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J. Differential Equations 226 (2006) 180–209

Journal of Differential Equations

www.elsevier.com/locate/jde

# Forced two layer beta-plane quasigeostrophic flow. Part I: Long-time existence and uniqueness of weak solutions

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Received 14 February 2005; revised 20 September 2005

Available online 14 November 2005

**Abstract**

We consider a model of quasigeostrophic turbulence that has proven useful in theoretical studies of large scale heat transport and coherent structure formation in planetary atmospheres and oceans. The model consists of a coupled pair of hyperbolic PDEs with a forcing which represents domain-scale thermal energy source. Although the use to which the model is typically put involves gathering information from very long numerical integrations, little of a rigorous nature is known about long-time properties of solutions to the equations. In this first paper we define a notion of weak solution, and show using Galerkin methods the long-time existence and uniqueness of such solutions.


*Keywords:* Two-layer quasigeostrophic flow; Existence and uniqueness

## 1. Introduction

Among several challenging aspects of weather prediction, one recognized very early was the large range of time and space scales involved if attempts are based on fundamen-

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0022-0396/$ – see front matter 
doi:10.1016/j.jde.2005.10.002

tal equations of continuum mechanics. “Weather” here refers to motions of relatively low frequency when compared with sound or gravity waves. Pioneering attempts [3,4] with the first computers to predict extra-tropical weather patterns on spatial scales of order 1000 km used a series of observationally motivated approximations to derive a system of equations which “filtered” out relatively high-frequency motions, thereby substantially reducing the range of timescales and easing the computational burden to the point where the goal of a useful forecast came within reach. The assumptions and approximations, now collectively called quasigeostrophic theory, placed special emphasis on observations that the evolution of the horizontal velocity and pressure gradient fields appeared to nearly preserve a “geostrophic” balance between Coriolis and pressure gradients forces, on large space scales and time scales exceeding a day. While computational technology now allows forecasts using equations derived under less restrictive assumptions, and the theory is now but one of a class based on geophysically relevant “balances” (see [16,17]), quasigeostrophic theory and its numerical models remain of interest to meteorologists and oceanographers because they capture a number of physically important features while possessing a structure amenable to mathematical analysis and extensive numerical experimentation.

This paper concerns a simple quasigeostrophic model used by one of authors [14] to study a problem of pattern formation believed to be import in climate studies. The same model has been used for other purposes [8,9,11,12]. It is a coupled pair of 2D vorticity equations, in which the coupling term has the physical interpretation of a temperature field and is of central importance to its use. The system is forced by stimulation of a geophysically important instability present in the system. Numerical integrations indicate that the instability is typically arrested by nonlinearity, and all variables of interest come *eventually* to fluctuate irregularly about a *suitably defined* average value. Different variables take differing amounts of integration time to reveal this behavior; if this occurs for all variables of interest, the system is judged to be at “statistically steady state.” Statistically steady states are not always observed: for some choices of model parameters the system energy grows without bound and integrations must be stopped because of exponential overflow. No analysis has been done that explains this experience.

The model is typically used when many long-time numerical integrations of geophysical turbulence are required for purposes related to climate studies, purposes for which use of a climate models would be unnecessarily (and often prohibitively) demanding of computational time. Reliance on the model has been based on the convincing representation it gives of certain observed phenomena. Data from long numerical integrations are subjected to various averaging procedures to extract information about statistically steady states; these averages constitute the “climate” of the model, and sensitivity of these averages to parametric changes in the model is of interest to theories of climate behavior. No analytical guidance exists for the proper construction, or interpretation, of these averages.

*Our primary motivation in undertaking this study is to put on a firm mathematical ground the calculations in* [14]. We expect that this analytic study will clarify the theoretical difficulties referred to in the preceding paragraph. Also, as the reader will see in the next section, the model system sits in an interesting position between 2D and 3D Navier–Stokes, so the problem may have some independent interest. The most closely related analytical work appears to be that of [1], which establishes finite-time existence and uniqueness for the quasigeostrophic model proposed by [2], with estimates of that finite time based on the size of initial data and the size of the forcing. (We mention recent work on a less closely related equation in the next section.) What we report here is the infinite time existence and uniqueness of a particular kind of weak solution to the equations of [14] (and [8,9,11,12]). Subsequent papers will discuss regularity of such a solution, examine in detail its long-time behavior, and numerical methods.

The plan of the paper is as follows. In Section 2 we present the model in physical space variables, place it in context with recent related work, give some discussion of the forcing, and motivate an energy norm chosen for the subsequent analysis. In Section 3 we reformulate the model in wave-vector space, define relevant function spaces and norms, and present our notion of a weak solution. Sections 4 and 5 follow an approach presented in [7] for study of the Navier–Stokes equations. In Section 4 we define a sequence of approximating Galerkin systems. Each system is a finite set of ODEs with quadratic nonlinearity, constructed by truncating the full wave-vector system at a wavenumber $N$. The long time existence of a classical solution (called there an $N$-solution) for each such system follows from the theory of ordinary differential equations. Key steps involve obtaining bound on energy injection by the forcing and certain algebraic observations that are analogues of integration-by-parts arguments. Section 5 then establishes (Theorem 5.3) the existence of a weak solution by first verifying equicontinuity and uniform boundedness of the family of $N$-solutions, for a fixed wavenumber and time interval $[0, T]$ of integer length $T$. Applications of the Arzela–Ascoli theorem, diagonalizing over wave-numbers and $T$, produces a limit which is then shown to be a weak solution. Section 6 demonstrates the uniqueness of the weak solution. In each of these sections the main effort is to control the nonlinear term: key steps in the proof of Theorem 6.3 involve combinations of Holder's and Ladyzhenskaya's inequalities with a Gronwall argument. In [13] we will show that our unique weak solution is in fact a classical solution. In addition we will prove that the mentioned solution is time and space analytic. Meantime, L. Panetta, E. Titi and M. Ziane have announced in [15] existence and uniqueness results (as well as a dissipativity property) for the strong solutions of our system under a more restrictive condition on the dissipative terms of the system.

## 2. The model system

In this section we employ non-dimensionalizations that we do not discuss. Details can be found in [14,16,17]. Common to all versions of quasigeostrophic theory is the assumption that the horizontal velocity field has a stream-function

$$\vec{u} = \nabla^{\perp}\psi \tag{1}$$

(a non-dimensional form of geostrophic balance), together with an evolution equation for a quantity $Q$

$$\frac{\partial Q}{\partial t} + \frac{\partial \psi}{\partial x_1}\frac{\partial Q}{\partial x_2} - \frac{\partial \psi}{\partial x_2}\frac{\partial Q}{\partial x_1} = F[\psi] + D[\psi]. \tag{2}$$

Here $\nabla^{\perp}\psi = (-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1})$, $(x_1, x_2)$ are horizontal coordinates, $F$ and $D$ are forcing and dissipation terms, and $Q$ is related to $\psi$ by a linear differential operator $L$ in space variables

$$Q = L[\psi]. \tag{3}$$

Different choices for $L$ give different versions of the theory: the general form is

$$L[\psi] = \beta x_2 + \Delta\psi + a(x_3)\frac{\partial}{\partial x_3}\left(b(x_3)\frac{\partial}{\partial x_3}\psi\right). \tag{4}$$

Here $\Delta \equiv \frac{\partial}{\partial x_1^2} + \frac{\partial}{\partial x_2^2}$, $\beta \geqslant 0$ is a constant and $a(x_3)$, $b(x_3)$ are functions related to a reference state density structure which is not explained by the theory. In this form $Q$ is called the continuously stratified version of potential vorticity; in numerical models the vertical dependence is expressed in terms of fluid layers or modes, with appropriate treatments of the vertical derivatives. Thorough discussions from different points of view are given by [16] and [17].

The quantity

$$\tau \equiv \frac{\partial\psi}{\partial x_3} \tag{5}$$

appearing in (4) plays an important role in the theory: it is a representation of temperature (or buoyancy), and in view of (1), its horizontal gradient is related to vertical shear:

$$\frac{\partial\vec{u}}{\partial x_3} = \nabla^{\perp}\tau. \tag{6}$$

The presence of nonzero $\tau$ also allows a form of vorticity generation not present in 2D flow. Versions of the theory that assume $\tau \equiv 0$, are called barotropic, and ones that do not are called baroclinic. (Note that barotropic versions with $\beta = 0$ are simply 2D incompressible Navier–Stokes equations.) For baroclinic versions, an equation for evolution of temperature on the boundary is included. Recent interest has in fact focused on the model that emerges when $Q$ is assumed constant within the interior of the domain, and the evolution equation (2) is replaced by one governing boundary temperature field: this model, with $L[\psi] = -(-\Delta)^{-1/2}\psi$ is called "surface geostrophic theory" and presents an interesting connection with the 3D Euler and Navier–Stokes equations [5,6,10].

The model we study here uses the same vertical discretization of (2), (4) used in the early forecast attempts [4], but with the periodic boundary conditions motivated by [2] and with a special form of forcing that we describe briefly. Details are in [8,14]. The model is defined in terms of a pair of stream-functions $(\psi_1, \psi_2)$. In the physical interpretation, the flow given by $\psi_1$ is at a greater altitude ($x_3$ value) than that given by $\psi_2$. The analogue of the temperature variable (5) is

$$\hat{\psi} = \frac{\psi_1 - \psi_2}{2} \tag{7}$$

and there is a relation corresponding naturally to (6) between horizontal derivatives of $\hat{\psi}$ and vertical velocity differences. It is assumed that the flow takes place in the presence of an imposed, horizontally uniform temperature gradient, with a strength sufficient to excite an exponential instability at a number of scales. This gradient, like the reference stratification, cannot be altered by the flow evolution. It is a stronger physical assumption than a simple imposition of a temperature drop across the domain. What actually appears in the equations is the vertical velocity difference related to the temperature gradient, which we denote in this section by $2\hat{U}$. The equations are

$$\frac{\partial q_1}{\partial t} + \frac{\partial \psi_1}{\partial x_1}\frac{\partial q_1}{\partial x_2} - \frac{\partial \psi_1}{\partial x_2}\frac{\partial q_1}{\partial x_1} = -\left[2\hat{U}\frac{\partial q_1}{\partial x_1} + (\beta + \hat{U})\frac{\partial \psi_1}{\partial x_1}\right] - \nu(-\Delta)^p q_1, \tag{8}$$

$$\frac{\partial q_2}{\partial t} + \frac{\partial \psi_2}{\partial x_1}\frac{\partial q_2}{\partial x_2} - \frac{\partial \psi_2}{\partial x_2}\frac{\partial q_2}{\partial x_1} = -\left[(\beta - \hat{U})\frac{\partial \psi_2}{\partial x_1}\right] - \nu(-\Delta)^p q_2 - \kappa_M \Delta\psi_2. \tag{9}$$

Here the $q_i$ are related to the $\psi_i$ by

$$q_1 = \Delta\psi_1 - \hat{\psi}, \tag{10}$$

$$q_2 = \Delta\psi_2 + \hat{\psi} \tag{11}$$

Solutions $(\psi_1(x_1, x_2, t), \psi_2(x_1, x_2, t))$ to these equations are sought which are *periodic* on the domain $\Omega \equiv [0, 2\pi\hat{L}]^2$, where $\hat{L}$ is a non-dimensional real number. It is also assumed in [14] that such solutions have vanishing horizontal average. (Note: the velocity difference $2\hat{U}$ is actually used to non-dimensionalize the equations in [8,14], and so should be replaced by the value $1/2$. We keep it, in this section alone, to mark terms related to the forcing and to show below how the imposed temperature gradient enters in the energy equation.)

The linear term involving $\beta$ is a representation in this planar geometry of an effect of sphericity in planetary scale flow [16,17]; nonzero $\beta$ is crucial to the formation of jets and introduces long timescales in the solutions [14]. (Getting estimates regarding this effect is one of our aims.) The term involving $\kappa_M$ is a parameterization of a boundary layer effect called Ekman pumping [16,17]. In the terms involving $\nu$, choices of $p > 1$ are not as directly based on physical principles, and have more to do with expectations regarding energy and enstrophy cascades, and most often are made for computational convenience: they are designed to produce dissipative terms, and to concentrate the dissipation processes in simulations at the smallest small spatial scales included in the calculation. The hope is that this does not affect in any important way nonlinear interactions at larger scales. When $p > 1$ the value of $\nu$ has only phenomenological justification. (We note that in [14] the high order Laplacian operator is not applied to the $q_i$, but instead to the $\psi_i$. The analysis we present for the equations here differs inessentially from what would be needed in that case. We choose this form of the equations because it the one being used in currently ongoing numerical studies, and it also agrees with [9,11,12].)

A useful view of the roles of the terms on the right-hand sides of (8), (9) comes from deriving the energy equation for the model. To do this, each layer equation is multiplied by its stream-function, the equations are integrated horizontally, and the results are added. Using the notation (in this section alone)

$$\langle F\rangle = \int_{\Omega} F(x_1, x_2, t)\, dx_1\, dx_2$$

what results after several integrations by parts and uses of periodicity is

$$\frac{\partial E}{\partial t} = 2\hat{U}\left\langle \frac{\partial \tilde{\psi}}{\partial x_1}\hat{\psi}\right\rangle - \kappa_M\left\langle |\nabla \psi_2|^2\right\rangle - \nu P \tag{12}$$

where $\tilde{\psi} \equiv \frac{\psi_1+\psi_2}{2}$ and the total energy $E$ is defined by

$$E = \frac{\langle |\nabla \psi_1|^2 + |\nabla \psi_2|^2\rangle}{2} + \left\langle \hat{\psi}^2\right\rangle \tag{13}$$

is the sum of terms representing the kinetic energies in each layer and the model's form of potential energy. The term $P$ is positive definite:

$$P = \begin{cases} \langle (\Delta^{m+1}\psi_1)^2 + (\Delta^{m+1}\psi_2)^2 + 2|\nabla(\Delta^m\hat{\psi})|^2\rangle & \text{if } p = 2m+1, \\ \langle |\nabla(\Delta^m\psi_1)|^2 + |\nabla(\Delta^m\psi_2)|^2 + 2(\Delta^m\hat{\psi})^2\rangle & \text{if } p = 2m. \end{cases} \tag{14}$$

The only term not clearly sign-definite is that involving $\hat{U}$ and is the energy source term for the model. It corresponds to the net flux of heat down the mean temperature gradient represented by the imposed vertical shear $\hat{U}$. This is as in models of thermal convection, where the energy generation for turbulent motions may also be related to the net down-gradient heat flux.

