by

$$\mathbf{b}_{\mathbb{R}_\pm} \mathbf{v}_\pm = \pm [j_\pm \mathbf{v}_\pm], \quad \mathbf{b}_{\mathbb{R}} \mathbf{v} = \mathbf{b}_{\mathbb{R}_+}(\mathbf{v}|_{\mathbb{C}_+}) = \mathbf{b}_{\mathbb{R}_-}(\mathbf{v}|_{\mathbb{C}_-}), \tag{6.1}$$

where $\mathbf{v}_\pm \in \mathcal{A}(\mathbb{R}_\pm)$, $\mathbf{v} \in \mathcal{A}(\mathbb{R})$, and $\mathbf{v}|_{\mathbb{C}_\pm}$ are the restrictions of $\mathbf{v}$ to $\mathbb{C}_\pm$ (note that $\mathcal{A}(\mathbb{R}_\pm) = H(\mathbb{C}_\pm)$ and $\mathcal{A}(\mathbb{R}) = H(\mathbb{C})$). By Theorem 3.10 and the definition of $\mathcal{U}(\mathbb{R})$, the Laplace operators $\mathcal{L}_K$ determine an isomorphic mapping $\mathcal{L} : \mathcal{U}(\mathbb{R}) \to \underrightarrow{\lim}_{K \in \mathcal{P}(\mathbb{R})} \mathcal{A}(\operatorname{int} K^*)$. The set $\mathcal{P}(\mathbb{R})$ contains only three elements: $\bar{\mathbb{R}}_+$, $\bar{\mathbb{R}}_-$, and $\{0\}$. By definition of the inductive limit, we have

$$\underrightarrow{\lim}_{K \in \mathcal{P}(\mathbb{R})} \mathcal{A}(\operatorname{int} K^*) = (\mathcal{A}(\mathbb{R}) \oplus \mathcal{A}(\mathbb{R}_+) \oplus \mathcal{A}(\mathbb{R}_-))/N,$$

where N is the subspace of $\mathcal{A}(\mathbb{R}) \oplus \mathcal{A}(\mathbb{R}_+) \oplus \mathcal{A}(\mathbb{R}_-)$ spanned by the vectors of the form $(\mathbf{v}, -\mathbf{v}|_{\mathbb{C}_+}, 0)$ and $(\mathbf{v}, 0, -\mathbf{v}|_{\mathbb{C}_-})$ with $\mathbf{v} \in \mathcal{A}(\mathbb{R})$. Let the mapping $\tilde{s} : \mathcal{A}(\mathbb{R}) \oplus \mathcal{A}(\mathbb{R}_+) \oplus \mathcal{A}(\mathbb{R}_-) \to \mathcal{B}(\mathbb{R})$ be defined by the relation

$$\tilde{s}(\mathbf{v}, \mathbf{v}_+, \mathbf{v}_-) = \mathbf{b}_{\mathbb{R}} \mathbf{v} + \mathbf{b}_{\mathbb{R}_+} \mathbf{v}_+ + \mathbf{b}_{\mathbb{R}_-} \mathbf{v}_- \tag{6.2}$$

and let $s : \underrightarrow{\lim}_{K \in \mathcal{P}(\mathbb{R})} \mathcal{A}(\operatorname{int} K^*) \to \mathcal{B}(\mathbb{R})$ be the mapping induced by $\tilde{s}$. By definition of the Fourier operator $\mathcal{F}$, we have $\mathcal{F} = s\mathcal{L}$. Thus, to prove the bijectivity of $\mathcal{F}$, it suffices to show that s is one-to-one. In other words, we have to show that $(\mathbf{v}, \mathbf{v}_+, \mathbf{v}_-) \in$

N if $\tilde{s}(\mathbf{v}, \mathbf{v}_+, \mathbf{v}_-) = 0$. In view of (6.1) and (6.2), the last condition means that $[j_+(\mathbf{v}|_{\mathbb{C}_+})] + [j_+\mathbf{v}_+] - [j_-\mathbf{v}_-] = 0$. In other words, there is a $\mathbf{u} \in \mathcal{A}(\mathbb{R})$ such that $\mathbf{v}|_{\mathbb{C}_+} + \mathbf{v}_+ = \mathbf{u}|_{\mathbb{C}_+}$ and $-\mathbf{v}_- = \mathbf{u}|_{\mathbb{C}_-}$. This implies that

$$(\mathbf{v}, \mathbf{v}_+, \mathbf{v}_-) = (\mathbf{v} - \mathbf{u}, -(\mathbf{v} - \mathbf{u})|_{\mathbb{C}_+}, 0) + (\mathbf{u}, 0, -\mathbf{u}|_{\mathbb{C}_-}).$$

Thus, $(\mathbf{v}, \mathbf{v}_+, \mathbf{v}_-) \in N$ and Theorem 3.11 is proved for the case $k = 1$.

Let us now consider the general case. With every set $x^1, \ldots, x^l$ of vectors in $\mathbb{R}^k$ we associate the cone $K(x^1, \ldots, x^l) = \{x \in \mathbb{R}^k \,|\, x = t_1x^1 + \cdots + t_lx^l,\ t_i \geqslant 0\}$. Let $x^1, \ldots, x^{k+1}$ be vectors in $\mathbb{R}^k$ such that $K(x^1, \ldots, x^{k+1}) = \mathbb{R}^k$. For $i, j = 1, \ldots, k+1$, $i \neq j$, we set

$$\begin{aligned} E_i &= \{\xi \in \mathbb{R}^k \,|\, \langle \xi, x_i \rangle \geqslant 0\}, \\ K_i &= K(x^1, \ldots, \hat{x}^i, \ldots x^{k+1}), \quad K_{ij} = K(x^1, \ldots, \hat{x}^i, \ldots, \hat{x}^j, \ldots x^{k+1}), \\ \Gamma_i &= E_1 \cap \cdots \cap \hat{E}_i \cap \cdots \cap E_{k+1}, \quad V_{ij} = E_1 \cap \cdots \cap \hat{E}_i \cap \cdots \cap \hat{E}_j \cap \cdots \cap E_{k+1}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the symmetric nondegenerate bilinear form on $\mathbb{R}^k$ entering in the definitions of the Fourier and Laplace transformations and the hat means that an element is omitted. It is easy to see that $\bigcup_{i=1}^{k+1} K_i = \mathbb{R}^k$ and $K_i \cap K_j = K_{ij}$. Furthermore, we have $\Gamma_i = \operatorname{int} K_i^*$ and $V_{ij} = \operatorname{int} K_{ij}^*$. Let I denote the finite set consisting of all possible intersections of the cones $K_1, \ldots, K_{k+1}$. By Lemma 5.4 and Theorem 5.6, there is a natural isomorphism $l: \mathcal{U}(\mathbb{R}^k) \to \varinjlim_{K \in I} \mathcal{U}(K)$. By Theorem 3.10, the Laplace operators $\mathcal{L}_K$ determine an isomorphic mapping $\mathcal{L}: \varinjlim_{K \in I} \mathcal{U}(K) \to \varinjlim_{K \in I} \mathcal{A}(\operatorname{int} K^*)$. Let $\mathcal{N}$ be the subspace of $\oplus_{i=1}^{k+1} \mathcal{A}(\Gamma_i)$ consisting of the elements $(\mathbf{v}_1, \ldots, \mathbf{v}_{k+1})$ such that

