Iwasawa invariants for the False–Tate extension and congruences between modular forms

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**ABSTRACT**

For an ordinary prime $p \geq 3$, we consider the Hida family associated to modular forms of a fixed tame level, and their Selmer groups defined over certain Galois extensions of $\mathbb{Q}(\mu_p)$ whose Galois group is $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_p$. For Selmer groups defined over the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}(\mu_p)$, we show that if the $\mu$-invariant of one member of the Hida family is zero, then so are the $\mu$-invariants of the other members, while the $\lambda$-invariants remain the same only in a branch of the Hida family. We use these results to study the behavior of some invariants from non-commutative Iwasawa theory in the Hida family.

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1. Introduction

The main aim of this paper is to study non-commutative Iwasawa theory along Hida families. Let $f_k = \sum a_n(f_k)q^n$ be a modular cusp form of weight $k$, level $Np^i$ with $(N, p) = 1$ and nebentypus $\chi$. We further assume that $f_k$ is a $p$-ordinary (i.e., $a_p(f_k)$ is a $p$-adic unit) and $p$-stabilized newform. Consider the Hida family associated to $f_k$ which consists of all $p$-ordinary and $p$-stabilized cusp forms whose associated mod $p$ residual Galois representations are all isomorphic. We denote this mod $p$ residual representation by $\bar{\rho}$, and the Hida family by $H(\bar{\rho})$ (see Remark 3.3).
Let \( \mu_{p^n} \) be the group of \( p^n \)-th roots of 1 and \( \mu_{p^\infty} = \bigcup_n \mu_{p^n} \). The False–Tate extension of \( \mathbb{Q} \) is the extension

\[
F_\infty := \bigcup_n \mathbb{Q}(\mu_{p^n}, m^{1/p^n})
\]

where \( m \) is an integer such that \( v_q(m) \) is not divisible by \( p \), for any prime \( q \) dividing \( m \); here \( v_q \) denotes the \( q \)-adic order valuation on \( \mathbb{Q} \). It is a non-abelian pro-\( p \), \( p \)-adic Lie extension of \( F := \mathbb{Q}(\mu_p) \).

Clearly, \( \mathbb{Q}(\mu_{p^\infty}) \subset F_\infty \), and we write \( \mathcal{H} := G(F_\infty/\mathbb{Q}(\mu_{p^\infty})) \). For any \( p \)-adic Lie group \( G \) and a finite integral extension \( \mathcal{O} \) of \( \mathbb{Z}_p \), we denote by \( \Lambda_\mathcal{O}(G) \) the Iwasawa algebra of \( G \) defined by

\[
\Lambda_\mathcal{O}(G) = \lim_{\leftarrow} \mathcal{O}[G/U]
\]

where \( U \) runs over the open, normal subgroups of \( G \), and the inverse limit is taken with respect to the canonical projection maps. We simply put \( \Lambda \) (respectively \( \Lambda_\mathcal{O} \)), for the Iwasawa algebra \( \mathbb{Z}_p[[\Gamma]] \) (resp. \( \mathcal{O}[[\Gamma]] \)), where \( \Gamma \) is the Galois group of the cyclotomic \( \mathbb{Z}_p \)-extension of any number field.

For each modular form \( g \in \mathbf{H}(\tilde{\rho}) \) consider the Selmer group attached to the \( p \)-adic Galois representations described in the \( p \)-extension over the False–Tate extension, the definition of which is given in Definition 2.4. Such Selmer groups over \( p \)-adic Lie extensions for elliptic curves have been studied by Coates and Howson [CH], Hachimori and Venjakob [HV] in the context of non-commutative Iwasawa theory. In particular, Howson has defined the analogue of the classical \( \lambda \)-invariant in [H], the definition of which we recall in Section 4. In the case of the Selmer groups attached to the False–Tate extension it is just the \( \Lambda(\mathcal{H}) \)-corank. We study the variation of these non-commutative invariants when the modular forms vary over a Hida family (see Theorem 4.12). Such a variation has been studied by Emerton, Pollack and Weston in [EPW] for the classical Iwasawa invariants of Selmer groups associated to ordinary \( p \)-adic Galois representations of modular forms over the cyclotomic \( \mathbb{Z}_p \)-extension of \( \mathbb{Q} \). Tom Weston studied the variation of these classical Iwasawa invariants in more general Galois deformations in [W]. Their studies are largely inspired by the paper of Greenberg and Vatsal [GV].

Our approach is based on the precise link between the non-commutative algebraic Iwasawa invariants mentioned above and the classical Iwasawa invariants for the cyclotomic-\( \mathbb{Z}_p \) extension. It is inspired by the methods in [CSS] and [HV]. In addition, we also use results from [EPW]. The other important ingredients are the description of the images of the decomposition groups at various primes, level lowering results and a deep result of Kato that the Selmer groups of modular forms defined over cyclotomic \( \mathbb{Z}_p \)-extensions of abelian number fields are cotorsion over the Iwasawa algebra.

Here is the plan of the paper. In Section 2, we set some notations and recall some preliminaries, and in Section 3 we recall results from Hida theory that we require later. In Section 4 we compute the \( \Lambda(\mathcal{H}) \)-coranks of the Selmer groups defined over a False–Tate extension. Finally, in Section 5 we give a nice numerical example, where we show that the \( \mu \)-invariant of the Selmer group of the Ramanujan Delta function over the field \( \mathbb{Q}(\mu_{11^{\infty}}) \) is zero. We then use it to illustrate the results we prove.

### 2. Notation and preliminaries

Let \( \bar{\mathbb{Q}} \) and \( \bar{\mathbb{Q}}_l \) be fixed algebraic closures of \( \mathbb{Q} \) and \( \mathbb{Q}_l \) respectively. Throughout, \( p \) will denote an odd prime. We fix an embedding \( \mathbb{Q} \hookrightarrow \bar{\mathbb{Q}}_l \) for every prime \( l \). Let \( G_l \) denote a decomposition subgroup above a prime of \( l \) given by this embedding and \( I_l \) denote its inertia subgroup. Let \( \text{Frob}_l \) denote the Frobenius element in \( G_l/I_l \). The \( p \)-adic cyclotomic character is denoted by \( \chi_p \).

Throughout, we use the notation \( F = \mathbb{Q}(\mu_p) \). For any number field \( L \), we denote by \( L^{\text{cycl}} \) the cyclotomic \( \mathbb{Z}_p \)-extension of \( L \) and by \( L_\infty := \bigcup_n L(\mu_{p^n}, m^{1/p^n}) \), the False–Tate extension of \( L \). Then, as the extension \( F^{\text{cycl}} \) is totally ramified over \( \mathbb{Q} \), by abuse of notation, we denote the unique prime of \( F \) and \( F^{\text{cycl}} \) lying above \( p \) by \( v_p \). Let \( \mathbb{Z}_{p,N}^\times = \mathbb{Z}_p^\times \times (\mathbb{Z}/N)^\times \), where \( (N,p) = 1 \), and let \( \mathbb{Z}_{p,N}^\times \hookrightarrow \mathbb{Z}_p[[\mathbb{Z}_{p,N}^\times]] \) be the natural inclusion map. We recall that \( \Lambda \) is non-canonically isomorphic to the power series
ring $\mathbb{Z}_p[[T]]$, the isomorphism being obtained by choosing a topological generator $\gamma$ of $I := 1 + p\mathbb{Z}_p$
and mapping $(\gamma)_p$ to $1 + T$.

Let $f = \sum a_nq^n \in S_k(Np^r, \chi)$, with $(N, p) = 1$, be a newform of weight $k \geq 2$, and ordinary at $p$.
Let $K_f$ denote the number field generated by the Fourier coefficients of $f$. Let $\varphi$ denote a prime of $K_f$
above $p$ determined by the embedding $\iota_p$. Let $K_{f, \varphi}$ denote the completion of $K_f$ at $\varphi$, $\mathcal{O}_{f, \varphi}$ denote
its ring of integers, and $k$ denote the residue field of $\mathcal{O}_{f, \varphi}$. By Class Field theory, the nebentype $\chi$
can be viewed as a character of $G(\mathbb{Q}(\mu_{Np^r})/\mathbb{Q})$. Then we have the following theorem due to Eichler,
Deligne, Shimura, Mazur–Wiles, Wiles, which may be found conveniently in [Wi].

**Theorem 2.1.** Let $f = \sum a(n)q^n$ be a newform, of weight $k \geq 2$, level $Np^r$, $(N, p) = 1$, nebentype $\chi$, with $K_{f, \varphi}$
as above. Then there exists a 2-dimensional Galois representation over $K_{f, \varphi}$,
\[ \rho_f : G_\mathbb{Q} \to \text{GL}_2(K_{f, \varphi}) \]
with the following properties:

(i) The representation $\rho_f$ is unramified at all primes $l \nmid Np$, and
\[ \text{Trace}(\rho_f(\text{Frob}_l)) = a_l, \quad \text{det}(\rho_f(\text{Frob}_l)) = \chi(l)\chi_p(\text{Frob}_l)^{k-1}. \]

By the Chebotarev Density Theorem it follows that $\text{det}(\rho_f) = \chi \chi_p^{k-1}$.

(ii) The restriction of $\rho_f$ to $G_p$ is upper-triangular. More precisely,
\[ \rho_f|_{G_p} \sim \left( \begin{array}{cc} \delta & u \\ 0 & \epsilon \end{array} \right) \]

where $\delta, \epsilon : G_p \to \mathcal{O}_{f, \varphi}^\times$ are characters with $\epsilon$ unramified, and $u$ is a continuous function. Moreover
$\epsilon(\text{Frob}_p)$ is equal to the unique $p$-adic unit root of the polynomial $X^2 - a_p X + \chi(p)p^{k-1}$ if $r = 0$; and is
equal to $a_p$ if $r \geq 1$.

For primes dividing $N$, we have the following results which are due, under varying degrees of
generality, to Eichler, Shimura, Langlands and Carayol [C1,L].

**Theorem 2.2.** For a modular form as in Theorem 2.1, let $C$ be the conductor of the nebentype $\chi$ and let $l$ be
any prime which divides $N$. Then:

(i) (Ramified Principal Series). If $\text{ord}_l(N) = \text{ord}_l(C) > 0$, then $|a_l|^2 = l^{k-1}$, and $\rho_f$ restricted to the decom-
position subgroup $G_l$ at $l$ is equivalent to a diagonal representation
\[ \rho_f|_{G_l} \sim \left( \begin{array}{cc} \chi & 0 \\ 0 & \delta_l \end{array} \right) \]

where $\chi$ is a Dirichlet character, which is regarded as a Galois character $\chi : G_\mathbb{Q} \to \mathcal{O}_{f, \varphi}^\times$ by Class Field
theory, and $\delta_l$ is an unramified character with $\delta_l(\text{Frob}_l) = \text{the eigenvalue of the Hecke operator } U_l$.