Notice that formal use of Cauchy–Schwartz and Poincaré inequalities (recall the assumption of zero horizontal average for the $\psi_i$) gives the crude estimate

$$\begin{aligned} \left\langle \frac{\partial \tilde{\psi}}{\partial x_1}\hat{\psi}\right\rangle &\leqslant \left(\left\langle |\nabla\tilde{\psi}|^2\right\rangle\right)^{1/2}\left(\left\langle |\hat{\psi}|^2\right\rangle\right)^{1/2} \leqslant \hat{L}\left(\left\langle |\nabla\tilde{\psi}|^2\right\rangle\right)^{1/2}\left(\left\langle |\nabla\hat{\psi}|^2\right\rangle\right)^{1/2} \\ &= \frac{\hat{L}}{2}\frac{\langle |\nabla\psi_1|^2 + |\nabla\psi_2|^2\rangle}{2} \leqslant \frac{\hat{L}}{2}E. \end{aligned}$$

So from the energy equation (12) we get

$$\frac{\partial E}{\partial t} + \nu P + \kappa_M\left\langle |\nabla\psi_2|^2\right\rangle \leqslant \hat{U}\hat{L}E. \tag{15}$$

An analogue of this argument will be used in Section 4. Notice that no mention of the parameter $\beta$ occurs in this estimate of the domain-integrated energy. (It does, however, appear in the equation for enstrophy equation.) Nevertheless, experience with the model has indicated that the presence of the term $\beta$ fundamentally affects the manner in which energy transfers within the domain occur, and the timescales present in numerical solutions.

We now drop further mention of the constant $\hat{U}$, using instead its value $1/2$.

## 3. The equations

Let $\hat{L} > 0$, $\Omega$ be the square $[0, 2\pi\hat{L}]^2 \subset \mathbb{R}^2$ and $\alpha$ be an arbitrary nonnegative real number. We consider the Eqs. (8)–(11) in $\Omega$ with periodic boundary conditions:

$$\frac{\partial q_1}{\partial t} + \left(\frac{\partial \psi_1}{\partial x_1}\frac{\partial q_1}{\partial x_2} - \frac{\partial \psi_1}{\partial x_2}\frac{\partial q_1}{\partial x_1}\right) = -\frac{\partial q_1}{\partial x_1} - \left(\beta + \frac{1}{2}\right)\frac{\partial \psi_1}{\partial x_1} - \nu(-\Delta)^{1+\alpha} q_1, \tag{16}$$

$$\frac{\partial q_2}{\partial t} + \left(\frac{\partial \psi_2}{\partial x_1}\frac{\partial q_2}{\partial x_2} - \frac{\partial \psi_2}{\partial x_2}\frac{\partial q_2}{\partial x_1}\right) = -\kappa_M \Delta\psi_2 - \left(\beta - \frac{1}{2}\right)\frac{\partial \psi_2}{\partial x_1} - \nu(-\Delta)^{1+\alpha} q_2, \tag{17}$$

where

$$q_1 = \Delta\psi_1 - \frac{\psi_1 - \psi_2}{2} \quad \text{and} \quad q_2 = \Delta\psi_2 + \frac{\psi_1 - \psi_2}{2}. \tag{18}$$

If $\varphi$ is a $2\pi\hat{L}$-periodic complex-valued scalar or vector function which is integrable over $\Omega$, we define its Fourier coefficients by

$$\varphi(\mathbf{k}) = \frac{1}{(2\pi\hat{L})^2}\int_\Omega e^{-(i/\hat{L})\mathbf{k}\cdot\mathbf{x}}\varphi(\mathbf{x})\,d\mathbf{x}, \quad \mathbf{k} \in \mathbb{Z}^2.$$

Its Fourier series will then be

$$\sum_{\mathbf{k}\in\mathbb{Z}^2} \varphi(\mathbf{k}) e^{(i/\hat{L})\mathbf{k}\cdot\mathbf{x}}.$$

Moreover, if $\varphi = \varphi(\mathbf{x}, t)$: $\mathbb{R}^2 \times [0, T]$ (or $[0, \infty)$) $\to \mathbb{C}^d$, $d \in \mathbb{N}$, is $2\pi\hat{L}$-periodic in the plane variable, we denote by $\{\varphi(\mathbf{k}, t)\}_{\mathbf{k}\in\mathbb{Z}^2}$ the Fourier coefficients of $\varphi(\cdot, t)$.

By formally replacing in (16)–(18) $\psi_j(\mathbf{x}, t)$ with $\sum_{\mathbf{k}\in\mathbb{Z}^2}\psi_j(\mathbf{k}, t)e^{(i/\hat{L})\mathbf{k}\cdot\mathbf{x}}$ and $q_j(\mathbf{x}, t)$ with $\sum_{\mathbf{k}\in\mathbb{Z}^2} q_j(\mathbf{k}, t)e^{(i/\hat{L})\mathbf{k}\cdot\mathbf{x}}$, $j = 1, 2$, and identifying the corresponding Fourier coefficients we obtain the following equations:

$$\begin{aligned}&\frac{d}{dt}q_1(\mathbf{k}, t) + \frac{1}{\hat{L}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}(h_2 l_1 - h_1 l_2)\psi_1(\mathbf{h}, t)q_1(\mathbf{l}, t)\\ &\quad = -\frac{i}{\hat{L}}k_1 q_1(\mathbf{k}, t) - \left(\beta + \frac{1}{2}\right)\frac{i}{\hat{L}}k_1\psi_1(\mathbf{k}, t) - \nu\left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)} q_1(\mathbf{k}, t),\end{aligned} \tag{19}$$

$$\begin{aligned}&\frac{d}{dt}q_2(\mathbf{k}, t) + \frac{1}{\hat{L}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}(h_2 l_1 - h_1 l_2)\psi_2(\mathbf{h}, t)q_2(\mathbf{l}, t)\\ &\quad = \kappa_M\frac{|\mathbf{k}|^2}{\hat{L}^2}\psi_2(\mathbf{k}, t) - \left(\beta - \frac{1}{2}\right)\frac{i}{\hat{L}}k_1\psi_2(\mathbf{k}, t) - \nu\left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)} q_2(\mathbf{k}, t),\end{aligned} \tag{20}$$

with

$$q_1(\mathbf{k},t) = -\frac{|\mathbf{k}|^2}{\hat{L}^2}\psi_1(\mathbf{k},t) - \frac{\psi_1(\mathbf{k},t)-\psi_2(\mathbf{k},t)}{2}, \tag{21}$$

$$q_2(\mathbf{k},t) = -\frac{|\mathbf{k}|^2}{\hat{L}^2}\psi_2(\mathbf{k},t) + \frac{\psi_1(\mathbf{k},t)-\psi_2(\mathbf{k},t)}{2}, \tag{22}$$

for every $\mathbf{k} \in \mathbb{Z}^2$. Since $\psi_j(\mathbf{x},t)$, $j = 1,2$, are real-valued functions we have that

$$\psi_j(-\mathbf{k},t) = \overline{\psi_j(\mathbf{k},t)}, \quad \mathbf{k} \in \mathbb{Z}^2,\ j = 1,2, \tag{23}$$

where for a complex number $z$ we denote by $\bar{z}$ the complex conjugate of $z$. Equations (19)–(23) are called the wave-vectors formulation of Eqs. (16)–(18) for plane $2\pi\hat{L}$-periodic solutions. Let

$$\mathcal{K} := \bigl\{\vec{\psi} = \bigl(\{\psi_1(\mathbf{k})\}_{\mathbf{k}\in\mathbb{Z}^2}, \{\psi_2(\mathbf{k})\}_{\mathbf{k}\in\mathbb{Z}^2}\bigr)\colon\ \psi_j(\mathbf{k}) \in \mathbb{C}, \psi_j(-\mathbf{k}) = \overline{\psi_j(\mathbf{k})},\ j = 1,2,\ \mathbf{k} \in \mathbb{Z}^2, \psi_1(\mathbf{0}) + \psi_2(\mathbf{0}) = 0\bigr\} \tag{24}$$

and

$$H := \Bigl\{\vec{\psi} \in \mathcal{K}\colon\ |\vec{\psi}|^2 := \sum_{\mathbf{k}\in\mathbb{Z}^2} E(\vec{\psi})(\mathbf{k}) < \infty\Bigr\}, \tag{25}$$

where

$$E(\vec{\psi})(\mathbf{k}) := \frac{|\mathbf{k}|^2}{\hat{L}^2}\bigl(\bigl|\psi_1(\mathbf{k})\bigr|^2 + \bigl|\psi_2(\mathbf{k})\bigr|^2\bigr) + \frac{|\psi_1(\mathbf{k}) - \psi_2(\mathbf{k})|^2}{2}.$$

The space $\mathcal{K}$ with the metric

$$d(\vec{\psi},\vec{\varphi}) := \sum_{\mathbf{k}\in\mathbb{Z}^2}\left(\sum_{j=1}^{2}\frac{|\psi_j(\mathbf{k}) - \varphi_j(\mathbf{k})|}{1 + |\psi_j(\mathbf{k}) - \varphi_j(\mathbf{k})|}\right)2^{-|\mathbf{k}|^2}, \tag{26}$$

is a Fréchet space, and $H$ with the norm (as above) given by the scalar product

$$\langle\vec{\psi},\vec{\varphi}\rangle := \sum_{\mathbf{k}\in\mathbb{Z}^2}\left[\frac{|\mathbf{k}|^2}{\hat{L}^2}\bigl(\psi_1(\mathbf{k})\overline{\varphi_1(\mathbf{k})} + \psi_2(\mathbf{k})\overline{\varphi_2(\mathbf{k})}\bigr) + \frac{(\psi_1(\mathbf{k}) - \psi_2(\mathbf{k}))(\overline{\varphi_1(\mathbf{k})} - \overline{\varphi_2(\mathbf{k})})}{2}\right]$$

is a Hilbert space. For each $\gamma > 0$ define

$$V_\gamma := \Bigl\{\vec{\psi} \in H\colon\ |\vec{\psi}|_\gamma^2 := \sum_{\mathbf{k}\in\mathbb{Z}^2}\left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2\gamma} E(\vec{\psi})(\mathbf{k}) < \infty\Bigr\}. \tag{27}$$

We denote by $C([0,\infty),\mathcal{K})$ the space of all $\mathcal{K}$-valued continuous functions on $[0,\infty)$, where the continuity is with respect to the metric defined by (26). We also define the spaces $L^\infty_{\text{loc}}([0,\infty),H)$ and $L^2_{\text{loc}}([0,\infty),V_\gamma)$ by the following:

$$L^\infty_{\text{loc}}\big([0,\infty),H\big)=\Big\{\vec{\psi}:[0,\infty)\to H\colon \operatorname*{ess\text{-}sup}_{0\leqslant t\leqslant T}\big|\vec{\psi}(t)\big|<\infty,\ \text{for every } T\in[0,\infty)\Big\}$$

and

$$L^2_{\text{loc}}\big([0,\infty),V_\gamma\big)=\left\{\vec{\psi}:[0,\infty)\to V_\gamma\colon \int_0^T \big|\vec{\psi}(t)\big|_\gamma^2\,dt<\infty,\ \text{for every } T\in[0,\infty)\right\}.$$

Now we are ready to give the definition of a weak solution for (19)–(23) with initial data $\vec{\psi}^0\in H$.

**Definition 3.1.** Let $\vec{\psi}^0\in H$. A $H$-valued function $\vec{\psi}$ is called weak solution for Eqs. (19)–(23) with initial data $\vec{\psi}^0$ if it has the following properties:

(1) $\vec{\psi}\in C\big([0,\infty),\mathcal{K}\big)\cap L^\infty_{\text{loc}}\big([0,\infty),H\big)\cap L^2_{\text{loc}}\big([0,\infty),V_{1+\alpha}\big)$,

(2)
$$q_1(\mathbf{k},t)=q_1(\mathbf{k},0)-\int_0^t\left\{\frac{1}{\hat{L}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}(h_2l_1-h_1l_2)\psi_1(\mathbf{h},\tau)q_1(\mathbf{l},\tau)+\frac{i}{\hat{L}}k_1q_1(\mathbf{k},\tau)\right.$$
$$\left.+\left(\beta+\frac{1}{2}\right)\frac{i}{\hat{L}}k_1\psi_1(\mathbf{k},\tau)+\nu\left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)}q_1(\mathbf{k},\tau)\right\}d\tau,$$
$$q_2(\mathbf{k},t)=q_2(\mathbf{k},0)-\int_0^t\left\{\frac{1}{\hat{L}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}(h_2l_1-h_1l_2)\psi_2(\mathbf{h},\tau)q_2(\mathbf{l},\tau)-\kappa_M\frac{|\mathbf{k}|^2}{\hat{L}^2}\psi_2(\mathbf{k},\tau)\right.$$
$$\left.+\left(\beta-\frac{1}{2}\right)\frac{i}{\hat{L}}k_1\psi_2(\mathbf{k},\tau)+\nu\left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)}q_2(\mathbf{k},\tau)\right\}d\tau,\quad \forall t\in[0,\infty),$$
$$\forall\mathbf{k}\in\mathbb{Z}^2,\quad\text{where}\quad q_1(\mathbf{k},t)=-\frac{|\mathbf{k}|^2}{\hat{L}^2}\psi_1(\mathbf{k},t)-\frac{\psi_1(\mathbf{k},t)-\psi_2(\mathbf{k},t)}{2},$$
$$q_2(\mathbf{k},t)=-\frac{|\mathbf{k}|^2}{\hat{L}^2}\psi_2(\mathbf{k},t)+\frac{\psi_1(\mathbf{k},t)-\psi_2(\mathbf{k},t)}{2},\quad\forall\mathbf{k}\in\mathbb{Z}^2,\quad\text{and}$$