$$\mathbf{v}_i = \sum_{j=1}^{k+1} \mathbf{v}_{ij},$$

where $\mathbf{v}_{ij} = -\mathbf{v}_{ji}$ belong to $\mathcal{A}(V_{ij})$. By Corollary 5.10, we have a natural isomorphism $m: \varinjlim_{K \in I} \mathcal{A}(\operatorname{int} K^*) \to \oplus_{i=1}^{k+1} \mathcal{A}(\Gamma_i)/\mathcal{N}$. Let $\tilde{b}$ be the mapping from $\oplus_{i=1}^{k+1} \mathcal{A}(\Gamma_i)$ to $\mathcal{B}(\mathbb{R}^k)$ defined by the relation

$$\tilde{b}(\mathbf{v}_1, \ldots, \mathbf{v}_{k+1}) = \sum_{i=1}^{k+1} \mathbf{b}_{\Gamma_i} \mathbf{v}_i,$$

where $\mathbf{b}_{\Gamma_i}$ are the boundary value operators. Obviously, we have the inclusion $\mathcal{N} \subset \ker \tilde{b}$ and, therefore, $\tilde{b}$ determines a mapping $b: \oplus_{i=1}^{k+1} \mathcal{A}(\Gamma_i)/\mathcal{N} \to \mathcal{B}(\mathbb{R}^k)$. From the definition of the Fourier operator $\mathcal{F}$, it easily follows that $\mathcal{F} = bm\mathcal{L}l$. Thus, it suffices to establish that b is a one-to-one mapping. Let $\tilde{B}$ be the mapping from $\oplus_{i=1}^{k+1}\mathcal{A}(\Gamma_i)$ to $\mathcal{B}(\mathbb{R}^k)$ defined by the relation

$$\tilde{B}(\mathbf{v}_1,\ldots,\mathbf{v}_{k+1}) = \sum_{i=1}^{k+1}(-1)^i\,\mathbf{b}_{\Gamma_i}\mathbf{v}_i.$$

Let the mapping $\delta\colon \oplus_{i<j}\mathcal{A}(V_{ij}) \to \oplus_i \mathcal{A}(\Gamma_i)$ be defined by

$$(\delta\mathbf{v})_j = \sum_{1\leqslant i<j}(-1)^i\mathbf{v}_{ij} + \sum_{j<i\leqslant k+1}(-1)^{i+1}\mathbf{v}_{ji}, \quad \mathbf{v} = \{\mathbf{v}_{ij}\}_{i<j}.$$

It is easy to see that $\operatorname{Im}\delta \subset \operatorname{Ker}\tilde{B}$ and, consequently, $\tilde{B}$ determines a mapping $B\colon \oplus_{i=1}^{k+1}\mathcal{A}(\Gamma_i)/\operatorname{Im}\delta \to \mathcal{B}(\mathbb{R}^k)$. As shown in [9] (see formula 2.5 of Chapter 7 and Corollary 7.4.6) this mapping is one-to-one. Let τ be the isomorphic mapping from $\oplus_{i=1}^{k+1}\mathcal{A}(\Gamma_i)$ onto itself defined by

$$(\tau\mathbf{v})_i = (-1)^i\mathbf{v}_i, \quad \mathbf{v} = (\mathbf{v}_1,\ldots,\mathbf{v}_{k+1}).$$

Then we have $\tilde{b} = \tau\tilde{B}$ and $\operatorname{Im}\delta = \tau(\mathcal{N})$. Therefore, the bijectivity of B implies that of b. Theorem 3.11 is proved.

Remark 6.1. Let

$$\mathfrak{V} = (\mathbb{C}^k, T^{E_1},\ldots,T^{E_{k+1}}), \quad \mathfrak{V}' = (T^{E_1},\ldots,T^{E_{k+1}}),$$

where $T^{E_j} = \mathbb{R}^k + iE_j$. The above isomorphism $B\colon \oplus_{i=1}^{k+1}\mathcal{A}(\Gamma_i)/\operatorname{Im}\delta \to \mathcal{B}(\mathbb{R}^k)$ gives the Čech cohomology representation of $\mathcal{B}(\mathbb{R}^k)$ if $(\mathfrak{V},\mathfrak{V}')$ is used as a relative Stein open covering of $(\mathbb{C}^k, \mathbb{C}^k\setminus\mathbb{R}^k)$ (see details in [9, Section 7.2]).

7. Conclusion

The obtained results suggest the way of constructing "nontrivializations" of some seemingly trivial generalized function spaces. We conclude this paper by indicating some possible results of this type in the framework of the Gurevich spaces W_M^{Ω} described in Chapter I of the book [1]. Let Ω and M be monotone convex nonnegative differentiable indefinitely increasing functions defined on the positive real semi-axis and satisfying the condition $\Omega(0) = M(0) = 0$. The space $W_M^{\Omega}(\mathbb{R}^k)$ is the union (inductive limit) with respect to $A, B > 0$ of the Banach spaces consisting of entire analytic functions on $\mathbb{C}^k$ with the finite norm

$$\sup_{z=x+iy\in\mathbb{C}^k} |f(z)|\exp[M(Ax) - \Omega(By)].$$

If Ω and M grow faster than any linear function, then the Fourier transformation isomorphically maps the space $W_M^\Omega(\mathbb{R}^k)$ onto the space $W_{\Omega_*}^{M_*}(\mathbb{R}^k)$, where

$$M_*(s) = \sup_{t\geqslant 0}(st - M(t)), \quad \Omega_*(s) = \sup_{t\geqslant 0}(st - \Omega(t))$$

are the dual functions of Ω and M in the sense of Young. For $0 < \alpha \leqslant 1$ and $0 \leqslant \beta < 1$, the space $S_\alpha^\beta(\mathbb{R}^k)$ coincides with the space $W_M^\Omega(\mathbb{R}^k)$ with $\Omega(s) = s^{1/(1-\beta)}$ and $M(s) = s^{1/\alpha}$. In particular, $S_1^0(\mathbb{R}^k) = W_M^\Omega(\mathbb{R}^k)$, where $\Omega(s) = M(s) = s$. By analogy with Definitions 2.1 and 3.1, one can make

Definition 7.1. Let U be a cone in $\mathbb{R}^k$ and Ω and M be functions with the properties specified above. The Banach space $W_{M,A}^{\Omega,B}(U)$ consists of entire analytic functions on $\mathbb{C}^k$ with the finite norm

$$\sup_{z=x+iy\in\mathbb{C}^k} |f(z)| \exp(M(|x/A|) - \Omega(\delta_U(Bx)) - \Omega(|By|)),$$

where $\delta_U(x) = \inf_{x'\in U}|x - x'|$ is the distance from x to U. The space $W_M^\Omega(U)$ is defined by the relation

$$W_M^\Omega(U) = \bigcup_{A,B>0,\,\tilde{U}\supset U} W_{M,A}^{\Omega,B}(\tilde{U}),$$

where $\tilde{U}$ runs over all conic neighborhoods of U and the union is endowed with the inductive limit topology.

Further, we can introduce the following definition analogous to Definition 3.4:

Definition 7.2. Let K be a closed cone in $\mathbb{R}^k$. The space $\mathcal{U}_M^\Omega(K)$ is defined to be the inductive limit $\varinjlim_{K'\in\mathcal{P}(K)} W_M'^\Omega(K)$, where $\mathcal{P}(K)$ is the set of all nonempty proper closed cones contained in K. A closed cone K is said to be a carrier cone of an element $u \in \mathcal{U}_M^\Omega(\mathbb{R}^k)$ if the latter belongs to the image of the canonical mapping from $\mathcal{U}_M^\Omega(K)$ to $\mathcal{U}_M^\Omega(\mathbb{R}^k)$.