(ii) (Steinberg). If $\text{ord}_l(N) = 1$ and $\text{ord}_l(C) = 0$, then $a_l^2 = \chi(l)l^{k-2}$, and $\rho_f$ restricted to the decomposition
group $G_l$ is ramified with
\[ \rho_f|_{G_l} \sim \left( \begin{array}{cc} \eta \chi_p & * \\ 0 & \eta \end{array} \right) \]

where $\eta$ is an unramified character with $\eta(\text{Frob}_l) = a_l$. 
(iii) (Others) If $\text{ord}_l(N) \geq 2$ and $\text{ord}_l(N) > \text{ord}_l(C)$, then $a_l = 0$ and $\rho_f|_{G_l}$ is absolutely irreducible. Further, if $l \neq 2$, then $\rho_f|_{G_l}$ is induced from a character of a quadratic extension of $\mathbb{Q}_l$.

**Remark 2.3.** It is known that the image of the inertia group at $l$ is finite except in case (ii) (see [C1]).

Let $V_f$ denote the representation space for $\rho_f$. The compactness of the Galois group $G_{\mathbb{Q}}$ allows us to choose a lattice $T_f$, which is stable under the action of $G_{\mathbb{Q}}$. Put $A_f = V_f/T_f$. Then, by Theorem 2.1(ii) there exists a filtration of $A_f$, $0 \subset F^+A_f \subset A_f$ where $G_p$ acts on $F^+A_f$ via the character $\delta$ and on the quotient $\tilde{A}_f := A_f/F^+A_f$ by the unramified character $\epsilon$. We shall henceforth assume that the corresponding residual representation $\tilde{\rho}_f : G_{\mathbb{Q}} \rightarrow \text{GL}_2(k)$ is absolutely irreducible. It follows that, up to conjugation by $\text{GL}_2(O_{f,p})$, there is a unique integral model $\rho_f : G_{\mathbb{Q}} \rightarrow \text{GL}_2(O_{f,p})$ which we now fix (cf. [C2]).

Let $\mathcal{L}$ be any pro-$p$, $p$-adic Lie extension of $F$. Let $\Sigma = \text{any finite set of places of F containing the prime above } p$, all archimedean places, all primes ramified in $\mathcal{L}$ and all places dividing the tame level $N$ of $f$. We denote the maximal extension of $\mathcal{L}$ which is unramified outside the set $\Sigma$ by $\mathcal{L}^\Sigma$. For a prime $v$ of $\mathcal{L}$ we write $v \mid \Sigma$ if $v$ lies above a prime in $\Sigma$. For a prime $v$ of $\mathcal{L}$, we fix a compatible set of restrictions of $v$ at all finite layers of $\mathcal{L}$ over $F$. We denote by $\mathcal{L}_v$ the union of the corresponding completions at the finite layers.

We then define

$$H^1_s(\mathcal{L}_v, A_f) = \begin{cases} H^1(\mathcal{L}_v, A_f) & \text{for } v \nmid p, \\ \text{im}[H^1(\mathcal{L}_v, A_f) \rightarrow H^1(\mathcal{I}_v, A_f/F^+A_f)] & \text{for } v \mid p, \end{cases}$$

where $\mathcal{I}_v$ is the inertia subgroup of $G_{\mathcal{L}_v}$.

**Definition 2.4.** The Selmer group of $A_f$ over the $p$-adic Lie extension $\mathcal{L}/F$ is defined as

$$\text{Sel}(\mathcal{L}, A_f) = \ker[H^1(\mathcal{L}^\Sigma/\mathcal{L}, A_f) \rightarrow \prod_{v \mid \Sigma} H^1_s(\mathcal{L}_v, A_f)].$$

(4)

**Remark 2.5.** The Selmer group $\text{Sel}(\mathcal{L}, A_f)$ depends upon the lattice which we have chosen.

We regard $\text{Sel}(\mathcal{L}, A_f)$ as an $O_{f,p}[[G(\mathcal{L}/F)]]$-module via the natural action of $G(\mathcal{L}/F)$. These Selmer groups are discrete so that their Pontryagin duals, denoted by $X(\mathcal{L}, A_f)$, are compact groups. It is a deep theorem of Kato [K] that the Selmer groups $\text{Sel}(F^{\text{cyc}}, A_f)$ are cotorsion over the Iwasawa algebra $O_{f,p}[[\Gamma]]$, where $\Gamma := \text{G}(F^{\text{cyc}}/F)$. Classical structure theorem then allows us to define the following invariants.
Definition 2.6.

(i) The \( \mu \)-invariant of \( \text{Sel}(F^{\text{ov}}, A_f) \), denoted by \( \mu_f \), is defined to be the largest power of \( \pi \) dividing the characteristic power series of the \( \Lambda_{\mathcal{O}_L} \)-dual of \( \text{Sel}(F^{\text{ov}}, A_f) \), where \( \pi \) is a uniformizer of \( \mathcal{O}_{f, \wp} \).

(ii) The \( \lambda \)-invariant of \( \text{Sel}(F^{\text{ov}}, A_f) \), denoted by \( \lambda_f \), is defined to be the \( \mathcal{O}_{f, \wp} \)-rank of \( X(F^{\text{ov}}, A_f) \).

Remark 2.7. If \( \mu_f = 0 \), then the Selmer group is cofinitely generated as an \( \mathcal{O}_{f, \wp} \)-module and if it does not have any pseudonull submodules, then it is in fact \( \mathcal{O}_{f, \wp} \)-cofree.

3. Hida theory

In this section, we briefly recall some facts from Hida theory, the main references are [Hi1,Hi2, EPW]. Let \( L \) be a finite extension of \( \mathbb{Q}_p \), and \( \mathcal{O}_L \) be its ring of integers. Let \( L \) denote the quotient field of \( \Lambda_{\mathcal{O}_L} \). As in [Hi1, Section 3], let \( p(N, \mathcal{O}_L) \) (denoted \( T_{N}^{\text{lev}} \) in [EPW, Section 2.1]) be the universal Hecke algebra of newforms. It follows from [Hi1, Corollary 3.7] that the height one primes of \( p(N, \mathcal{O}_L) \) correspond to normalized eigenforms of tame conductor \( N \), which are \( p \)-ordinary. Further, Hida showed that there is a \( G_{\mathbb{Q}} \) representation that encodes each of the \( G_{\mathbb{Q}} \) representations associated to each of the modular forms that come from the ideal \( \mathcal{O} \).

Proposition 2.2.4), it follows that the injection, \( \mathcal{O} \rightarrow \mathcal{O} \), is an isomorphism. Let \( I \) denote the space of \( \mathcal{O} \)-coinvariants of \( \rho \) and \( \rho \) acts on this space via the operator \( U \). More precisely, we have (see [Hi2, Section 2] for details)

**Theorem 3.1.** (See [Hi2, Theorem 2.1].) There is a continuous Galois representation

\[
\rho : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathcal{O}(N, \mathcal{O}_L) \otimes_{\Lambda} L)
\]

such that

(i) \( \rho \) is unramified outside \( Np \);

(ii) if \( \mathfrak{l} \nmid Np \), then \( \rho(\text{Frob}_\mathfrak{l}) \) has characteristic polynomial equal to

\[
X^2 - T_\mathfrak{l}X + (\mathfrak{l}Np)^{-1} \in \mathcal{O}(N, \mathcal{O}_L)[X];
\]

(iii) the space of \( \mathfrak{p} \)-coinvariants of \( \rho \) is free of rank one over \( \mathcal{O}(N, \mathcal{O}_L) \otimes_{\Lambda} L \) and \( \text{Frob}_\mathfrak{p} \) acts on this space via the operator \( U_\mathfrak{p} \).

For a minimal prime ideal \( \mathfrak{a} \) of \( \mathcal{O}(N, \mathcal{O}_L) \), \( \rho \) induces the representation

\[
\rho_\mathfrak{a} : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathcal{O}(N, \mathcal{O}_L)_{\mathfrak{a}})
\]

where \( \mathcal{O}(N, \mathcal{O}_L)_{\mathfrak{a}} \) denotes the localization at the prime ideal \( \mathfrak{a} \). Let \( K(\mathfrak{a}) \) denote the quotient \( \mathcal{O}(N, \mathcal{O}_L)/\mathfrak{a} \), and \( I(\mathfrak{a}) \) denote the integral closure of \( K(\mathfrak{a}) \) in its quotient field (this is denoted by \( \mathfrak{T} \) in the proof of [EPW, Proposition 2.2.4]).

Let \( \mathcal{V} \) denote a two-dimensional vector space over the fraction field of \( K(\mathfrak{a}) \), denoted by \( Q(\mathcal{V}) \), on which \( \rho_\mathfrak{a} \) acts via its residual representation, and let \( M \) be a rank two \( I(\mathfrak{a}) \)-lattice in \( \mathcal{V} \) invariant under \( G_{\mathbb{Q}} \). If \( \mathcal{P} \) is a classical height one prime of \( \mathcal{O}(N, \mathcal{O}_L) \) containing \( \mathfrak{a} \), then (cf. [EPW, proof of Proposition 2.2.4]), it follows that the injection,

\[
I(\mathfrak{a})_{\mathcal{P}} \rightarrow (\mathfrak{a})_{\mathcal{P}}
\]

is an isomorphism. Let \( \mathcal{V}(\mathcal{P}) = (M/\mathcal{P}M)[1/p] \) denote the two-dimensional \( G_{\mathbb{Q}} \) representation space over the residue field of \( I(\mathfrak{a})_{\mathcal{P}} \). Recall [Hi1, Corollary 3.7] that corresponding to the classical height one prime \( \mathcal{P} \), there is a normalized eigenform, say \( f_{\mathcal{P}} \), of tame conductor \( N \). Then the representation \( \mathcal{V}(\mathcal{P}) \) is equivalent to the representation \( \hat{\rho}_{\mathcal{P}} \) attached to the newform \( f_{\mathcal{P}} \). In other words, the representation associated to \( f_{\mathcal{P}} \) comes from the residual representation of \( \rho_\mathfrak{a} \).
Definition 3.2. The set of newforms associated to the classical height one primes containing a minimal prime ideal $a$ of $p(N, O_L)$, as above, is called the branch associated to $a$.