(3) $\psi_j(\mathbf{k},0)=\psi_j^0(\mathbf{k}),\quad j=1,2,\ \forall\mathbf{k}\in\mathbb{Z}^2.$

## 4. Galerkin approximations

In order to prove the existence of a weak solution for Eqs. (19)–(23) we will use the Galerkin approximations technique. Notice that

$$\begin{pmatrix} q_1(\mathbf{k}) \\ q_2(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} -\big(\frac{|\mathbf{k}|^2}{\hat{L}^2}+\frac{1}{2}\big) & \frac{1}{2} \\ \frac{1}{2} & -\big(\frac{|\mathbf{k}|^2}{\hat{L}^2}+\frac{1}{2}\big) \end{pmatrix} \begin{pmatrix} \psi_1(\mathbf{k}) \\ \psi_2(\mathbf{k}) \end{pmatrix}.$$

Denote

$$A_{\mathbf{k}} = \begin{pmatrix} -\big(\frac{|\mathbf{k}|^2}{\hat{L}^2}+\frac{1}{2}\big) & \frac{1}{2} \\ \frac{1}{2} & -\big(\frac{|\mathbf{k}|^2}{\hat{L}^2}+\frac{1}{2}\big) \end{pmatrix}$$

and note that $A_{\mathbf{k}}$ is invertible for every $\mathbf{k} \neq \mathbf{0}$. For every $\mathbf{k} \in \mathbb{Z}^2 \backslash \{\mathbf{0}\}$, Eqs. (19) and (20) become

$$\begin{aligned} &\frac{d}{dt}\begin{pmatrix} \psi_1(\mathbf{k},t) \\ \psi_2(\mathbf{k},t) \end{pmatrix} \\ &\quad = A_{\mathbf{k}}^{-1} \begin{pmatrix} -\frac{1}{\hat{L}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}(h_2 l_1 - h_1 l_2)\psi_1(\mathbf{h},t) q_1(\mathbf{l},t) \\ -\frac{1}{\hat{L}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}(h_2 l_1 - h_1 l_2)\psi_2(\mathbf{h},t) q_2(\mathbf{l},t) \end{pmatrix} \\ &\qquad + A_{\mathbf{k}}^{-1} \begin{pmatrix} -\frac{i}{\hat{L}} k_1 q_1(\mathbf{k},t) - \big(\beta+\frac{1}{2}\big)\frac{i}{\hat{L}} k_1 \psi_1(\mathbf{k},t) - \nu\big(\frac{|\mathbf{k}|}{\hat{L}}\big)^{2(1+\alpha)} q_1(\mathbf{k},t) \\ \kappa_M \frac{|\mathbf{k}|^2}{\hat{L}^2}\psi_2(\mathbf{k},t) - \big(\beta-\frac{1}{2}\big)\frac{i}{\hat{L}} k_1 \psi_2(\mathbf{k},t) - \nu\big(\frac{|\mathbf{k}|}{\hat{L}}\big)^{2(1+\alpha)} q_2(\mathbf{k},t) \end{pmatrix}. \end{aligned} \tag{28}$$

For $N \in \mathbb{N}$ fixed we consider the system:

$$\begin{aligned} &\frac{d}{dt}\begin{pmatrix} \varphi_1(\mathbf{k},t) \\ \varphi_2(\mathbf{k},t) \end{pmatrix} \\ &\quad = A_{\mathbf{k}}^{-1} \begin{pmatrix} -\frac{1}{\hat{L}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}(h_2 l_1 - h_1 l_2)\varphi_1(\mathbf{h},t) r_1(\mathbf{l},t) \\ -\frac{1}{\hat{L}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}(h_2 l_1 - h_1 l_2)\varphi_2(\mathbf{h},t) r_2(\mathbf{l},t) \end{pmatrix} \\ &\qquad + A_{\mathbf{k}}^{-1} \begin{pmatrix} -\frac{i}{\hat{L}} k_1 r_1(\mathbf{k},t) - \big(\beta+\frac{1}{2}\big)\frac{i}{\hat{L}} k_1 \varphi_1(\mathbf{k},t) - \nu\big(\frac{|\mathbf{k}|}{\hat{L}}\big)^{2(1+\alpha)} r_1(\mathbf{k},t) \\ \kappa_M \frac{|\mathbf{k}|^2}{\hat{L}^2}\varphi_2(\mathbf{k},t) - \big(\beta-\frac{1}{2}\big)\frac{i}{\hat{L}} k_1 \varphi_2(\mathbf{k},t) - \nu\big(\frac{|\mathbf{k}|}{\hat{L}}\big)^{2(1+\alpha)} r_2(\mathbf{k},t) \end{pmatrix}, \end{aligned} \tag{29}$$

for $\mathbf{k} \neq \mathbf{0}$, $|\mathbf{k}| \leqslant N$, and

$$\frac{d}{dt}\begin{pmatrix} \varphi_1(\mathbf{0},t) \\ \varphi_2(\mathbf{0},t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{30}$$

where

$$r_1(\mathbf{k},t) = -\frac{|\mathbf{k}|^2}{\hat{L}^2}\varphi_1(\mathbf{k},t) - \frac{\varphi_1(\mathbf{k},t)-\varphi_2(\mathbf{k},t)}{2}, \tag{31}$$

$$r_2(\mathbf{k},t) = -\frac{|\mathbf{k}|^2}{\hat{L}^2}\varphi_2(\mathbf{k},t) + \frac{\varphi_1(\mathbf{k},t)-\varphi_2(\mathbf{k},t)}{2}. \tag{32}$$

We will be referring to Eqs. (29) and (30) together with (31) and (32) as the $N$-system.

**Definition 4.1.** Let $\mathbb{Z}^2_N = \{\mathbf{k} \in \mathbb{Z}^2 \mid |\mathbf{k}| \leqslant N\}$. A $N$-solution is a family of functions $\{(\varphi_1(\mathbf{k},\cdot), \varphi_2(\mathbf{k},\cdot))\}_{\mathbf{k}\in\mathbb{Z}^2_N}$ satisfying the $N$-system.

**Lemma 4.2.** *Let $N \in \mathbb{N}$ and $\vec{\psi}^0 \in H$. Then*

(a) *there exist $t_0 > 0$ and $\{(\varphi_1(\mathbf{k},\cdot), \varphi_2(\mathbf{k},\cdot))\}_{\mathbf{k}\in\mathbb{Z}^2_N}$ such that*
 (i) *$\varphi_j(\mathbf{k},\cdot) \in C^\infty([0,t_0];\mathbb{C})$,*
 (ii) *$\{(\varphi_1(\mathbf{k},\cdot), \varphi_2(\mathbf{k},\cdot))\}_{\mathbf{k}\in\mathbb{Z}^2_N}$ is a $N$-solution with $\varphi_j(\mathbf{k},0) = \psi_j^0(\mathbf{k})$, $\forall|\mathbf{k}| \leqslant N$, $j = 1,2$, and*
 (iii) *$\overline{\varphi_j(\mathbf{k},t)} = \varphi_j(-\mathbf{k},t)$, $\forall|\mathbf{k}| \leqslant N$, $j = 1,2$,*

(b) *for every $T \in (0,\infty)$ with the property that the above solution exists on $[0,T)$ there exists $M > 0$ such that*

$$\left|\varphi_j(\mathbf{k},t)\right| \leqslant M, \quad \forall t \in [0,T),\ \forall|\mathbf{k}| \leqslant N,\ j = 1,2. \tag{33}$$

*Moreover, the $N$-solution $\{(\varphi_1(\mathbf{k},\cdot), \varphi_2(\mathbf{k},\cdot))\}_{\mathbf{k}\in\mathbb{Z}^2_N}$ with initial data $\vec{\psi}^0$ is unique in the interval of existence.*

**Proof.** Part (a) follows immediately from the classical theory of systems of ordinary differential equations and the fact that $\{\overline{\varphi_j(\mathbf{k},t)}\}_{|\mathbf{k}|\leqslant N, j=1,2}$ and $\{\varphi_j(-\mathbf{k},t)\}_{|\mathbf{k}|\leqslant N, j=1,2}$ are solutions for the same system of ODEs with the same initial data (since $\overline{\psi_j^0(\mathbf{k})} = \psi_j^0(-\mathbf{k})$, for every $\mathbf{k} \in \mathbb{Z}^2$). For (b) we start by noticing that from (30) we have that

$$\varphi_j(\mathbf{0},t) = \psi_j^0(\mathbf{0}), \quad \forall t \in [0,T),\ j = 1,2. \tag{34}$$

Using Eqs. (31) and (32) we also get

$$\begin{aligned}
&\mathrm{Re} \sum_{|\mathbf{k}|\leqslant N} \left(\left(\frac{d}{dt} r_1(\mathbf{k},t)\right)\overline{\varphi_1(\mathbf{k},t)} + \left(\frac{d}{dt} r_2(\mathbf{k},t)\right)\overline{\varphi_2(\mathbf{k},t)}\right) \\
&\quad = \mathrm{Re} \sum_{|\mathbf{k}|\leqslant N} \left\{-\frac{|\mathbf{k}|^2}{\hat{L}^2}\left(\frac{d}{dt}\varphi_1(\mathbf{k},t)\right)\overline{\varphi_1(\mathbf{k},t)} - \frac{|\mathbf{k}|^2}{\hat{L}^2}\left(\frac{d}{dt}\varphi_2(\mathbf{k},t)\right)\overline{\varphi_2(\mathbf{k},t)}\right. \\
&\qquad \left. - \left(\frac{d}{dt}\left(\frac{\varphi_1(\mathbf{k},t) - \varphi_2(\mathbf{k},t)}{2}\right)\right)\overline{\varphi_1(\mathbf{k},t)} + \left(\frac{d}{dt}\left(\frac{\varphi_1(\mathbf{k},t) - \varphi_2(\mathbf{k},t)}{2}\right)\right)\overline{\varphi_2(\mathbf{k},t)}\right\} \\
&\quad = -\frac{1}{2}\frac{d}{dt} \sum_{|\mathbf{k}|\leqslant N} \left\{\frac{|\mathbf{k}|^2}{\hat{L}^2}\left(\left|\varphi_1(\mathbf{k},t)\right|^2 + \left|\varphi_2(\mathbf{k},t)\right|^2\right) + \frac{|\varphi_1(\mathbf{k},t) - \varphi_2(\mathbf{k},t)|^2}{2}\right\}.
\end{aligned} \tag{35}$$

We will extend a $N$-solution in a natural way to a function $\vec{\psi}_N$ such that for every $t$ in the interval of existence of our $N$-solution we have $\vec{\psi}_N(t) \in \mathcal{K}$, namely,

$$\psi_N(\mathbf{k},t) = \varphi(\mathbf{k},t), \quad \text{if } |\mathbf{k}| \leqslant N \quad \text{and} \quad \psi_N(\mathbf{k},t) = \begin{pmatrix}0\\0\end{pmatrix}, \quad \text{if } |\mathbf{k}| > N. \tag{36}$$

For $\vec{\psi}_N$ we then obtain from (35) that

$$\frac{1}{2}\frac{d}{dt}\left|\vec{\psi}_N(t)\right|^2 = -\operatorname{Re}\sum_{|\mathbf{k}|\leqslant N}\left(\frac{d}{dt}r_1(\mathbf{k},t)\right)\overline{\varphi_1(\mathbf{k},t)} + \left(\frac{d}{dt}r_2(\mathbf{k},t)\right)\overline{\varphi_2(\mathbf{k},t)},$$

and using (29) we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left|\vec{\psi}_N(t)\right|^2 = -\operatorname{Re}\sum_{|\mathbf{k}|\leqslant N}\Bigg\{&\Bigg(-\frac{1}{\hat{L}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}(h_2l_1-h_1l_2)\varphi_1(\mathbf{h},t)r_1(\mathbf{l},t) \\
&-\frac{i}{\hat{L}}k_1r_1(\mathbf{k},t) - \left(\beta+\frac{1}{2}\right)\frac{i}{\hat{L}}k_1\varphi_1(\mathbf{k},t) - \nu\left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)} r_1(\mathbf{k},t)\Bigg)\overline{\varphi_1(\mathbf{k},t)} \\
&+\Bigg(-\frac{1}{\hat{L}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}(h_2l_1-h_1l_2)\varphi_2(\mathbf{h},t)r_2(\mathbf{l},t) + \kappa_M\frac{|\mathbf{k}|^2}{\hat{L}^2}\varphi_2(\mathbf{k},t) \\
&-\left(\beta-\frac{1}{2}\right)\frac{i}{\hat{L}}k_1\varphi_2(\mathbf{k},t) - \nu\left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)} r_2(\mathbf{k},t)\Bigg)\overline{\varphi_2(\mathbf{k},t)}\Bigg\},
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left|\vec{\psi}_N(t)\right|^2 = \frac{1}{\hat{L}^2}\operatorname{Re}\sum_{|\mathbf{k}|\leqslant N}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}&(h_2l_1-h_1l_2)\big(\varphi_1(\mathbf{h},t)r_1(\mathbf{l},t)\overline{\varphi_1(\mathbf{k},t)} \\
&+\varphi_2(\mathbf{h},t)r_2(\mathbf{l},t)\overline{\varphi_2(\mathbf{k},t)}\big) + \operatorname{Re}\left(\frac{i}{\hat{L}}\sum_{|\mathbf{k}|\leqslant N}k_1r_1(\mathbf{k},t)\overline{\varphi_1(\mathbf{k},t)}\right) \\
&-\kappa_M\sum_{|\mathbf{k}|\leqslant N}\frac{|\mathbf{k}|^2}{\hat{L}^2}\left|\varphi_2(\mathbf{k},t)\right|^2 \\
&+\nu\operatorname{Re}\sum_{|\mathbf{k}|\leqslant N}\left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)}\big(r_1(\mathbf{k},t)\overline{\varphi_1(\mathbf{k},t)} + r_2(\mathbf{k},t)\overline{\varphi_2(\mathbf{k},t)}\big). \qquad (37)
\end{aligned}$$

