Permanence for a delayed periodic predator–prey model with prey dispersal in multi-patches and predator density-independent

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Abstract

In this paper, we study two species time-delayed predator–prey Lotka–Volterra type dispersal systems with periodic coefficients, in which the prey species can disperse among \( n \) patches, while the density-independent predator species is confined to one of patches and cannot disperse. Sufficient conditions on the boundedness, permanence and existence of positive periodic solution for this systems are established. The theoretical results are confirmed by a special example and numerical simulations.

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1. Introduction

Predator–prey models without diffusion between patches have been extensively and deeply investigated. With regard to their qualitative analysis, particularly the analyses on the properties with sound ecological meanings, such as boundedness, permanence, extinction, stability and existence of periodic solution for survival of predator and prey species, far more important results have been obtained and collected in some monographs (e.g., [2,5,6,10,11,20–24,26, 29,31] and references cited therein). Owing to many natural and man-made factors such as low birth rate, high death rate, hunting, decreasing habitats, aggravating living environment, etc., some predator animals become very rare and even liable to extinction. Hence, in many articles (e.g., [5,24,29] and references cited therein), authors usually assumed that the predator’s density is regulated only by predation and is density-independent, which is much identical with the real biological background.

However, due to the spatial heterogeneity and the increasing spread of human activities, the habitats of biological species have been smashed to partitioned habitat patches, thus resulting in the apparent imbalance in distribution of natural resources and life-form among these patches (e.g., the patches may be divided into food-rich patches and food-poor patches, the prey species can disperse between patches with less food and patches with abundant food...
but with predation, these models can be found in many papers, such as [3,9]). Therefore, the phenomenon of patchy environment has been a potential threat to the survival and diversity of organisms living in these patches. Recently, the effect of dispersion on the possibility of species survival has been an important subject in population biology.

Two species predator–prey dispersal systems have been extensively studied, and many good results with biological meanings have been achieved (e.g., [3,7–9,12,13,19,27,28,32,37,39] and references cited therein).

However, in all of these investigated models of populations dispersing among patches in a heterogeneous predator–prey environment considered so far, it has been assumed that the predator species is strictly density-dependent. This assumption is obviously too idealistic to be true for many animals. To our best knowledge, up to now, the dynamic behavior of predator species with density-independence in a nonautonomous predator–prey dispersal system has not been discussed. Furthermore, for the sake of objectivity, the prey species may be losing during the course of dispersion, some authors introduced the loss rate of prey species during the course of dispersal into their research models (e.g., [30]). On the other hand, the effects of a periodically varying environment and time delay play an important role in the permanence and extinction of population dynamic systems (e.g., [1,4,5,14,15,18]). Thus, the assumptions of periodicity of the parameters and time delay of species during the course of dispersion and conversion of nutrients into the reproduction are effective way to characterize and investigate dispersal population systems. Moreover, in [2], the author discussed the quantity of periodic solutions for a nonautonomous periodic predator–prey system without dispersal and gave an example to interpret that the nonautonomous predator-prey systems have at least two periodic solutions, which enlighten us whether the nonautonomous periodic predator–prey systems with dispersal have not only one periodic solutions.

Motivated by the calculation hereinbefore, in this paper, we consider the following delayed periodic two species predator–prey Lotka–Volterra type dispersal system with n patches:

\[
\frac{dx_i(t)}{dt} = x_i(t) \left[ a_i(t)(1) - b_i(t)x_i(t) - c(t) \int_{-\infty}^{0} k_{12}(s)y(t+s) \, ds \right] + \sum_{j=1}^{n} \left[ \alpha_{ij}(t)d_{ij}(t)x_j(t) - d_{j1}(t)x_1(t) \right],
\]

\[
\frac{dy_i(t)}{dt} = y_i(t) \left[ a_i(t)(2) - b_i(t)x_i(t) \right] + \sum_{j=1}^{n} \left[ \alpha_{ij}(t)d_{ij}(t)x_j(t) - d_{j1}(t)x_1(t) \right], \quad i = 2, 3, \ldots, n,
\]

\[
\frac{dy(t)}{dt} = y(t) \left[ -e(t) + f(t) \int_{-\infty}^{0} k_{21}(s)x_1(t+s) \, ds \right],
\]

(1.1)

where \( t \in R_{+0} = [0, \infty), x_i (i \in I = \{1, 2, \ldots, n\}) \) denote the population density of prey species in \( i \)th patch and \( y \) is the population density of predator species.

For above system (1.1), we see that the predator species \( y \) which is required to be density-independent is confined to patch 1 (food rich patch) and cannot disperse to other \( n - 1 \) patches (food poor patches), while the prey species \( x \) can disperse among \( n \) patches. In this paper, we always assume functions \( a_i(t), b_i(t), \tau_{ij}(t), \alpha_{ij}(t), d_{ij}(t) \) for all \( t \in R_{+0} \) with common period \( \omega > 0 \) and \( d_{ii}(t) = 0 \) for \( i \in I \), \( i \neq j \), \( c(t), e(t), f(t) \) are continuous and periodic defined on \( R_{+0} \) with common period \( \omega > 0 \), and the functions \( k_{ij}(s) \) for \( i = 1, 2 \) defined on \( R_{-} = (-\infty, 0) \) are nonnegative and integrable, \( \int_{-\infty}^{0} k_{ij}(s) \, ds = 1 \). \( \tau_{ij}(t) \) denote the requisite time that the prey species \( x \) reaches patch \( i \) from patch \( j \) at time \( t \). \( 1 - a_{ij}(t) \) (\( i, j \in I, i \neq j \)) the loss rate for the prey species \( x \) during dispersion from patch \( j \) to \( i \), \( a_{ij}(t) \) (\( i \in I \)) the intrinsic growth rate of prey species, \( b_{ij}(t) \) (\( i \in I \)) the density-dependent coefficient of the prey species, \( c(t) \) the predation rate of predator species for prey species in the 1th patch, \( e(t) \) the death rate of predator species, \( f(t) \) the rate of conversion of nutrients into the reproduction rate of the predator species, \( d_{ij}(t) \) (\( i \in I, i \neq j \)) the dispersal rate of prey species from the \( i \)th patch to the \( j \)th patch. We assume that there is a delay effect of predation on predator’s growth, and the predator both has an instantaneous predation and a memory predation in the past.

Our main purpose is to establish a series of criteria on the ultimate boundedness, permanence, extinction and the existence of the periodic solution of the prey and predator species for system (1.1). The method used in this paper is motivated by the works on the permanence and extinction for periodic predator–prey systems in patchy environment.
given by Teng and Chen in [32], the effect of dispersal on single species nonautonomous dispersal model with delays given by Teng and Lu in [33], and the permanence criteria for the general nonautonomous predator–prey Kolmogorov systems given by Teng, Li and Jiang in [34].

The organization of this paper is as follows. In Section 2 some basic assumptions and useful lemmas will be presented. Section 3 is to state our main results. From Sections 4–6 we will give the proofs of main results. A special example and numerical simulations to confirm the validity of our theoretical results will be given in Section 7.

2. Preliminaries

Before going into details, we draw some notations and assumptions, furthermore, quote some useful lemmas. Let $h_0(s)$ be a given continuous function defined on $\mathbb{R}_{-0} = (-\infty, 0]$, $h_0(s) > 0$ for all $s < 0$ and $\int_{-\infty}^{0} h_0(s) \, ds < \infty$. We define the phase space $C$ of system (1.1) as follows. $C$ is the space of continuous functions $\phi(s) : \mathbb{R}_- \to \mathbb{R}^{n+1}$ with the norm

$$\|\phi\| = \int_{-\infty}^{0} h_0(s) \sup_{s \leq t \leq 0} |\phi(t)| \, ds.$$ 

In [36], Wang and Huang have proved that the space $C$ is a Banach space and satisfies all of the fundamental axioms described by Hale and Kato [17] for functional differential equations with infinite delay. Let set $C_+ = \{\phi = (\phi_1, \phi_2, \ldots, \phi_{n+1}) \in C : \phi_i \ (i = 1, 2, \ldots, n + 1)$ is nonnegative and bounded on $\mathbb{R}_-$ and $\phi_i(0) > 0\}$. Motivated by the biological background of system (1.1), in this paper we always assume that solutions of system (1.1) satisfy the following initial conditions:

$$x_i(s) = \phi_i(s), \quad y(s) = \phi_{n+1}(s) \quad \text{for all} \ s \in \mathbb{R}_{-0}, \ i \in I,$$

(2.1)

where $\phi = (\phi_1, \phi_2, \ldots, \phi_{n+1}) \in C_+$. It is not hard to prove that the functional of the right side of system (1.1) is continuous and satisfies a local Lipschitz condition with respect to $\phi$ in the space $\mathbb{R} \times C$. Therefore, by the fundamental theory of functional differential equations with infinite delay (see [16,17,36]), for any $\phi \in C_+$, system (1.1) has a unique solution $(x(t, \phi), y(t, \phi)) = (x_1(t, \phi), x_2(t, \phi), \ldots, x_n(t, \phi), y(t, \phi))$ satisfying the initial condition (2.1). It is also easy to prove that the solution $(x(t, \phi), y(t, \phi))$ is positive, that is $x_i(t, \phi) > 0 \ (i \in I)$ and $y(t, \phi) > 0$ in its maximal interval of the existence. In this paper, such a solution of system (1.1) is called a positive solution.