Remark 3.3. Let $\hat{\rho}: G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathbb{F})$ be an absolutely irreducible residual representation into a finite field $\mathbb{F}$, which is associated to a $p$-ordinary, $p$-stabilized newform $f$ of weight $k$, level $Np^s$, and nebentype $\chi$. Then the primitive local ring (see [Hi2, Section 1, p. 554] for the definition) to which $f$ belongs may be viewed under suitable conditions (cf. [Hi3, Section 3.2.4]) as the universal deformation ring parameterizing lifts of $\hat{\rho}$ which are $p$-ordinary, $p$-stabilized and whose tame level is $N$. We refer to the set of these modular forms as the Hida family of $\hat{\rho}$, denoted by $\mathbf{H}(\hat{\rho})$ (see [EPW, Definition 2.4.3] also). In particular, all the residual representations of the modular forms in $\mathbf{H}(\hat{\rho})$ are isomorphic to $\hat{\rho}$.

As in [EPW], we want to compare the Iwasawa invariants of $\text{Sel}(F^{\text{cyc}}, A_f)$ and $\text{Sel}(F^{\text{cyc}}, A_g)$, where $f$ and $g$ are two modular forms in $\mathbf{H}(\hat{\rho})$. I would like to thank the referee for following approach leading to the proof of Proposition 3.7. In fact, the following lemmas required to prove this proposition are well known.

Lemma 3.4. Let $\Delta = G(F/\mathbb{Q})$. Then the restriction map

$$\text{Sel}(F^{\text{cyc}}, A_f) \rightarrow (\text{Sel}(F^{\text{cyc}}, A_f))^\Delta$$

has finite kernel and cokernel.

Proof. This follows directly from the fact that $A_f$ is cofinitely generated; cf. [HM, Lemma 3.3] for details. In fact, using the fact order of $\Delta$ is coprime to $p$ we even have an isomorphism, but we do not require this in what follows. \(\square\)

Now, let $\omega$ denote the mod $p$-cycloctonic character of $\Delta = G(\mathbb{Q}(\mu_p)/\mathbb{Q})$. Then by giving $A_f \otimes \omega^i$ the filtration induced from $A_f$ at the prime dividing $p$, we can define the Selmer groups $\text{Sel}(\mathbb{Q}^{\text{cyc}}, A_f \otimes \omega^i)$. This is a module over $\Lambda_{\mathbb{Q}^{\text{cyc}}}$, and by the theorem of Kato in [K] they are cotorsion over $\Lambda_{\mathbb{Q}^{\text{cyc}}}$. The Iwasawa invariants of $\text{Sel}(\mathbb{Q}^{\text{cyc}}, A_f \otimes \omega^i)$ are similarly defined as in Definition 2.6.

Lemma 3.5. Let $\mu_F(\text{Sel}(F^{\text{cyc}}, A_f)) = 0$. Then

$$\text{Sel}(F^{\text{cyc}}, A_f) \cong \bigoplus_{i=0}^{p-2} (\text{Sel}(F^{\text{cyc}}, A_f) \otimes \mathcal{O}_{f, p^i}(\omega^i))^\Delta. \quad (5)$$

Proof. As modules over $\mathcal{O}_{f, p^i}[G_{F^{\text{cyc}}}]$, we have

$$A_f \cong A_f \otimes \omega^i.$$ 

From this, it follows easily that

$$(\text{Sel}(F^{\text{cyc}}, A_f) \otimes \mathcal{O}_{f, p^i}(\omega^i))^\Delta = (\text{Sel}(F^{\text{cyc}}, A_f \otimes \omega^i))^\Delta. \quad (6)$$

Since $\mu_F(\text{Sel}(F^{\text{cyc}}, A_f)) = 0$, therefore $\text{Sel}(F^{\text{cyc}}, A_f)$ is cofinitely generated over $\mathcal{O}_{f, p^i}$. Now using the fact that the irreducible representations of $\Delta$ are given by $\omega^i$ for $i = 0, \ldots, p-2$, we get the following decomposition:

$$\text{Sel}(F^{\text{cyc}}, A_f) \cong \bigoplus_{i=0}^{p-2} (\text{Sel}(F^{\text{cyc}}, A_f) \otimes \mathcal{O}_{f, p^i}(\omega^i))^\Delta. \quad (7)$$
Corollary 3.6. Let $\mu_{f}(\text{Sel}(F^{\text{cyc}}, A_f)) = 0$. Then the map

$$\bigoplus_{i=0}^{p-2} \text{Sel}(\mathbb{Q}^{\text{cyc}}, A_f \otimes \omega^i) \rightarrow \text{Sel}(F^{\text{cyc}}, A_f)$$

(8)

obtained from the restriction map as in Lemma 3.4, (6) and (7) has finite kernel and cokernel.

Proof. Applying Lemma 3.4 to each of the twist $A_f \otimes \omega^i$ and using the isomorphisms in (6) and (7), the corollary follows. □

Proposition 3.7. If $f$ and $g$ are in the same Hida family $\mathbf{H}(\bar{\rho})$ and $\mu_f = 0$, then $\mu_g = 0$. Further, if they are in the same branch, then $\lambda_f = \lambda_g$.

Proof. Consider the Selmer groups $\text{Sel}(F^{\text{cyc}}, A_f[\pi])$ which are defined by taking $A_f[\pi]$ instead of $A_f$ and taking $F^+A_f[\pi]$ for the local condition at the place above $p$. Then it is easy to see that $\text{Sel}(F^{\text{cyc}}, A_f[\pi])$ and $\text{Sel}(F^{\text{cyc}}, A_f)[\pi]$ differ by a finite set. Now, note that $A_f[\pi]$ is nothing but the residual representation associated to $A_f$ and hence isomorphic to the residual representation of $A_g$. Since $\mu_f = 0$, therefore $\text{Sel}(F^{\text{cyc}}, A_f[\pi])$ is finite. Therefore, if $\pi'$ denotes a uniformizer of $O_{g, g}$, then it follows that $\text{Sel}(F^{\text{cyc}}, A_g[\pi'])$ is finite. Hence, $\text{Sel}(F^{\text{cyc}}, A_g)[\pi']$ is finite and $\mu_g = 0$.

For the second part of the proposition, by [EPW, Theorem 4.3.4], $\text{Sel}(\mathbb{Q}^{\text{cyc}}, A_f \otimes \omega^j)$ and $\text{Sel}(\mathbb{Q}^{\text{cyc}}, A_g \otimes \omega^j)$ have the same $\lambda$-invariants, therefore it follows from Corollary 3.6 that $\text{Sel}(F^{\text{cyc}}, A_f)$ and $\text{Sel}(F^{\text{cyc}}, A_g)$ have the same $\lambda$-invariants. □

4. Iwasawa invariants over the extension $F_\infty/F$

Recall that $F = \mathbb{Q}^{\mu_p}$, $F_{\infty} = \bigcup_n \mathbb{Q}(\mu_{p^n}, m^{1/p^n})$, where $m$ is $p$-power free. We have the following diagram of fields.

![Diagram](image-url)

Here $\mathcal{H} \cong \mathbb{Z}_p$ and $G(F_{\infty}/F) \cong \mathbb{Z}_p \rtimes \mathbb{Z}_p$, with $\Gamma$ acting on $\mathcal{H}$ by the cyclotomic character. Let $S_N, S_m, S_{\infty}$ denote set of primes in $F$ lying over $N, m$ and the infinite prime respectively. As in the previous section, we let $v_p$ denote the unique prime above $p$ in the totally ramified extensions $F$ and $F^{\text{cyc}}$. For a prime $v$ of $F_\infty$, let $G(F_{\infty}/F)_v$ denote the decomposition group of $G(F_{\infty}/F)$, and $\mathcal{H}_v$ denote the decomposition subgroup of $\mathcal{H}$. We state a lemma of Hachimori–Venjakob which determines the dimension of the decomposition subgroups of $\mathcal{H}$ as a $p$-adic Lie group [HV, Lemma 3.9].
Lemma 4.1.

(i) If $S = S_m \cup \{ v_p \} \cup S_{\infty}$, then $F_{\infty}$ is unramified outside $S$, and if $v$ is a prime of $F_{\infty}$ not lying above any prime in $S$, then $G(F_{\infty}/F)$ is a pro-$p$ group topologically generated by the Frobenius and hence $\mathcal{H}_v = 0$.

(ii) If $v = v_p$, then $\dim \mathcal{H}_v = 1$ with $\mathcal{H}_v \cong \mathbb{Z}_p$. Moreover, $F_{\infty}/\mathbb{Q}$ is totally ramified at $p$.

(iii) If $v \in S_m$, $v \neq v_p$, then for all places $w_{\infty}$ of $F_{\infty}$ lying over $v$, the local extension $F_{\infty,w_{\infty}}/F_{w_{\infty}}$, where $w$ denotes the place of $F_{\text{cycl}}$ induced by $w_{\infty}$, is a totally ramified $\mathbb{Z}_p$-extension. As a $p$-adic Lie group, $\dim G_v = \dim G = 2$, so that $\dim \mathcal{H}_v = 1$, and the places of $S_m \setminus \{ v_p \}$ decompose only into finitely many places in $F_{\infty}$.

For a modular form $f$ we retain the notations $K_{f,v}, \mathcal{O}_{f,v}, V_f, A_f$, etc., of the previous sections, with $k$ the residue field of $\mathcal{O}_{f,v}$. We consider the Selmer groups $\text{Sel}(F_{\infty}, A_f)$ as defined in Section 2 by taking $F_{\infty}$ to be the $p$-adic Lie extension and $\Sigma := S_N \cup S_m \cup \{ v_p \} \cup S_{\infty}$, and refer to them as the False–Tate Selmer groups. We denote the Pontryagin dual of $\text{Sel}(F_{\infty}, A_f)$ by $X(F_{\infty}, A_f)$. These are compact $\Lambda_{\mathcal{O}_{f,v}}[[G(F_{\infty}/F)]]$-modules.

Henceforth, we assume the following hypothesis.

Hypothesis 1. There exists a modular form $g \in H(\bar{\rho})$, such that the Selmer group $\text{Sel}(F_{\infty}/F, A_g)$ is cofinitely generated as a $\Lambda_{\mathcal{O}_{f,v}}(\mathcal{H})$-module.