Using (iii) from part (a) of Lemma 4.2 we deduce that

$$\begin{aligned}
S_1 :=& \sum_{|\mathbf{k}|\leqslant N}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}(h_2l_1-h_1l_2)\varphi_1(\mathbf{h},t)r_1(\mathbf{l},t)\overline{\varphi_1(\mathbf{k},t)} \\
=& \sum_{\mathbf{h}+\mathbf{l}+\mathbf{k}=\mathbf{0},\ |\mathbf{h}|,|\mathbf{l}|,|\mathbf{k}|\leqslant N}(h_2l_1-h_1l_2)\varphi_1(\mathbf{h},t)r_1(\mathbf{l},t)\varphi_1(\mathbf{k},t),
\end{aligned}$$

and after we interchange **h** with **k** we obtain

$$
\begin{aligned}
S_1 &= \sum_{\mathbf{h}+\mathbf{l}+\mathbf{k}=\mathbf{0},\ |\mathbf{h}|,|\mathbf{l}|,|\mathbf{k}|\leqslant N} (k_2 l_1 - k_1 l_2)\varphi_1(\mathbf{k},t) r_1(\mathbf{l},t)\varphi_1(\mathbf{h},t) \\
&= \sum_{\mathbf{h}+\mathbf{l}+\mathbf{k}=\mathbf{0},\ |\mathbf{h}|,|\mathbf{l}|,|\mathbf{k}|\leqslant N} \big((-h_2 - l_2)l_1 - (-h_1 - l_1)l_2\big)\varphi_1(\mathbf{k},t) r_1(\mathbf{l},t)\varphi_1(\mathbf{h},t) \\
&= \sum_{\mathbf{h}+\mathbf{l}+\mathbf{k}=\mathbf{0},\ |\mathbf{h}|,|\mathbf{l}|,|\mathbf{k}|\leqslant N} (-h_2 l_1 + h_1 l_2)\varphi_1(\mathbf{k},t) r_1(\mathbf{l},t)\varphi_1(\mathbf{h},t) = -S_1.
\end{aligned}
$$

Therefore, $S_1 = 0$. Similarly,

$$
S_2 := \sum_{|\mathbf{k}|\leqslant N} \sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (h_2 l_1 - h_1 l_2)\varphi_2(\mathbf{h},t) r_2(\mathbf{l},t)\overline{\varphi_2(\mathbf{k},t)} = 0.
$$

Thus, (37) becomes

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big|\vec{\psi}_N(t)\big|^2 + \kappa_M \sum_{|\mathbf{k}|\leqslant N} \frac{|\mathbf{k}|^2}{\hat{L}^2}\big|\varphi_2(\mathbf{k},t)\big|^2 \\
&\quad = \operatorname{Re}\left(\frac{i}{\hat{L}} \sum_{|\mathbf{k}|\leqslant N} k_1 r_1(\mathbf{k},t)\overline{\varphi_1(\mathbf{k},t)}\right) \\
&\qquad + \nu \operatorname{Re} \sum_{|\mathbf{k}|\leqslant N} \left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)} \big(r_1(\mathbf{k},t)\overline{\varphi_1(\mathbf{k},t)} + r_2(\mathbf{k},t)\overline{\varphi_2(\mathbf{k},t)}\big).
\end{aligned} \tag{38}
$$

From (31) and (32) we easily get that

$$
r_1(\mathbf{k},t)\overline{\varphi_1(\mathbf{k},t)} + r_2(\mathbf{k},t)\overline{\varphi_2(\mathbf{k},t)} = -E(\vec{\varphi})(\mathbf{k}) \quad \text{and} \tag{39}
$$

$$
\begin{aligned}
S_3 &:= \operatorname{Re}\left(\frac{i}{\hat{L}} \sum_{|\mathbf{k}|\leqslant N} k_1 r_1(\mathbf{k},t)\overline{\varphi_1(\mathbf{k},t)}\right) \\
&= \operatorname{Re}\frac{i}{\hat{L}}\left(-\sum_{|\mathbf{k}|\leqslant N} k_1\left(\frac{|\mathbf{k}|^2}{\hat{L}^2} + \frac{1}{2}\right)\big|\varphi_1(\mathbf{k},t)\big|^2 + \frac{1}{2}\sum_{|\mathbf{k}|\leqslant N} k_1\varphi_2(\mathbf{k},t)\overline{\varphi_1(\mathbf{k},t)}\right) \\
&= \operatorname{Re}\left(\frac{i}{2\hat{L}} \sum_{|\mathbf{k}|\leqslant N} k_1\varphi_2(\mathbf{k},t)\overline{\varphi_1(\mathbf{k},t)}\right).
\end{aligned} \tag{40}
$$

Using (39) and (40), (38) becomes

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big|\vec{\psi}_N(t)\big|^2 + \kappa_M \sum_{|\mathbf{k}|\leqslant N} \frac{|\mathbf{k}|^2}{\hat{L}^2}\big|\varphi_2(\mathbf{k},t)\big|^2 \\
&\quad = \operatorname{Re}\left(\frac{i}{2\hat{L}} \sum_{|\mathbf{k}|\leqslant N} k_1\varphi_2(\mathbf{k},t)\overline{\varphi_1(\mathbf{k},t)}\right) - \nu \sum_{|\mathbf{k}|\leqslant N} \left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)} E(\vec{\psi}_N)(\mathbf{k}).
\end{aligned} \tag{41}
$$

Next we notice that

$$\begin{aligned}
&\left|\frac{i}{2\hat{L}}\sum_{|\mathbf{k}|\leqslant N} k_1\varphi_2(\mathbf{k},t)\overline{\varphi_1(\mathbf{k},t)}\right| \\
&\quad\leqslant \frac{1}{2\hat{L}}\sum_{|\mathbf{k}|\leqslant N} |\mathbf{k}|\left|\varphi_2(\mathbf{k},t)\right|\left|\varphi_1(\mathbf{k},t)\right| \\
&\quad\leqslant \frac{\hat{L}}{2}\sum_{|\mathbf{k}|\leqslant N}\left(\frac{|\mathbf{k}|}{\hat{L}}\left|\varphi_1(\mathbf{k},t)\right|\right)\left(\frac{|\mathbf{k}|}{\hat{L}}\left|\varphi_2(\mathbf{k},t)\right|\right) \\
&\quad\leqslant \frac{\hat{L}}{4}\sum_{|\mathbf{k}|\leqslant N}\frac{|\mathbf{k}|^2}{\hat{L}^2}\left(\left|\varphi_1(\mathbf{k},t)\right|^2+\left|\varphi_2(\mathbf{k},t)\right|^2\right)\leqslant \frac{\hat{L}}{4}\left|\vec{\psi}_N(t)\right|^2.
\end{aligned} \tag{42}$$

From (41) and (42) we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left|\vec{\psi}_N(t)\right|^2+\kappa_M\sum_{|\mathbf{k}|\leqslant N}\frac{|\mathbf{k}|^2}{\hat{L}^2}\left|\varphi_2(\mathbf{k},t)\right|^2+\nu\sum_{|\mathbf{k}|\leqslant N}\left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)}E(\vec{\psi}_N)(\mathbf{k}) \\
&\quad\leqslant \frac{\hat{L}}{4}\left|\vec{\psi}_N(t)\right|^2, \quad \forall t\in[0,T).
\end{aligned} \tag{43}$$

Therefore, $\frac{1}{2}\frac{d}{dt}|\vec{\psi}_N(t)|^2 \leqslant \frac{\hat{L}}{4}|\vec{\psi}_N(t)|^2$, $\forall t\in[0,T)$ which implies that $|\vec{\psi}_N(t)|^2 \leqslant e^{(\hat{L}/2)t}|\vec{\psi}^0|^2 \leqslant e^{\hat{L}/2)T}|\vec{\psi}^0|^2$, $\forall t\in[0,T)$. From here and relation (34) we easily get that $\exists M>0$ such that $|\varphi_j(\mathbf{k},t)|\leqslant M$, $\forall t\in[0,T)$, $\forall|\mathbf{k}|\leqslant N$, $j=1,2$. □

From Lemma 4.2(b) and the classical theory of ODEs it follows immediately:

**Corollary 4.3.** *For given $\vec{\psi}^0\in H$ there exists a unique $N$-solution with initial data $\vec{\psi}^0$ defined on* $[0,\infty)$.

**Corollary 4.4.** *The function $\vec{\psi}_N$ defined by* (36) *belongs to* $C([0,\infty),\mathcal{K})$.

**Proof.** Recall that $\overline{\varphi_j(\mathbf{k},t)}=\varphi_j(-\mathbf{k},t)$, $\forall|\mathbf{k}|\leqslant N$, $j=1,2$, and notice also that $\varphi_1(\mathbf{0},t)+\varphi_2(\mathbf{0},t)=\psi_1^0(\mathbf{0})+\psi_2^0(\mathbf{0})=0$, $\forall t\in[0,\infty)$. Therefore,

$$\vec{\psi}_N(t)\in\mathcal{K}, \quad \forall t\in[0,\infty).$$

Since $\varphi_j(\mathbf{k},\cdot)$ is continuous on $[0,\infty)$, $\forall|\mathbf{k}|\leqslant N$, $j=1,2$, we see that $\vec{\psi}_N(\cdot)\in C([0,\infty),\mathcal{K})$. □

## 5. Existence of weak solutions

Applying the process in Section 4 for every $N \in \mathbb{N}$ we get the sequence $\{\vec{\psi}_N(\cdot)\}_{N\in\mathbb{N}} \subset C([0,\infty),\mathcal{K})$. On the space $C([0,\infty),\mathcal{K})$ we define the metric

$$\operatorname{dist}\bigl(\vec{\psi}(\cdot),\vec{\varphi}(\cdot)\bigr) = \sum_{T=1,2,\dots} \frac{1}{2^T}\frac{\sup\{d(\vec{\psi}(t),\vec{\varphi}(t)):\ 0\leqslant t\leqslant T\}}{1+\sup\{d(\vec{\psi}(t),\vec{\varphi}(t)):\ 0\leqslant t\leqslant T\}}.$$

**Remark 5.1.** The convergence $\operatorname{dist}(\vec{\varphi}_m(\cdot),\vec{\varphi}(\cdot)) \to 0$ as $m\to\infty$ is equivalent to, for every $\mathbf{k}\in\mathbb{Z}^2$ and $t_0\in[0,\infty)$, $\varphi_{m,j}(\mathbf{k},t)\to\varphi_j(\mathbf{k},t)$ uniformly on $[0,t_0]$, $j=1,2$.

The proof of the existence of weak solutions for (19)–(23) with initial data $\vec{\psi}^0 \in H$ will be split in two parts. First we prove that there exists a subsequence $\{\vec{\psi}_{N_p}(\cdot)\}_{p\in\mathbb{N}}$ of $\{\vec{\psi}_N(\cdot)\}_{N\in\mathbb{N}}$ converging to some $\vec{\psi}(\cdot)$ in $C([0,\infty),\mathcal{K})$. After that we will show that the limit $\vec{\psi}(\cdot)$ is our desired weak solution. The first part is covered by the following lemma.

**Lemma 5.2.** *There exist a subsequence $\{\vec{\psi}_{N_p}(\cdot)\}_{p\in\mathbb{N}}$ of $\{\vec{\psi}_N(\cdot)\}_{N\in\mathbb{N}}$ and a function $\vec{\psi}(\cdot)\in C([0,\infty),\mathcal{K})$ such that* $\lim_{p\to\infty}\operatorname{dist}(\vec{\psi}_{N_p},\vec{\psi})=0$.

**Proof.** Let $T$, $N\in\mathbb{N}$ be fixed. Using (29) we can write

$$\begin{aligned}
&\frac{d}{dt}\bigl(r_1(\mathbf{k},t)+r_2(\mathbf{k},t)\bigr)\\
&\quad= -\frac{1}{\hat{L}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k},\ |\mathbf{h}|,|\mathbf{l}|\leqslant N}(h_2l_1-h_1l_2)\bigl(\varphi_1(\mathbf{h},t)r_1(\mathbf{l},t)+\varphi_2(\mathbf{h},t)r_2(\mathbf{l},t)\bigr)\\
&\qquad-\frac{i}{\hat{L}}k_1r_1(\mathbf{k},t)-\left(\beta+\frac{1}{2}\right)\frac{i}{\hat{L}}k_1\varphi_1(\mathbf{k},t)-\left(\beta-\frac{1}{2}\right)\frac{i}{\hat{L}}k_1\varphi_2(\mathbf{k},t)\\
&\qquad+\kappa_M\frac{|\mathbf{k}|^2}{\hat{L}^2}\varphi_2(\mathbf{k},t)-\nu\left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)}\bigl(r_1(\mathbf{k},t)+r_2(\mathbf{k},t)\bigr).
\end{aligned}\tag{44}$$

Next we add (31) with (32) and we divide by $-\frac{|\mathbf{k}|^2}{\hat{L}^2}$. With the use of (44) we get

$$\begin{aligned}
&\frac{d}{dt}\bigl(\varphi_1(\mathbf{k},t)+\varphi_2(\mathbf{k},t)\bigr)\\
&\quad= \frac{1}{|\mathbf{k}|^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k},\ |\mathbf{h}|,|\mathbf{l}|\leqslant N}(h_2l_1-h_1l_2)\bigl(\varphi_1(\mathbf{h},t)r_1(\mathbf{l},t)+\varphi_2(\mathbf{h},t)r_2(\mathbf{l},t)\bigr)\\
&\qquad+\frac{i\hat{L}k_1}{|\mathbf{k}|^2}r_1(\mathbf{k},t)+\left(\beta+\frac{1}{2}\right)\frac{i\hat{L}k_1}{|\mathbf{k}|^2}\varphi_1(\mathbf{k},t)+\left(\beta-\frac{1}{2}\right)\frac{i\hat{L}k_1}{|\mathbf{k}|^2}\varphi_2(\mathbf{k},t)\\
&\qquad-\kappa_M\varphi_2(\mathbf{k},t)-\nu\left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)}\bigl(\varphi_1(\mathbf{k},t)+\varphi_2(\mathbf{k},t)\bigr).
\end{aligned}\tag{45}$$

