Optimal Landau–Kolmogorov Inequalities for Dissipative Operators in Hilbert and Banach Spaces

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1. INTRODUCTION

In 1939 Kolmogorov [12] proved that if k and n are integers with 0 ≤ k ≤ n, then there are finite constants C_{n,k}(∞) such that the following inequality holds for any real-valued function f on the real line, where D is differentiation:

\[ \| D^k f \|_\infty \leq C_{n,k}(\infty) \| f \|_l^{1-k/n} \| D^n f \|_l^{k/n}. \]  (1)

Of course this is "merely" a Sobolev inequality, but Kolmogorov discovered a remarkable explicit formula for the best constants C_{n,k}(∞). The formula shows, among other things, that [C_{n,k}(∞)]^n is a rational number. For example, C_{2,1}(∞) = 2^{1/2}. Analogous inequalities hold for functions on the half line, and the corresponding best constants C_{n,k}^+(∞) were determined in 1970 by Schoenberg and Cavaretta [20]. Unlike Kolmogorov, they lack explicit formulas except in a few cases (see [21]), but they do present an effective numerical algorithm for calculating the constants. The special case C_{2,1}^+(∞) = 2 was established in 1913 by Landau with an elementary argument.

The same sort of inequalities hold in the L^p norm, but in most cases the best constants C_{n,k}(p) and C_{n,k}^+(p) are not known. (See [6] for a discussion of this and related problems.) When p = 2, D is a skew-adjoint operator on L^2(−∞, ∞), and the spectral theorem immediately shows that C_{n,k}(2) = 1. But it is by no means trivial to determine the half-line L^2 constants C_{n,k}^+(2). Hardy and Littlewood [7] showed that C_{2,1}^+(2) = 2^{1/2}. The general case was dealt with by Ljubič in 1960 [15]; a recent independent solution is due to Kupcov [13]. These authors give asymptotic formulas for the constants C_{n,k}^+(2) as well as finite algorithms by which they may be calculated explicitly. It follows from this work that C_{n,k}^+(2) is an algebraic number. For example,

\[ C_{3,1}^+(2) = 3^{1/2}/[2(2^{1/2} - 1)]^{1/3}. \]

In the last decade there have been a number of extensions of these inequalities to an abstract operator setting, beginning with Kallman and Rota [10], who

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showed that, if $A$ is the generator of a $(C_0)$ contraction semigroup on a Banach space $E$ (that is, $A$ is $m$-dissipative [16]), then

$$\| Ax \| \leq 2 \| x \|^{1/2} \| A^2 x \|^{1/2} \tag{2}$$

for $x \in \mathcal{D}(A^2)$. Note that the constant 2 is the Landau constant $C_{2,1}(\infty)$, so is the best possible. If $E$ is a Hilbert space, then by a theorem of Kato [11] the constant 2 in the Kallman–Rota inequality can be replaced by $2^{1/2}$, the Hardy-Littlewood constant $C_{2,1}(2)$.

Gindler and Goldstein [6] showed that there are universal constants $M_{n,k}$ such that, for $0 < k < n$,

$$\| A^k x \| \leq M_{n,k} \| x \|^{1-k/n} \| A^n x \|^{k/n}, \tag{3}$$

where is any dissipative operator on a Banach space. However they did not get very good estimates for the constants. The case of $m$-dissipative $A$ was considered earlier by Hille [9], whose approach has been simplified and extended to the general dissipative case by Protter [18].

For $m$-dissipative $A$, Ditzian [4] showed that the best constants $M_{n,k}$ in (3) are precisely the constants $C_{n,k}(\infty)$ for differentiation on $L^\infty([0, \infty))$. This result was also obtained independently by Certain and Kurtz [3] and by the author (unpublished). The argument shows that if in addition $-A$ is $m$-dissipative, so that $A$ generates a group of isometries, then the constants $M_{n,k}$ may be taken to be $C_{n,k}(\infty)$. The proof is extremely simple. Suppose that $x \in \mathcal{D}(A^n)$. Let $\varphi \in E^*$ be a normalized linear functional with $q(A^n) = \| A^n x \|$. Define $f(t) = \varphi(e^{tA}x)$. This is a $C^n$ function on $[0, \infty)$, hence

$$\| D^kf \|_\infty \leq C_{n,k}(\infty) \| f \|^{1-k/n}_\infty \| D^n f \|^{k/n}_\infty. \tag{4}$$

Now $D^nf(t) = \varphi(e^{tA}A^n x)$, so $\| D^n f \|_\infty \leq \| A^n x \|$. Similarly $\| f \|_\infty \leq \| x \|$, while $\| A^n x \| = | D^n f(0) |$. Substituting into (4) we thus obtain (3) with $M_{n,k} = C_{n,k}(\infty)$. The case of a group generator is quite similar. In Section 3 we will extend this argument to deal with fractional powers of $A$.