Theorem 4.2. Assume Hypothesis 1 holds, i.e., for one member $g \in H(\bar{\rho})$, $\text{Sel}(F_{\infty}, A_g)$ is $\Lambda_{\mathcal{O}_{f,v}}(\mathcal{H})$-cofinitely generated. Then, for any other member $h$ of $H(\bar{\rho})$, the Selmer group $\text{Sel}(F_{\infty}, A_h)$ is cofinitely generated as $\Lambda_{\mathcal{O}_{f,v}}(\mathcal{H})$-module.

Proof. From [HV], we know that $\text{Sel}(F_{\infty}, A_g)$ is $\Lambda(\mathcal{H})$-cofinitely generated if and only if $\text{Sel}(F_{\text{cycl}}, A_g)$ is cofinitely generated over $\mathcal{O}_{g,0}$, but this is equivalent to saying that $\text{Sel}(F_{\text{cycl}}, A_g)$ is $\Lambda_{\mathcal{O}_{f,v}}(\Gamma)$-cotorsion with $\mu_g = 0$. Hence, as a consequence of Kato’s theorem and Proposition 3.7 the theorem follows. □

Under Hypothesis 1, the analogue of the $\lambda$-invariant for these Selmer groups, as proposed by Howson [H], is the $\Lambda(\mathcal{H})$-rank of the dual of the Selmer group $\text{Sel}(F_{\infty}, A_f)$, and it is given by the following “Euler characteristic” formula:

$$\text{corank}_{\Lambda(\mathcal{H})}(\text{Sel}(F_{\infty}, A_f)) = \sum_{i \geq 0} (-1)^i \text{corank}_{\mathcal{O}_{f,v}} H^i(\mathcal{H}, \text{Sel}(F_{\infty}, A_f)).$$  \hspace{1cm} (9)

Our aim in this section is to study the behavior of these non-commutative Iwasawa theoretic invariants in the Hida family $H(\bar{\rho})$. We will use analogues of the results in [CSS] to reduce the study of the $\Lambda(\mathcal{H})$ ranks to the study of the cyclotomic-$\lambda$ invariants.

Proposition 4.3. Assume Hypothesis 1 holds. Then in the sequence defining the Selmer group, $\text{Sel}(F_{\infty}, A_f)$ (see Definition 2.4), the last map is surjective, and we have $H^1(\mathcal{H}, \text{Sel}(F_{\infty}, A_f)) = 0$.

Proof. The surjectivity is shown in [HV, Section 7.1], and $H^1(\mathcal{H}, \text{Sel}(F_{\infty}, A_f)) = 0$ follows as in [CSS, Lemma 2.5] under the assumption that $X(F_{\text{cycl}}, A_f)$ is $\Lambda_{\mathcal{O}_{f,v}}(\Gamma)$-torsion where $\Gamma = G(F_{\text{cycl}}/F)$. □

Recall the notation $\tilde{A}_f = A_f/F^+ A_f$ from Section 2. In view of Proposition 4.3, we then have the following fundamental diagram.
Sel \(F_{\infty}, A_f) \to H^1(F^\Sigma/F_{\infty}, A_f) \to (\prod_{v | \Sigma} H^1((F_{v}, A_f)) \to 0 \]

\[\begin{array}{ccc}
0 & \rightarrow & \text{Sel}(F^\Sigma/F_{\infty}, A_f) \\
\alpha & \rightarrow & \beta \\
0 & \rightarrow & \bigoplus_{v | \Sigma} H^1(F_{v}, A_f) \to 0
\end{array}\]

Lemma 4.4. In the above fundamental diagram \(\ker \gamma_p\) is finite.

Proof. Let \(I_{\infty,p}\) and \(I_{\infty,p}^{\text{cy}}\) denote the inertia subgroups of \(G_{F_{\infty}, v}\) and \(G_{F_{\infty}, v}^{\text{cy}}\) respectively. Let \(G' = G_{F_{\infty}, v}/I_{\infty,v} \cong \mathbb{Z}.\) Similarly, let \(G = G_{F_{v}^{\text{cy}}, v}/I_{\infty,v} \cong \mathbb{Z}.\) Recall that the inertia subgroup at \(p\) acts trivially on \(\tilde{A}_f\), so that both \(I_{\infty,v}\) and \(I_{\infty,v}^{\text{cy}}\) act trivially on \(\tilde{A}_f\). In particular, this implies that

\[H^0(G, \tilde{A}_f) = \tilde{A}_f(F_{v}^{\text{cy}}) = H^0(\mathcal{H}_{v}^{p}, \tilde{A}_f(F_{\infty,v}^{p})).\] (10)

By Definition 3, \(H^1(I_{\infty,v}^{\text{cy}}, A_f) = \text{im}[H^1(I_{\infty,v}^{\text{cy}}, A_f) \to H^1(I_{\infty,v}^{\text{cy}}, \tilde{A}_f)].\) This map is also the composite of the following maps

\[H^1(I_{\infty,v}^{\text{cy}}, A_f) \to H^1(I_{\infty,v}^{\text{cy}}, \tilde{A}_f) \to H^1(I_{\infty,v}^{\text{cy}}, \tilde{A}_f),\]

of which the first map \(\eta\) is surjective as \(H^2(I_{\infty,v}^{\text{cy}}, F^+ A_f) = 0\) (cf. proof of [G4, Theorem 2.9]). Therefore the image of the restriction map \(\text{res}\) is nothing but \(H^1(I_{\infty,v}^{\text{cy}}, \tilde{A}_f)^{G'}\) as \(H^2(G', \tilde{A}_f^{I_{\infty,v}}) = 0.\) Hence,

\[\text{im}[H^1(I_{\infty,v}^{\text{cy}}, A_f) \to H^1(I_{\infty,v}^{\text{cy}}, \tilde{A}_f)] = H^1(I_{\infty,v}^{\text{cy}}, \tilde{A}_f)^{G'} .\]

Similarly,

\[\text{im}[H^1(I_{\infty,v}^{\text{cy}}, A_f) \to H^1(I_{\infty,v}^{\text{cy}}, \tilde{A}_f)] = H^1(I_{\infty,v}^{\text{cy}}, \tilde{A}_f).\]

Now, the above two equalities along with the inflation–restriction sequence and the fact that \(H^2(G, \tilde{A}) = 0\), gives the following commutative diagram.

\[\begin{array}{ccc}
0 & \rightarrow & H^1(G^\Sigma, \tilde{A}_f)^{\mathcal{H}_{v}^{p}} \\
\gamma_p & \rightarrow & \bigoplus_{v | \Sigma} H^1(F_{v}^{\text{cy}}, \tilde{A}_f) \\
0 & \rightarrow & H^1(G, \tilde{A}_f) \to H^1(F_{v}^{\text{cy}}, \tilde{A}_f) \to H^1(F_{v}^{\text{cy}}, A_f) \to 0
\end{array}\]

Now, noting that \(I_{\infty,v} \subset G_{F_{\infty}, v} \subset G_{F_{v}^{\text{cy}}, v}\) let \(E := G_{F_{v}^{\text{cy}}, v}/I_{\infty,v}.\) We claim that \(H^2(E, \tilde{A}_f) = 0.\) By local Tate duality (cf. [NSW, Theorem 7.2.6]), we have

\[H^2(E, \tilde{A}_f[\pi^n]) \cong \tilde{A}_f[\pi^n](F_{v}^{\text{cy}})^{\mathcal{H}_{v}^{p}} \cong H^2(F_{v}^{\text{cy}}, \tilde{A}_f[\pi^n]), \quad \forall n,\]

where \(\tilde{A}_f[\pi^n] = \text{Hom}(A_f[\pi^n], \mathbb{Q}/\mathbb{Z}_p(1)).\) As \(F_{v}^{\text{cy}}\) is a ramified \(\mathbb{Z}_p\)-extension of \(F_{v}^{p}\), it follows that \(H^2(F_{v}^{\text{cy}}, \tilde{A}_f[\pi^n]) = 0\) (cf. proof of [G4, Theorem 2.9]). Therefore, \(H^2(E, \tilde{A}_f[\pi^n]) = 0\) for all \(n.\)
Then by taking direct limit we get $H^2(\mathcal{E}, \tilde{A}_f) = 0$. Hochschild–Serre spectral sequence now gives $H^1(\mathcal{H}_{v_p}, H^1(G', \tilde{A}_f)) = 0$. By Lemma 4.1(ii), $\mathcal{H}_{v_p} \cong \mathbb{Z}_p$, from which we conclude that

$$\text{corank}_{O_{f,p}} H^0(\mathcal{H}_{v_p}, H^1(G', \tilde{A}_f)) = 0. \quad (11)$$

Therefore, $\text{corank}_{O_{f,p}}(\text{coker } r) = 0$ and $\text{corank}_{O_{f,p}} \ker r = \text{corank}_{O_{f,p}} H^1(G, \tilde{A}_f)$. We also have

$$\ker s = H^1(\mathcal{H}_{v_p}, \tilde{A}_f(F_{v_p})) \quad (12)$$

Applying snake lemma to the previous commutative diagram gives us

$$\text{corank}_{O_{f,p}} \ker \gamma_{v_p} = \text{corank}_{O_{f,p}} \ker s - \text{corank}_{O_{f,p}} \ker r \quad \text{(since } \text{coker } r \text{ is finite)}$$

$$= \text{corank}_{O_{f,p}} H^1(\mathcal{H}_{v_p}, \tilde{A}_f(F_{v_p})) - \text{corank}_{O_{f,p}} H^1(G, \tilde{A}_f) \quad \text{(by } (11), \text{ and } (12)).$$

Since $\mathcal{H}_{v_p}$ and $G$ are procyclic, this gives

$$\text{corank}_{O_{f,p}} \ker \gamma_{v_p} = \text{corank}_{O_{f,p}} H^0(\mathcal{H}_{v_p}, \tilde{A}_f(F_{v_p})) - \text{corank}_{O_{f,p}} H^0(G, \tilde{A}_f)$$

$$- \text{corank}_{O_{f,p}} H^0(G, \tilde{A}_f) - \text{corank}_{O_{f,p}} H^0(G, \tilde{A}_f) \quad \text{(by } (10)) = 0.$$

This completes the proof of the lemma. □

**Proposition 4.5.** Let $X(F_{\infty}, A_f) = \text{Sel}(F_{\infty}, A_f)^{\vee}$, the Pontryagin dual. Then

$$\text{rank}_{A_{O_{f,p}}(\mathcal{H})} X(F_{\infty}, A_f) = \lambda_f + \text{corank}_{O_{f,p}} \left[ \bigoplus_{v \mid \Sigma, v \neq v_p} H^1(\mathcal{H}_{v_p}, A_f(F_{v_p})) \oplus \ker \gamma_{v_p} \right]. \quad (13)$$

**Proof.** The proof follows by an easy diagram chase in the fundamental diagram on observing that $\ker \beta$ and $\text{coker } \beta$ are finite. □

We next study the $O_{f,p}$-coranks of the local cohomology groups and show that they are the same for all the members of the Hida family. This is done by determining the $O_{f,p}$-coranks of each summand in the direct sum.