Now define $\tilde{\varphi}(\mathbf{k},t) := \varphi_1(\mathbf{k},t) + \varphi_2(\mathbf{k},t)$ and $\hat{\varphi}(\mathbf{k},t) := \varphi_1(\mathbf{k},t) - \varphi_2(\mathbf{k},t)$. For $s,t \in [0,T]$, $s<t$, from (45) we obtain

$$\begin{aligned}
&\left|\tilde{\varphi}(\mathbf{k},t) - \tilde{\varphi}(\mathbf{k},s)\right| \\
&\quad \leqslant \frac{1}{|\mathbf{k}|^2}\left|\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k},\ |\mathbf{h}|,|\mathbf{l}|\leqslant N} \int_s^t (h_2l_1 - h_1l_2)\big(\varphi_1(\mathbf{h},\tau)r_1(\mathbf{l},\tau) + \varphi_2(\mathbf{h},\tau)r_2(\mathbf{l},\tau)\big)\,d\tau\right| \\
&\qquad + \frac{\hat{L}}{|\mathbf{k}|}\int_s^t \left|r_1(\mathbf{k},\tau)\right|d\tau + \left(\beta + \frac{1}{2}\right)\frac{\hat{L}}{|\mathbf{k}|}\int_s^t \left|\varphi_1(\mathbf{k},\tau)\right|d\tau \\
&\qquad + \left(\beta - \frac{1}{2}\right)\frac{\hat{L}}{|\mathbf{k}|}\int_s^t \left|\varphi_2(\mathbf{k},\tau)\right|d\tau + \kappa_M\int_s^t \left|\varphi_2(\mathbf{k},\tau)\right|d\tau \\
&\qquad + \nu\left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)}\int_s^t \big(\left|\varphi_1(\mathbf{k},\tau)\right| + \left|\varphi_2(\mathbf{k},\tau)\right|\big)\,d\tau.
\end{aligned} \tag{46}$$

Recall that we proved that

$$\begin{aligned}
\left|\vec{\psi}_N(t)\right|^2 &= \sum_{|\mathbf{k}|\leqslant N}\left(\frac{|\mathbf{k}|^2}{\hat{L}^2}\big(\left|\varphi_1(\mathbf{k},t)\right|^2 + \left|\varphi_2(\mathbf{k},t)\right|^2\big) + \frac{|\varphi_1(\mathbf{k},t) - \varphi_2(\mathbf{k},t)|^2}{2}\right) \\
&\leqslant e^{\hat{L}T/2}\left|\vec{\psi}^0\right|^2,
\end{aligned}$$

for every $t \in [0,T]$. Therefore, for $\mathbf{k} \neq \mathbf{0}$, we have

$$\left|\varphi_j(\mathbf{k},t)\right| \leqslant \frac{\hat{L}}{|\mathbf{k}|}e^{(\hat{L}T)/4}\left|\vec{\psi}^0\right|, \quad \forall t \in [0,T],\ j = 1,2.$$

Using this we see that all the integrals from the right-hand side of (46) except the first one are bounded by $c_1(t-s)$ where $c_1$ is some positive constant which does not depend on $N$. From (31) we obtain

$$\begin{aligned}
&\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k},\ |\mathbf{h}|,|\mathbf{l}|\leqslant N} (h_2l_1 - h_1l_2)\varphi_1(\mathbf{h},\tau)r_1(\mathbf{l},\tau) \\
&\quad = -\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k},\ |\mathbf{h}|,|\mathbf{l}|\leqslant N} (h_2l_1 - h_1l_2)\frac{|\mathbf{l}|^2}{\hat{L}^2}\varphi_1(\mathbf{h},\tau)\varphi_1(\mathbf{l},\tau) \\
&\qquad - \frac{1}{2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k},\ |\mathbf{h}|,|\mathbf{l}|\leqslant N} (h_2l_1 - h_1l_2)\varphi_1(\mathbf{h},\tau)\big(\varphi_1(\mathbf{l},\tau) - \varphi_2(\mathbf{l},\tau)\big).
\end{aligned} \tag{47}$$

We notice that

$$
\begin{aligned}
S_4 &:= \sum_{\mathbf{h}+\mathbf{l}=\mathbf{k},\ |\mathbf{h}|,|\mathbf{l}|\leqslant N} (h_2 l_1 - h_1 l_2)\frac{|\mathbf{l}|^2}{\hat{L}^2}\varphi_1(\mathbf{h},\tau)\varphi_1(\mathbf{l},\tau) \\
&= \sum_{\mathbf{h}+\mathbf{l}=\mathbf{k},\ |\mathbf{h}|,|\mathbf{l}|\leqslant N} (h_2 l_1 - h_1 l_2)\frac{\mathbf{l}\cdot(\mathbf{k}-\mathbf{h})}{\hat{L}^2}\varphi_1(\mathbf{h},\tau)\varphi_1(\mathbf{l},\tau) \\
&= \frac{1}{\hat{L}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k},\ |\mathbf{h}|,|\mathbf{l}|\leqslant N} (h_2 l_1 - h_1 l_2)(\mathbf{l}\cdot\mathbf{k})\varphi_1(\mathbf{h},\tau)\varphi_1(\mathbf{l},\tau) \\
&\quad - \frac{1}{\hat{L}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k},\ |\mathbf{h}|,|\mathbf{l}|\leqslant N} (h_2 l_1 - h_1 l_2)(\mathbf{l}\cdot\mathbf{h})\varphi_1(\mathbf{h},\tau)\varphi_1(\mathbf{l},\tau). \qquad (48)
\end{aligned}
$$

By interchanging **h** with **l** in the last sum of (48) we get that the indicated sum is 0, and, therefore

$$
\begin{aligned}
S_4 &= \frac{1}{\hat{L}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k},\ |\mathbf{h}|,|\mathbf{l}|\leqslant N} (h_2 l_1 - h_1 l_2)(\mathbf{l}\cdot\mathbf{k})\varphi_1(\mathbf{h},\tau)\varphi_1(\mathbf{l},\tau) \\
&= \frac{1}{\hat{L}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k},\ |\mathbf{h}|,|\mathbf{l}|\leqslant N} \big(h_2(k_1-h_1) - h_1(k_2-h_2)\big)(\mathbf{l}\cdot\mathbf{k})\varphi_1(\mathbf{h},\tau)\varphi_1(\mathbf{l},\tau) \\
&= \frac{1}{\hat{L}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k},\ |\mathbf{h}|,|\mathbf{l}|\leqslant N} (h_2 k_1 - h_1 k_2)(\mathbf{l}\cdot\mathbf{k})\varphi_1(\mathbf{h},\tau)\varphi_1(\mathbf{l},\tau). \qquad (49)
\end{aligned}
$$

From (47) and (49) we deduce that

$$
\begin{aligned}
&\left|\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k},\ |\mathbf{h}|,|\mathbf{l}|\leqslant N} (h_2 l_1 - h_1 l_2)\varphi_1(\mathbf{h},\tau) r_1(\mathbf{l},\tau)\right| \\
&\quad\leqslant \frac{1}{\hat{L}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k},\ |\mathbf{h}|,|\mathbf{l}|\leqslant N} |\mathbf{h}||\mathbf{k}|^2|\mathbf{l}|\big|\varphi_1(\mathbf{h},\tau)\big|\big|\varphi_1(\mathbf{l},\tau)\big| \\
&\quad\quad + \frac{\hat{L}^2}{2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k},\ |\mathbf{h}|,|\mathbf{l}|\leqslant N} \frac{|\mathbf{h}|}{\hat{L}}\big|\varphi_1(\mathbf{h},\tau)\big|\frac{|\mathbf{l}|}{\hat{L}}\big|\varphi_2(\mathbf{l},\tau)\big|,
\end{aligned}
$$

and using Cauchy–Schwartz inequality we get that the left term in the above inequality is less or equal then

$$
|\mathbf{k}|^2\bigg(\sum_{|\mathbf{h}|\leqslant N}\frac{|\mathbf{h}|^2}{\hat{L}^2}\big|\varphi_1(\mathbf{h},\tau)\big|^2\bigg)^{\frac{1}{2}}\bigg(\sum_{|\mathbf{l}|\leqslant N}\frac{|\mathbf{l}|^2}{\hat{L}^2}\big|\varphi_1(\mathbf{l},\tau)\big|^2\bigg)^{\frac{1}{2}}
$$

$$+\frac{\hat{L}^2}{2}\left(\sum_{|\mathbf{h}|\leqslant N}\frac{|\mathbf{h}|^2}{\hat{L}^2}\left|\varphi_1(\mathbf{h},\tau)\right|^2\right)^{\frac{1}{2}}\left(\sum_{|\mathbf{l}|\leqslant N}\frac{|\mathbf{l}|^2}{\hat{L}^2}\left|\varphi_2(\mathbf{l},\tau)\right|^2\right)^{\frac{1}{2}}$$

$$\leqslant\left(|\mathbf{k}|^2+\frac{\hat{L}^2}{2}\right)\left|\vec{\psi}_N(\tau)\right|^2\leqslant\left(|\mathbf{k}|^2+\frac{\hat{L}^2}{2}\right)e^{\hat{L}T/2}\left|\vec{\psi}^0\right|^2,\quad\forall\tau\in[0,T].\tag{50}$$

Similarly,

$$\left|\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k},\ |\mathbf{h}|,|\mathbf{l}|\leqslant N}(h_2l_1-h_1l_2)\varphi_2(\mathbf{h},\tau)r_2(\mathbf{l},\tau)\right|\leqslant c_2,\quad\forall\tau\in[0,T],\tag{51}$$

where $c_2$ is a positive constant which does not depend on $N$. Thus, there exists $\tilde{c}>0$ which does not depend on $N$ such that

$$\left|\tilde{\varphi}(\mathbf{k},t)-\tilde{\varphi}(\mathbf{k},s)\right|\leqslant\tilde{c}(t-s),\quad\forall t,s\in[0,T],\ s<t.\tag{52}$$

Using the same idea we can easily get that $\hat{c}>0$ such that

$$\left|\hat{\varphi}(\mathbf{k},t)-\hat{\varphi}(\mathbf{k},s)\right|\leqslant\hat{c}(t-s),\quad\forall t,s\in[0,T],\ s<t.\tag{53}$$

From (52) and (53) we obtain that there exists $c>0$ which does not depend on $N$ such that

$$\left|\psi_{N,j}(\mathbf{k},t)-\psi_{N,j}(\mathbf{k},s)\right|\leqslant c(t-s),\quad\forall t,s\in[0,T],\ j=1,2.\tag{54}$$

Notice that we can choose $c$ such that the following is also true

$$\left|\psi_{N,j}(\mathbf{k},0)\right|=\left|\psi_j^0(\mathbf{k})\right|\leqslant c\left|\vec{\psi}^0\right|.\tag{55}$$

The relations (54) and (55) allow us to apply Arzela–Ascoli theorem for the sequence $\{\psi_{N,j}(\mathbf{k},\cdot)\}_{N\in\mathbb{N}}$. We get that for $T=1$ and a fixed $\mathbf{k}\in\mathbb{Z}^2$ there exist a subsequence

$$\left\{\psi_{N_h,j}(\mathbf{k},\cdot)\right\}_{h\in\mathbb{N}}\quad\text{of}\quad\left\{\psi_{N,j}(\mathbf{k},\cdot)\right\}_{N\in\mathbb{N}}$$

and a function $\psi_{1,j}(\mathbf{k},\cdot)\in C([0,1],\mathbb{C})$ such that $\{\psi_{N_h,j}(\mathbf{k},\cdot)\}_{h\in\mathbb{N}}$ converges to $\psi_{1,j}(\mathbf{k},\cdot)$ uniformly on $[0,1]$. By applying Cantor's diagonal method for $\mathbf{k}\in\mathbb{Z}^2$ (written as a sequence) we prove the existence of a subsequence of $\{\vec{\psi}_N(\cdot)\}_{N\in\mathbb{N}}$ which converges to a function $\vec{\psi}_1(\cdot)$ in $C([0,1],\mathcal{K})$. For this subsequence we repeat the above argument with $T=2$ to get another subsequence which converges to a function $\vec{\psi}_2(\cdot)$ in $C([0,2],\mathcal{K})$. We continue with $T=3,4,\ldots,$ and we apply Cantor's diagonal method to obtain that there exist a subsequence $\{\vec{\psi}_{N_p}(\cdot)\}_{p\in\mathbb{N}}$ of $\{\vec{\psi}_N(\cdot)\}_{N\in\mathbb{N}}$ and $\vec{\psi}(\cdot)\in C([0,\infty),\mathcal{K})$ such that $\{\vec{\psi}_{N_p}(\cdot)\}_{p\in\mathbb{N}}$ converges to $\vec{\psi}(\cdot)$ in $C([0,\infty),\mathcal{K})$. □

Now we are ready to prove the main theorem of this section.