The results obtained in this paper suggest the following conjecture:

Hypothesis 7.3. Let the defining functions Ω and M be such that $M(s) \leqslant \Omega(as)$ for some $a > 0$. Then the following statements hold:

(1) The space $\mathcal{U}_M^\Omega(\mathbb{R}^k)$ is nontrivial regardless of the triviality or nontriviality of $W_M^\Omega(\mathbb{R}^k)$.

(2) If $W_M^{\Omega}(\mathbb{R}^k)$ is nontrivial, then $\mathcal{U}_M^{\Omega}(\mathbb{R}^k)$ is canonically isomorphic to the space $W_M'^{\Omega}(\mathbb{R}^k)$.
(3) Theorems 3.6–3.8 are valid for the spaces $\mathcal{U}_M^{\Omega}(K)$.
(4) One can canonically define the Fourier transformation that isomorphically maps $\mathcal{U}_M^{\Omega}(\mathbb{R}^k)$ onto $\mathcal{U}_{\Omega_*}^{M_*}(\mathbb{R}^k)$.

Note that the Fourier transformation on $\mathcal{U}_M^{\Omega}(\mathbb{R}^k)$ cannot be constructed as that of ultrafunctionals because the elements of $\mathcal{U}_M^{\Omega}(\mathbb{R}^k)$ grow faster than exponentially and their Laplace transformation is not well defined.

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Appendix A. Proof of Theorem 5.7

This appendix is organized as follows. We first introduce some additional notation concerning inductive systems, which will also be used in proving Lemma 5.9 in Appendix B. Then we derive several auxiliary results (Lemmas A.1–A.5) and, finally, prove Theorem 5.7.

Let $\mathcal{X}$ be an inductive system over a partially ordered set A. For $I \subset A$, we denote by $T_I^{\mathcal{X}}$ the set of triples (x, α, α') such that $\alpha, \alpha' \in I$, $\alpha \leqslant \alpha'$, and $x \in \mathcal{X}(\alpha)$. If $(x, \alpha, \alpha') \in T_A^{\mathcal{X}}$, then we set $\sigma^{\mathcal{X}}(x, \alpha, \alpha') = \iota_{\alpha}^{\mathcal{X}} x - \iota_{\alpha'}^{\mathcal{X}} \rho_{\alpha\alpha'}^{\mathcal{X}} x$ (recall that $\iota_{\alpha}^{\mathcal{X}}$ is the canonical embedding of $\mathcal{X}(\alpha)$ into $\oplus_{\alpha' \in A} \mathcal{X}(\alpha')$). We denote by $N_I^{\mathcal{X}}$ the subspace of $\oplus_{\alpha' \in A} \mathcal{X}(\alpha')$ spanned by all $\sigma^{\mathcal{X}}(x, \alpha, \alpha')$ with $(x, \alpha, \alpha') \in T_I^{\mathcal{X}}$. For $I \subset A$, we denote by $M_I^{\mathcal{X}}$ the subspace $\oplus_{\alpha \in I} \mathcal{X}(\alpha)$ of the space $\oplus_{\alpha \in A} \mathcal{X}(\alpha)$. Obviously, the space $\varinjlim \mathcal{X}^I$ is isomorphic to $M_I^{\mathcal{X}} / N_I^{\mathcal{X}}$. We denote by $j_I^{\mathcal{X}}$ the canonical surjection from $M_I^{\mathcal{X}}$ onto $\varinjlim \mathcal{X}^I$. If $I \subset J \subset A$, then we have

$$\tau_{I,J}^{\mathcal{X}} j_I^{\mathcal{X}} x = j_J^{\mathcal{X}} x, \quad x \in M_I^{\mathcal{X}}. \tag{A.1}$$

We say that a subset I of a partially ordered set A is hereditary if the relations $\alpha \in I$ and $\alpha' \leqslant \alpha$ imply that $\alpha' \in I$.

Lemma A.1. *Let $\mathcal{X}$ be a prelocalizable inductive system of vector spaces over a distributive quasi-lattice A. If I is a hereditary subset of A, then $N_A^{\mathcal{X}} \cap M_I^{\mathcal{X}} = N_I^{\mathcal{X}}$.*

Proof. The inclusion $N_I^{\mathcal{X}} \subset N_A^{\mathcal{X}} \cap M_I^{\mathcal{X}}$ is obvious. To prove the converse inclusion, it suffices to show that $N_J^{\mathcal{X}} \cap M_I^{\mathcal{X}} \subset N_I^{\mathcal{X}}$ for every finite $\wedge$-closed $J \subset A$. For

$\alpha \in J$, we denote by $k(\alpha)$ the cardinality $|J_\alpha|$ of the set $J_\alpha = \{\alpha' \in J \mid \alpha' \geqslant \alpha\}$. It is obvious that $\alpha = \inf J_\alpha$. Therefore, if $\alpha, \alpha' \in J$, $\alpha \neq \alpha'$, and $k(\alpha') \geqslant k(\alpha)$, then we have $J_\alpha \neq J_{\alpha'}$ and, consequently, $k(\alpha \wedge \alpha') = |J_{\alpha\wedge\alpha'}| \geqslant |J_\alpha \cup J_{\alpha'}| > |J_\alpha| = k(\alpha)$. For $n \in \mathbb{N}$, set $C_n = \{\alpha \in J \mid k(\alpha) \geqslant n\}$. We have $J = C_1 \supset C_2 \supset \cdots \supset C_{|J|} = \{\tilde\alpha\}$, where $\tilde\alpha = \inf J$, and $C_n = \emptyset$ for $n > |J|$. We shall say that an $x \in N_J^{\mathcal{X}} \cap M_I^{\mathcal{X}}$ admits a decomposition of order n if there are a family of vectors $x_{\alpha\alpha'} \in \mathcal{X}(\alpha)$ indexed by the set $\{(\alpha, \alpha') : \alpha \in C_n,\ \alpha' \in J \setminus I,\ \alpha < \alpha'\}$ and an element $\tilde x \in N_I^{\mathcal{X}}$ such that[7]

$$x = \tilde x + \sum_{\alpha\in C_n,\ \alpha'\in J\setminus I,\ \alpha<\alpha'} \sigma^{\mathcal{X}}(x_{\alpha\alpha'}, \alpha, \alpha'). \tag{A.2}$$

If x has a decomposition of order $> |J|$, then $x \in N_I^{\mathcal{X}}$. Therefore, the lemma will be proved as soon as we show that every $x \in N_J^{\mathcal{X}} \cap M_I^{\mathcal{X}}$ admits a decomposition of order n for any $n \in \mathbb{N}$. Since I is hereditary, every $x \in N_J^{\mathcal{X}} \cap M_I^{\mathcal{X}}$ has a decomposition of order 1, and we have to show that x has a decomposition of order $n+1$ supposing it has a decomposition of the form (A.2) of order n. To this end, it suffices to establish that $\sigma^{\mathcal{X}}(x_{\alpha\alpha'}, \alpha, \alpha')$ has a decomposition of order $n+1$ for every $\alpha \in C_n$, $\alpha' \in J \setminus I$ such that $\alpha < \alpha'$ and $k(\alpha) = n$. Let $\Lambda = \{\beta \in C_n \mid \beta < \alpha',\ \beta \neq \alpha\}$. Since $\alpha' \notin I$, the α'-component of x is equal to zero and by (A.2) we have

$$\rho^{\mathcal{X}}_{\alpha\alpha'} x_{\alpha\alpha'} + \sum_{\beta\in\Lambda} \rho^{\mathcal{X}}_{\beta\alpha'} x_{\beta\alpha'} = 0. \tag{A.3}$$