**Lemma 4.6.** Let $v \mid p$, $\pi$ be a uniformizer of $O_{f,p}$, and $\epsilon : G_{F_v} \to O_{f,p}^{\times}$ be an unramified character. Then $\epsilon|_{G_{F_v}^{\text{cyc}}}$ is trivial if and only if $\epsilon \equiv 1 \pmod{\pi}$.

**Proof.** Let $l$ denote the prime in $\mathbb{Q}$ lying below $v$ in $F$, and $w$ be the prime above $v$ in $F^{\text{cyc}}$. Let $G_v = G(F_v/F_w)$, $H' = G(F_{v}^{ur}/F_{w}^{cy})$, and $\Gamma = G(F_{w}^{cy}/F_{v})$. Then the lemma follows from the fact that $G_v \cong \prod_{l \neq p} \mathbb{Z}_l$, $H' \cong \prod_{l \neq p} \mathbb{Z}_l$ and $\Gamma \cong \mathbb{Z}_p$. □

Using this lemma, we compute the $O_{f,p}$-coranks of the local cohomology groups in Proposition 4.5. Below we use Theorems 2.1, 2.2 to compute the $O_{f,p}$-coranks. We also use the explicit matrix representation of $\rho_f$. For a prime $v$ of $F_{\infty}$, we use the same notation $v$ for the prime lying below $v$ in $F^{\text{cyc}}$ and $F$. Given a character $\phi : G_{\mathbb{Q}} \to O_{f,p}^{\times}$, we put $\phi|_{v} := \phi|_{G_{F_v}}$ and $\phi|_{\text{cyc}} := \phi|_{G_{F_v}^{\text{cyc}}}$ for a prime $v$ of $F = \mathbb{Q}(\mu_p)$ lying above $q$, and put $r_q := [F_v : \mathbb{Q}_q]$. 

---

*Note: The document is a mathematical text discussing cohomology groups and their properties, including diagrams and explicit matrix representations.*
Lemma 4.7 (Good). Let \( q \nmid Np \) and \( \nu \) be a prime of \( F_\infty \) above \( q \) such that \( \nu \mid S_m, \nu \neq \nu_p \). Then for any member \( f \) of \( \mathcal{H}(\hat{\rho}) \) we have

\[
\text{corank}_{O_{f,\nu}} H^1(\mathcal{H}_\nu, A_f(F_\infty, \nu)) = \begin{cases} 
2 & \text{if } \hat{\rho}|_{G_{\nu^{\infty}}} = 1, \\
1 & \text{if } \hat{\rho}|_{G_{\nu^{\infty}}} \neq 1, \ H^0(F_{\nu^{\infty}}, A_f[\pi]) \neq 0, \\
0 & \text{if } \hat{\rho}|_{G_{\nu^{\infty}}} \neq 1, \ H^0(F_{\nu^{\infty}}, A_f[\pi]) = 0.
\end{cases}
\]  
\[\text{(14)}\]

Proof. If \( \nu \nmid S_m \) with \( \nu \neq \nu_p \), then by Lemma 4.1(i), \( \mathcal{H}_\nu \) is finite, hence

\[
\text{corank}_{O_{f,\nu}} H^1(\mathcal{H}_\nu, A_f(F_\infty, \nu)) = 0.
\]

On the other hand, if \( \nu \mid S_m \) and \( \nu \neq \nu_p \), then by Lemma 4.1(iii), \( \mathcal{H}_\nu \cong \mathbb{Z}_p \), therefore

\[
\text{corank}_{O_{f,\nu}} H^1(\mathcal{H}_\nu, A_f(F_\infty, \nu)) = \text{corank}_{O_{f,\nu}} H^0(\mathcal{H}_\nu, A_f(F_\infty, \nu)) = \text{corank}_{O_{f,\nu}} A_f(F_{\nu^{\infty}}).
\]

We now determine \( \text{corank}_{O_{f,\nu}} A_f(F_{\nu^{\infty}}) \). As \( A_f \) is unramified at \( q \), so \( A_f^0 = A_f \), which is \( \pi \)-divisible. Further, as \( \nu \neq \nu_p \) and \( I_q \subset G_{\nu^{\infty}} \) with the profinite order of \( G_{\nu^{\infty}}/I_q \) prime to \( p \), it follows that \( H^0(G_{\nu^{\infty}}/I_q, A_f) \) is also \( \pi \)-divisible. Therefore the exact sequence

\[
0 \to A_f[\pi] \to A_f \xrightarrow{\pi} A_f \to 0
\]

induces the following exact sequence

\[
0 \to H^0(G_{\nu^{\infty}}/I_q, A_f[\pi]) \to H^0(G_{\nu^{\infty}}/I_q, A_f) \xrightarrow{\pi} H^0(G_{\nu^{\infty}}/I_q, A_f) \to 0.
\]  
\[\text{(15)}\]

If \( \hat{\rho}|_{G_{\nu^{\infty}}} = \text{the identity matrix, then we have } H^0(G_{\nu^{\infty}}/I_q, A_f[\pi]) \cong A_f(F_{\nu^{\infty}}) \cong (O_{f,\nu}/\pi)^2. \) From this, by noting that \( A_f(F_{\nu^{\infty}}) \) is \( \pi \)-divisible, it follows that \( A_f(F_{\nu^{\infty}}) \cong (K_{f,\nu}/O_{f,\nu})^2. \) Hence \( \text{corank}_{O_{f,\nu}} A_f(F_{\nu^{\infty}}) = 2. \)

Now, suppose that \( \hat{\rho}|_{G_{\nu^{\infty}}} \neq 1 \), then \( \dim_k H^0(F_{\nu^{\infty}}, A_f[\pi]) = 1 \) or \( 0 \). If \( \hat{\rho}|_{G_{\nu^{\infty}}} \neq 1 \) with \( \dim_k H^0(F_{\nu^{\infty}}, A_f[\pi]) = 1 \). Then from the exact sequence (15), it follows that \( \text{corank}_{O_{f,\nu}} A_f(F_{\nu^{\infty}}) = 1. \)

On the other hand, if \( \hat{\rho}|_{G_{\nu^{\infty}}} \neq 1 \) with \( \dim_k H^0(F_{\nu^{\infty}}, A_f[\pi]) = 0 \), then again from the exact sequence (15), it is clear that \( \text{corank}_{O_{f,\nu}} A_f(F_{\nu^{\infty}}) = 0. \) Summing up we have shown the following

\[
\text{corank}_{O_{f,\nu}} A_f(F_{\nu^{\infty}}) = \begin{cases} 
2 & \text{if } \hat{\rho}|_{G_{\nu^{\infty}}} = 1, \\
1 & \text{if } \hat{\rho}|_{G_{\nu^{\infty}}} \neq 1, \ H^0(F_{\nu^{\infty}}, A_f[\pi]) \neq 0, \\
0 & \text{if } \hat{\rho}|_{G_{\nu^{\infty}}} \neq 1, \ H^0(F_{\nu^{\infty}}, A_f[\pi]) = 0.
\end{cases}
\]  
\[\text{(16)}\]

This completes the proof of the lemma. \( \square \)

Lemma 4.8 (Ramified Principal Series). Let \( q \neq p \) and \( \text{ord}_q(N) = \text{ord}_q(C) > 0 \), where \( C \) is the conductor of the nebentype, and let \( \delta_{q,f} \) be the unramified character which occurs in the representation \( \rho_f|_{G_{q}} \) (see Theorem 2.2). Let \( \nu \) be a prime of \( F_\infty \) above \( q \). Then

\[
\text{corank}_{O_{f,\nu}} H^1(\mathcal{H}_\nu, A_f(F_\infty, \nu)) \cong \begin{cases} 
1 & \text{if } \delta_{q,f}|\nu \equiv 1 \text{ mod } \pi, \chi|_{G_{\nu^{\infty}}} \neq 1 \text{ and } \nu \mid S_m, \\
0 & \text{otherwise.}
\end{cases}
\]
In particular, if \( g \) is another member of \( \mathbf{H}(\bar{\rho}) \), then

\[
\text{corank}_{\mathcal{O}_{f',\phi}'} H^1(\mathcal{H}_v, A_f(F_{\infty,v})) = \text{corank}_{\mathcal{O}_{g',\phi}'} H^1(\mathcal{H}_v, A_g(F_{\infty,v})).
\] (17)

**Proof.** By Theorem 2.2 on restricting to the decomposition subgroup \( G_q \), we have

\[
\rho_f|_{G_q} \sim \begin{pmatrix} \psi_f & 0 \\ 0 & \delta_{q,f} \end{pmatrix}
\]

with \( \delta_{q,f} \) the unramified character and \( \delta_{q,f}(\text{Frob}_q) = a_q(f) \). Therefore

\[
\rho_f|_{G_{q,G}} \sim \begin{pmatrix} (\chi \delta_{q,f})|_{\text{cyc}} & 0 \\ 0 & \delta_{q,f}|_{\text{cyc}} \end{pmatrix}
\]

as \( (\delta_{q,f} \psi_f)|_{\text{cyc}} = \chi|_{\text{cyc}} \). Since \( G(F_{q,G}^v/F_v) \) is a \( \mathbb{Z}_p \) extension, by Lemma 4.6 one gets \( \delta_{q,f}|_{\text{cyc}} = 1 \) if and only if \( \delta_{q,f}|_v \equiv 1 \pmod{\pi} \), and in this case \( \chi \delta_{q,f}^{-1}|_{\text{cyc}} = \chi|_{\text{cyc}} \) with

\[
\rho_f|_{G_{q,G}^v} \sim \begin{pmatrix} \chi|_{G_{q,G}^v} & 0 \\ 0 & 1 \end{pmatrix}.
\]