**Theorem 5.3.** *The function $\vec{\psi}$ provided by Lemma* 5.2 *is a weak solution for* (19)–(23) *with initial data $\vec{\psi}^0$.*

**Proof.** Since $\{\vec{\psi}_{N_p}\}_{p\in\mathbb{N}}$ converges to $\vec{\psi}$ in $C([0,\infty),\mathcal{K})$ we get that for every $T \in [0,\infty)$

$$\psi_{N_p,j}(\mathbf{k},t) \to \psi_j(\mathbf{k},t) \quad \text{uniformly on } [0,T],\ j=1,2. \tag{56}$$

If $\vec{\theta} \in \mathcal{K}$ and $M \in \mathbb{N}$ define $P_M\vec{\theta} \in \mathcal{K}$ by

$$(P_M\vec{\theta})(\mathbf{k}) = \theta(\mathbf{k}) \quad \text{if } |\mathbf{k}| \leqslant M, \quad \text{and} \quad (P_M\vec{\theta})(\mathbf{k}) = \mathbf{0}, \quad \text{if } |\mathbf{k}| > M.$$

Then, if $M \in \mathbb{N}$ and $N_p \geqslant M$ we have

$$\left|P_M\vec{\psi}_{N_p}(t)\right| \leqslant \left|\vec{\psi}_{N_p}(t)\right| \leqslant e^{\hat{L}T/4}\left|\vec{\psi}^0\right|, \quad \forall t \in [0,T].$$

Letting $p \to \infty$ and using (56) we obtain

$$\left|P_M\vec{\psi}(t)\right| \leqslant e^{\hat{L}T/4}\left|\vec{\psi}^0\right|, \quad \forall t \in [0,T],\ \forall M \in \mathbb{N},$$

and by letting $M \to \infty$ we get

$$\left|\vec{\psi}(t)\right| \leqslant e^{\hat{L}T/4}\left|\vec{\psi}^0\right|, \quad \forall t \in [0,T],$$

which shows that $\vec{\psi} \in L^\infty_{\mathrm{loc}}([0,\infty),H)$. By integrating (43) we deduce that

$$\begin{aligned}\nu\int_0^T \sum_{|\mathbf{k}|\leqslant N_p}\left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)} E(\vec{\psi}_N)(\mathbf{k})\,dt &\leqslant \frac{1}{2}\left|\vec{\psi}^0\right|^2 + \frac{\hat{L}}{4}\int_0^T \left|\vec{\psi}_N(t)\right|^2 dt \\ &\leqslant \left(\frac{1}{2} + \frac{\hat{L}T}{4}e^{\hat{L}T/2}\right)\left|\vec{\psi}^0\right|^2. \end{aligned}\tag{57}$$

If $M \in \mathbb{N}$ and $N_p \geqslant M$ we have

$$\begin{aligned}\nu\int_0^T \sum_{|\mathbf{k}|\leqslant M}\left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)} E(\vec{\psi}_{N_p})(\mathbf{k})\,dt &\leqslant \nu\int_0^T \sum_{|\mathbf{k}|\leqslant N_p}\left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)} E(\vec{\psi}_{N_p})(\mathbf{k})\,dt \\ &\leqslant \left(\frac{1}{2} + \frac{\hat{L}T}{4}e^{\hat{L}T/2}\right)\left|\vec{\psi}^0\right|^2. \end{aligned}$$

Therefore, by using (56), if $p \to \infty$ we obtain

$$\nu\int_0^T \sum_{|\mathbf{k}|\leqslant M}\left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)} E(\vec{\psi}_{N_p})(\mathbf{k})\,dt \leqslant \left(\frac{1}{2} + \frac{\hat{L}T}{4}e^{\hat{L}T/2}\right)\left|\vec{\psi}^0\right|^2, \quad \forall M \in \mathbb{N}.$$

Now we apply Beppo–Levi theorem to get that

$$\int_0^T \left|\vec{\psi}(t)\right|_{1+\alpha}^2 dt \leqslant \frac{1}{\nu}\left(\frac{1}{2}+\frac{\hat{L}T}{4}e^{\hat{L}T/2}\right)\left|\vec{\psi}^0\right|^2, \quad \forall T \in [0,\infty),$$

which proves that $\vec{\psi} \in L^2_{\mathrm{loc}}([0,\infty), V_{1+\alpha})$. Thus $\vec{\psi}$ satisfies condition (1) from Definition 3.1. From (29) and (30) we easily get that

$$\begin{aligned} q_{N_p,1}(\mathbf{k},t) = q_{N_p,1}(\mathbf{k},0) - \int_0^t \Biggl\{ &\frac{1}{\hat{L}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (h_2l_1 - h_1l_2)\psi_{N_p,1}(\mathbf{h},\tau)q_{N_p,1}(\mathbf{l},\tau) \\ &+ \frac{i}{\hat{L}}k_1 q_{N_p,1}(\mathbf{k},\tau) + \left(\beta+\frac{1}{2}\right)\frac{i}{\hat{L}}k_1\psi_{N_p,1}(\mathbf{k},\tau) \\ &+ \nu\left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)} q_{N_p,1}(\mathbf{k},\tau)\Biggr\}\,d\tau, \quad \forall N_p \geqslant |\mathbf{k}|. \end{aligned} \tag{58}$$

Using (56) it is clear that all the terms under the integral except the first one converge to the corresponding ones for $\vec{\psi}$. We need to show that

$$\delta := \int_0^t \sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (h_2l_1 - h_1l_2)\bigl(\psi_{N_p,1}(\mathbf{h},\tau)q_{N_p,1}(\mathbf{l},\tau) - \psi_1(\mathbf{h},\tau)q_1(\mathbf{l},\tau)\bigr)\,d\tau \to 0 \tag{59}$$

as $p \to \infty$. For this we have

$$\begin{aligned} &\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (h_2l_1 - h_1l_2)\bigl(\psi_{N_p,1}(\mathbf{h},\tau)q_{N_p,1}(\mathbf{l},\tau) - \psi_1(\mathbf{h},\tau)q_1(\mathbf{l},\tau)\bigr) \\ &\quad= \sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (h_2l_1 - h_1l_2)\frac{|\mathbf{l}|^2}{\hat{L}^2}\bigl(\psi_{N_p,1}(\mathbf{h},\tau)\psi_{N_p,1}(\mathbf{l},\tau) - \psi_1(\mathbf{h},\tau)\psi_1(\mathbf{l},\tau)\bigr) \\ &\qquad+ \frac{1}{2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (h_2l_1 - h_1l_2)\bigl(\psi_{N_p,1}(\mathbf{h},\tau)\psi_{N_p,1}(\mathbf{l},\tau) - \psi_1(\mathbf{h},\tau)\psi_1(\mathbf{l},\tau)\bigr) \\ &\qquad+ \frac{1}{2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (h_2l_1 - h_1l_2)\bigl(\psi_{N_p,1}(\mathbf{h},\tau)\psi_{N_p,2}(\mathbf{l},\tau) - \psi_1(\mathbf{h},\tau)\psi_2(\mathbf{l},\tau)\bigr). \end{aligned}$$

The first sum on the right-hand side is equal to

$$\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (h_2l_1 - h_1l_2)\frac{\mathbf{l}\cdot(\mathbf{k}-\mathbf{h})}{\hat{L}^2}\bigl(\psi_{N_p,1}(\mathbf{h},\tau)\psi_{N_p,1}(\mathbf{l},\tau) - \psi_1(\mathbf{h},\tau)\psi_1(\mathbf{l},\tau)\bigr)$$

$$= \sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (h_2 l_1 - h_1 l_2) \frac{\mathbf{l}\cdot\mathbf{k}}{\hat{L}^2} \big(\psi_{N_p,1}(\mathbf{h},\tau)\psi_{N_p,1}(\mathbf{l},\tau) - \psi_1(\mathbf{h},\tau)\psi_1(\mathbf{l},\tau)\big)$$
$$- \sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (h_2 l_1 - h_1 l_2) \frac{\mathbf{l}\cdot\mathbf{h}}{\hat{L}^2} \big(\psi_{N_p,1}(\mathbf{h},\tau)\psi_{N_p,1}(\mathbf{l},\tau) - \psi_1(\mathbf{h},\tau)\psi_1(\mathbf{l},\tau)\big)$$

and since the last sum is zero we get that

$$\begin{aligned}
&\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (h_2 l_1 - h_1 l_2) \frac{|\mathbf{l}|^2}{\hat{L}^2} \big(\psi_{N_p,1}(\mathbf{h},\tau)\psi_{N_p,1}(\mathbf{l},\tau) - \psi_1(\mathbf{h},\tau)\psi_1(\mathbf{l},\tau)\big) \\
&\quad = \sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (h_2 l_1 - h_1 l_2) \frac{\mathbf{l}\cdot\mathbf{k}}{\hat{L}^2} \big(\psi_{N_p,1}(\mathbf{h},\tau)\psi_{N_p,1}(\mathbf{l},\tau) - \psi_1(\mathbf{h},\tau)\psi_1(\mathbf{l},\tau)\big) \\
&\quad = \sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} \big(h_2(k_1 - h_1) - h_1(k_2 - h_2)\big) \frac{\mathbf{l}\cdot\mathbf{k}}{\hat{L}^2} \big(\psi_{N_p,1}(\mathbf{h},\tau)\psi_{N_p,1}(\mathbf{l},\tau) - \psi_1(\mathbf{h},\tau)\psi_1(\mathbf{l},\tau)\big) \\
&\quad = \sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (h_2 k_1 - h_1 k_2) \frac{\mathbf{l}\cdot\mathbf{k}}{\hat{L}^2} \big(\psi_{N_p,1}(\mathbf{h},\tau)\psi_{N_p,1}(\mathbf{l},\tau) - \psi_1(\mathbf{h},\tau)\psi_1(\mathbf{l},\tau)\big).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\bigg|\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (h_2 l_1 - h_1 l_2)\big(\psi_{N_p,1}(\mathbf{h},\tau) q_{N_p,1}(\mathbf{l},\tau) - \psi_1(\mathbf{h},\tau) q_1(\mathbf{l},\tau)\big)\bigg| \\
&\quad \leqslant \bigg|\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (h_2 k_1 - h_1 k_2) \frac{\mathbf{l}\cdot\mathbf{k}}{\hat{L}^2} \big(\psi_{N_p,1}(\mathbf{h},\tau)\psi_{N_p,1}(\mathbf{l},\tau) - \psi_1(\mathbf{h},\tau)\psi_1(\mathbf{l},\tau)\big)\bigg| \\
&\qquad + \frac{1}{2}\bigg|\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (h_2 l_1 - h_1 l_2)\big(\psi_{N_p,1}(\mathbf{h},\tau)\psi_{N_p,1}(\mathbf{l},\tau) - \psi_1(\mathbf{h},\tau)\psi_1(\mathbf{l},\tau)\big)\bigg| \\
&\qquad + \frac{1}{2}\bigg|\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (h_2 l_1 - h_1 l_2)\big(\psi_{N_p,1}(\mathbf{h},\tau)\psi_{N_p,2}(\mathbf{l},\tau) - \psi_1(\mathbf{h},\tau)\psi_2(\mathbf{l},\tau)\big)\bigg| \\
&\quad \leqslant \left(|\mathbf{k}|^2 + \frac{\hat{L}^2}{2}\right) \sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} \bigg(\frac{|\mathbf{h}|}{\hat{L}}\big|\psi_{N_p,1}(\mathbf{h},\tau) - \psi_1(\mathbf{h},\tau)\big|\frac{|\mathbf{l}|}{\hat{L}}\big|\psi_{N_p,1}(\mathbf{l},\tau)\big| \\
&\qquad + \frac{|\mathbf{h}|}{\hat{L}}\big|\psi_1(\mathbf{h},\tau)\big|\frac{|\mathbf{l}|}{\hat{L}}\big|\psi_{N_p,1}(\mathbf{l},\tau) - \psi_1(\mathbf{l},\tau)\big|\bigg) \\
&\qquad + \frac{\hat{L}^2}{2} \sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} \bigg(\frac{|\mathbf{h}|}{\hat{L}}\big|\psi_{N_p,1}(\mathbf{h},\tau) - \psi_1(\mathbf{h},\tau)\big|\frac{|\mathbf{l}|}{\hat{L}}\big|\psi_{N_p,2}(\mathbf{l},\tau)\big| \\
&\qquad + \frac{|\mathbf{h}|}{\hat{L}}\big|\psi_1(\mathbf{h},\tau)\big|\frac{|\mathbf{l}|}{\hat{L}}\big|\psi_{N_p,2}(\mathbf{l},\tau) - \psi_2(\mathbf{l},\tau)\big|\bigg) \\
&\quad \leqslant \big(|\mathbf{k}|^2 + \hat{L}^2\big)\big(\big|\vec{\psi}_{N_p}(\tau) - \vec{\psi}(\tau)\big|\big|\vec{\psi}_{N_p}(\tau)\big| + \big|\vec{\psi}(\tau)\big|\big|\vec{\psi}_{N_p}(\tau) - \vec{\psi}(\tau)\big|\big).
\end{aligned}$$

Using the last estimate and Holder's inequality we get that

$$\delta \leqslant 2\big(|\mathbf{k}|^2 + \hat{L}^2\big)\left(\int_0^t \big|\vec{\psi}_{N_p}(\tau) - \vec{\psi}(\tau)\big|^2\,d\tau\right)^{1/2} \sqrt{t}\,e^{\hat{L}t/4}\big|\vec{\psi}^0\big|.$$

Now we can see that in order to prove (59) it suffices to show that

$$\int_0^t \big|\vec{\psi}_{N_p}(\tau) - \vec{\psi}(\tau)\big|^2\,d\tau \to 0 \quad \text{as } p \to \infty.$$

Since $\psi_{N_p,j}(\mathbf{k},\tau) \to \psi_j(\mathbf{k},\tau)$ uniformly for $\tau \in [0,t]$, for each fixed $\mathbf{k} \in \mathbb{Z}^2$, we have for each $M = 1, 2, \ldots$

$$\begin{aligned}
\lambda &:= \lim\sup_{p\to\infty} \int_0^t \big|\vec{\psi}_{N_p}(\tau) - \vec{\psi}(\tau)\big|^2\,d\tau \\
&= \lim\sup_{p\to\infty} \int_0^t \big|(I - P_M)\big(\vec{\psi}_{N_p}(\tau) - \vec{\psi}(\tau)\big)\big|^2\,d\tau \\
&\leqslant \lim\sup_{p\to\infty}\left[2\int_0^t \big|(I - P_M)\vec{\psi}_{N_p}(\tau)\big|^2\,d\tau\right] + 2\int_0^t \big|(I - P_M)\vec{\psi}(\tau)\big|^2\,d\tau. \qquad (60)
\end{aligned}$$

We also have

$$\begin{aligned}
\big|(I - P_M)\vec{\psi}_{N_p}(\tau)\big|^2 &= \sum_{|\mathbf{k}|>M} E\big(\vec{\psi}_{N_p}(\tau)\big)(\mathbf{k}) \leqslant \frac{\hat{L}}{M}\sum_{|\mathbf{k}|>M}\frac{|\mathbf{k}|}{\hat{L}}E\big(\vec{\psi}_{N_p}(\tau)\big)(\mathbf{k}) \\
&\leqslant \frac{\hat{L}}{M}\big|\vec{\psi}_{N_p}(\tau)\big|_{1+\alpha}^2. \qquad (61)
\end{aligned}$$

From (57) and (61) we get

$$\int_0^t \big|(I - P_M)\vec{\psi}_{N_p}(\tau)\big|^2\,d\tau \leqslant \frac{\hat{L}}{M}\int_0^t \big|\vec{\psi}_{N_p}(\tau)\big|_{1+\alpha}^2\,d\tau \leqslant \frac{\hat{L}}{\nu M}\left(\frac{1}{2} + \frac{\hat{L}t}{4}e^{\frac{\hat{L}t}{2}}\right)\big|\vec{\psi}^0\big|^2. \qquad (62)$$

Applying Lebesgue's dominated convergence theorem we have that

$$\lim_{M\to\infty}\int_0^t \big|(I - P_M)\vec{\psi}(\tau)\big|^2\,d\tau = 0. \qquad (63)$$

Using (62) and (63) we let $M \to \infty$ in (60) and we obtain that $\lambda = 0$. Next we let $p \to \infty$ in (58) to get the first equation of (2) in Definition 3.1. In a similar fashion we deduce the second equation of (2) in Definition 3.1. It is easy to see that $\psi_j(\mathbf{k}, 0) = \psi_j^0(\mathbf{k})$, $\forall \mathbf{k} \in \mathbb{Z}^2$, $j = 1, 2$, and the proof that $\vec{\psi}$ is a weak solution for (19)–(23) with initial data $\vec{\psi}^0$ is complete. □

## 6. Uniqueness

In this section we prove that $\vec{\psi}$ (the weak solution found in the previous section) is the unique weak solution for (19)–(23). For this we need a few preliminary results.