In this paper, system (1.1) is said to be permanent, if there are positive constants $m$ and $M$ such that for any positive solution $(x_1(t), x_2(t), \ldots, x_n(t), y(t))$ of system (1.1)

$$m \leq \liminf_{t \to +\infty} x_i(t) \leq \limsup_{t \to +\infty} x_i(t) \leq M, \quad i \in I,$$

and

$$m \leq \liminf_{t \to +\infty} y(t) \leq \limsup_{t \to +\infty} y(t) \leq M.$$

Let $f(t)$ be an $\omega$-periodic continuous function defined on $\mathbb{R}_{+0}$. We define

$$A_{\omega}(f) = \omega^{-1} \int_{0}^{\omega} f(t) \, dt, \quad f^m = \max_{t \in \mathbb{R}^+} f(t), \quad f^l = \min_{t \in \mathbb{R}^+} f(t).$$

For system (1.1), we first introduce the following basic assumptions:

(H1) $a_i(t) + \sum_{j=1}^{n} [\alpha_{ij}(t) d_{ij}(t) - d_{ji}(t)] > 0 \ \text{for all} \ t \in R_{+0}, \ i \in I.$

(H2) $a^l_i > 0$ and $d^l_{ij} > 0 \ \text{for all} \ i, j \in I \ \text{and} \ i \neq j; \ c^l > 0, \ f^l \geq 0 \ \text{and} \ A_{\omega}(e) > 0.$

Let $C[-\tau, 0]$ be the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with supremum norm $\|\phi\| = \sup_{t \in [-\tau, 0]} |\phi(\theta)|$ for any $\phi \in C[-\tau, 0]$. For any $\phi, \psi \in C[-\tau, 0]$, we denote $\phi \leq \psi \ (\phi < \psi)$ if $\phi(\theta) \leq \psi(\theta) \ (\phi(\theta) < \psi(\theta))$ for all $\theta \in [-\tau, 0]$. Consider a functional differential equation
\[
\frac{dx(t)}{dt} = f(t, x_t),
\]  

where \( f(t, \phi) : \mathbb{R} \times \mathbb{C}[-\tau, 0] \to \mathbb{R}^n \) is continuous and satisfies the local Lipshitz condition with respect to \( \phi \). By the fundamental theory of functional differential equations (see [16]), for any \( t_0 \in \mathbb{R} \) and \( \phi \in \mathbb{C}[-\tau, 0] \), there exists a unique solution \( x(t, t_0, \phi) \) of system (2.2) through \( (t_0, \phi) \). System (2.2) is said to be cooperative if the following quasimonotone increasing condition holds:

For any \( \phi \) and \( \psi \in \mathbb{C}[-\tau, 0] \), if \( \phi \leq \psi \) and \( \phi(t) = \psi(t) \) for some \( t_0 \in I \), then \( f(t, \phi(t)) \leq f(t, \psi(t)) \).

Let \( u(t) = (u_1(t), u_2(t), \ldots, u_n(t)) : [t_0, \sigma] \to \mathbb{R}^n \) be a continuously differentiable function and \( x(t, t_0, \phi) = (x_1(t, t_0, \phi), x_2(t, t_0, \phi), \ldots, x_n(t, t_0, \phi)) \) be a solution of Eq. (2.2) defined on \([t_0, \sigma] \). We have the following well-known result.

**Lemma 1.** Suppose that system (2.2) is quasimonotone. If

\[
\frac{du(t)}{dt} \leq f(t, u_t) \text{ for all } t \in [t_0, \sigma],
\]

and \( u_{t_0} \leq \phi \), then we have \( u_i(t) \leq x_i(t, t_0, \phi) \) for all \( t \in [t_0, \sigma] \) and \( i \in I \). If

\[
\frac{du(t)}{dt} \geq f(t, u_t) \text{ for all } t \in [t_0, \sigma],
\]

and \( u_{t_0} \geq \phi \), then we have \( u_i(t) \geq x_i(t, t_0, \phi) \) for all \( t \in [t_0, \sigma] \) and \( i \in I \).

Lemma 1 can be found in [25]. Further, we consider the following single logistic dispersal system with \( n \) patches:

\[
\frac{dx_i(t)}{dt} = x_i[t\left(a_i(t) - b_i(t)x_i(t)\right) + \sum_{j=1}^{n} \left[\alpha_{ij}(t)d_{ij}(t)x_j(t - \tau_{ij}(t)) - d_{ji}(t)x_i(t)\right]], \quad i \in I,
\]

where \( x_i \) is the population density of species \( x \) in the \( i \)th patch and the coefficients \( \tau_{ij}(t), a_i(t), b_i(t) \) and \( d_{ij}(t) \) (\( i, j \in I \), \( i \neq j \)) are given as in system (1.1). Let \( \tau = \max \{\tau_{ij}(t): i, j = 1, \ldots, n, t \in \mathbb{R}_+\} \). It is easy to testify that system (2.3) is cooperative on \( \mathbb{C}[-\tau, 0] \). We have the following result.

**Lemma 2.** If assumption (H_1) holds, then Eq. (2.3) has a unique globally asymptotically stable positive \( \omega \)-periodic solution.

Lemma 2 can be found in [38,40].

**Remark 1.** In above system (2.3), we obtain the unique positive periodic solution under the assumption that all of parameters with common periodicity, however, considering all parameters fluctuating in time with the same period is unrealistic because it will be more realistic if we allow time fluctuations with different period or even nonperiodic within some of patches, i.e., almost periodic environment, which will be more identical with the sound ecosystem. Therefore, there is a very important open question that is whether the same result given in Lemma 2 will be true under the assumption that the parameters in system (2.3) be almost periodic.

### 3. Main results

Firstly, concerning the persistence, permanence and extinction of the prey species \( x \) for system (1.1), we have the following general result.

**Theorem 1.** Suppose that all of positive solutions of system (1.1) is ultimately bounded.

(a) If there is a constant \( m > 0 \) such that for any positive solution \( (x_1(t, \phi), x_2(t, \phi), \ldots, x_n(t, \phi), y(t, \phi)) \) of system (1.1), \( \liminf_{t \to \infty} x_j(t) > m \) for some \( j \in I \), then there is a constant \( \rho > 0 \) and \( \rho \) is independent of any positive solutions, such that \( \liminf_{t \to \infty} x_i(t, \phi) > \rho \) for all \( i \in I \) and \( i \neq j \).
(b) If for any positive solution \((x_1(t, \phi), x_2(t, \phi), \ldots, x_n(t, \phi), y(t, \phi))\) of system (1.1) there is \(j \in I\) such that \(\liminf_{t \to \infty} x_j(t, \phi) > 0\), then \(\liminf_{t \to \infty} x_i(t, \phi) > 0\) for each \(i \in I\) and \(i \neq j\).

(c) For any constant \(\epsilon > 0\) there is \(\delta > 0\) such that for any positive solution \((x_1(t, \phi), x_2(t, \phi), \ldots, x_n(t, \phi), y(t, \phi))\) of system (1.1) there is \(T > 0\), when \(x_j(t, \phi) < \delta\) for all \(t \geq T\) for some \(j \in I\), then \(x_i(t, \phi) < \epsilon\) for all \(t \geq T\), \(i \in I\) and \(i \neq j\).

(d) If for any positive solution \((x_1(t, \phi), x_2(t, \phi), \ldots, x_n(t, \phi), y(t, \phi))\) of system (1.1) there is \(j \in I\) such that \(\lim_{t \to \infty} x_j(t, \phi) = 0\), then \(\lim_{t \to \infty} x_i(t, \phi) = 0\) for each \(i \in I\) and \(i \neq j\).

(e) If for any positive solution \((x_1(t, \phi), x_2(t, \phi), \ldots, x_n(t, \phi), y(t, \phi))\) of system (1.1) there is \(j \in I\) such that \(\limsup_{t \to \infty} x_j(t, \phi) > 0\), then \(\limsup_{t \to \infty} x_i(t, \phi) > 0\) for all \(i \in I\) and \(i \neq j\).