Now we claim that \( \chi|_{G_{q,G}^v} \neq 1 \). For this, recall that the nebentype \( \chi \) can be viewed, by Class Field theory, as a character of \( G(\mathbb{Q}(\mu_{N_p^r})/\mathbb{Q}) \cong (\mathbb{Z}/Np^r)^\times \). As the conductor of \( \chi \) is \( C \), it must factor through \( (\mathbb{Z}/C)^\times \cong G(\mathbb{Q}(\mu_{C})/\mathbb{Q}) \). Thus if \( \chi|_{G_{q,G}^v} = 1 \), then \( G_{f,v}^{G} \) is contained in \( G(\mathbb{Q}(\mu_{C})/\mathbb{Q}) \), so that \( F_{v}^{G} \) contains \( \mathbb{Q}(\mu_{C}) \). In fact, \( \mathbb{Q}(\mu_{C}) \) is contained in \( \mathbb{Q}(\mu_{p^r}) \), for some \( n \). But then the only roots of unity in \( \mathbb{Q}(\mu_{p^r}) \) are \( \mu_{(q-1)} \) for some \( s \) and \( \mu_{p^r} \), so that \( \mu_{C} \subset \mu_{(q^r-1)} \cup \mu_{p^r} \). As \( q \nmid C \), so \( \mu_{q} \subset \mu_{C} \), and hence \( \mu_{q} \subset \mu_{p^n} \). But this is not possible as \( q \neq p \). Therefore the claim that \( \chi|_{G_{q,G}^v} \neq 1 \). Hence,

\[
A_f(F_{v}^{G}) \cong (K_{f,\phi}/\mathcal{O}_{f,\phi}) \oplus (\text{finite set}).
\]

In case, if \( v \mid S_m \), then as \( v \neq v_p \), by Lemma 4.1(i) we see that \( H^1(\mathcal{H}_v, A_f(F_{\infty,v})) \) has \( \mathcal{O}_{f,\phi} \)-corank equal to zero.

Let \( v \mid S_m \), then as \( v \neq v_p \), by Lemma 4.1(iii), we have \( H_v \cong \mathbb{Z}_p \). Therefore,

\[
\text{corank}_{\mathcal{O}_{f,\phi}} H^1(\mathcal{H}_v, A_f(F_{\infty,v})) = \text{corank}_{\mathcal{O}_{f,\phi}} H^0(\mathcal{H}_v, A_f(F_{\infty,v})) = \text{corank}_{\mathcal{O}_{f,\phi}} A_f(F_{v}^{G}).
\]

Hence, we have the following

\[
\text{corank}_{\mathcal{O}_{f,\phi}} H^1(\mathcal{H}_v, A_f(F_{\infty,v})) = \begin{cases} 1 & \text{if } \delta_{q,f}|_v \equiv 1 \pmod{\pi}, \chi|_{G_{f,v}^{G}} \neq 1 \text{ and } v \mid S_m, \\ 0 & \text{otherwise}. \end{cases}
\]

Next, for any other form \( g \) in the Hida family \( \mathbf{H}(\bar{\rho}) \), note that \( g \) has level \( Np^r \), for some \( r \geq 1 \). It follows from [Hi2, Corollary 1.6], that the nebentypus of \( g \) is \( \chi \tilde{f}_p^t \), for some \( t \), where \( \tilde{f}_p \) denotes the Teichmüller character which has conductor \( p \). Let its conductor be \( C_g \). Then it is easy to see that \( \text{ord}_q(Np^r) = \text{ord}_q(C_g), \) as \( (N, p) = 1 \), and hence \( q \) is a prime of Ramified Principal Series type for \( \rho_{g} \) as well. Let the characters of \( \rho_{g}|_q \) be \( \psi_g \) and \( \delta_{g,g} \), with \( \delta_{g,g} \) unramified and \( \delta_{g,g}(\text{Frob}_q) = a_q(g) \). Let \( \pi' \) denote a uniformizer of \( \mathbb{O}_{g,\phi}/\mathbb{K} \equiv \mathcal{O}_{f,\phi}/\mathbb{P} \). Further, as the mod \( p \) residual Galois representations of \( f \) and \( g \) are isomorphic the images of the Fourier coefficients, \( a_q(f) \) and \( a_q(g) \),
in $k$ are the same; and so we have $\delta, f|_{G_{F_{\rho^c}}} = 1 \Leftrightarrow \delta, f|_v \equiv 1 \pmod{\pi} \Leftrightarrow a_q(f)|_v \equiv 1 \pmod{\pi} \Leftrightarrow a_q(g)|_v \equiv 1 \pmod{\pi} \Leftrightarrow \delta, q, g|_v \equiv 1 \pmod{\pi} \Leftrightarrow \delta, q, g|_{G_{F_{\rho^c}}} = 1$. From this, as done above, it follows that

$$\text{corank}_{G_q} H^1(\mathcal{H}_v, A_g(F_{\infty, v})) \cong \begin{cases} 1 & \text{if } \delta, g|_v \equiv 1 \pmod{\pi}, \chi|_{G_{F_{\rho^c}}} \neq 1 \text{ and } v \mid S_m. \\ 0 & \text{otherwise}. \end{cases} \quad \Box$$

**Lemma 4.9** (Steinberg). Let $q \neq p$, $\text{ord}_q(N) = 1$, $\text{ord}_q(C) = 0$, and $\eta_f$ be the unramified character occurring in the representation $\rho_f | G_q$ (see Theorem 2.2(ii)). Suppose $v$ be a prime of $F_{\infty}$ above $q$, then

$$\text{corank}_{G_q} H^1(\mathcal{H}_v, A(F_{\infty, v})) = \begin{cases} 1 & \text{if } \eta_f|_v \equiv 1 \pmod{\pi} \text{ and } v \mid S_m. \\ 0 & \text{otherwise}. \end{cases} \quad (18)$$

Moreover, if $g$ is another member of $\mathcal{H}(\tilde{\rho})$, then

$$\text{corank}_{G_q} H^1(\mathcal{H}_v, A_f(F_{\infty, v})) = \text{corank}_{G_q} H^1(\mathcal{H}_v, A_{g}(F_{\infty, v})). \quad (19)$$

**Proof.** By Theorem 2.2, $\rho|_{G_q}$ is ramified and is equivalent to an upper triangular matrix

$$\rho_f | G_q \sim \begin{pmatrix} \eta_f X_p & * \\ 0 & \eta_f \end{pmatrix}$$

such that $\eta_f(\text{Frob}_q) = a_q(f)$ and $(\eta_f^2 X_p)|_{\text{cyc}} = (\chi X_p^{-1})|_{\text{cyc}}$. The image of the inertia subgroup is known to be infinite in this case, so that the entry $*$ is not zero on restricting to $G_{F_{\rho^c}}$. As $\eta_f$ is an unramified character $\rho|_{G_{F_{\rho^c}}}$ is determined by its congruence mod $\pi$. Therefore, if $\eta_f|_v \equiv 1 \pmod{\pi}$, then

$$\rho_f | G_{F_{\rho^c}} \sim \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}. $$

Hence, $A_f(F_{\rho^c}) \cong K_{f, p}/O_{f, p}$. On the other hand, if $\eta_f|_v \not\equiv 1 \pmod{\pi}$, then $A_f(F_{\rho^c})$ is finite. Therefore, as in Lemma 4.8

$$\text{corank}_{G_q} H^1(\mathcal{H}_v, A_f(F_{\infty, v})) = \begin{cases} 1 & \text{if } \eta_f|_v \equiv 1 \pmod{\pi} \text{ and } v \mid S_m, \\ 0 & \text{otherwise}. \end{cases}$$

Next, let $g$ be any other form in the same Hida family of $\tilde{\rho}$. Then, as shown in the proof of Lemma 4.8, it follows that $q$ is a prime where the local Galois representation of $g$ is also Steinberg at the prime $q$. If

$$\rho_g | G_q \sim \begin{pmatrix} \eta_g X_p & * \\ 0 & \eta_g \end{pmatrix},$$

then the fact that $a_q(f) \equiv a_q(g)$ in $k$ implies that $\eta_f|_v \equiv \eta_g|_v$ in $k$. Therefore,

$$\text{corank}_{G_q} H^1(\mathcal{H}_v, A_g(F_{\infty, v})) = \begin{cases} 1 & \text{if } \eta_g|_v \equiv 1 \pmod{\pi} \text{ and } v \mid S_m, \\ 0 & \text{otherwise}, \end{cases}$$

and the lemma is proved. \Box
Lemma 4.10 (Others). Let \( q \neq p, \, \text{ord}_q(N) \geq 2, \, \text{ord}_q(N) > \text{ord}_q(C) \). Let \( v \) be a prime of \( F_\infty \) lying above \( q \). Then

\[
\text{corank}_{\mathcal{O}_{F,v}}^{} H^1(\mathcal{H}_v, A_f(F_{\infty,v})) = 0.
\] (20)

Proof. In this case it is known that the inertia subgroup \( I_q \) acts irreducibly on \( V_f \). Therefore \( V^0_f = 0 \). Hence \( A_f(F^{\text{cyc}}_v) = A^{\text{cyc}}_f \) is finite. If \( v \nmid S_m \), then by Lemma 4.1, \( \mathcal{H}_v \) is finite and hence \( \text{corank}_{\mathcal{O}_{F,v}}^{} H^1(\mathcal{H}_v, A_f(F_{\infty,v})) = 0 \). On the other hand, if \( v \mid S_m \), then by Lemma 4.1, \( \mathcal{H}_v \) has \( p \)-cohomological dimension one, so that \( \text{corank}_{\mathcal{O}_{F,v}}^{} H^1(\mathcal{H}_v, A_f(F_{\infty,v})) = \text{corank}_{\mathcal{O}_{F,v}}^{} H^0(\mathcal{H}_v, A_f(F_{\infty,v})) = 0 \). \( \square \)

This finally completes the comparison of the \( \mathcal{O}_{F,v} \)-coranks of the local terms which we sought to compare when \( g \) varies over all the members of \( \mathcal{H}(\bar{\rho}) \), and we have the following theorem.