If $\Lambda = \emptyset$, then the injectivity of $\rho^{\mathcal{X}}_{\alpha\alpha'}$ implies that $x_{\alpha\alpha'} = 0$ and $\sigma^{\mathcal{X}}(x_{\alpha\alpha'}, \alpha, \alpha') = 0$. Therefore, in this case, $\sigma^{\mathcal{X}}(x_{\alpha\alpha'}, \alpha, \alpha')$ admits decompositions of all orders. Now let $\Lambda \neq \emptyset$ and $\tilde\beta = \sup \Lambda$ (the element $\tilde\beta$ is well defined because Λ is a finite set whose elements do not exceed α'; note that $\tilde\beta$ does not necessarily belong to J). Set $y = \sum_{\beta\in\Lambda} \rho^{\mathcal{X}}_{\beta\tilde\beta} x_{\beta\alpha'}$. Then it follows from (A.3) that $\rho^{\mathcal{X}}_{\alpha\alpha'} x_{\alpha\alpha'} + \rho^{\mathcal{X}}_{\tilde\beta\alpha'} y = 0$. Hence, by (III) there is a $z \in \mathcal{X}(\tilde\beta \wedge \alpha)$ such that $x_{\alpha\alpha'} = \rho^{\mathcal{X}}_{\tilde\beta\wedge\alpha,\,\alpha} z$. Because the quasi-lattice A is distributive, we have $\tilde\beta \wedge \alpha = \sup_{\beta\in\Lambda} \beta \wedge \alpha$ and by (II), there is a family $\{z_\beta\}_{\beta\in\Lambda}$ such that $z_\beta \in \mathcal{X}(\beta \wedge \alpha)$ and $z = \sum_{\beta\in\Lambda} \rho^{\mathcal{X}}_{\beta\wedge\alpha,\,\tilde\beta\wedge\alpha} z_\beta$. We thus have $x_{\alpha\alpha'} = \sum_{\beta\in\Lambda} \rho^{\mathcal{X}}_{\beta\wedge\alpha,\,\alpha} z_\beta$ and, consequently,

$$\sigma^{\mathcal{X}}(x_{\alpha\alpha'}, \alpha, \alpha') = \sum_{\beta\in\Lambda} [\sigma^{\mathcal{X}}(z_\beta, \alpha\wedge\beta, \alpha') - \sigma^{\mathcal{X}}(z_\beta, \alpha\wedge\beta, \alpha)]. \tag{A.4}$$

If $\alpha \in I$, then we set $\tilde y = -\sigma^{\mathcal{X}}(z_\beta, \alpha\wedge\beta, \alpha)$ and $y_{\gamma\gamma'} = \delta_{\gamma',\alpha'} \sum_{\beta\in\Lambda,\ \alpha\wedge\beta=\gamma} z_\beta$, where $\gamma, \gamma' \in J$ and $\delta_{\gamma',\alpha'} = 1$ for $\gamma' = \alpha'$ and $\delta_{\gamma',\alpha'} = 0$ for $\gamma' \neq \alpha'$. If $\alpha \notin I$, then we set

[7] Here and below, we assume that the sum of a family of vectors indexed by the empty set is equal to zero.

$\tilde{y} = 0$, $y_{\gamma\gamma'} = \delta_{\gamma',\alpha'} \sum_{\beta\in\Lambda,\,\alpha\wedge\beta=\gamma} z_\beta - \delta_{\gamma',\alpha} \sum_{\beta\in\Lambda,\,\alpha\wedge\beta=\gamma} z_\beta$. Since $k(\alpha\wedge\beta) > k(\alpha) = n$ for $\beta \in \Lambda$, it follows from (A.4) that

$$\sigma^{\mathcal{X}}(x_{\alpha\alpha'}, \alpha, \alpha') = \tilde{y} + \sum_{\gamma\in C_{n+1},\,\gamma'\in J\setminus I,\,\gamma<\gamma'} \sigma^{\mathcal{X}}(y_{\gamma\gamma'}, \gamma, \gamma'),$$

i.e., $\sigma^{\mathcal{X}}(x_{\alpha\alpha'}, \alpha, \alpha')$ admits a decomposition of order $n+1$. The lemma is proved. $\square$

Corollary A.2. *Let A be a distributive quasi-lattice, $\mathcal{X}$ be a prelocalizable inductive system over A, and $I \subset J \subset A$. If I is a hereditary subset of A, then the canonical mapping $\tau^{\mathcal{X}}_{I,J}\colon \varinjlim \mathcal{X}^I \to \varinjlim \mathcal{X}^J$ is injective.*

Proof. Let $x \in \varinjlim \mathcal{X}^I$ and $\tau^{\mathcal{X}}_{I,J} x = 0$. By the surjectivity of $j^{\mathcal{X}}_I$, there is an $\tilde{x} \in M^{\mathcal{X}}_I$ such that $x = j^{\mathcal{X}}_I \tilde{x}$. It follows from (A.1) that $j^{\mathcal{X}}_J \tilde{x} = 0$, i.e., $\tilde{x} \in N^{\mathcal{X}}_J$. Therefore, $\tilde{x} \in M^{\mathcal{X}}_I \cap N^{\mathcal{X}}_J$ and in view of Lemma A.1 we conclude that $\tilde{x} \in N^{\mathcal{X}}_I$ and $x = j^{\mathcal{X}}_I \tilde{x} = 0$. The corollary is proved. $\square$

Lemma A.3. *Let $\mathcal{X}$ be an inductive system over a partially ordered set A and I_1 and I_2 be hereditary subsets of A. Then for every $x \in N^{\mathcal{X}}_{I_1\cup I_2}$, there are $x_{1,2} \in N^{\mathcal{X}}_{I_{1,2}}$ such that $x = x_1 + x_2$.*

Proof. Let Λ be the set of all pairs (α, α') such that $\alpha, \alpha' \in I_1 \cup I_2$ and $\alpha \leqslant \alpha'$. By definition of $N^{\mathcal{X}}_{I_1\cup I_2}$ there is a family $\{x_{\alpha\alpha'}\}_{(\alpha,\alpha')\in\Lambda}$ such that $x_{\alpha\alpha'} \in \mathcal{X}(\alpha)$ and $x = \sum_{(\alpha,\alpha')\in\Lambda} \sigma^{\mathcal{X}}(x_{\alpha\alpha'}, \alpha, \alpha')$. We have $x = x_1 + x_2$, where

$$x_1 = \sum_{(\alpha,\alpha')\in\Lambda,\,\alpha'\in I_1} \sigma^{\mathcal{X}}(x_{\alpha\alpha'}, \alpha, \alpha'), \quad x_2 = \sum_{(\alpha,\alpha')\in\Lambda,\,\alpha'\in I_2\setminus I_1} \sigma^{\mathcal{X}}(x_{\alpha\alpha'}, \alpha, \alpha').$$