Theorem 1 shows an interesting fact on the effect of dispersal for persistence and extinction in two species predator–prey systems with prey dispersal. That is, if the prey species \(x\) is persistent in all other patches; if \(x\) is also persistent in all other patches; if \(x\) is extinct in a patch, then, owing to the effect of dispersal, it is also extinction in all other patches.

Next, on the ultimate boundedness of all positive solutions for system (1.1) we have following result.

**Theorem 2.** Suppose that assumptions \((H_1)\) and \((H_2)\) hold, then there is a constant \(M > 0\) such that

\[
\limsup_{t \to \infty} x_i(t, \phi) \leq M \quad (i \in I), \quad \limsup_{t \to \infty} y(t, \phi) \leq M
\]

for any positive solution \((x_1(t, \phi), x_2(t, \phi), \ldots, x_n(t, \phi), y(t, \phi))\) of system (1.1).

Let us see the biological meaning of Theorem 2. In fact, if the predator species is not ultimately bounded, then the population density of predator species will expand unlimitedly. Since the predation rate of predator species for prey species is strictly positive (i.e., \(c^I > 0\) in assumption \((H_2)\) in 1th patch, the prey species will become extinct in the 1th patch because of the massive preying by the predator species. Since the survival of predator species is absolutely dependent on the prey species in the 1th patch, as an opposite result the predator species will become extinct too.

However, if the predation rate \(c(t)\) of the predator species is not strictly positive, that is \(c^I = 0\), then it cannot lead to extinction when the population density of predator species expands unlimitedly. Therefore, an important open question is whether we can still obtain the boundedness of predator species which is density-independent when \(c^I = 0\).

Further, on the permanence of prey species \(x\) for system (1.1) we have the following results.

**Theorem 3.** Suppose that assumptions \((H_1)\) and \((H_2)\) hold. Then the prey species \(x\) of system (1.1) is permanent in each patch \(i \in I\).

Theorem 3 show that if we guarantee the assumption \((H_1)\) and \((H_2)\) hold then the prey species must be permanent. In fact, if the prey species \(x\) is not permanent, then it may be extinct, as a result the predator species \(y\) will be extinct too because its survival is absolutely dependent on \(x\). However, when \(y\) became extinct, \(x\) will not turn to extinction because assumption \((H_1)\) shows that \(x\) has a total positive average growth rate in \(n\) patches.

Finally, on the permanence of predator species \(y\) for system (1.1) we have the following results.

**Theorem 4.** Suppose that assumptions \((H_1)\) and \((H_2)\) hold, and let \((x_1^*(t), x_2^*(t), \ldots, x_n^*(t))\) be the unique positive \(\omega\)-periodic solution of Eq. (2.3). If

\[
\int_{0}^{\omega} \left[ -e(t) + f(t) \int_{-\infty}^{0} k_{21}(s)x_1^*(t + s)ds \right] dt > 0.
\]

then the predator species \(y\) of system (1.1) is permanent.

From Lemma 2, we know, when there is not the predator species \(y\), the prey species \(x\) will approach a positive periodic stable state \(x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))\). Therefore, Theorem 4 shows that, if the positive periodic stable
state $x^*(t)$ of $x$ can guarantee that $y$ obtain a positive total average growth rate, i.e., condition (3.1), then $y$ will be permanent.

Theorem 4 can be proved by using the same method given by Teng and Chen in Claims 5 and 6 in the proof of Theorem 1 in [32], so we omit it here.

Lastly, from Theorems 3 and 4, and Theorem 1 given by Teng and Chen in [35] on the existence of positive periodic solutions for general delayed periodic $n$-species Kolmogorov systems, we have the following result.

**Corollary 1.** Suppose the assumptions (H$_1$) and (H$_2$) hold. If

$$
\int_0^\infty \left[ -a(t) + \sum_{i=1}^n b_i(t) - b_i(t)M - 2c^*M \right] dt > 0,
$$

then system (1.1) has a positive $\omega$-periodic solution.

### 4. Proof of Theorem 1

**Proof.** The proofs of conclusions (a), (b), (d) and (e) of Theorem 1 are similar to the proofs of Theorem 2 in [33], here we omit them. Now, we only prove the conclusion (c) of Theorem 1. Suppose that the conclusion is not true, then there is a constant $\epsilon > 0$ such that for any constant $\delta > 0$ there is a positive solution $(x_1(t), x_2(t, \phi), \ldots, x_n(t, \phi), y(t, \phi))$ of system (1.1) such that $x_j(t, \phi) \leq \delta$ for all $t \geq T_0$ for some $i \in I$ and $T_0 \geq 0$, but there is a time sequence $\left\{t_k\right\}$ with $\lim_{k \to \infty} t_k = \infty$ and some $j \in I$ with $j \neq i$ such that

$$
x_j(t_k, \phi) > \epsilon, \quad k = 1, 2, \ldots.
$$

By the ultimate boundedness of solution of system (1.1), there is a constant $T_1 \geq T_0$ such that

$$
x_j(t, \phi) \leq M, \quad y(t, \phi) \leq M \quad \text{for all} \quad t \geq T_1, \quad i \in I,
$$

where $M > 0$ is a constant and independent of any positive solution of system (1.1). Hence, there is a constant $B > 0$ and $B$ is also independent of any positive solution of system (1.1) such that

$$
\left| \frac{dx_j(t, \phi)}{dt} \right| \leq B \quad \text{for all} \quad t \in R_+.
$$

From this, we can obtain that there is a constant $\eta > 0$, and $\eta$ is independent of any positive solution of system (1.1) and $k = 1, 2, \ldots$, such that

$$
x_j(t, \phi) \geq \frac{1}{2} \epsilon \quad \text{for all} \quad t \in [t_k - \eta, t_k + \eta], \quad k = 1, 2, \ldots.
$$

Without loss of generality, we can assume

$$
[t_k - \eta, t_k + \eta] \cap [t_m - \eta, t_m + \eta] = \Phi
$$

for any integers $k, m$ and $k \neq m$. By the arbitrariness of $\delta$ we can choose a small enough $\delta > 0$ and $\delta < \frac{1}{2} d_1 \epsilon \eta$, such that

$$
x_i(t, \phi) \left[ a_i(t) - \sum_{j=1}^n d_{ij}(t) - b_i(t)M - 2c^*M \right] + \frac{1}{4} d_1 \epsilon > 0
$$

for all $x_i(t) \leq \delta$ and $t \geq T_1$, where $d_1 = \min_{t \in R} \{a_{ij}(t) - d_{ij}(t) \} : i, j \in I, \quad j \neq i$, $c^* = \max_{t \in R} \{c(t)\}$. Choose an integer $K_0 > 0$ and a constant $0 < \sigma < \eta$ such that $t_k - \eta > T_1$ and

$$
H_1 \int_{-\sigma}^{\infty} k_{12}(s) ds < M
$$

for all $k \geq K_0$, where $H_1 = \sup \{x_1(t + s), y(t + s) : t \geq t_0, \quad s \leq 0\}$. Hence, for any $k \geq K_0$ and $t \in [t_k - \eta + \sigma, t_k + \eta]$, we have
\[
\frac{dx_i(t, \phi)}{dt} \geq x_i(t, \phi) \left[ a_i(t) - \sum_{j=1}^{n} d_{ji}(t) - b_i(t)M - c^* \int_{-\sigma}^{\sigma} k_{12}(s)y(t+s, \phi)\,ds \right] \\
\quad - c^* \int_{-\sigma}^{0} k_{12}(s)y(t+s, \phi)\,ds + \frac{1}{2} d_1 \epsilon \\
> x_i(t, \phi) \left[ a_i(t) - \sum_{j=1}^{n} d_{ji}(t) - b_i(t)M - 2c^*M \right] + \frac{1}{2} d_1 \epsilon > \frac{1}{4} d_1 \epsilon.
\]

Therefore, we finally have \(x_i(t_k + \eta, \phi) \geq \frac{1}{2} d_1 \epsilon \eta + x_i(t_k - \eta + \sigma, \phi)\), i.e., \(\delta \geq \frac{1}{4} d_1 \epsilon \eta\) for all \(k \geq K_0\). This lead to a contradiction. This completes the proof of Theorem 1.