In Section 2 we will generalize Kato’s theorem by showing that if $A$ is a dissipative operator on a Hilbert space then (3) holds with $M_{n,k} = C_{n,k}(2)$. Thus the $L^2$ differentiation constants are universal for dissipative operators on Hilbert space. We will prove this for fractional powers as well in Section 3.

2. DISSIPATIVE OPERATORS ON HILBERT SPACES

Any contraction semigroup on a Banach space $E$ can be embedded in the translation semigroup on the space $C([0, \infty); E)$ of bounded $E$-valued uniformly continuous functions. This fact underlies the proof given above that the $L^\infty$
constants are universal for generators of contraction semigroups. In the Hilbert space case we will use an analogous, though less trivial, embedding theorem to reduce the general dissipative operator to differentiation on vector-valued $L^2$ functions. Accordingly we first show that an inequality valid for scalar functions can be “inflated” to vector functions. The following lemma establishes that fact in a general setting.

**Lemma 2.1.** Let $A$ and $B$ be closed, densely defined operators on a Hilbert space $H$. Suppose that $\lambda \in [0, 1]$ and $C < \infty$ are constants such that

$$\| Ax \| \leq C \| x \|^{1-\lambda} \| Bx \|^{\lambda}$$

(5)

for all $x \in \mathcal{D}(B)$.

Let $K$ be another Hilbert space, and let $A$ and $B$ denote the closures of the operators $A \otimes I$ and $B \otimes I$ on $H \otimes K$. Then inequality (5) is valid for $A$ and $B$.

**Proof.** The domain of $B \otimes I$ is the algebraic tensor product $\mathcal{D}(B) \otimes K$. It is enough to prove the analog of (5) for vectors in this domain, since one can pass automatically to the closure. (A proof that $A \otimes I$ and $B \otimes I$ are closable, as well as other basic facts about tensor products of operators, can be found in [19, VIII.10].)

A typical $y \in \mathcal{D}(B) \otimes K$ can be written as a finite sum

$$y = \sum_n x_n \otimes e_n,$$

where $x_n \in \mathcal{D}(B)$, and the vectors $e_n$ in $K$ are orthonormal. Then, using (5) and applying Hölder’s inequality, we compute

$$\| Ay \|^2 = \left\| \sum_n Ax_n \otimes e_n \right\|^2 = \sum_n \| Ax_n \|^2 \leq C^2 \sum_n \| x_n \|^{2(1-\lambda)} \| Bx_n \|^{2\lambda} \leq C^2 \left( \sum_n \| x_n \|^2 \right)^{1-\lambda} \left( \sum_n \| Bx_n \|^2 \right)^{\lambda} = C^2 \| y \|^{2(1-\lambda)} \| By \|^{2\lambda}.$$

Thus

$$\| Ay \| \leq C \| y \|^{1-\lambda} \| By \|^{\lambda}.$$ 

Now let $H = L^2(0, \infty)$ and let $K$ be any Hilbert space, so that $H \otimes K = L^2(0, \infty; K)$. If $0 \leq k \leq n$ are integers, take $A = D^k$ and $B = D^n$. Then
$A = D^k$ on the space of $K$-valued $L^2$ functions; likewise $B = D^n$ on this space. Hence we have the following corollary.

**Corollary 2.2.** Let $K$ be a Hilbert space. Let $f$ be a square-integrable $K$-valued function on $[0, \infty)$. Then

$$||D^k f||_a \leq C_{n,k}^+(2) ||f||_a^{1-k/n} ||D^n f||_a^{1-k/n}. \tag{6}$$

(Incidentally, it would suffice to know inequality (6) for real valued $L^2$ functions, since by the lemma we can always tensor with $C$ to conclude (6) for complex functions as well.)

**Theorem 2.3.** Let $A$ be a densely defined dissipative operator on a Hilbert space $H$. Let $0 < k < n$ be integers. Then for all $x \in D(A^n)$,

$$||A^k x|| \leq C_{n,k}^+(2) ||x||^{1-k/n} ||A^n x||^{k/n}. \tag{7}$$

These constants are best possible.