Since $I_{1,2}$ are hereditary, we conclude that $x_{1,2} \in N^{\mathcal{X}}_{I_{1,2}}$. The lemma is proved. $\square$

Lemma A.4. *Let A be a distributive quasi-lattice, $\mathcal{X}$ be a prelocalizable inductive system over A. Let $J \subset A$, and I_1, I_2 be hereditary subsets of A contained in J. Suppose $x_{1,2} \in \varinjlim \mathcal{X}^{I_{1,2}}$ are such that $\tau^{\mathcal{X}}_{I_1,J} x_1 = \tau^{\mathcal{X}}_{I_2,J} x_2$. Then there is an $x \in \varinjlim \mathcal{X}^{I_1\cap I_2}$ such that $x_1 = \tau^{\mathcal{X}}_{I_1\cap I_2, I_1} x$ and $x_2 = \tau^{\mathcal{X}}_{I_1\cap I_2, I_2} x$.*

Proof. Let $\tilde{x}_{1,2} \in M^{\mathcal{X}}_{I_{1,2}}$ be such that $x_{1,2} = j^{\mathcal{X}}_{I_{1,2}} \tilde{x}_{1,2}$. We have

$$\tau^{\mathcal{X}}_{I_1\cup I_2, J} \tau^{\mathcal{X}}_{I_1, I_1\cup I_2} x_1 = \tau^{\mathcal{X}}_{I_1,J} x_1 = \tau^{\mathcal{X}}_{I_2,J} x_2 = \tau^{\mathcal{X}}_{I_1\cup I_2, J} \tau^{\mathcal{X}}_{I_2, I_1\cup I_2} x_2.$$

Since the sets $I_{1,2}$ are hereditary, the set $I_1 \cup I_2$ is also hereditary and by Corollary A.2, the mapping $\tau^{\mathcal{X}}_{I_1\cup I_2,\,J}$ is injective. Therefore, $\tau^{\mathcal{X}}_{I_1,\,I_1\cup I_2} x_1 = \tau^{\mathcal{X}}_{I_2,\,I_1\cup I_2} x_2$ and using (A.1), we obtain $j^{\mathcal{X}}_{I_1\cup I_2}(\tilde{x}_1 - \tilde{x}_2) = 0$. This means that $\tilde{x}_1 - \tilde{x}_2 \in N^{\mathcal{X}}_{I_1\cup I_2}$. By Lemma A.3, there are $y_{1,2} \in N^{\mathcal{X}}_{I_{1,2}}$ such that $\tilde{x}_1 - \tilde{x}_2 = y_1 + y_2$. Set $\tilde{x} = \tilde{x}_1 - y_1 = \tilde{x}_2 + y_2$. Then $\tilde{x} \in M^{\mathcal{X}}_{I_1} \cap M^{\mathcal{X}}_{I_2} = M^{\mathcal{X}}_{I_1\cap I_2}$. Set $x = j^{\mathcal{X}}_{I_1\cap I_2}\tilde{x}$. Then $x \in \varinjlim \mathcal{X}^{I_1\cap I_2}$ and it follows from (A.1) that

$$\tau^{\mathcal{X}}_{I_1\cap I_2,\,I_1} x = \tau^{\mathcal{X}}_{I_1\cap I_2,\,I_1} j^{\mathcal{X}}_{I_1\cap I_2}\tilde{x} = j^{\mathcal{X}}_{I_1}(\tilde{x}_1 - y_1) = x_1,$$
$$\tau^{\mathcal{X}}_{I_1\cap I_2,\,I_2} x = \tau^{\mathcal{X}}_{I_1\cap I_2,\,I_2} j^{\mathcal{X}}_{I_1\cap I_2}\tilde{x} = j^{\mathcal{X}}_{I_2}(\tilde{x}_2 + y_2) = x_2.$$

The lemma is proved. $\square$

Lemma A.5. *Let A be a quasi-lattice, B be a lattice, and $\lambda: A \to B$ be an injective quasi-lattice morphism such that any element $\beta \in B$ is representable in the form $\beta = \lambda(\alpha_1) \vee \cdots \vee \lambda(\alpha_n)$, where $\alpha_1, \ldots, \alpha_n \in A$. Then we have*

(1) *If $\alpha, \alpha' \in A$ and $\lambda(\alpha') \leqslant \lambda(\alpha)$, then $\alpha' \leqslant \alpha$.*
(2) *If $\beta, \beta' \in B$, $\beta' \leqslant \beta$, and $\beta = \lambda(\alpha)$ for an $\alpha \in A$, then there is a unique $\alpha' \in A$ such that $\beta' = \lambda(\alpha')$.*
(3) *If $A' \subset A$ has an infimum in A, then $\lambda(A')$ has an infimum in B, and $\lambda(\inf A') = \inf \lambda(A')$.*

Proof. (1) We have $\lambda(\alpha \wedge \alpha') = \lambda(\alpha) \wedge \lambda(\alpha') = \lambda(\alpha')$. In view of the injectivity of λ it hence follows that $\alpha \wedge \alpha' = \alpha'$. This means that $\alpha' \leqslant \alpha$.

(2) Let $\alpha_1, \ldots, \alpha_n \in A$ be such that $\beta' = \lambda(\alpha_1) \vee \cdots \vee \lambda(\alpha_n)$. Since $\lambda(\alpha_j) \leqslant \beta$, in view of (1) we have $\alpha_j \leqslant \alpha$ for any $j = 1, \ldots, n$. Therefore, the element $\alpha' = \alpha_1 \vee \cdots \vee \alpha_n$ is well defined and satisfies the relation $\lambda(\alpha') = \lambda(\alpha_1) \vee \cdots \vee \lambda(\alpha_n) = \beta'$. The uniqueness of α' follows from the injectivity of λ.

(3) Obviously, $\lambda(\inf A') \leqslant \beta'$ for any $\beta' \in \lambda(A')$. Let $\beta \in B$ be such that $\beta \leqslant \beta'$ for all $\beta' \in \lambda(A')$. Then by (2), there is an $\alpha \in A$ such that $\beta = \lambda(\alpha)$, and in view of (1) we have $\alpha \leqslant \alpha'$ for every $\alpha' \in A'$. This implies that $\alpha \leqslant \inf A'$ and $\beta \leqslant \lambda(\inf A')$ and so $\lambda(\inf A') = \inf \lambda(A')$.