5. Proof of Theorem 2

**Proof.** Let \((x_1(t), x_2(t), \ldots, x_n(t), y(t))\) is any nonnegative solution of system (1.1) satisfied the initial condition (2.1). We first prove that the interval of existence of the solution is \([0, \infty)\). Otherwise, let the maximal interval of existence of the solution be \([0, T)\) with \(T < \infty\). Then, the solution is unbounded as \(t \to T\). Since

\[
\frac{dx_i(t)}{dt} \leq a_i(t) - b_i(t)x_i(t) + \sum_{j=1}^{n} \left[ a_{ij}(t)d_{ij}(t)x_j(t - \tau_{ij}(t)) - d_{ji}(t)x_i(t) \right]
\]

for all \(t \in [0, T)\) and \(i \in I\), by Lemma 1, we obtain

\[
x_i(t) \leq \bar{x}_i(t)\quad \text{for all } t \in [0, T), \ i \in I,
\]

where \((\bar{x}_1(t), \bar{x}_2(t), \ldots, \bar{x}_n(t))\) is the solution of Eq. (2.3) with the initial condition \(\psi(\theta) = \phi(\theta)\) for all \(\theta \in [-\tau, 0]\). Under assumption \((H_1)\), from Lemma 2 we obtain \(\bar{x}_i(t) \to x^*_i(t) (i \in I)\) as \(t \to \infty\), where \((x^*_1(t), x^*_2(t), \ldots, x^*_n(t))\) is the globally asymptotically stable positive \(\omega\)-periodic solution of Eq. (2.3). Hence, \(\bar{x}_i(t) (i \in I)\) is bounded on \(R_+\).

Consequently, from (5.2), \(x_i(t) (i \in I)\) is also bounded on \([0, T)\). Further from system (1.1), we have

\[
y(t) = y(0) \exp \int_{0}^{t} \left[ -e(u) + f(u) \int_{-\infty}^{0} k_{21}(s)x_1(u + s)\,ds \right] du
\]

for all \(t \in [0, T)\). From the boundedness of \(x_i(t) (i \in I)\) on \([0, T)\), we can easily obtain that \(y(t)\) is bounded on \([0, T)\), which leads to a contradiction.

We choose a constant \(M_1 > \max_{t \in R} |x^*(t)|\), where \(|x^*(t)| = \sum_{i=1}^{n} x^*_i(t)\). Since \(\bar{x}_i(t) \to x^*_i(t) (i \in I)\) as \(t \to \infty\), there is \(T_1 > 0\) such that \(\bar{x}_i(t) \leq M_1\) for all \(t \geq T_1\), \(i \in I\).

Hence, by (5.2) we have

\[
x_i(t) \leq M_1\quad \text{for all } t \geq T_1, \ i \in I.
\]

Consequently,

\[
\limsup_{t \to \infty} x_i(t) \leq M_1, \quad i \in I.
\]

Further, we prove that there is a constant \(M_2 > 0\) such that

\[
\limsup_{t \to \infty} y(t) \leq M_2.
\]

From assumption \((H_2)\) we can choose the constants \(M_0 > M_1\) and \(0 < \epsilon < M_1\) such that
\[ 
\epsilon (a_1(t) - c(t)M_0) + \sum_{j=1}^{n} \alpha_{1j}(t)d_{1j}(t)M_1 < -\epsilon 
\]

(5.6) for all \( t \in R_{++0} \) and

\[ 
A_\omega(-\epsilon(t) + 2f(t)\epsilon) < -\epsilon. 
\]

(5.7) We first prove

\[ 
\liminf_{t \to \infty} y(t) \leq M_0. 
\]

(5.8) Otherwise, there is a constant \( T_2 \geq T_1 \) such that \( y(t) > M_0 \) for all \( t \geq T_2 \). If \( x_1(t) \geq \epsilon \) for all \( t \geq T_2 \), then directly from system (1.1), we have

\[ 
x_1(t) \leq x_1(T_2) \exp \left[ \int_{T_2}^{t} \left[ a_1(s) - c(s) \left( \int_{-\infty}^{s} k_{12}(u - s)y(u) \, du + \sum_{j=1}^{n} \alpha_{1j}(s)d_{1j}(s) \frac{x_j(s - \tau_{1j}(s))}{x_1(s)} \right) \right] \, ds \right] \]

\[ 
\leq x_1(T_2) \exp \left[ \int_{T_2}^{t} \frac{1}{\epsilon} \left[ \epsilon (a_1(s) - c(s) \int_{T_2}^{s} k_{12}(u - s)y(u) \, du) + \sum_{j=1}^{n} \alpha_{1j}(s)d_{1j}(s)M_1 \right] \, ds \right] 
\]

\[ 
< x_1(T_2) \exp \left[ \int_{T_2}^{t} \frac{1}{\epsilon} \left[ \epsilon (a_1(s) - c(s)M_0) + \sum_{j=1}^{n} \alpha_{1j}(s)d_{1j}(s)M_1 \right] \, ds \right] 
\]

\[ 
< x_1(T_2) \exp \left( -\epsilon(t - T_2) \right). 
\]

Hence, we obtain \( x_1(t) \to 0 \) as \( t \to \infty \). This leads to a contradiction with \( x_1(t) \geq \epsilon \) for all \( t \geq T_2 \).

Therefore, there is \( t_1 > T_2 \) such that \( x_1(t_1) < \epsilon \). Now, we prove \( x_1(t) < \epsilon \) for all \( t \geq t_1 \). Otherwise, there is \( t_2 > t_1 \) such that \( x_1(t_2) = \epsilon \) and \( x_1(t) < \epsilon \) for all \( t \in (t_1, t_2) \). Hence, we have \( \frac{dx_1(t_2)}{dt} \geq 0 \). On the other hand, directly from system (1.1) we obtain

\[ 
\frac{dx_1(t_2)}{dt} \leq x_1(t_2) \left( a_1(t_2) - c(t_2) \int_{-\infty}^{0} k_{12}(s)y(t_2 + s) \, ds \right) + \sum_{j=1}^{n} \alpha_{1j}(t_2)d_{1j}(t_2)x_j(t_2 - \tau_{1j}(t_2)) 
\]

\[ 
= x_1(t_2) \left( a_1(t_2) - c(t_2) \left( \int_{-\infty}^{t_1} + \int_{t_1}^{T_2} \right) k_{12}(u - t_2)y(u) \, du \right) + \sum_{j=1}^{n} \alpha_{1j}(t_2)d_{1j}(t_2)x_j(t_2 - \tau_{1j}(t_2)) 
\]

\[ 
\leq \epsilon (a_1(t_2) - c(t_2)M_0) + \sum_{j=1}^{n} \alpha_{1j}(t_2)d_{1j}(t_2)M_1 < -\epsilon. 
\]

This leads to a contradiction. Therefore, \( x_1(t) < \epsilon \) for all \( t \geq t_1 \). For any \( t \geq t_1 \), we can choose an integer \( p_t \geq 0 \) such that \( t \in [t_1 + p_t\omega, t_1 + (p_t + 1)\omega) \). Obviously, we have \( p_t \to \infty \) as \( t \to \infty \). Let \( H_0 = \sup\{x_1(t + s) : t \geq t_0, s \leq 0\} \). Further, choose a constant \( \tau > 0 \) such that

\[ 
H_0 \int_{-\infty}^{-\tau} k(s) \, ds < \epsilon, 
\]

(5.9) where \( k(s) = k_{12}(s) + k_{21}(s) \). For any \( t \geq t_1 + \tau \), directly from system (1.1) we obtain

\[ 
y(t) = y(t_1 + \tau) \exp \left[ \int_{t_1 + \tau}^{t} \left[ -\epsilon(s) + f(s) \int_{-\infty}^{0} k_{21}(\mu)x_1(s + \mu) \, d\mu \right] \, ds \right] 
\]
\[
\leq y(t_1 + \tau) \exp \left[ \int_{t_1 + \tau}^{t} \left[ -e(s) + f(s) \int_{-\tau}^{0} k_{21}(\mu)x_1(s + \mu) d\mu + f(s)\epsilon \right] ds \right] \\
\leq y(t_1 + \tau) \exp \left\{ \int_{t_1 + \tau + p_1 \omega}^{t} + \int_{t_1 + \tau + p_1 \omega}^{t} \left[ -e(s) + 2f(s)\epsilon \right] ds \right\} \\
\leq y(t_1 + \tau) \exp (r^* \omega) \exp \left( p_t \int_{0}^{\omega} \left[ -e(s) + 2f(s)\epsilon \right] ds \right) .
\]

(5.10)

where \( r^* = \max_{0 \leq t \leq \omega} \{|e(t)| + 2f(t)\epsilon\} \). Hence, from (5.7) we finally obtain \( y(t) \to 0 \) as \( t \to \infty \). This leads to a contradiction with \( y(t) > M_0 \) for all \( t \geq T_2 \). Therefore, (5.8) holds.