**Proof.** We may always extend $A$ to a maximal dissipative operator. But a maximal dissipative operator on a Hilbert space is $m$-dissipative (see [17]; this is not the case in an arbitrary Banach space [16]). So we may assume that $A$ is the generator of a $(C_0)$ contraction semigroup $e^{tA}$ on $H$. Moreover, replacing $A$ by $A - \epsilon I$ for $\epsilon > 0$, we can assume that $\|e^{tA}\| \to 0$ as $t \to \infty$. (If we can prove (7) in this case we then simply let $\epsilon \to 0$ to deduce it in general.)

We now invoke a well-known representation theorem for such contraction semigroups (see, for example, [14, p. 67]): There exists a Hilbert space $K$ so that $H$ can be represented isometrically as a left-translation invariant subspace of $L^2(0, \infty; K)$ such that $e^{tA}$ corresponds to left translation $e^{tD}$. Accordingly $A$ may be identified with the generator of the restriction of $D$ to some subspace of $L^2(0, \infty; K)$.

Then inequality (7) follows immediately from the corresponding inequality (6) satisfied by $D$. \bbox

The corresponding inequality for generators of groups is valid as well, but in the Hilbert case it is trivial. For the generator of a group of isometries is skew adjoint, and so the optimal constants are all equal to 1.

The technique used to prove Theorem 2.3 of course can be applied to transfer many other inequalities from $D$ on $L^2(0, \infty)$ to arbitrary dissipative operators. Rather than stating a formal general theorem we prefer to give an illustrative example. Everitt [5, p. 158] has shown that, for functions $f$ in $L^2[0, \infty)$ and $\mu \geq 0$,

$$||Df||_2^2 - \mu ||f||_2^2 \leq 2 ||f||_2 ||D^2 f + \mu f||_2. \tag{8}$$
When $\mu = 0$, (8) reduces to the Hardy–Littlewood inequality. By using an appropriate modification of the inflation lemma (2.1) together with the representation theorem for contraction semigroups, we deduce the following generalization of Everitt’s inequality.

**Theorem 2.4.** Let $A$ be a densely defined, dissipative operator on a Hilbert space. Let $\mu$ be a nonnegative constant. Then for all $x \in \mathcal{D}(A^2)$,

$$
||Ax||^2 - \mu ||x||^2 \leq 2 ||x|| ||A^2x + \mu x||.
$$

(9)

As another sort of application of Theorem 2.3 we have a proof of an interesting symmetry property of the coefficients $C_{n,k}^+(2)$ (cf. [15, p. 71; 13, p. 110]).

**Corollary 2.5 (of 2.3).** For $0 < k < n$,

$$
C_{n,k}^+(2) = C_{n,n-k}^+(2).
$$

**Proof.** We abbreviate the coefficients by $C_{n,k}$. Then for any dissipative $A$ on Hilbert space

$$
||A^kx|| \leq C_{n,k} ||x||^{1-k/n} ||A^n x||^{k/n}.
$$

Suppose now that $A$ is bounded and invertible. Then $A^{-1}$ is also bounded, invertible, and dissipative. Replace $A$ by $A^{-1}$ and then replace $x$ by $A^n x$. Then the last inequality becomes

$$
||A^{-k}x|| \leq C_{n,k} ||A^n x||^{1-k/n} ||x||^{k/n}.
$$

(10)

We have established (10) for dissipative operators which are bounded and invertible. But any dissipative $A$ may be approximated by such operators. Indeed, assume $A$ is maximal dissipative. Then (10) holds for $A_c = A(I - \epsilon A)^{-1} - \epsilon I$. If we let $\epsilon$ tend to 0 we get (10) for $A$.

But $C_{n,n-k}$ is the optimal constant for inequality (10). Accordingly $C_{n,n-k} \leq C_{n,k}$. By symmetry the reverse inequality holds. 

**Remark.** The $L^\infty$ constants $C_{n,k}^+(\infty)$ lack this symmetry property. For example $C_{n,1}^+(\infty)^8 = 243/8$ while $C_{n,2}^+(\infty)^8 = 24$ [21, p. 152]. From the vantage point of the proof of Corollary 2.5, this reflects the fact that in general Banach spaces the inverse of a dissipative operator need not be dissipative. (The Kolmogorov constants $C_{n,k}(\infty)$ are asymmetric for the same reason.)

3. Fractional Powers

In 1935, Hardy et al. [8, Theorem 4] proved inequalities like (1) for fractional powers of the differentiation operator $D$ in $L^p$ norms. They used the classical Riemann–Liouville integral formula to define $D^\alpha$ for nonintegral $\alpha$. They did
not investigate the best constants in these inequalities. In 1971, Trebels and Westphal [22] showed by a direct argument that corresponding inequalities are valid for fractional powers of dissipative semigroup generators on Banach spaces. That is, they proved the following.