The lemma is proved. $\square$

Proof of Theorem 5.7. Let $\mathcal{Z} = \lambda(\mathcal{X})$. Note that A_β is a hereditary subset of A for any $\beta \in B$. The fulfilment of conditions (I) and (III) for $\mathcal{Z}$ therefore follows from Corollary A.2 and from Lemma A.4, respectively. Let $\beta_{1,2} \in B$, $\beta = \beta_1 \vee \beta_2$, and $x \in \mathcal{Z}(\beta)$. Since $\mathcal{Z}(\beta) = \varinjlim \mathcal{X}^{A_\beta}$, there are $\alpha_1, \ldots, \alpha_m \in A$ and $x_1 \in \mathcal{X}(\alpha_1), \ldots, x_m \in \mathcal{X}(\alpha_m)$ such that $\lambda(\alpha_j) \leqslant \beta$ and $x = \sum_{j=1}^m \rho^\beta_{\alpha_j} x_j$, where $\rho^\beta_{\alpha_j}$ is the canonical mapping from $\mathcal{X}(\alpha_j)$ into $\varinjlim \mathcal{X}^{A_\beta}$. Choose $\gamma_1^1, \ldots, \gamma_1^s, \gamma_2^1, \ldots, \gamma_2^t \in A$ such that

$$\beta_1 = \lambda(\gamma_1^1) \vee \cdots \vee \lambda(\gamma_1^s), \quad \beta_2 = \lambda(\gamma_2^1) \vee \cdots \vee \lambda(\gamma_2^t).$$

The distributivity of B implies that

$$\lambda(\alpha_j) = \lambda(\alpha_j) \wedge (\beta_1 \vee \beta_2) = \lambda((\alpha_j \wedge \gamma_1^1) \vee \cdots \vee (\alpha_j \wedge \gamma_2^t)), \quad j = 1, \ldots, m$$

and by the injectivity of λ, we have $\alpha_j = (\alpha_j \wedge \gamma_1^1) \vee \cdots \vee (\alpha_j \wedge \gamma_2^t)$. Since $\mathcal{X}$ satisfies condition (II), for any $j = 1, \ldots, m$ there are $y_j^1 \in \mathcal{X}(\alpha_j \wedge \gamma_1^1), \ldots, y_j^s \in \mathcal{X}(\alpha_j \wedge \gamma_1^s)$ and $z_j^1 \in \mathcal{X}(\alpha_j \wedge \gamma_2^1), \ldots, z_j^t \in \mathcal{X}(\alpha_j \wedge \gamma_2^t)$ such that

$$x_j = \sum_{l=1}^{s} \rho^{\mathcal{X}}_{\alpha_j \wedge \gamma_1^l,\, \alpha_j} y_j^l + \sum_{l=1}^{t} \rho^{\mathcal{X}}_{\alpha_j \wedge \gamma_2^l,\, \alpha_j} z_j^l.$$

Set $y = \sum_{j=1}^m \sum_{l=1}^s \rho^{\beta_1}_{\alpha_j \wedge \gamma_1^l} y_j^l$, $z = \sum_{j=1}^m \sum_{l=1}^t \rho^{\beta_2}_{\alpha_j \wedge \gamma_2^l} z_j^l$. Then $y \in \mathcal{Z}(\beta_1)$, $z \in \mathcal{Z}(\beta_2)$ and we have

$$\begin{aligned}
\rho^{\mathcal{Z}}_{\beta_1,\,\beta}\, y + \rho^{\mathcal{Z}}_{\beta_2,\,\beta}\, z &= \sum_{j=1}^{m} \sum_{l=1}^{s} \rho^{\beta}_{\alpha_j \wedge \gamma_1^l}\, y_j^l + \sum_{j=1}^{m} \sum_{l=1}^{t} \rho^{\beta}_{\alpha_j \wedge \gamma_2^l}\, z_j^l \\
&= \sum_{j=1}^{m} \rho^{\beta}_{\alpha_j} \left[\sum_{l=1}^{s} \rho^{\mathcal{X}}_{\alpha_j \wedge \gamma_1^l,\, \alpha_j} y_j^l + \sum_{l=1}^{t} \rho^{\mathcal{X}}_{\alpha_j \wedge \gamma_2^l,\, \alpha_j} z_j^l \right] = \sum_{j=1}^{m} \rho^{\beta}_{\alpha_j} x_j = x.
\end{aligned}$$

Thus, the inductive system $\mathcal{X}$ satisfies the condition (II) and, consequently, is prelocalizable.

We now suppose that the lattice B is infinitely distributive and $\mathcal{X}$ is a localizable inductive system and check that $\mathcal{Z}$ satisfies condition (III$'$). Let $\{\beta_\omega\}_{\omega \in \Omega}$ be a nonempty family of elements of B bounded above by a $\beta \in B$, and let $\{x_\omega\}_{\omega \in \Omega}$ be a family such that $x_\omega \in \mathcal{Z}(\beta_\omega)$ and $y = \rho^{\mathcal{Z}}_{\beta_\omega,\,\beta} x_\omega$ does not depend on ω.

We first prove the statement for the case when $\beta_{\omega_0} = \lambda(\alpha_0)$ for some $\omega_0 \in \Omega$ and $\alpha_0 \in A$. For brevity, we write $\beta_0 = \beta_{\omega_0}$ and $x_0 = x_{\omega_0}$. Set $\beta'_\omega = \beta_\omega \wedge \beta_0$. Since $\beta'_\omega \leqslant \beta_0$, by Lemma A.5 there are (uniquely defined) $\alpha_\omega \in A$ such that $\alpha_\omega \leqslant \alpha_0$ and $\beta'_\omega = \lambda(\alpha_\omega)$. Because $\mathcal{Z}$ satisfies condition (III), there are $x'_\omega \in \mathcal{Z}(\beta'_\omega)$ such that $\rho^{\mathcal{Z}}_{\beta'_\omega,\,\beta_0} x'_\omega = x_0$ and $\rho^{\mathcal{Z}}_{\beta'_\omega,\,\beta_\omega} x'_\omega = x_\omega$ for every $\omega \in \Omega$. The canonical mapping $\rho^{\lambda(\alpha)}_{\alpha}$ from $\mathcal{X}(\alpha)$ into $\mathcal{Z}(\lambda(\alpha)) = \varinjlim \mathcal{X}^{A_{\lambda(\alpha)}}$ is isomorphic for any $\alpha \in A$ because $\lambda(\alpha)$ is the biggest element of the set $A_{\lambda(\alpha)}$. Therefore, for any $\omega \in \Omega$ there exists a unique $\tilde{x}_\omega \in \mathcal{X}(\alpha_\omega)$ such that $x'_\omega = \rho^{\beta'_\omega}_{\alpha_\omega} \tilde{x}_\omega$. We have $\rho^{\mathcal{Z}}_{\lambda(\alpha'),\,\lambda(\alpha)} \rho^{\lambda(\alpha')}_{\alpha'} = \rho^{\lambda(\alpha)}_{\alpha} \rho^{\mathcal{X}}_{\alpha'\alpha}$ for any $\alpha', \alpha \in A$ such that $\alpha' \leqslant \alpha$. Hence $\rho^{\beta_0}_{\alpha_0} \rho^{\mathcal{X}}_{\alpha_\omega,\,\alpha_0} \tilde{x}_\omega = \rho^{\mathcal{Z}}_{\beta'_\omega,\beta_0} x'_\omega = x_0$ and, consequently, $\rho^{\mathcal{X}}_{\alpha_\omega,\,\alpha_0} \tilde{x}_\omega = \left(\rho^{\beta_0}_{\alpha_0}\right)^{-1} x_0$ does not depend on ω. Let $\tilde{\alpha} = \inf_{\omega \in \Omega} \alpha_\omega$ and $\tilde{\beta} = \lambda(\tilde{\alpha})$. By Lemma A.5, we have $\tilde{\beta} = \inf_{\omega \in \Omega} \beta'_\omega = \inf_{\omega \in \Omega} \beta_\omega$. In view of the localizability of