Now, we prove that (5.5) is true. Otherwise, there is a sequence of initial functions \( \{\phi_k\} \subset C_+ \) such that

\[
\limsup_{t \to \infty} y(t, \phi_k) > (2M_0 + 1)k \quad \text{for all } k = 1, 2, \ldots .
\]

In view of (5.8), we can obtain that, for each \( k \), there are two time sequences \( \{s^{(k)}_q\} \) and \( \{t^{(k)}_q\} \), satisfying

\[
0 < s^{(k)}_1 < t^{(k)}_1 < s^{(k)}_2 < t^{(k)}_2 < \cdots < s^{(k)}_q < t^{(k)}_q < \cdots
\]

and \( s^{(k)}_q \to \infty \) as \( q \to \infty \), such that

\[
y(s^{(k)}_q, \phi_k) = 2M_0, \quad y(t^{(k)}_q, \phi_k) = (2M_0 + 1)k
\]

and

\[
2M_0 < y(t, \phi_k) < (2M_0 + 1)k \quad \text{for all } t \in (s^{(k)}_q, t^{(k)}_q).
\]

(5.11)

(5.12)

Further, choose a constant \( \tau > 0 \) such that

\[
H^{(k)}_1 \int_{-\infty}^{\tau} k(s) ds < M_1,
\]

(5.13)

where \( H^{(k)}_1 = \sup \{(x_1(t + s, \phi_k), y(t + s, \phi_k)): t \geq t_0, \ s \leq 0\} \). By the ultimate boundedness of \( x_1(t, \phi_k), x_2(t, \phi_k), \ldots, x_n(t, \phi_k) \), for each \( k \) there is a constant \( T^{(k)} > 0 \) such that \( x_1(t, \phi_k) < M_1 \) for all \( t \geq T^{(k)} \). Further, for each \( k \) there is \( K^{(k)} > 0 \) such that \( s^{(k)}_q > T^{(k)} + \tau \) for all \( q \geq K^{(k)} \). Hence, for any \( q \geq K^{(k)} \) directly from system (1.1) we obtain

\[
y(t^{(k)}_q, \phi_k) = y(s^{(k)}_q, \phi_k) \exp \int_{s^{(k)}_q}^{t^{(k)}_q} \left[ -e(t) + f(t) \left\{ \int_{-\tau}^{0} + \int_{-\tau}^{\tau} k_{21}(s)x_1(t + s, \phi_k) ds \right\} \right] \\
\leq y(s^{(k)}_q, \phi_k) \exp \int_{s^{(k)}_q}^{t^{(k)}_q} \left( -e(t) + 2f(t)M_1 \right) dt \\
\leq y(s^{(k)}_q, \phi_k) \exp [r_1(t^{(k)}_q - s^{(k)}_q)]
\]

where \( r_1 = \max_{t \in [0, \omega]} \{|e(t)| + 2f(t)M_1\} \). Consequently, by (5.11) we have

\[
t^{(k)}_q - s^{(k)}_q \geq \frac{\ln k}{r_1} \quad \text{for all } q \geq K^{(k)}.
\]
Hence, for any constant $L > 0$ there is $N_L > 0$ such that $t_q^{(k)} > s_q^{(k)} + 2L$ for all $k \geq N_L$ and $q \geq K^{(k)}$. For any fixed $k \geq N_L$ and $q \geq K^{(k)}$, we prove that there must be $\tilde{t}_1 \in [s_q^{(k)}, s_q^{(k)} + L]$ such that $x_1(\tilde{t}_1, \phi_k) < \epsilon$. Otherwise, $x_1(t, \phi_k) \geq \epsilon$ for all $t \in [s_q^{(k)}, s_q^{(k)} + L]$.

Directly from system (1.1) and (5.12) we have

\[
x_1(s_q^{(k)} + L, \phi_k) \leq x_1(s_q^{(k)}, \phi_k) \exp \int_{s_q^{(k)}}^{s_q^{(k)} + L} \left[ a_1(t) - c(t) \int_{-\infty}^{0} k_{12}(s)y(t + s, \phi_k) \, ds \right] \, dt
\]

\[
+ \sum_{j=1}^{n} \alpha_j(t) d_1 j(t) x_j(t - \tau_1 j(t), \phi_k) \left[ a_1(t) - c(t) \int_{-\infty}^{0} k_{12}(u - t) y(u, \phi_k) \, du \right] \, dt
\]

\[
= x_1(s_q^{(k)}, \phi_k) \exp \int_{s_q^{(k)}}^{s_q^{(k)} + L} \left[ a_1(t) - c(t) \int_{-\infty}^{0} k_{12}(s)y(t + s, \phi_k) \, ds \right] \, dt
\]

\[- c(t) \int_{-\infty}^{t} k_{12}(u - t) y(u, \phi_k) \, du + \sum_{j=1}^{n} \alpha_j(t) d_1 j(t) x_j(t - \tau_1 j(t), \phi_k) \right] \, dt
\]

\[
\leq M_1 \exp \int_{s_q^{(k)}}^{s_q^{(k)} + L} \left[ \frac{1}{e} \left( a_1(t) - c(t)M_0 + \sum_{j=1}^{n} \alpha_j(t) d_1 j(t) M_1 \right) \right] \, dt
\]

\[
\leq M_1 \exp(-L).
\]

(5.14)

We can choose enough large $L > 0$ such that $M_1 \exp(-L) < \epsilon$. Then from (5.14) we obtain a contradiction. Next, we prove $x_1(t) < \epsilon$ for all $t \in (\tilde{t}_1, t_q^{(k)})$. Otherwise, there is $\tilde{t}_2 > \tilde{t}_1$ such that $x_1(\tilde{t}_2, \phi_k) = \epsilon$ and $x_1(t, \phi_k) < \epsilon$ for all $t \in (\tilde{t}_1, \tilde{t}_2)$. Hence, we obtain $\frac{dx_1(\tilde{t}_2, \phi_k)}{dt} \geq 0$. On the other hand, directly from system (1.1) we have

\[
\frac{dx_1(\tilde{t}_2, \phi_k)}{dt} \leq x_1(\tilde{t}_2, \phi_k) \left( a_1(\tilde{t}_2) - c(\tilde{t}_2) \int_{-\infty}^{0} k_{12}(u - \tilde{t}_2) y(u, \phi_k) \, du \right)
\]

\[
+ \sum_{j=1}^{n} \alpha_j(\tilde{t}_2) d_1 j(\tilde{t}_2) x_j(\tilde{t}_2 - \tau_1 j(\tilde{t}_2), \phi_k)
\]

\[
\leq \epsilon \left( a_1(\tilde{t}_2) - c(\tilde{t}_2)M_0 + \sum_{j=1}^{n} \alpha_j(\tilde{t}_2) d_1 j(\tilde{t}_2) M_1 \right)
\]

\[
< -\epsilon.
\]

This leads to a contraction. Therefore, $x_1(t, \phi_k) < \epsilon$ for all $t \in [s_q^{(k)} + L, t_q^{(k)}]$. Further from (5.9), (5.11) and (5.13) we have

\[
(2M_0 + 1)k = y(t_q^{(k)}, \phi_k) = y(s_q^{(k)} + L, \phi_k) \exp \int_{s_q^{(k)} + L}^{t_q^{(k)}} \left( -e(t) + f(t) \int_{-\infty}^{0} k_{21}(s)x_1(t + s, \phi_k) \, ds \right) \, dt
\]

\[
< (2M_0 + 1)k \exp \int_{s_q^{(k)} + L}^{t_q^{(k)}} \left( -e(t) + 2f(t)\epsilon \right) \, dt.
\]

(5.15)
By (5.7), we can choose enough large \( L > 0 \) such that
\[
\exp \left( \int_{t}^{t+L} \left( -e(t) + 2f(t)\epsilon \right) dt \right) < 1
\]
for all \( t \in \mathbb{R}_{+} \). Hence, from (5.15) we finally obtain a contradiction as follows
\[
(2M_0 + 1)k < (2M_0 + 1)k.
\]

Therefore, (5.5) holds. Choose a constant \( M > \max\{M_1, M_2\} \). Then we see the conclusion of Theorem 2 is true. This completes the proof of Theorem 2. \( \square \)

6. Proof of Theorem 3

**Proof.** From the positivity of \( x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t)) \) and assumption (H2) we can choose small enough positive constants \( \epsilon \), such that
\[
x_i^*(t) - \epsilon > \epsilon, \quad i \in I,
\]
and
\[
\int_{0}^{\omega} \left[ -e(t) + 2f(t)\epsilon \right] dt < -\epsilon.
\]

From conclusion (c) of Theorem 1, we have that for above constant \( \epsilon > 0 \) there is \( \alpha_1 > 0 \) and \( \alpha_1 < \epsilon \) such that for any positive solution \( (x_1(t), x_2(t), \ldots, y(t)) \) of system (1.1) there is \( T_0 > 0 \), if \( x_i(t) < \alpha_1 \) for all \( t \geq T_0 \) for some \( i \in I \), then we have
\[
x_j(t) < \epsilon \quad \text{for all } t \geq T_0, \quad j \in I, \quad j \neq i.
\]

For above constant \( \alpha_1 > 0 \), there is a constant \( \alpha_0 \in (0, \alpha_1) \) such that for any \( \alpha \in (0, \alpha_0) \),
\[
\phi_\alpha(t) > 0,
\]
where \( \phi_\alpha(t) = \min_{i \in I} \left\{ a_i(t) - 2c(t)\alpha - \sum_{j=1}^{n} d_{ij}(t) + \sum_{j=1}^{n} \alpha_{ij}(t)d_{ij}(t) \right\} \).