**Lemma 6.1.** *Let $\varphi_0, \psi_0 \in \mathbb{R}^d$, $f, g \in L^2([0, T]; \mathbb{R}^d)$ and let*

$$\varphi(t) = \varphi_0 + \int_0^t f(\tau)\, d\tau, \qquad \psi(t) = \psi_0 + \int_0^t g(\tau)\, d\tau, \quad \forall 0 \leqslant t \leqslant T. \tag{64}$$

*Then*

$$\varphi(t) \cdot \psi(t) = \varphi_0 \cdot \psi_0 + \int_0^t \big(f(\tau) \cdot \psi(\tau) + \varphi(\tau) \cdot g(\tau)\big)\, d\tau, \quad \forall 0 \leqslant t \leqslant T. \tag{65}$$

**Proof.** If $f, g \in C([0, T]; \mathbb{R}^d)$ then from (64) we get that $\varphi'(t) = f(t)$, $\psi'(t) = g(t)$ and (65) is easily obtained by integrating

$$\frac{d}{dt}(\varphi \cdot \psi) = \frac{d\varphi}{dt} \cdot \psi + \varphi \cdot \frac{d\psi}{dt}. \tag{66}$$

The proof is complete by noticing that $C([0, T]; \mathbb{R}^d)$ is dense in $L^2([0, T]; \mathbb{R}^d)$. □

**Corollary 6.2.** *Let $\varphi_0 \in \mathbb{C}$, $f \in L^2([0, T]; \mathbb{C})$ and let*

$$\varphi(t) = \varphi_0 + \int_0^t f(\tau)\, d\tau, \quad \forall 0 \leqslant t \leqslant T.$$

*Then*

$$\big|\varphi(t)\big|^2 = |\varphi_0|^2 + 2\,\mathrm{Re} \int_0^t f(\tau)\overline{\varphi(\tau)}\, d\tau, \quad \forall 0 \leqslant t \leqslant T.$$

The next result that we will use in the proof of uniqueness is the following variant of Ladyzhenskaya's inequality.

With $\Omega = [0, 2\pi\hat{L}]^2 \subset \mathbb{R}^2$ there exists $c_L > 0$ such that

$$\|u\|_{L^4(\Omega)} \leqslant c_L \|u\|^{1/2}_{L^2(\Omega)} \|\nabla u\|^{1/2}_{(L^2(\Omega))^2}, \tag{67}$$

for every $u$ in

$$H^1_{\mathrm{per}}(\Omega) = \left\{ v \in L^2(\Omega)\colon \sum_{\mathbf{k}\in\mathbb{Z}^2} \frac{|\mathbf{k}|^2}{\hat{L}^2} \left|v(\mathbf{k})\right|^2 < \infty \right\} \tag{68}$$

with average zero (i.e., $\int_\Omega u(x)\,dx = 0$). (Recall that in (68), $\{v(\mathbf{k})\}_{\mathbf{k}\in\mathbb{Z}^2}$ are the Fourier coefficients of $v$.)

Next we prove the main theorem of this section.

**Theorem 6.3.** *For every given initial data in $H$ Eqs.* (17)–(21) *have a unique weak solution.*

**Proof.** Suppose that $\vec{\varphi}$ is another weak solution for (19)–(23) with initial data $\vec{\psi}^0$. Let $\vec{w} = \vec{\psi} - \vec{\varphi}$ and $\vec{y} = \vec{q} - \vec{r}$, where

$$r_j(\mathbf{k}, t) = -\frac{|\mathbf{k}|^2}{\hat{L}^2}\varphi_j(\mathbf{k}, t) + (-1)^j \frac{\varphi_1(\mathbf{k}, t) - \varphi_2(\mathbf{k}, t)}{2}, \quad j = 1, 2.$$

Since $\vec{\psi}$ and $\vec{\varphi}$ are weak solutions we have that

$$\begin{aligned} q_1(\mathbf{k}, t) = q_1(\mathbf{k}, 0) - \int_0^t \Bigg\{ &\frac{1}{\hat{L}^2} \sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (h_2 l_1 - h_1 l_2)\psi_1(\mathbf{h}, \tau) q_1(\mathbf{l}, \tau) + \frac{i}{\hat{L}} k_1 q_1(\mathbf{k}, \tau) \\ &+ \left(\beta + \frac{1}{2}\right)\frac{i}{\hat{L}} k_1 \psi_1(\mathbf{k}, \tau) + \nu \left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)} q_1(\mathbf{k}, \tau) \Bigg\}\, d\tau \end{aligned} \tag{69}$$

and

$$\begin{aligned} r_1(\mathbf{k}, t) = r_1(\mathbf{k}, 0) - \int_0^t \Bigg\{ &\frac{1}{\hat{L}^2} \sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (h_2 l_1 - h_1 l_2)\varphi_1(\mathbf{h}, \tau) r_1(\mathbf{l}, \tau) + \frac{i}{\hat{L}} k_1 r_1(\mathbf{k}, \tau) \\ &+ \left(\beta + \frac{1}{2}\right)\frac{i}{\hat{L}} k_1 \varphi_1(\mathbf{k}, \tau) + \nu \left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)} r_1(\mathbf{k}, \tau) \Bigg\}\, d\tau. \end{aligned} \tag{70}$$

By subtracting (70) from (69) we get that

$$y_1(\mathbf{k},t)=y_1(\mathbf{k},0)-\int_0^t\left\{\frac{1}{\hat L^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}(h_2l_1-h_1l_2)\big(\psi_1(\mathbf{h},\tau)q_1(\mathbf{l},\tau)-\varphi_1(\mathbf{h},\tau)r_1(\mathbf{l},\tau)\big)\right.$$
$$\left.+\frac{i}{\hat L}k_1y_1(\mathbf{k},\tau)+\left(\beta+\frac12\right)\frac{i}{\hat L}k_1w_1(\mathbf{k},\tau)+\nu\left(\frac{|\mathbf{k}|}{\hat L}\right)^{2(1+\alpha)}y_1(\mathbf{k},\tau)\right\}d\tau. \quad (71)$$

Similarly,

$$q_2(\mathbf{k},t)=q_2(\mathbf{k},0)-\int_0^t\left\{\frac{1}{\hat L^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}(h_2l_1-h_1l_2)\psi_2(\mathbf{h},\tau)q_2(\mathbf{l},\tau)-\kappa_M\frac{|\mathbf{k}|^2}{\hat L^2}\psi_2(\mathbf{k},\tau)\right.$$
$$\left.+\left(\beta-\frac12\right)\frac{i}{\hat L}k_1\psi_2(\mathbf{k},\tau)+\nu\left(\frac{|\mathbf{k}|}{\hat L}\right)^{2(1+\alpha)}q_2(\mathbf{k},\tau)\right\}d\tau,$$
$$r_2(\mathbf{k},t)=r_2(\mathbf{k},0)-\int_0^t\left\{\frac{1}{\hat L^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}(h_2l_1-h_1l_2)\varphi_2(\mathbf{h},\tau)r_2(\mathbf{l},\tau)\right.$$
$$\left.-\kappa_M\frac{|\mathbf{k}|^2}{\hat L^2}\varphi_2(\mathbf{k},\tau)+\left(\beta-\frac12\right)\frac{i}{\hat L}k_1\varphi_2(\mathbf{k},\tau)+\nu\left(\frac{|\mathbf{k}|}{\hat L}\right)^{2(1+\alpha)}r_2(\mathbf{k},\tau)\right\}d\tau,$$

and

$$y_2(\mathbf{k},t)=y_2(\mathbf{k},0)-\int_0^t\left\{\frac{1}{\hat L^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}(h_2l_1-h_1l_2)\big(\psi_2(\mathbf{h},\tau)q_2(\mathbf{l},\tau)-\varphi_2(\mathbf{h},\tau)r_2(\mathbf{l},\tau)\big)\right.$$
$$\left.-\kappa_M\frac{|\mathbf{k}|^2}{\hat L^2}w_2(\mathbf{k},\tau)+\left(\beta-\frac12\right)\frac{i}{\hat L}k_1w_2(\mathbf{k},\tau)+\nu\left(\frac{|\mathbf{k}|}{\hat L}\right)^{2(1+\alpha)}y_2(\mathbf{k},\tau)\right\}d\tau. \quad (72)$$

Next we define $\widetilde w=w_1+w_2$ and $\widehat w=w_1-w_2$. An easy calculation gives us that

$$y_1(\mathbf{k})+y_2(\mathbf{k})=-\frac{|\mathbf{k}|^2}{\hat L^2}\widetilde w(\mathbf{k})\quad\text{and}\quad y_1(\mathbf{k})-y_2(\mathbf{k})=-\left(\frac{|\mathbf{k}|^2}{\hat L^2}+1\right)\widehat w(\mathbf{k}).$$

Adding (71) and (72) we obtain that

$$\frac{|\mathbf{k}|^2}{\hat L^2}\widetilde w(\mathbf{k},t)=\frac{|\mathbf{k}|^2}{\hat L^2}\widetilde w(\mathbf{k},0)+\int_0^t\left\{\frac{1}{\hat L^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}(h_2l_1-h_1l_2)\big(\psi_1(\mathbf{h},\tau)q_1(\mathbf{l},\tau)\right.$$
$$-\varphi_1(\mathbf{h},\tau)r_1(\mathbf{l},\tau)+\psi_2(\mathbf{h},\tau)q_2(\mathbf{l},\tau)-\varphi_2(\mathbf{h},\tau)r_2(\mathbf{l},\tau)\big)+\frac{i}{\hat L}k_1y_1(\mathbf{k},\tau)$$

$$+\left(\beta+\frac{1}{2}\right)\frac{i}{\hat{L}}k_1 w_1(\mathbf{k},\tau)-\kappa_M\frac{|\mathbf{k}|^2}{\hat{L}^2}w_2(\mathbf{k},\tau)+\left(\beta-\frac{1}{2}\right)\frac{i}{\hat{L}}k_1 w_2(\mathbf{k},\tau)$$
$$+\nu\left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)}\left(-\frac{|\mathbf{k}|^2}{\hat{L}^2}\widetilde{w}(\mathbf{k},\tau)\right)\Bigg\}\,d\tau. \tag{73}$$

Applying Corollary 6.2 with $\varphi(t)=\frac{|\mathbf{k}|}{\hat{L}}\widetilde{w}(\mathbf{k},t)$, for every $\mathbf{k}\neq\mathbf{0}$ we obtain

$$\frac{|\mathbf{k}|^2}{\hat{L}^2}\left|\widetilde{w}(\mathbf{k},t)\right|^2$$
$$=\frac{|\mathbf{k}|^2}{\hat{L}^2}\left|\widetilde{w}(\mathbf{k},0)\right|^2+2\operatorname{Re}\int_0^t\Bigg\{\frac{1}{\hat{L}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}(h_2l_1-h_1l_2)$$
$$\times\big(\psi_1(\mathbf{h},\tau)q_1(\mathbf{l},\tau)-\varphi_1(\mathbf{h},\tau)r_1(\mathbf{l},\tau)+\psi_2(\mathbf{h},\tau)q_2(\mathbf{l},\tau)-\varphi_2(\mathbf{h},\tau)r_2(\mathbf{l},\tau)\big)\overline{\widetilde{w}(\mathbf{k},\tau)}$$
$$+\frac{i}{\hat{L}}k_1y_1(\mathbf{k},\tau)\overline{\widetilde{w}(\mathbf{k},\tau)}+\left(\beta+\frac{1}{2}\right)\frac{i}{\hat{L}}k_1w_1(\mathbf{k},\tau)\overline{\widetilde{w}(\mathbf{k},\tau)}-\kappa_M\frac{|\mathbf{k}|^2}{\hat{L}^2}w_2(\mathbf{k},\tau)\overline{\widetilde{w}(\mathbf{k},\tau)}$$
$$+\left(\beta-\frac{1}{2}\right)\frac{i}{\hat{L}}k_1w_2(\mathbf{k},\tau)\overline{\widetilde{w}(\mathbf{k},\tau)}-\nu\left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(2+\alpha)}\left|\widetilde{w}(\mathbf{k},\tau)\right|^2\Bigg\}\,d\tau. \tag{74}$$