$\mathcal{X}$ there is an $\tilde{x} \in \mathcal{X}(\tilde{\alpha})$ such that $\tilde{x}_\omega = \rho^{\mathcal{X}}_{\tilde{\alpha},\alpha_\omega}\tilde{x}$ for all $\omega \in \Omega$. Set $x = \rho^{\tilde{\beta}}_{\tilde{\alpha}}\tilde{x}$. Then $x \in \mathcal{Z}(\tilde{\beta})$ and we have

$$\rho^{\mathcal{Z}}_{\tilde{\beta},\beta_\omega} x = \rho^{\mathcal{Z}}_{\beta'_\omega,\beta_\omega}\rho^{\mathcal{Z}}_{\tilde{\beta},\beta'_\omega}\rho^{\tilde{\beta}}_{\tilde{\alpha}}\tilde{x} = \rho^{\mathcal{Z}}_{\beta'_\omega,\beta_\omega}\rho^{\beta'_\omega}_{\alpha_\omega}\rho^{\mathcal{X}}_{\tilde{\alpha},\alpha_\omega}\tilde{x} = \rho^{\mathcal{Z}}_{\beta'_\omega,\beta_\omega} x'_\omega = x_\omega.$$

We now consider the general case. Let $\tilde{\beta} = \inf_{\omega\in\Omega}\beta_\omega$ and J be a finite $\wedge$-closed subset of B such that $J \subset \lambda(A)$ and $\beta = \sup_{\alpha\in J}\lambda(\alpha)$. As in the proof of Lemma A.1, we denote by J_γ ($\gamma \in J$) the set $\{\gamma' \in J \mid \gamma' \geqslant \gamma\}$. For $n \in \mathbb{N}$, set $C_n = \{\gamma \in J \mid |J_\gamma| \geqslant n\}$. We have $J = C_1 \supset C_2 \supset \cdots \supset C_{|J|} = \{\tilde{\gamma}\}$, where $\tilde{\gamma} = \inf J$, and $C_n = \emptyset$ for $n > |J|$. It suffices to show that for any $n \in \mathbb{N}$, there is a family $\{y_\gamma\}_{\gamma\in C_n}$ such that $y_\gamma \in \mathcal{Z}(\gamma)$ and

$$y = \rho^{\mathcal{Z}}_{\tilde{\beta},\beta}\,\tilde{y} + \sum_{\gamma\in C_n}\rho^{\mathcal{Z}}_{\gamma,\beta}\,y_\gamma, \tag{A.5}$$

where $\tilde{y} \in \mathcal{Z}(\tilde{\beta})$. We prove this statement by induction on n. For $n = 1$, the existence of a decomposition of form (A.5) follows from condition (II). Therefore, it suffices to show that if (A.5) holds for some $n \in \mathbb{N}$, then for any $\gamma \in C_n$ there is a family $\{y'_{\gamma'}\}_{\gamma'\in C_{n+1}}$ such that $y_{\gamma'} \in \mathcal{Z}(\gamma')$ and

$$y_\gamma = \rho^{\mathcal{Z}}_{\tilde{\beta}\wedge\gamma,\gamma}\,\tilde{y}_\gamma + \sum_{\gamma'\in C_{n+1}}\rho^{\mathcal{Z}}_{\gamma',\gamma}\,y'_{\gamma'}, \tag{A.6}$$

where $\tilde{y}_\gamma \in \mathcal{Z}(\tilde{\beta}\wedge\gamma)$. Let Ω' be the disjoint union of Ω and a one-element set $\{\varkappa\}$ ($\varkappa \notin \Omega$). Set $\beta'_\varkappa = \gamma$ and $\beta'_\omega = \beta_\omega \vee \sup(C_n \setminus \{\gamma\})$ for $\omega \in \Omega$ (if $C_n \setminus \{\gamma\} = \emptyset$, then we assume $\beta'_\omega = \beta_\omega$). For every $\omega \in \Omega'$, we define an element $x'_\omega \in \mathcal{Z}(\beta'_\omega)$ setting $x'_\varkappa = y_\gamma$ and

$$x'_\omega = \rho^{\mathcal{Z}}_{\beta_\omega,\beta'_\omega}\,x_\omega - \rho^{\mathcal{Z}}_{\tilde{\beta},\beta'_\omega}\,\tilde{y} - \sum_{\gamma'\in C_n\setminus\{\gamma\}}\rho^{\mathcal{Z}}_{\gamma',\beta'_\omega}\,y_{\gamma'}$$

for $\omega \in \Omega$. It follows from (A.5) that the element $\rho^{\mathcal{Z}}_{\beta'_\omega,\beta}\,x'_\omega$ does not depend on $\omega \in \Omega'$. Let $\beta' = \inf_{\omega\in\Omega'}\beta'_\omega$. Since $\beta'_\varkappa \in \lambda(A)$, we can apply the result of the preceding paragraph and find an $x' \in \mathcal{Z}(\beta')$ such that $y_\gamma = x'_\varkappa = \rho^{\mathcal{Z}}_{\beta',\gamma}\,x'$. Because the lattice B is infinitely distributive, we have

$$\beta' = (\tilde{\beta}\wedge\gamma) \vee \sup_{\gamma'\in C_n\setminus\{\gamma\}}(\gamma'\wedge\gamma), \quad (\beta' = \tilde{\beta}\wedge\gamma \text{ for } C_n \setminus \{\gamma\} = \emptyset).$$

Hence, by II, there are $\tilde{y}_\gamma \in \mathcal{Z}(\tilde{\beta}\wedge\gamma)$ and a family $\{z_{\gamma'}\}_{\gamma'\in C_n\setminus\{\gamma\}}$ such that $z_{\gamma'} \in \mathcal{Z}(\gamma'\wedge\gamma)$ and $y_\gamma = \rho^{\mathcal{Z}}_{\tilde{\beta}\wedge\gamma,\gamma}\tilde{y}_\gamma + \sum_{\gamma'\in C_n\setminus\{\gamma\}}\rho^{\mathcal{Z}}_{\gamma'\wedge\gamma,\gamma}y'_{\gamma}$. Because $\gamma'\wedge\gamma \in C_{n+1}$ for $\gamma',\gamma \in C_n$ and $\gamma' \neq \gamma$, we can rewrite the last decomposition in form (A.6). Theorem 5.7 is proved. $\square$

Appendix B. Proof of Lemma 5.9

In what follows, we use the notation introduced in the beginning of Appendix A.

Let l be the linear mapping from $\oplus_{\tau\in T}\mathcal{X}(\lambda(\tau))$ to $\oplus_{\alpha\in A}\mathcal{X}(\alpha)$ such that $lj_\tau = \iota^{\mathcal{X}}_{\lambda(\tau)}$ for any $\tau \in T$. The operator l carries vector (5.2) to the element

$$\iota^{\mathcal{X}}_{\alpha}\rho^{\mathcal{X}}_{\alpha\wedge\alpha',\alpha}x - \iota^{\mathcal{X}}_{\alpha'}\rho^{\mathcal{X}}_{\alpha\wedge\alpha',\alpha'}x = \sigma^{\mathcal{X}}(x,\alpha\wedge\alpha',\alpha') - \sigma^{\mathcal{X}}(x,\alpha\wedge\alpha',\alpha), \tag{B.1}$$

where $\alpha = \lambda(\tau)$ and $\alpha' = \lambda(\tau')$. This implies that $l(\mathcal{N}) \subset N^{\mathcal{X}}_A$ and hence $\mathcal{N} \subset \operatorname{Ker} j^{\mathcal{X}}l$. The mapping $j^{\mathcal{X}}l$ therefore uniquely determines a mapping $m: \oplus_{\tau\in T}\mathcal{X}(\lambda(\tau))/\mathcal{N} \to \varinjlim \mathcal{X}$. To prove the lemma, we have to show that m is an isomorphism. To this end, it suffices to establish the opposite inclusion