For above \( \alpha > 0 \), we consider following auxiliary system:
\[
\frac{dx_1(t)}{dt} = x_1(t)\left[ a_1(t) - b_1(t)x_1(t) - 2c(t)\alpha + \sum_{j=1}^{n} \alpha_{1j}(t)d_{1j}(t)x_j(t - \tau_{1j}(t)) - d_{1j}(t)x_1(t) \right],
\]
\[
\frac{dx_i(t)}{dt} = x_i(t)\left[ a_i(t) - b_i(t)x_i(t) + \sum_{j=1}^{n} \alpha_{ij}(t)d_{ij}(t)x_j(t - \tau_{ij}(t)) - d_{ij}(t)x_i(t) \right], \quad i = 2, \ldots, n.
\]

Hence, by Lemma 2 we obtain that Eq. (6.5) has a unique globally asymptotically stable positive \( \omega \)-periodic solution \( x^*_\alpha(t) = (x_{1\alpha}^*(t), x_{2\alpha}^*(t), \ldots, x_{n\alpha}^*(t)) \). Let \( x_\alpha(t) = (x_{1\alpha}(t), x_{2\alpha}(t), \ldots, x_{n\alpha}(t)) \) be the solution of Eq. (6.5) with the initial condition \( x_\alpha(0) = x^*(0) \). By the continuity of solutions with respect to parameters, we can obtain that when \( \alpha \rightarrow 0 \), \( x^*_\alpha(t) \) uniformly for \( t \in [0, \omega] \) converges to \( x^*(t) \). Hence, there is \( \alpha^* \in (0, \alpha_0) \) such that
\[
\left| x_{i\alpha^*}^*(t) - x_i^*(t) \right| < \frac{1}{2}\epsilon \quad \text{for all } t \in [0, \omega], \quad i \in I.
\]

By (6.6) and the periodicity of \( x_{i\alpha^*}^*(t) \) and \( x_i^*(t) \), we finally have
\[
\left| x_{i\alpha^*}^*(t) - x_i^*(t) \right| < \frac{1}{2}\epsilon \quad \text{for all } t \in \mathbb{R}, \quad i \in I.
\]
\[
\limsup_{t \to \infty} x_i(t) \geq \alpha^*, \quad i \in I,
\]
for any positive solution \((x_1(t), x_2(t), \ldots, x_n(t), y(t))\) of system (1.1).

Suppose this conclusion is not true. Then, there is a positive solution \((x_1(t), x_2(t), \ldots, x_n(t), y(t))\) of system (1.1) such that for some \(j \in I,\)
\[
\limsup_{t \to \infty} x_j(t) \leq \alpha^*.
\]
Hence, there is \(T_1 > 0\) such that \(x_j(t) < \alpha^* < \alpha\) for all \(t \geq T_1\). Thus, from (6.3), we obtain
\[
x_i(t) < \epsilon \quad \text{for all } t \geq T_1, \quad i \in I, \quad i \neq j.
\]
Further, we choose a constant \(\tau > 0\) such that
\[
H_1 \int_{-\infty}^{-\tau} k(s) \, ds < \alpha^* < \epsilon.
\]
Hence, for all \(t > T_1 + \tau\), we have
\[
\frac{dy(t)}{dt} = y(t) \left[-e(t) + f(t) \int_{-\infty}^{-\tau} k_{21}(s)x_1(t+s) \, ds + f(t) \int_{-\tau}^{0} k_{21}(s)x_1(t+s) \, ds \right] \\
\leq y(t) \left[-e(t) + 2f(t)\epsilon \right].
\]
By (6.2), we directly have \(y(t) \to 0\) as \(t \to \infty\). Hence, there is \(T_2 > T_1 + \tau\) such that \(y(t) < \alpha^*\) for all \(t \geq T_2\). Therefore, for any \(t \geq T_2\), we have
\[
\frac{dx_1(t)}{dt} = x_1(t) \left(a_1(t) - b_1(t)x_1(t) - c(t) \left\{ \int_{-\infty}^{-\tau} + \int_{-\tau}^{0} \right. k_{12}(s)y(t+s) \, ds \right\} \right) \\
+ \sum_{j=1}^{n} \left[ \alpha_{1j}(t)d_{1j}(t)x_j(t - \tau_{1j}(t)) - d_{1j}(t)x_1(t) \right] \\
\geq x_1(t)(a_1(t) - b_1(t)x_1(t) - 2c(t)\alpha^*) + \sum_{j=1}^{n} \left[ \alpha_{1j}(t)d_{1j}(t)x_j(t - \tau_{1j}(t)) - d_{1j}(t)x_1(t) \right],
\]
\[
\frac{dx_i(t)}{dt} = x_i(t)(a_i(t) - b_i(t)x_i(t)) + \sum_{j=1}^{n} \left[ \alpha_{ij}(t)d_{ij}(t)x_j(t - \tau_{ij}(t)) - d_{ij}(t)x_i(t) \right], \quad i = 2, 3, \ldots, n,
\]
for all \(t \geq T_2\). Hence, by Lemma 1 we have
\[
x_i(t) \geq x_{i\alpha^*}(t) \quad \text{for all } t \geq T_2, \quad i \in I,
\]
where \(x_{\alpha^*}(t) = (x_{1\alpha^*}(t), x_{2\alpha^*}(t), \ldots, x_{n\alpha^*}(t))\) is the solution of Eq. (6.5) with initial condition \(x_{\alpha^*}(T_2) = x(T_2)\) with \(\alpha = \alpha^*\). By the asymptotic stability of the positive \(\omega\)-periodic solution \(x_{\alpha^*}(t)\), we obtain that there is \(T_3 > T_2\) such that
\[
|x_{i\alpha^*}(t) - x_{i\alpha^*}(t)| < \frac{1}{2} \epsilon \quad \text{for all } t \geq T_3, \quad i \in I.
\]
Hence, by (6.7), (6.13) and (6.14) we obtain
\[
x_i(t) \geq x_i(t) - \epsilon \quad \text{for all } t \geq T_3, \quad i \in I.
\]
Finally, from (6.1) we obtain
\[
x_i(t) > \epsilon \quad \text{for all } t \geq T_3, \quad i \in I.
\]
This lead to a contradiction with (6.10). Thus (6.8) holds.
Next, we prove that there is a constant $\sigma > 0$ such that
\[ \liminf_{t \to \infty} x_i(t) > \sigma \quad (6.17) \]
for any positive solution $(x_1(t), x_2(t), \ldots, x_n(t), y(t))$ of system (1.1).