Theorem 4.11. Let \( f \) be a modular form in the Hida family \( \mathcal{H}(\bar{\rho}) \) which satisfies the standing Hypothesis 1 and \( \mathcal{P}_{pr}, \mathcal{P}_{st} \) denote the primes at which the local Galois representations are Ramified Principal Series and Steinberg respectively. Let \( w_q \) denote a prime of \( F \) lying above \( q \) with \( q \neq p \). Write \( r_q := [F_{w_q} : \mathbb{Q}_q] \) as above. Further, let

- \( m_{gd} = \#\{v_q \in F_\infty \mid q \nmid N, \, v_q \mid S_m, \, \bar{\rho}|_{\mathcal{G}_{F^{\text{cyc}}}} = 1 \text{ in } k\} \),
- \( n_{gd} = \#\{v_q \in F_\infty \mid q \nmid N, \, v_q \mid S_m, \, \bar{\rho}|_{\mathcal{G}_{F^{\text{cyc}}}} \neq 1 \text{ in } k, \, \text{and } A_f[\pi](F^{\text{cyc}}_v) \neq 0\} \),
- \( m_{pr} = \#\{v_q \in F_\infty \mid q \in \mathcal{P}_{pr}, \, v_q \mid S_m, \, a_q(f)^q \equiv 1 \text{ in } k, \, \chi|_{\mathcal{P}_{pr}} \neq 1\} \),
- \( m_{st} = \#\{v_q \in F_\infty \mid q \in \mathcal{P}_{st}, \, v_q \mid S_m, \, a_q(f)^q \equiv 1 \text{ in } k\} \).

then:

(i) \( \text{rk}_{\mathcal{O}_{F,v}}(\mathcal{H}) \, X(F_\infty, A_f) = \lambda_f + 2m_{gd} + n_{gd} + m_{pr} + m_{st} \).

(ii) Under the Hypothesis 1 there exists a \( \Lambda_{\mathcal{O}_{F,v}}(\mathcal{H}) \)-homomorphism

\[
X(F_\infty, A_f) \hookrightarrow \Lambda_{\mathcal{O}_{F,v}}(\mathcal{H})^{\lambda_f + 2m_{gd} + n_{gd} + m_{pr} + m_{st}}
\]

with finite cokernel.

Proof. The first statement is nothing but a collection of all the cases considered under the various headings of primes which are Good, Ramified Principal, Steinberg, and Others, and using them in Eq. (13). The second statement follows from the usual structure theorem for torsion modules over the Iwasawa algebra \( \Lambda_{\mathcal{O}_{F,v}}(\mathcal{H}) \cong \mathcal{O}_{\mathcal{P}_{v}}(\mathbb{Z}_p) \). \( \square \)

Theorem 4.12. Let \( g, h \) be two modular forms in the Hida family \( \mathcal{H}(\bar{\rho}) \), such that the invariant \( \mu_g \) over the cyclotomic \( \mathbb{Z}_p \) extension of \( F = \mathbb{Q}(\mu_p) \) is zero, then \( \mu_h = 0 \), and if they are in the same branch, then

\[
\text{rank}_{\mathcal{O}_{\mathcal{P},v}}(\mathcal{H}) X(F_\infty, A_g) = \text{rank}_{\mathcal{O}_{\mathcal{P},v}}(\mathcal{H}) X(F_\infty, A_h).
\]

Proof. The only thing to be noted here is that for \( g, h \) in the same branch of \( \mathcal{H}(\bar{\rho}) \), we have shown in Proposition 3.7 that \( \lambda_g = \lambda_h \), while for \( m_{gd}, n_{gd}, m_{pr}, m_{st} \) we have already shown above, in the course of the case by case examination, that they do not depend upon the members of \( \mathcal{H}(\bar{\rho}) \). \( \square \)
5. Example

In this final section, we give an explicit numerical example to illustrate the results of the paper. Dokchitser has made extensive computations of the $\mu$ invariants of elliptic curves over a field of the type $\mathbb{Q}(\mu_p)$ in [D]. However we give an example which is not covered in his table. We use a result of Greenberg and the modular symbols of Manin along with a very deep result of Kato to exhibit an elliptic curve $E/\mathbb{Q}$ such that for $F = \mathbb{Q}(\mu_p)$ the $\mu$-invariant of the dual Selmer group $X(F^\text{cyc}, E_p)$ is zero. We, in fact, show that the Main conjecture holds for this example. We now state the theorem due to Kato (see [S,R]) which goes a long way towards proving the Main conjecture.

**Theorem 5.1.** (See [K].) Let $E$ be an elliptic curve over $\mathbb{Q}$, and $F = \mathbb{Q}(\mu_p)$. Let $f_E$ be the cusp form of weight two associated to $E$ and $L(f_E, F, s)$ be its $L$-function. Then:

(i) $\text{Sel}(F^\text{cyc}, E_p)$ is $\Lambda = \mathbb{Z}_p[[\Gamma]]$-cotorsion.
(ii) If $L(f_E, F, 1) \neq 0$, then $\text{Sel}(F, E_p)$ is finite.
(iii) Let $g(T)$ denote the characteristic polynomial for $X(F^\text{cyc}, E_p)$, and $L_p(f_E, F)(T)$ denote the $p$-adic $L$-function for $E$ over $F$ as defined in [MSwD]. Then

(a) $g(T) | L_p(f_E, F)(T)$, if $E$ has good ordinary reduction at $p$,
(b) $Tg(T) | L_p(f_E, F)(T)$ if $E$ has split multiplicative reduction at $p$.

**Remark 5.2.** If $\text{Sel}(F, E_p)$ is finite, then by the Control Theorem of Greenberg in [G3,G4], $\text{Sel}(F^\text{cyc}, E_p)^\Gamma$ is finite and $g(0) \neq 0$. This implies that $\text{Sel}(F^\text{cyc}, E_p)^\Gamma$ is finite [G3, Lemma 4.2]. If $E(F)_p = 0$, then by [G3, Proposition 4.14], $X(F^\text{cyc}, E_p)$ has no non-zero pseudonull submodules, and hence $X(F^\text{cyc}, E_p)^\Gamma = 0$, i.e., $\text{Sel}(F^\text{cyc}, E_p)^\Gamma = 0$.

From the Euler characteristic formula in [G3, Lemma 4.2], we have

$$g(0) \sim |\text{Sel}(F^\text{cyc}, E_p)|^\Gamma / |\text{Sel}(F^\text{cyc}, E_p)^\Gamma|$$

(21)

where the symbol $\sim$ means that the two terms differ by a $p$-adic unit. The following lemma follows easily from [G3, Proposition 4.14]. As a corollary, we have the following theorem.

**Theorem 5.3.** Under the condition that $\text{Sel}(F, E_p)$ is finite and $E(F)_p = 0$, we have:

(i) If $p$ is a prime of good ordinary reduction and $L_p(f_E, F)(0)$ is a $p$-adic unit, then $\text{Sel}(F^\text{cyc}, E_p) = 0$.
(ii) If $p$ is a prime of split multiplicative reduction such that $(T^{-1} L_p(f_E, F))(0)$ is a $p$-adic unit, then $\text{Sel}(F^\text{cyc}, E_p) = 0$.

**Proof.** (i) By the remark above, $g(0) \neq 0$. Together with Theorem 5.1 it follows that $g(0)$ is a $p$-adic unit. From (21), the remark above and the fact that $|\text{Sel}(F^\text{cyc}, E_p)^\Gamma|$ is a power of $p$ we see that $\text{Sel}(F^\text{cyc}, E_p)^\Gamma = 0$. From this we conclude that $\text{Sel}(F^\text{cyc}, E_p) = 0$ by Nakayama’s lemma.

(ii) For the $p$-adic $L$-function of $E$ over $F$ recall from [MSwD] that

$$L_p(f_E, F)(T) = \prod_{\psi} L_p(f_E, \psi, \mathbb{Q})(T)$$

(22)

where $\psi$ runs over all the characters of $G(F/\mathbb{Q})$ and $f_E \otimes \psi$ denotes the modular form $f_E$ twisted by the character $\psi$. Now, as $p$ is a prime of split multiplicative reduction for $F/\mathbb{Q}$, the $p$-adic $L$-function $L_p(f_E, \mathbb{Q})(T)$ has a trivial zero. Therefore $g(T) | T^{-1} L_p(f_E, F)(T)$. By the hypothesis, it now follows that $g(0)$ is a $p$-adic unit and hence as above $\text{Sel}(F^\text{cyc}, E_p) = 0$. □
We recall some results on Modular symbols due to Manin [M1] (see [M1] for details). These will enable us to compute the special values of L-functions. From now onwards we take \( E \) to be a modular curve \( X_0(N) \) of genus 1. We continue to denote the cusp form associated to \( E \) by \( f_E \) and its twist by a character \( \psi \) by \( f_E \otimes \psi \). Let \( \Omega^\pm = \int_{\gamma \in \omega} \), where \( \omega \) is the Neron differential of the curve, and \( \gamma^+ \) (resp. \( \gamma^- \)) \( \in H_1(E(\mathbb{C}), \mathbb{Z}) \) is a generator of cycles invariant (resp. anti-invariant) under complex conjugation. Further, \( \mathbb{P}^1(\mathbb{Z}/N) = \{ \text{classes of pairs } \bar{c} : \bar{d} \mid \bar{c} \equiv c \mod N, \bar{d} \equiv d \mod N, (c,d) = 1 \} \). Consider the function

\[
\xi_N : \mathbb{P}^1(\mathbb{Z}/N) \to H_1(X_0(N)(\mathbb{C}), \mathbb{R})
\]

defined by

\[
\xi_N(\bar{c} : \bar{d}) = \left\{ \frac{a}{b} : \frac{c}{d} \right\}_N, \quad \text{for any } ad - bc = 1.
\]