$$\mathcal{N} \supset \operatorname{Ker} j^{\mathcal{X}}l. \tag{B.2}$$

Set $I = \lambda(T)$. Let a mapping $\lambda': I \to T$ be such that $\lambda(\lambda'(\alpha)) = \alpha$ for any $\alpha \in I$ and let the mapping $l': M^{\mathcal{X}}_I \to \oplus_{\tau\in T}\mathcal{X}(\lambda(\tau))$ be defined by the relations $l'\iota^{\mathcal{X}}_\alpha x = j_{\lambda'(\alpha)}x$ for any $\alpha \in I$ and $x \in \mathcal{X}(\alpha)$. Clearly, $\operatorname{Im} l'$ coincides with the subspace E of $\oplus_{\tau\in T}\mathcal{X}(\lambda(\tau))$ spanned by all elements $j_\tau x$ with $\tau \in \lambda'(I)$ and $x \in \mathcal{X}(\lambda(\tau))$. Moreover, we have $l'lx = x$ for any $x \in E$. Let us show that every $x \in \oplus_{\tau\in T}\mathcal{X}(\lambda(\tau))$ can be decomposed as $x = n + x'$, where $n \in \mathcal{N}$ and $x' \in E$. It suffices to consider the case $x = j_\tau y$, where $\tau \in T$ and $y \in \mathcal{X}(\lambda(\tau))$. Let $\tau' = \lambda'(\lambda(\tau))$. Then we have $\lambda(\tau) = \lambda(\tau')$ and, consequently, the element $n = j_\tau y - j_{\tau'}y$ belongs to $\mathcal{N}$. Setting $x' = x - n = j_{\tau'}y$, we obtain the desired decomposition because $\tau' \in \lambda'(I)$.

Let $\tilde{\mathcal{N}}$ be the subspace of $M^{\mathcal{X}}_I$ spanned by all vectors of the form $\iota^{\mathcal{X}}_\alpha\rho^{\mathcal{X}}_{\alpha\wedge\alpha',\alpha}x - \iota^{\mathcal{X}}_{\alpha'}\rho^{\mathcal{X}}_{\alpha\wedge\alpha',\alpha'}x$ with $\alpha,\alpha' \in I$ and $x \in \mathcal{X}(\alpha\wedge\alpha')$. We obviously have

$$l(\mathcal{N}) = \tilde{\mathcal{N}}, \quad l'(\tilde{\mathcal{N}}) \subset \mathcal{N}. \tag{B.3}$$

Inclusion (B.2) can be easily derived from the equality

$$N^{\mathcal{X}}_A \cap M^{\mathcal{X}}_I = \tilde{\mathcal{N}} \tag{B.4}$$

which will be proved a little bit later. Indeed, let $x \in \operatorname{Ker} j^{\mathcal{X}}l$. Then we have $lx \in \operatorname{Ker} j^{\mathcal{X}} = N^{\mathcal{X}}_A$ and in view of the obvious inclusion $\operatorname{Im} l \subset M^{\mathcal{X}}_I$ it follows from (B.4) that $lx \in \tilde{\mathcal{N}}$. According to the above we can write $x = n + x'$, where $n \in \mathcal{N}$ and $x' \in E$. By (B.3), we have $ln \in \tilde{\mathcal{N}}$ and, therefore, $lx' \in \tilde{\mathcal{N}}$. Since $x' \in E$, we have $x' = l'lx'$ and it follows from (B.3) that $x' \in \mathcal{N}$. Thus, $x \in \mathcal{N}$ and the implication (B.4) $\Rightarrow$ (B.2) is proved.

It remains to prove (B.4). The inclusion $\tilde{\mathcal{N}} \subset N_A^{\mathcal{X}} \cap M_I^{\mathcal{X}}$ obviously follows from (B.1) and we have to verify that $x \in \tilde{\mathcal{N}}$ supposing $x \in N_A^{\mathcal{X}} \cap M_I^{\mathcal{X}}$. Let $\alpha', \alpha \in A$ be such that $\alpha' \leqslant \alpha$ and let $y \in \mathcal{X}(\alpha')$. Since the set I is cofinal in A, there is a $\beta \in I$ such that $\beta \geqslant \alpha$, and we have $\sigma^{\mathcal{X}}(y, \alpha', \alpha) = \sigma^{\mathcal{X}}(y, \alpha', \beta) - \sigma^{\mathcal{X}}(\rho^{\mathcal{X}}_{\alpha'\alpha} y, \alpha, \beta)$. Therefore, when writing sums of the elements of the form $\sigma^{\mathcal{X}}(y, \alpha', \alpha)$, we can always assume that $\alpha \in I$. In particular, since $x \in N_A^{\mathcal{X}}$, we can write

$$\begin{aligned} x &= \sum_{(\alpha',\alpha)\in A\times I,\, \alpha'\leqslant\alpha} \sigma^{\mathcal{X}}(x_{\alpha'\alpha}, \alpha', \alpha) \\ &= \sum_{\alpha'\in A\setminus I} \sum_{\alpha\in C(\alpha')} \sigma^{\mathcal{X}}(x_{\alpha'\alpha}, \alpha', \alpha) + \sum_{(\alpha',\alpha)\in I\times I,\, \alpha'\leqslant\alpha} \sigma^{\mathcal{X}}(x_{\alpha'\alpha}, \alpha', \alpha), \end{aligned} \tag{B.5}$$

where $C(\alpha') = \{\alpha \in I \mid \alpha' \leqslant \alpha\}$ and the family $\{x_{\alpha'\alpha}\}_{(\alpha',\alpha)\in A\times I}$ contains only finite number of nonzero elements. It is obvious that the second sum in the right-hand side belongs to $\tilde{\mathcal{N}}$. Therefore, it suffices to show that $y_{\alpha'} = \sum_{\alpha\in C(\alpha')} \sigma^{\mathcal{X}}(x_{\alpha'\alpha}, \alpha', \alpha)$ belongs to $\tilde{\mathcal{N}}$ for any given $\alpha' \in A \setminus I$. Since the α'-component of x is equal to zero, equality (B.5) implies that $\sum_{\alpha\in C(\alpha')} x_{\alpha'\alpha} = 0$. Fixing an $\tilde{\alpha} \in C(\alpha')$, we therefore obtain

$$y_{\alpha'} = \sum_{\alpha\in C(\alpha')\setminus\{\tilde{\alpha}\}} (\sigma^{\mathcal{X}}(x_{\alpha'\alpha}, \alpha', \alpha) - \sigma^{\mathcal{X}}(x_{\alpha'\alpha}, \alpha', \tilde{\alpha})).$$

Using (B.1) and the relation

$$\sigma^{\mathcal{X}}(x_{\alpha'\alpha}, \alpha', \alpha) - \sigma^{\mathcal{X}}(x_{\alpha'\alpha}, \alpha', \tilde{\alpha}) = \sigma^{\mathcal{X}}(z, \alpha \wedge \tilde{\alpha}, \alpha) - \sigma^{\mathcal{X}}(z, \alpha \wedge \tilde{\alpha}, \tilde{\alpha}),$$

where $z = \rho^{\mathcal{X}}_{\alpha',\, \alpha\wedge\tilde{\alpha}} x_{\alpha'\alpha}$, we conclude that $y_{\alpha'} \in \tilde{\mathcal{N}}$. The lemma is proved.

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