**Theorem** [22, p. 117]. Let \( A \) be an \( m \)-dissipative operator, and let \( 0 < \alpha < \gamma \) be real numbers. Then there is a constant \( M_{\gamma, \alpha} \), independent of \( A \), such that, for all \( X \in \mathcal{D}((-A)^\gamma) \),

\[
\|(-A)^{\alpha}X\| \leq M_{\gamma, \alpha} \|X\|^{|1-\alpha|/\gamma} \|(-A)^\gamma X\|^\alpha.
\] (11)

Trebels and Westphal gave a bound for \( M_{\gamma, \alpha} \) but they did not attempt to determine the optimal constants.

We want to make the observation that the optimal constants in (11) are the ones associated with \( D \) on \( L^p(0, \infty) \). Likewise the constants associated with \( D \) on \( L^2(0, \infty) \) are optimal for dissipative operators on Hilbert spaces. In other words, the previous arguments for integral powers extend to fractional powers. Of course, the numerical evaluation of these constants for fractional powers of \( D \) is an open, and undoubtedly quite difficult, problem.

If \( A \) generates the contraction semigroup \( e^{tA} \) and \( 0 < \alpha < 1 \), then we define the fractional power \((-A)^{\alpha}\) by the following formula [11]. If \( x \in \mathcal{D}(A) \),

\[
(-A)^{\alpha}x = \frac{1}{\Gamma(\alpha)} \int_0^\infty u^{-1-\alpha}[e^{uA} - I]x \, du.
\] (12)

There are analogous formulas for larger values of \( \alpha \). For the case of the differentiation operator on the space \( C[0, \infty) \) of bounded, uniformly continuous functions we obtain the following formula:

\[
(-D)^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^\infty u^{-(\alpha+1)}[f(t+u) - f(t)] \, du.
\] (13)

This is equivalent to the classical Riemann–Liouville formula used by Hardy et al.

Finally, suppose that \( f(t) = \varphi(e^{tA}x) \), where \( x \in \mathcal{D}(A) \) and \( \varphi \) is a linear functional. From (12) and (13) we find

\[
(-D)^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^\infty u^{-(\alpha+1)}[\varphi(e^{(t+u)A}x) - e^{tA}x] \, du
\]

\[
= \varphi \left( \frac{1}{\Gamma(\alpha)} \int_0^\infty u^{-(\alpha+1)}[e^{uA} - I] e^{tA}x \, du \right)
\]

\[
= \varphi(e^{tA}(-A)^{\alpha}x).
\] (14)
We have proved (14) for $0 < \alpha < 1$, but it is valid for all positive values of $\alpha$ by similar reasoning starting with the appropriate generalization of (12).

It is now easy to extend the reasoning of the integer power case to prove the expected theorem.

**Theorem 3.1.** Let $A$ be the generator of a contraction semigroup on a Banach space. Then if $0 < \alpha < \gamma$ and $x \in \mathcal{D}((-A)^\gamma)$ we have

$$\|(-A)^\alpha x\| \leq C_{\gamma,\alpha}(\infty) \| x \|^{1-\alpha/\gamma} \|(-A)^\gamma x\|^{\alpha/\gamma}. \tag{15}$$

Here the constants are the ones associated with the differentiation operator on the half-line in the $L^\infty$ norm.

If $A$ is the generator of a group of isometries then the constant in (15) may be replaced with $C_{\gamma,\alpha}(\infty)$, the constant for the whole line in the $L^\infty$ norm.

The corresponding Hilbert space theorem is equally straightforward. We need only observe that the inflation lemma (2.1) can be applied because the semigroup generated by $A \otimes I$ is $e^{tA} \otimes I$ and so the $\alpha$ power of $A \otimes I$ is $A^\alpha \otimes I$.

**Theorem 3.2.** Let $A$ be a densely defined, dissipative operator on a Hilbert space. If $0 < \alpha < \gamma$ and $x \in \mathcal{D}((-A)^\gamma)$ then

$$\|(-A)^\alpha x\| \leq C_{\gamma,\alpha}(2) \| x \|^{1-\alpha/\gamma} \|(-A)^\gamma x\|^{\alpha/\gamma}, \tag{16}$$

where the constants are those associated with the differentiation operator on $L^2(0, \infty)$.

Finally we remark that the symmetry property established in Corollary 2.5 obviously extends to the fractional power case: $C_{\gamma,\alpha}(2) = C_{\gamma,\alpha-\alpha}(2)$.

**References**