Further, \( \xi_N^\pm \) and \( x_N^\pm \) are defined as follows

\[
\xi_N(\bar{c} : \bar{d}) \mp \xi_N(\bar{d} : \bar{c}) = \xi_N^\pm(\bar{c} : \bar{d}) \gamma^\pm,
\]

\[
x_N^\pm \left( \frac{b}{a} \right) = \mp \sum_{k=1}^n \xi_N^\pm((-1)^{k-1} \bar{a}_k : \bar{a}_{k-1})
\]

where \( a_n = a, a_{n-1}, \ldots, a_0 = 1 \) are the successive convergents of \( b/a \). Then the following formula has been obtained in [M1,AV,V] (cf. [MTT])

\[
L(f_E, \psi, \mathbb{Q})(1) = \frac{1}{2} \frac{G(\psi)}{p} \left[ \sum_{b \mod p} \psi(b) x^\text{sign}(\psi) \left( \frac{b}{p} \right) \right] \Omega_2^\text{sign}(\psi)
\]

where \( G(\psi) \) is the Gauss sum for \( \psi \), and \( f_E \otimes \psi = \sum a(n) \psi(n) q^n \), the twisted form. This value is shown to be related to the \( p \)-adic L-function \( \mathcal{L}_p(f_E \otimes \psi, \mathbb{Q})(0) \) at \( T = 0 \) for \( f_E \otimes \psi \) with \( \chi \neq 1 \) in [AV,M2,V] (see [MTT] also), by the formula

\[
\mathcal{L}_p(f_E, \psi, \mathbb{Q})(0) = \begin{cases} 
\frac{1}{\alpha_p^T} \frac{p \cdot L(f_E \otimes \psi, \mathbb{Q})(1)}{\Omega_2^\text{sign}(\psi)} & \text{if } \psi \neq 1, \\
(1 - \alpha_p^{-1})^2 \frac{L(f_E, \mathbb{Q})(1)}{\Omega_2^\text{sign}(\psi)} & \text{otherwise},
\end{cases}
\]

with \( \alpha_p \) as the \( p \)-adic unit root of \( x^2 - a_p x + p \) if \( \psi \neq 1 \) and \( \alpha_p = a_p \) if \( \psi = 1 \). Here \( L(f_E \otimes \psi, \mathbb{Q})(1) \) is the usual L-function for \( f_E \otimes \psi \). Using the formula (25) get

\[
\mathcal{L}_p(f_E, \psi, \mathbb{Q})(0) = \begin{cases} 
\frac{1}{2 \alpha_p^T} \sum_{b \mod p} \psi(b) x^\text{sign}(\psi) \left( \frac{b}{p} \right) & \text{if } \psi \neq 1, \\
(1 - \alpha_p^{-1})^2 \frac{L(f_E, \mathbb{Q})(1)}{\Omega_2^\text{sign}(\psi)} & \text{otherwise}.
\end{cases}
\]

5.1. The curve \( X_0(11) \) and \( p = 11 \)

Let \( E = X_0(11) \), \( N = 11 \) and \( f_E \) denote the cusp form of weight 2 associated to \( E \). The prime \( p = 11 \) is a prime of split multiplicative reduction for \( E \) and it is an ordinary prime for \( f_E \). Let \( F = \mathbb{Q}(\mu_{11}) \) and put \( G = \text{Gal}(F/\mathbb{Q}) \), so that \( G \cong (\mathbb{Z}/11)^\times \). Let \( \psi \) be the character of \( G \) which maps the element \( 2 + 11\mathbb{Z} \) to \( \zeta \), where \( \zeta \) is a primitive 10th root of unity. Then
\[ \psi(-1) = -1, \quad \psi(1) = 1, \quad \psi(2) = \zeta, \quad \psi(3) = -\zeta^3, \quad \psi(4) = \zeta^2, \quad \psi(5) = (\zeta^3 - \zeta^2 + \zeta - 1). \]

The remaining values and powers of \( \psi \) can easily be determined from these values.

We compute the values of \( x_{11}^{\pm}(\frac{T}{11}) \) by using the formulas in (23) and (24) along with the values in the table in [M1] for \( X_0(11) \). Here we write \( \xi \) for \( \frac{b}{11} \). The values so obtained are tabulated below, wherein the first row displays the values of \( \frac{b}{11} \), and the corresponding values of \( x_{11}^{+} \) and \( x_{11}^{-} \) are written below them.

<table>
<thead>
<tr>
<th>( \frac{b}{11} )</th>
<th>1/11</th>
<th>2/11</th>
<th>3/11</th>
<th>4/11</th>
<th>5/11</th>
<th>6/11</th>
<th>7/11</th>
<th>8/11</th>
<th>9/11</th>
<th>10/11</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_{11}^{+} )</td>
<td>2/5</td>
<td>-8/5</td>
<td>-3/5</td>
<td>7/5</td>
<td>12/5</td>
<td>12/5</td>
<td>7/5</td>
<td>-3/5</td>
<td>-8/5</td>
<td>2/5</td>
</tr>
<tr>
<td>( x_{11}^{-} )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The various values of the \( p \)-adic \( L \)-functions at \( T = 0 \) computed using the formula in (27) are given below,

\[
\begin{align*}
\mathcal{L}_p(f_E, \psi, Q)(0) & \sim -2\zeta^3 + 2\zeta^2, \\
\mathcal{L}_p(f_E, \psi^3, Q)(0) & \sim 2\zeta^3 - 2\zeta^2 - 2, \\
\mathcal{L}_p(f_E, \psi^5, Q)(0) & \sim 4, \\
\mathcal{L}_p(f_E, \psi^7, Q)(0) & \sim 2\zeta^3 - 2\zeta^2 - 2, \\
\mathcal{L}_p(f_E, \psi^9, Q)(0) & \sim -2\zeta^3 + 2\zeta^2, \\
\mathcal{L}_p(f_E, \psi^2, Q)(0) & \sim 2\zeta^3 + 6\zeta^2 - 4\zeta + 2, \\
\mathcal{L}_p(f_E, \psi^4, Q)(0) & \sim 6\zeta^3 - 2\zeta^2 + 8\zeta - 4, \\
\mathcal{L}_p(f_E, \psi^6, Q)(0) & \sim -6\zeta^3 + 2\zeta^2 - 8\zeta + 4, \\
\mathcal{L}_p(f_E, \psi^8, Q)(0) & \sim -2\zeta^3 - 6\zeta^2 + 4\zeta - 2, \\
\mathcal{L}_p(f_E, Q)(0) & = 0.
\end{align*}
\]

It is easy to see that \( \prod_{i \neq 10} L(f_E \otimes \psi^i, Q)(1) \neq 0 \), and it is also known that \( L(f_E, Q)(1) \neq 0 \), therefore \( L(f_E, F)(1) \neq 0 \). Hence \( \text{Sel}(F, E_{11}) \) is finite and \( g(0) \neq 0 \).

Further, \( \prod_{i \neq 10} \mathcal{L}_p(f_E, \psi^i, Q)(0) \sim 128000 \), which is an 11-adic unit. Therefore, the element \( v(T) := \prod_{i \neq 10} \mathcal{L}_p(f_E, \psi^i, Q)(T), \) is a unit in \( \mathbb{Z}_p[[T]] \).

It is also known that \( \mathcal{L}_p(f_E, Q)(T) = T \cdot u(T) \), where \( u(T) \) is a unit in \( \mathbb{Z}_p[[T]] \). This follows from the Mazur, Tate and Teitelbaum conjecture (proved by Greenberg and Stevens in [GS], see [EPW, Example 5.3.1] also). Therefore, \( \mathcal{L}_p(f_E, F)(T) = T \cdot v(T) \cdot u(T) \). Combining this with Kato’s Theorem, which we have stated as Theorem 5.1(iii), it follows that the ideal generated by \( Tg(T) \) is the same as the ideal generated by \( \mathcal{L}_p(f_E, F)(T) \). Hence the \text{Main conjecture} holds for \( \text{Sel}(F_{11}, E_{11}) \).

It then follows from Theorem 5.3(ii) that the classical Selmer group \( \text{Sel}(F_{11}, E_{11}) = 0 \). As \( p = 11 \) is a prime of split-multiplicative reduction, this Selmer group coincides with the strict Selmer group defined by Greenberg in [G1], so that the strict Selmer group is also zero. Again, as in [G1, Section 2], we may show that the strict Selmer group is contained in the Selmer group \( \text{Sel}(F_{11}, A_{f_E}) \), where \( A_{f_E} = E_{11} \), with cokernel isomorphic to \( \mathbb{Q}_p / \mathbb{Z}_p \). Therefore the \( \lambda_{f_E} \)-invariant of \( \text{Sel}(F_{11}, A_{f_E}) \) is 1, while they have the same \( \mu \)-invariant.

Further, it follows from the fact that \( f_E \) is the only newform of weight 2 and level dividing 11 that the Galois representation associated to the Hida family of \( f_E \) is a Galois representation into \( GL_2(\Lambda) \) (see [EPW, Example 5.3.1]). Hence, the mod 11 residual representation \( \tilde{\rho} \) is

\[ \tilde{\rho} : G_\mathbb{Q} \to GL_2(\mathbb{F}_{11}). \]

The residual representation \( \tilde{\rho} \) is 11-distinguished. Thus, all the conditions on the residual representation, as in Section 4.1, that we require to define the residual Selmer groups are satisfied.

We have seen above that the \( \mu \)-invariant of \( \text{Sel}(F_{11}, A_{f_E}) \) is zero. Hence Hypothesis 1 holds. Now for \( k \geq 2 \), there is a unique newform \( f_k \) of weight \( k \) and level 11 that is congruent to \( f_E \) modulo 11. The weight 12 member of this family, say \( f_{12} \), is the 11-ordinary, 11-stabilized oldform of level 11
attached to the Ramanujan $\Delta$-function. Then by Theorem 4.12, the $\mu$-invariant of $\text{Sel}(F_{\text{cyc}}, A_{f_{12}})$ is zero.

Note that the tame level of $f_E$ is 1, therefore the terms $m_{p_T}, m_{gd}$ as defined in Theorem 4.11 for this family are all zero. This implies that the $A(H)$-rank is nothing but $1 + 2m_{gd} + n_{gd}$. As $f_E$ and $f_{12}$ are in the same branch it follows that the $A(H)$-rank of $\text{Sel}(F_{\text{cyc}}, A_{f_{12}})$ is also $1 + 2m_{gd} + n_{gd}$.

In general, it might be difficult to compute the numbers $m_{gd}$ and $n_{gd}$ for a given False–Tate extension. However, since the integer $m$ that occurs in the definition of the False–Tate extension is at our disposal, we can compute these invariants in some special cases. For example, take $m = 1123$. Then $m$ is prime with $m \equiv 1 \pmod{11}$, so that $m$ splits totally in $F$. Therefore, if $v$ denotes a prime of $F$ lying above $m$, then $F_v = \mathbb{Q}_m$. It is also easy to see that

$$\tau(1123) \equiv 2 \pmod{11}$$

where $\tau(n)$ denotes the $n$th Fourier coefficient of $\Delta$. Let $a_n$ denote the $n$th Fourier coefficient of $f_E$. Then $a_{1123} \equiv 2 \pmod{11}$. Thus the polynomial $x^2 - a_{1123}x + 1123 \equiv (x - 1)^2 \pmod{11}$. In other words, $\rho_{G_{f_v}} = 1$. Moreover, there are $[F : \mathbb{Q}] = 10$ primes of $F$ lying above $1123$. Hence $m_{gd} = 10$ and $n_{gd} = 0$. Therefore, the $A(H)$-rank of $\text{Sel}(F_{\text{cyc}}, A_{f_{12}})$ is 21.

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