Assume (6.17) is not true. Then there is a sequence initial functions $\{\phi_k\} \subset C_+$ such that for the solution $(x(t, \phi_k), y(t, \phi_k)) = (x_1(t, \phi_k), x_2(t, \phi_k), \ldots, x_n(t, \phi_k), y(t, \phi_k))$ of system (1.1),
\[ \liminf_{t \to \infty} x_i(t, \phi_k) < \frac{\alpha^*}{k^2} \quad \text{for all } k = 1, 2, \ldots. \]

From (6.8) we have $\limsup_{t \to \infty} x_i(t, \phi_k) > \alpha^*$. Hence, for each $k$ there are time sequences $\{s_q^{(k)}\}$ and $\{t_q^{(k)}\}$, satisfying $0 < s_1^{(k)} < s_2^{(k)} < t_1^{(k)} < s_3^{(k)} < \cdots < s_q^{(k)} < t_q^{(k)} < \cdots$ and $s_q^{(k)} \to \infty$ as $q \to \infty$, such that
\[ x_i(s_q^{(k)}, \phi_k) = \frac{\alpha^*}{k}, \quad x_i(t_q^{(k)}, \phi_k) = \frac{\alpha^*}{k^2} \quad (6.18) \]
and
\[ \frac{\alpha^*}{k^2} < x_i(t, \phi_k) < \frac{\alpha^*}{k} \quad \text{for all } t \in (s_q^{(k)}, t_q^{(k)}). \quad (6.19) \]

By Theorem 1, for each $k$ there is $T_1^{(k)} > 0$ such that
\[ x_i(t, \phi_k) < M, \quad y(t, \phi_k) < M \quad \text{for all } t \geq T_1^{(k)}, \quad i \in I. \quad (6.20) \]

From conclusion (c) of Theorem 1, we obtain that for above constant $\epsilon > 0$ there is $\alpha^* > 0$ such that for each positive solution $(x_1(t, \phi_k), x_2(t, \phi_k), \ldots, x_n(t, \phi_k), y(t, \phi_k))$ of system (1.1) there is $T_2^{(k)} > 0$, if $x_j(t, \phi_k) \leq \alpha^*$ for all $t \geq T_2^{(k)}$ for some $j \in I$, then
\[ x_i(t, \phi_k) \leq \epsilon \quad \text{for all } t \geq T_2^{(k)}, \quad i \in I, \quad i \neq j, \quad (6.21) \]
where $\epsilon > 0$ is given in (6.1). Further, choose a constant $\sigma^{(k)} > 0$ such that
\[ H_1^{(k)} \int_{-\infty}^{-\sigma^{(k)}} k(s) \, ds < M. \quad (6.22) \]

Moreover, there is $Q_1^{(k)} > 0$ such that $s_q^{(k)} > T^{(k)} + \sigma^{(k)}$, where $T^{(k)} = \max\{T_1^{(k)}, T_2^{(k)}\}$ for all $q \geq Q_1^{(k)}$. For any $t \in [s_q^{(k)}, t_q^{(k)}]$ and $q \geq Q_1^{(k)}$, by (6.18) and (6.19) there is a $j \in I$ such that
\[ x_j(t, \phi_k) \leq \frac{\alpha^*}{k} < \alpha^*. \]

Therefore, from (6.21) we further have
\[ x_i(t, \phi_k) < \epsilon \quad \text{for all } t \in [s_q^{(k)}, t_q^{(k)}] \quad \text{and } q \geq Q_1^{(k)}, \quad i \in I. \quad (6.23) \]

Then, for any $t \in [s_q^{(k)}, t_q^{(k)}]$, $q \geq Q_1^{(k)}$, by (6.22) and (6.23) and directly from system (1.1) we have
\[
\frac{dx_i(t, \phi_k)}{dt} \geq x_i(t, \phi_k) \left[ a_i(t) - b_i(t)x_i(t, \phi_k) - c^* \left\{ \int_{-\infty}^{-\sigma^{(k)}} + \int_{-\sigma^{(k)}}^0 k_{12}(s)y(t + s, \phi_k) \, ds \right\} 
+ \sum_{j=1}^n [a_{ij}(t)d_{ij}(t)x_j(t - \tau_{ij}(t), \phi_k(t)) - d_{ji}(t)x_i(t, \phi_k)] \right] \\
\geq x_i(t, \phi_k) \left[ a_i(t) - b_i(t)x_i(t, \phi_k) - 2c^*M - \sum_{j=1}^n d_{ji}(t)x_i(t, \phi_k) \right] \\
\geq -x_i(t, \phi_k)\psi_i^m, \]
where

$$\psi^m_i = \max_{t \in \mathbb{R}} \left\{ a_i(t) + b_i(t)M + 2c^*M + \sum_{j=1}^n d_{ji}(t) \right\}.$$ 

Integrating this inequality from $s^{(k)}_q$ to $t^{(k)}_q$, we obtain

$$x_i(t^{(k)}_q, \phi_k) \geq x_i(s^{(k)}_q, \phi_k) \exp \int_{s^{(k)}_q}^{t^{(k)}_q} (-\psi^m_i) \, dt.$$  \hspace{1cm} (6.24)$$

Consequently, from (6.24) we have

$$t^{(k)}_q - s^{(k)}_q \geq \frac{\ln k}{\psi^m_i} \text{ for all } q \geq Q^{(k)}_1.$$

(6.25)

By (6.2), we can choose a constant $P > 0$ such that for any $t \geq 0$ and $a \geq P$,

$$M \exp \int_t^{t+\alpha} \left[ -e(s) + f(s) \epsilon \right] ds < \alpha^*.$$  \hspace{1cm} (6.26)$$

On the other hand, since the positive $\omega$-periodic solution $x^*_\alpha(t) = (x^*_{1\alpha}, x^*_{2\alpha}, \ldots, x^*_{n\alpha})$ of Eq. (6.5) with $\alpha = \alpha^*$ is globally asymptotically stable, then by the periodicity of Eq. (6.5), we have $(x^*_{1\alpha}, x^*_{2\alpha}, \ldots, x^*_{n\alpha})$ is also globally uniformly asymptotically stable. Hence, for the constant $\epsilon > 0$ given in above, there is a constant $T > 0$ such that for any initial value $t_0 \geq 0$, $(x_{10}, x_{20}, \ldots, x_{n0}) \in C^*_+$ and $|x_{i0}| \leq M$ ($i \in I$) where the constant $M$ is given in Theorem 2, for any $t \geq t_0 + T$ we have

$$|x_{i\alpha*}(t, t_0, x_{0}) - x_{i\alpha*}(t)| < \frac{1}{2} \epsilon, \quad i \in I,$$

(6.27)

where $(x_{1\alpha*}, x_{2\alpha*}, \ldots, x_{n\alpha*})$ is the solution of Eq. (6.5) with $\alpha = \alpha^*$ satisfying the initial condition $x_{i\alpha*} = x_{i0}$ ($i \in I$) and $x_0 = (x_{10}, x_{20}, \ldots, x_{n0})$.

Further, we can choose constant $Q^{(k)}_2 > Q^{(k)}_1$ and $\tau > 0$ such that

$$H^{(k)}_0 \int_{-\infty}^{T^{(k)}-s^{(k)}_q} k(s) \, ds < \frac{1}{2} \alpha^* < \frac{1}{2} \epsilon \quad \text{for all } q \geq Q^{(k)}_2,$$

(6.28)

and

$$M \int_{-\infty}^{-\tau} k(s) \, ds < \frac{1}{2} \alpha^* < \frac{1}{2} \epsilon.$$  \hspace{1cm} (6.29)$$

From (6.25) and (6.26), we obtain there is a integer $K_0 > 0$ such that

$$t^{(k)}_q - s^{(k)}_q > P + T$$

(6.30)

for all $k \geq K_0$ and $q \geq Q^{(k)}_2$. Let $k \geq K_0$ and $q \geq Q^{(k)}_2$ in the following discussion. By (6.28) and (6.29), for any $t \in [s^{(k)}_q + \tau, t^{(k)}_q]$ we have

$$\frac{dy(t, \phi_k)}{dt} \leq y(t, \phi_k) \left[ -e(t) + f(t) H^{(k)}_0 \int_{-\infty}^{T^{(k)}-s^{(k)}_q} k_{21}(s) \, ds + f(t) M \int_{-\infty}^{-\tau} k_{21}(s) \, ds + f(t) \int_{-\infty}^{0} k_{21}(s) \, ds \right] < y(t, \phi_k) \left[ -e(t) + 2f(t) \epsilon \right].$$  \hspace{1cm} (6.31)$$
Integrating (6.31) from $s_q^{(k)} + \tau$ to $t$, we obtain

$$y(t, \phi_k) \leq y\left(s_q^{(k)} + \tau, \phi_k\right) \exp \int_{s_q^{(k)} + \tau}^{t} \left[-e(s) + 2f(s)\epsilon\right] ds.$$  

From (6.26) and (6.30) we further obtain when $t \in \left[s_q^{(k)} + \tau + P, t_q^{(k)}\right]$, 

$$y(t, \phi_k) \leq M \exp \int_{s_q^{(k)} + \tau + P}^{t} \left[-e(s) + 2f(s)\epsilon\right] ds < \alpha^*.$$  

(6.32)

