JOURNAL OF ALGEBRA 160, 179–191 (1993)

Finite Groups Having Chain Difference One

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Communicated by Walter Feit

Received July 24, 1991

INTRODUCTION

In a finite group, there are many intriguing connections between the lattice of subgroups and other group theoretic properties of the group. In this article, we consider lengths of maximal chains of subgroups.

Iwasawa [Iw] proved what is now a classical result along the theme we pursue. His result is that all maximal chains of subgroups of a finite group G have the same length if and only if G is supersolvable. We consider here groups in which any two maximal chains of subgroups differ in length by at most one. These are the groups in our title and our goal is to describe such groups.

At the beginning, we hoped to find a way to produce shortest maximal chains. The articles by Cameron, Solomon, and Turull [CST], Solomon and Turull [ST], and Seitz, Solomon, and Turull [SST] attack the problem of longest maximal chains in various classical groups. Shortest chains may, in general, be less tractable.

However, for solvable groups the picture is illuminated by the notion of critical maximal subgroups which appears in Section 4 of Carter and Hawkes [CH]. Kohler [Ko] has the result formulated well for our use in Section 2. Shortest maximal chains in solvable groups and, hence, solvable groups of chain difference one are apparent using Kohler's result.

0021-8693/93 $5.00

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Luckily, one need not know shortest chains in all simple groups to identify those of chain difference one. In Section 3 we classify the nonabelian finite simple groups with chain difference one, using the Classification of Finite Simple Groups, Dickson's Theorem on subgroups of $L_2(q)$, known involvement of one simple group in another and, of course, the Atlas [AT]. Not surprisingly, any such group is isomorphic to some $L_2(q)$ where q comes from a severely restricted set of prime powers.

For arbitrary finite groups, our concluding results in Section 4 show that a group with chain difference one can have at most one noncyclic chief factor which, if abelian, must have rank 2 or, if nonabelian, must be one of the simple groups with chain difference 1 identified in Section 3. Unfortunately this condition is not sufficient as we illustrate with $PGL(2, 29)$.

1. Notation and Preliminaries

All groups considered in this paper are finite. Group theoretic notation is standard, see [Hu].

For a positive integer n, $\Omega(n)$ is the number of prime divisors of n, counting multiplicity. For example, $\Omega(12) = 3$.

A maximal chain of subgroups of a group G is a chain of subgroups $1 = M_0 < M_1 < \cdots < M_k = G$ where each M_i is a maximal subgroup of M_{i+1}. The length of such a chain is k. We define the chain difference of a finite group G to be the length of a longest maximal chain of subgroups minus the length of a shortest maximal chain of subgroups and we denote it by $cd(G)$. With this notation, Iwasawa's characterization becomes

Iwasawa's Theorem 1.1. *A finite group G is supersolvable if and only if $cd(G) = 0$.*

The groups of interest to us are those having chain difference equal to *1*. Two lemmas about the chain difference of a group will be needed later on. The first one is obvious.

Lemma 1.2. *If G is a finite group, then $cd(G) = \max |r - s|$ where the maximum is taken over all r, s which are lengths of maximal chains of subgroups of G. Also, $cd(G) \leqslant \Omega(|G|)$.*

Lemma 1.3. *If G is a finite group, $B \leqslant G$ and A is a normal subgroup of B, then $cd(G) \geqslant cd(B/A) + cd(A)$. In particular, $cd(G) \geqslant cd(B)$ and $cd(G) \geqslant cd(B/A)$.*

Proof. Let $1 = M_0 < M_1 < \cdots < M_k = A$ be a longest maximal chain of subgroups of A and let $N_0/A < N_1/A < \cdots < N_j/A = B/A$ be a longest maximal chain of B/A, then $1 = M_0 < M_1 < \cdots < M_k = A = N_0 < N_1 < \cdots < N_j = B$ is a maximal chain of subgroups of B whose length is $k + j$. Similarly, by taking shortest maximal chains instead of longest ones we can construct a maximal chain of subgroups of B of length $m + n$, where m is the length of a shortest maximal chain of A and n is the length of a shortest maximal chain of A/B. By extending each of these maximal chains of B in the same way to maximal chains of G by adding, if necessary, subgroups between B and G, we obtain maximal chains of subgroups of G of length $k + j + r$ and $m + n + r$ for some $r \geqslant 0$. Thus, by Lemma 1.2, $cd(G) \geqslant (k + j + r) - (m + n + r) = (k - m) + (j - n) = cd(A) + cd(B/A)$ from which it follows that $cd(G) \geqslant cd(B)$ and $cd(G) \geqslant cd(B/A)$.

LEMMA 1.4. *Suppose G is a finite group and $N \leqslant Z(G)$, then for every maximal chain of subgroups of G, there is a maximal chain of G of the same length containing N as one of its terms.*

Proof. We induct on $|G|$. Let $1 = M_0 < M_1 < \cdots < M_k = G$ be any maximal chain of subgroups of G and let $K = M_{k-1}$, then K is a maximal subgroup of G and $1 = M_0 < M_1 < \cdots < M_{k-1} = K$ is a maximal chain of subgroups of K.