Subtracting (72) from (71) we get

$$\left(\frac{|\mathbf{k}|^2}{\hat{L}^2}+1\right)\widehat{w}(\mathbf{k},t)$$
$$=\left(\frac{|\mathbf{k}|^2}{\hat{L}^2}+1\right)\widehat{w}(\mathbf{k},0)+\int_0^t\Bigg\{\frac{1}{\hat{L}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}(h_2l_1-h_1l_2)\big(\psi_1(\mathbf{h},\tau)q_1(\mathbf{l},\tau)$$
$$-\varphi_1(\mathbf{h},\tau)r_1(\mathbf{l},\tau)-\psi_2(\mathbf{h},\tau)q_2(\mathbf{l},\tau)+\varphi_2(\mathbf{h},\tau)r_2(\mathbf{l},\tau)\big)$$
$$+\frac{i}{\hat{L}}k_1y_1(\mathbf{k},\tau)+\left(\beta+\frac{1}{2}\right)\frac{i}{\hat{L}}k_1w_1(\mathbf{k},\tau)+\kappa_M\frac{|\mathbf{k}|^2}{\hat{L}^2}w_2(\mathbf{k},\tau)$$
$$-\left(\beta-\frac{1}{2}\right)\frac{i}{\hat{L}}k_1w_2(\mathbf{k},\tau)+\nu\left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)}\left(-\left(\frac{|\mathbf{k}|^2}{\hat{L}^2}+1\right)\widehat{w}(\mathbf{k},\tau)\right)\Bigg\}\,d\tau.$$

Applying again Corollary 6.2 we deduce that

$$\left(\frac{|\mathbf{k}|^2}{\hat{L}^2}+1\right)\left|\widehat{w}(\mathbf{k},t)\right|^2$$
$$=\left(\frac{|\mathbf{k}|^2}{\hat{L}^2}+1\right)\left|\widehat{w}(\mathbf{k},0)\right|^2+2\operatorname{Re}\int_0^t\Bigg\{\frac{1}{\hat{L}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}(h_2l_1-h_1l_2)\big(\psi_1(\mathbf{h},\tau)q_1(\mathbf{l},\tau)$$

$$- \varphi_1(\mathbf{h},\tau) r_1(\mathbf{l},\tau) - \psi_2(\mathbf{h},\tau) q_2(\mathbf{l},\tau) + \varphi_2(\mathbf{h},\tau) r_2(\mathbf{l},\tau)\big)\overline{\widehat{w}(\mathbf{k},\tau)}$$
$$+ \frac{i}{\hat{L}} k_1 y_1(\mathbf{k},\tau)\overline{\widehat{w}(\mathbf{k},\tau)} + \left(\beta + \frac{1}{2}\right)\frac{i}{\hat{L}} k_1 w_1(\mathbf{k},\tau)\overline{\widehat{w}(\mathbf{k},\tau)} + \kappa_M \frac{|\mathbf{k}|^2}{\hat{L}^2} w_2(\mathbf{k},\tau)\overline{\widehat{w}(\mathbf{k},\tau)}$$
$$- \left(\beta - \frac{1}{2}\right)\frac{i}{\hat{L}} k_1 w_2(\mathbf{k},\tau)\overline{\widehat{w}(\mathbf{k},\tau)} - \nu\left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)}\left(\frac{|\mathbf{k}|^2}{\hat{L}^2} + 1\right)\left|\widehat{w}(\mathbf{k},\tau)\right|^2\bigg\} d\tau. \quad (75)$$

Recall that

$$|\vec{w}|^2 = \sum_{\mathbf{k}\in\mathbb{Z}^2}\left(\frac{|\mathbf{k}|^2}{\hat{L}^2}\left(\left|w_1(\mathbf{k})\right|^2 + \left|w_2(\mathbf{k})\right|^2\right) + \frac{|w_1(\mathbf{k}) - w_2(\mathbf{k})|^2}{2}\right)$$
$$= \frac{1}{2}\sum_{\mathbf{k}\in\mathbb{Z}^2}\left(\frac{|\mathbf{k}|^2}{\hat{L}^2}\left|\widetilde{w}(\mathbf{k})\right|^2 + \left(\frac{|\mathbf{k}|^2}{\hat{L}^2} + 1\right)\left|\widehat{w}(\mathbf{k})\right|^2\right).$$

Using this and relations (74) and (75), after summing over $\mathbf{k} \in \mathbb{Z}^2$ we obtain

$$\left|\vec{w}(t)\right|^2 = \left|\vec{w}(0)\right|^2 + 2\,\mathrm{Re}\int_0^t \bigg\{\frac{1}{\hat{L}^2}\sum_{\mathbf{k}\in\mathbb{Z}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (h_2 l_1 - h_1 l_2)\big(\psi_1(\mathbf{h},\tau) q_1(\mathbf{l},\tau)$$
$$- \varphi_1(\mathbf{h},\tau) r_1(\mathbf{l},\tau)\big)\overline{w_1(\mathbf{k},\tau)} + \frac{1}{\hat{L}^2}\sum_{\mathbf{k}\in\mathbb{Z}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (h_2 l_1 - h_1 l_2)\big(\psi_2(\mathbf{h},\tau) q_2(\mathbf{l},\tau)$$
$$- \varphi_2(\mathbf{h},\tau) r_2(\mathbf{l},\tau)\big)\overline{w_2(\mathbf{k},\tau)} + \frac{i}{\hat{L}}\sum_{\mathbf{k}\in\mathbb{Z}^2} k_1 y_1(\mathbf{k},\tau)\overline{w_1(\mathbf{k},\tau)}$$
$$- \kappa_M \sum_{\mathbf{k}\in\mathbb{Z}^2}\frac{|\mathbf{k}|^2}{\hat{L}^2}\left|w_2(\mathbf{k},\tau)\right|^2 - \nu\sum_{\mathbf{k}\in\mathbb{Z}^2}\left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)} E(\vec{w})(\mathbf{k})\bigg\}.$$

Therefore,

$$\left|\vec{w}(t)\right|^2 + \kappa_M\int_0^t \sum_{\mathbf{k}\in\mathbb{Z}^2}\frac{|\mathbf{k}|^2}{\hat{L}^2}\left|w_2(\mathbf{k},\tau)\right|^2 d\tau + \nu\int_0^t\sum_{\mathbf{k}\in\mathbb{Z}^2}\left(\frac{|\mathbf{k}|}{\hat{L}}\right)^{2(1+\alpha)} E(\vec{w})(\mathbf{k})\,d\tau$$
$$= 2\,\mathrm{Re}\,\frac{i}{\hat{L}}\int_0^t\sum_{\mathbf{k}\in\mathbb{Z}^2} k_1 y_1(\mathbf{k},\tau)\overline{w_1(\mathbf{k},\tau)}\,d\tau + 2\,\mathrm{Re}\,\frac{1}{\hat{L}^2}\sum_{j=1}^{2}\int_0^t\sum_{\mathbf{k}\in\mathbb{Z}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}(h_2 l_1 - h_1 l_2)$$
$$\times\left(w_j(\mathbf{h},\tau) q_j(\mathbf{l},\tau)\overline{w_j(\mathbf{k},\tau)} + \varphi_j(\mathbf{h},\tau) y_j(\mathbf{l},\tau)\overline{w_j(\mathbf{k},\tau)}\right) d\tau. \quad (76)$$

Using the same steps as when we proved that $S_1 = 0$ we can show that

$$\sum_{\mathbf{k}\in\mathbb{Z}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}(h_2 l_1 - h_1 l_2) w_j(\mathbf{h},\tau) q_j(\mathbf{l},\tau)\overline{w_j(\mathbf{k},\tau)} = 0, \quad j = 1, 2.$$

Also we have

$$\sum_{\mathbf{k}\in\mathbb{Z}^2}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}}(h_2l_1-h_1l_2)\varphi_1(\mathbf{h},\tau)y_1(\mathbf{l},\tau)\overline{w_1(\mathbf{k},\tau)}$$
$$=\sum_{\mathbf{h}+\mathbf{l}+\mathbf{k}=\mathbf{0}}(h_2l_1-h_1l_2)\varphi_1(\mathbf{h},\tau)w_1(\mathbf{k},\tau)\left(-\frac{|\mathbf{l}|^2}{\hat{L}^2}w_1(\mathbf{l},\tau)-\frac{w_1(\mathbf{l},\tau)-w_2(\mathbf{l},\tau)}{2}\right),$$

and

$$\left|\frac{1}{\hat{L}^2}\sum_{\mathbf{h}+\mathbf{l}+\mathbf{k}=\mathbf{0}}(h_2l_1-h_1l_2)\varphi_1(\mathbf{h},\tau)w_1(\mathbf{k},\tau)\frac{|\mathbf{l}|^2}{\hat{L}^2}w_1(\mathbf{l},\tau)\right|$$
$$=\left|\frac{1}{\hat{L}^2}\sum_{\mathbf{h}+\mathbf{l}+\mathbf{k}=\mathbf{0}}(h_2k_1-h_1k_2)\varphi_1(\mathbf{h},\tau)w_1(\mathbf{k},\tau)\frac{\mathbf{h}\cdot\mathbf{l}}{\hat{L}^2}w_1(\mathbf{l},\tau)\right|$$
$$\leqslant\sum_{\mathbf{h}+\mathbf{l}+\mathbf{k}=\mathbf{0}}\left(\frac{|\mathbf{h}|^2}{\hat{L}^2}\left|\varphi_1(\mathbf{h},\tau)\right|\right)\left(\frac{|\mathbf{k}|}{\hat{L}}\left|w_1(\mathbf{k},\tau)\right|\right)\left(\frac{|\mathbf{l}|}{\hat{L}}\left|w_1(\mathbf{l},\tau)\right|\right).$$

Next we define the auxiliary functions

$$f(x)=\sum_{\mathbf{k}\in\mathbb{Z}^2}\frac{|\mathbf{k}|^2}{\hat{L}^2}\left|\varphi_1(\mathbf{k},\tau)\right|e^{(i/\hat{L})\mathbf{k}\cdot\mathbf{x}}\quad\text{and}\quad g(x)=\sum_{\mathbf{k}\in\mathbb{Z}^2}\frac{|\mathbf{k}|}{\hat{L}}\left|w_1(\mathbf{k},\tau)\right|e^{(i/\hat{L})\mathbf{k}\cdot\mathbf{x}}.$$

Then,

$$S:=\sum_{\mathbf{h}+\mathbf{l}+\mathbf{k}=\mathbf{0}}\left(\frac{|\mathbf{h}|^2}{\hat{L}^2}\left|\varphi_1(\mathbf{h},\tau)\right|\right)\left(\frac{|\mathbf{k}|}{\hat{L}}\left|w_1(\mathbf{k},\tau)\right|\right)\left(\frac{|\mathbf{l}|}{\hat{L}}\left|w_1(\mathbf{l},\tau)\right|\right)$$
$$=\frac{1}{(2\pi\hat{L})^2}\int_{\Omega}f(x)g^2(x)\,dx.$$

Applying Holder's and Ladyzhenskaya's inequalities we obtain that

$$|S|\leqslant\frac{1}{(2\pi\hat{L})^2}\|f\|_{L^2(\Omega)}\|g\|^2_{L^4(\Omega)}\leqslant\frac{1}{(2\pi\hat{L})^2}c_L^2\|f\|_{L^2(\Omega)}\|g\|_{L^2(\Omega)}\|\nabla g\|_{(L^2(\Omega))^2}$$
$$\leqslant c\left|\vec{\varphi}(\tau)\right|_1\left|\vec{w}(\tau)\right|\left|\vec{w}(\tau)\right|_1,\quad\forall\tau\in[0,T].$$

Using the above estimate and similar estimates for the other terms, from (76) we get

$$\begin{aligned}\left|\vec{w}(t)\right|^2+\nu\int_0^t\left|\vec{w}(\tau)\right|_{1+\alpha}^2 d\tau \leqslant \left|\vec{w}(0)\right|^2+c\int_0^t\left|\vec{w}(\tau)\right|^2 d\tau\\ +c\int_0^t\left|\vec{\varphi}(\tau)\right|_1\left|\vec{w}(\tau)\right|\left|\vec{w}(\tau)\right|_1 d\tau, \qquad (77)\end{aligned}$$

for every $t\in[0,T]$, where $c>0$ depends on $T$. In the last integral of (77) we use the inequality $2ab\leqslant a^2+b^2$ to get

$$\begin{aligned}\left|\vec{w}(t)\right|^2+\nu\int_0^t\left|\vec{w}(\tau)\right|_{1+\alpha}^2 d\tau \leqslant \left|\vec{w}(0)\right|^2+c\int_0^t\left|\vec{w}(\tau)\right|^2 d\tau+\nu\int_0^t\left|\vec{w}(\tau)\right|_1^2 d\tau\\ +\tilde{c}\int_0^t\left|\vec{\varphi}(\tau)\right|_1^2\left|\vec{w}(\tau)\right|^2 d\tau,\end{aligned}$$

which implies that

$$\left|\vec{w}(t)\right|^2\leqslant\left|\vec{w}(0)\right|^2+\hat{c}\int_0^t\left|\vec{\varphi}(\tau)\right|_1^2\left|\vec{w}(\tau)\right|^2 d\tau.$$

Using Lemma 6.4 (below) we deduce that $|\vec{w}(t)|^2\leqslant|\vec{w}(0)|^2e^{\hat{c}\int_0^t|\vec{\varphi}(\tau)|_1^2\,d\tau}$, $\forall t\in[0,T]$. But $\vec{w}(0)=\mathbf{0}$, and thus, $\vec{w}(t)=\mathbf{0}$, $\forall t\in[0,T]$. Since $T$ was arbitrary we conclude that $\vec{\psi}(t)=\vec{\varphi}(t)$, $\forall t\in[0,\infty)$, and the proof is complete. □

The lemma below is a generalization of Gronwall's inequality. The proof is elementary and it is omitted.

**Lemma 6.4.** *Let* $f_0\geqslant 0$ *and* $f\in L^\infty([0,T],\mathbb{R})$, $g\in L^1([0,T],\mathbb{R})$ *be nonnegative functions such that*

$$f(t)\leqslant f_0+\int_0^t g(\tau)f(\tau)\,d\tau,\quad \forall t\in[0,T].$$

*Then*

$$f(t)\leqslant f_0e^{\int_0^t g(\tau)\,d\tau},\quad \forall t\in[0,T].$$

**Remark 6.5.** Since every limit point in $C([0,\infty),\mathcal{K})$ of $\{\vec{\psi}_N\}_{N\in\mathbb{N}}$ is a weak solution for (19)–(23) we easily get as a consequence of uniqueness that $\{\vec{\psi}_N\}_{N\in\mathbb{N}}$ converges to $\vec{\psi}$ in $C([0,\infty),\mathcal{K})$.

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