For any $t \in \left[s_q^{(k)} + \tau + P, t_q^{(k)}\right]$, by (6.32) and system (1.1) we obtain

$$\frac{dx_1(t, \phi_k)}{dt} = x_1(t, \phi_k) \left[a_1(t) - b_1(t)x_1(t, \phi_k) - c(t) \left(\int_{-\infty}^{T(k)} \int_{s_q^{(i)} + P}^{s_q^{(i)} + P + t} k_{12}(u-t)y(u, \phi_k) du\right)\right]$$

$$+ \sum_{j=1}^{n} \left[\alpha_{1j}(t)d_{1j}(t)x_j(t - \tau_{1j}(t), \phi_k) - d_{1j}(t)x_1(t, \phi_k)\right]$$

$$\geq x_1(t, \phi_k) \left[a_1(t) - b_1(t)x_1(t, \phi_k) - 2c(t)\alpha^*\right]$$

$$+ \sum_{j=1}^{n} \left[\alpha_{1j}(t)d_{1j}(t)x_j(t - \tau_{1j}(t), \phi_k) - d_{1j}(t)x_1(t, \phi_k)\right],$$

$$\frac{dx_i(t, \phi_k)}{dt} = x_i(t, \phi_k) \left[a_i(t) - b_i(t)x_i(t, \phi_k)\right] + \sum_{j=1}^{n} \left[\alpha_{ij}(t)d_{ij}(t)x_j(t - \tau_{ij}(t), \phi_k) - d_{ij}(t)x_i(t, \phi_k)\right].$$

Hence, by Lemma 1 we further obtain

$$x_i(t, \phi_k) \geq x_{i\alpha^*}(t) \quad \text{for all} \quad t \in \left[s_q^{(k)} + P, t_q^{(k)}\right], \quad i \in I,$$  

(6.33)

where $(x_{1\alpha^*}(t), x_{2\alpha^*}(t), \ldots, x_{n\alpha^*}(t))$ is the solution of Eq. (6.5) with $\alpha = \alpha^*$ and satisfying the initial condition $x_{i\alpha^*}(s_q^{(k)} + P + \tau) = x_i(s_q^{(k)} + P + \tau, \phi_k) \quad (i \in I)$. Since we have $0 \leq x_i(s_q^{(k)} + P + \tau, \phi_k) \leq M$ from (6.18), taking $t_0 = s_q^{(k)} + P + \tau$, $(x_{10}, x_{20}, \ldots, x_{n0}) = (x_1(s_q^{(k)} + P + \tau, \phi_k), x_2(s_q^{(k)} + P + \tau, \phi_k), \ldots, x_n(s_q^{(k)} + P + \tau, \phi_k))$, then from (6.27) we can obtain

$$\left|x_{i\alpha^*}(t) - x_{i\alpha^*}(t)\right| < \frac{1}{2} \epsilon \quad \text{for all} \quad t \in \left[s_q^{(k)} + T + P, t_q^{(k)}\right], \quad i \in I.$$  

(6.34)

Therefore, from (6.7), (6.33) and (6.34) we finally obtain

$$x_i(t, \phi_k) \geq x_{i*}(t) - \epsilon \quad \text{for all} \quad t \in \left[s_q^{(k)} + T + P, t_q^{(k)}\right], \quad i \in I.$$  

This show that by (6.1)

$$x_i(t, \phi_k) > \epsilon \quad \text{for all} \quad t \in \left[s_q^{(k)} + T + P, t_q^{(k)}\right], \quad i \in I.$$  

Which is a contradiction with (6.23). This contradiction shows that (6.17) is true. This completes the proof of Theorem 3. □
7. Example and numerical simulation

In this paper, we have investigated a class of periodic two species predator–prey Lotka–Volterra type systems with prey dispersal and predator density-independent. By means of analysis approach we give the criteria for the boundedness, permanence and existence of positive periodic solution.

In order to testify the validity of our results, we consider the following two species predator–prey Lotka–Volterra type systems with prey dispersal and predator density-independent in two patches:

\[
\frac{dx_1}{dt} = x_1(a_1(t) - b_1 x_1 - c(t)y(t - \tau_1)) + \alpha_{12}d_{12}x_2(t - \tau_1) - d_{21}x_1,
\]

\[
\frac{dx_2}{dt} = x_2(a_2(t) - b_2 x_1) + \alpha_{21}d_{21}x_1(t - \tau_3) - d_{12}x_2,
\]

\[
\frac{dy}{dt} = y(-e + f x_1(t - \tau_4)).
\]

(7.1)

Corresponding prey dispersal system is

\[
\frac{dx_1}{dt} = x_1(a_1(t) - b_1 x_1) + \alpha_{12}d_{12}x_2(t - \tau_1) - d_{21}x_1,
\]

\[
\frac{dx_2}{dt} = x_2(a_2(t) - b_2 x_1) + \alpha_{21}d_{21}x_1(t - \tau_3) - d_{12}x_2.
\]

(7.2)

In system (7.1) and system (7.2), let

\[a_1(t) = \frac{5}{4} + \sin(2\pi t), \quad a_2(t) = \frac{3}{4} + \frac{1}{2} \sin(2\pi t), \quad b_1 = 0.4, \quad b_2 = 0.4, \quad c(t) = 0.4, \quad d_{12} = 0.6, \quad d_{21} = 0.5, \quad \alpha_{12} = \alpha_{21} = 0.9, \quad e = 0.3, \quad f = 0.3, \quad \tau_1 = 0.8, \quad \tau_2 = 0.4, \quad \tau_3 = 0.3, \quad \tau_4 = 0.8.

We easily verify that assumptions (H_1) and (H_2) hold. Therefore, from Lemma 2, system (7.2) has a unique globally asymptotically stable positive periodic solution \(x^*(t) = (x_1^*(t), x_2^*(t))\). By numerical simulation, we get the upper and lower bounds of periodic function \(x_1^*(t)\) are 3.5 and 2.4, respectively, i.e., \(2.4 \leq x_1^*(t) \leq 3.5\) (see Fig. 1).

It is easy to verify that the condition (3.1) in Theorem 4 also holds. Therefore, from Theorems 2–4 and Corollary 1 we obtain that system (7.1) is ultimately bounded, permanent and has a positive periodic solution. Numerical simulation of the above results can be seen in Fig. 2.

In Fig. 3, we give a series of different initial points of system (7.1), by numerical simulations, we find that all of the solutions of system (7.1) which through these initial points will converge to the positive periodic solution. Therefore, we can guess that under the assumptions (H_1), (H_2) and condition (3.1) system (7.1) has a unique positive periodic solution which is globally asymptotically stable. Generally, it is an important open question whether or not...
Fig. 2. (a) Time-series of prey–predator population of system (7.1), (b) 3 dimension predator–prey population phase portrait of system (7.1). Here, we take the initial function \( \psi(s) = (\psi_1(s), \psi_2(s), \psi_3(s)) = (0.8, 0.4, 0.3) \) for all \( s \in [-0.8, 0] \).

Fig. 3. Dynamical behavior of system (7.1). Here, we take the initial function \( \psi_1(s) = (\psi_{11}(s), \psi_{12}(s), \psi_{13}(s)) = (1.8, 6.7, 0.9) \), \( \psi_2(s) = (\psi_{21}(s), \psi_{22}(s), \psi_{23}(s)) = (3.5, 6.9, 2.1) \), \( \psi_3(s) = (\psi_{31}(s), \psi_{32}(s), \psi_{33}(s)) = (5.8, 2.6, 1.5) \), \( \psi_4(s) = (\psi_{41}(s), \psi_{42}(s), \psi_{43}(s)) = (7.5, 3.5, 2.6) \) for all \( s \in [-0.8, 0] \).

two species periodic predator–prey system with prey dispersal has a unique positive periodic solution which is globally asymptotically stable.

However, if in system (7.1), \( a_i(t), b_i, \alpha_{ij}, d_{ij}, e \) and \( \tau_i \) \( (i, j = 1, 2, 3, i \neq j) \) are given as in above, but \( c(t) = |\sin(\pi t)| \) and \( f = 0.1 \). We easily verify that assumptions (H1) hold. Therefore, from Lemma 2 system (7.2) has a unique globally asymptotically stable positive periodic solution \( x^*(t) = (x_1^*(t), x_2^*(t), x_3^*(t)) \). By numerical simulation, we get the upper and lower bounds of periodic function \( x_1^*(t) \) are 2.5 and 1.7, respectively, i.e., \( 1.7 \leq x_1^*(t) \leq 2.5 \) (see Fig. 1). It is easy to verify that assumption (H2) condition (3.1) in Theorem 4 do not hold. Therefore, Theorem 4 and Corollary 1 do not hold, i.e., we cannot guarantee the permanence of all species (prey and predator). Numerical simulation of the above results can be seen in Fig. 4. From Fig. 4 we see that the prey species \( x \) is permanent, while the predator species \( y \) turn to extinction.
Fig. 4. (a) Time-series of prey–predator population of system (7.1), (b) 3 dimension predator–prey population phase portrait of system (7.1). Here, we take the initial function $\psi(s) = (\psi_1(s), \psi_2(s), \psi_3(s)) = (0.8, 0.4, 0.3)$ for all $s \in [-0.8, 0]$.

References