First assume $N \leqslant K$, then $N \leqslant Z(K)$. By induction, there is a maximal chain of subgroups of K of length of $k - 1$ containing N as a term. By adjoining G to that chain, we obtain maximal chain of G of length k with N as a term as was to be shown.

Now assume N is not contained in K, then $G = NK$. Since $N \leqslant Z(G)$, K is a normal subgroup of G. Hence, G/K has order p for some prime p. In addition, $N/K \cap N \cong G/K$ and so we see that $K \cap N$ is a maximal subgroup of N.

By induction again we obtain a maximal chain of subgroups of K, $1 = L_0 < L_1 < \cdots < L_i = K \cap N < \cdots < L_{k-1} = K$, having $K \cap N$ as a term. Clearly, $1 = L_i/K \cap N < L_{i+1}/K \cap N < \cdots < L_{k-1}/K \cap N = K/K \cap N$ is a maximal chain of subgroups of $K/K \cap N$ and, via the natural isomorphism $K/K \cap N$ to G/N, $1 = NL_i/N < \cdots < NL_{k-1}/N = G/N$ is a maximal chain of subgroups of G/N.

Recalling that $K \cap N$ is a maximal subgroup of N, we have $1 = L_0 < L_1 < \cdots < L_i = K \cap N < N < NL_{i-1} < \cdots < NL_{k-1} = G$ is a maximal chain of subgroups of G visibly containing N as a term and having length k. The proof of the lemma is complete.

COROLLARY 1.5. *For any finite group G, $cd(G) = cd(G/Z(G))$.*

Proof. By Lemma 1.4, there are longest and shortest maximal chains of subgroups of G each having $Z(G)$ as a term. By Iwasawa's Theorem 1.1, the number of terms contained in $Z(G)$ is the same in each chain and so the number of terms containing $Z(G)$ is, respectively, the length of a longest and shortest maximal chain of $G/Z(G)$. The corollary follows by subtraction.

2. Solvable Groups Having Chain Difference One

The chain difference of a finite solvable group is easy to compute using a theorem of J. Kohler.

Kohler's Theorem 2.1 [Ko, Theorem 2]. *The length of a shortest maximal chain in a finite solvable group G is the length of a chief series of G.*

Corollary 2.2. *If G is a finite solvable group, then $cd(G) = \sum(\text{rank } C - 1)$ where the sum is over all factors C in any fixed chief series of G. In particular, $cd(G) = 1$ if and only if any chief series of G has exactly one noncyclic chief factor, that factor having rank 2.*

Proof. Clearly a composition series of G is a longest maximal chain of G because its length is $\Omega(|G|)$. Take any chief series of G. Using Kohler's Theorem 2.1, $cd(G) = \Omega(|G|) - (\text{length of the chief series}) = \sum \text{rank } C - \sum 1 = \sum(\text{rank } C - 1)$, the sum taken over the factors C in the chief series. From the formula, we see $cd(G) = 1$ if and only if rank $C = 1$ except in one case wherein rank $C = 2$. Thus, every chief factor is cyclic except one which is of rank 2.

3. Nonabelian Simple Groups Having Chain Difference One

To determine which nonabelian simple groups have chain difference one, we use the classification of finite simple groups and simply check each family of simple groups. For low order and sporadic simple groups, the reader is frequently referred to the Atlas [AT] to check necessary details. The notation we use for the simple groups is that of the Atlas. Of course, by Iwasawa's Theorem 1.1, the chain difference of any nonabelian simple group is at least 1.

Lemma 3.1 *A simple group isomorphic to an alternating group A_n has chain difference 1 if and only if $n = 5$ or 6.*

Proof. It is easy to check (using the Atlas if nothing else) that A_5 and A_6 have chain difference one. Likewise, it is easy to check $cd(A_7) = 2$. Thus, by Lemma 1.3, $cd(A_n) \geqslant cd(A_7) = 2$ for $n \geqslant 7$.

LEMMA 3.2. *A simple group isomorphic to $L_2(q)$ has chain difference* 1 *if and only if*

(i) $q = 4, 5, 9$ (*i.e. the group is isomorphic to* A_5 *or* A_6), *or*

(ii) q *is an odd prime*, $5 | q^2 - 1$ *or* $16 | q^2 - 1$, *and* $3 \leqslant \Omega(q \pm 1) \leqslant 4$, *or*

(iii) q *is an odd prime*, $5 \cdot 16 \nmid q^2 - 1$ (*equivalently*, $q \equiv 3, 13, 27, 37$ (*mod* 40)), *and* $2 \leqslant \Omega(q \pm 1) \leqslant 3$.

Proof. First we show sufficiency. If (i) holds, then from above the group has chain difference 1. If (ii) holds then, by Dickson's Theorem, [Hu, II.8.27] the possible maximal subgroups are the Borel subgroups of order $q(q-1)/2$, dihedral subgroups of orders $q \pm 1$ and subgroups isomorphic to A_5 (when $5 | q^2 - 1$) or S_4 (when $16 | q^2 - 1$). All maximal chains of $L_2(q)$ through the Borel subgroups have length $\Omega(q-1)+1$ as do those through a dihedral subgroup of order $q-1$. Maximal chains through dihedral subgroups of order $q+1$ have length $\Omega(q+1)+1$ while those through a copy of A_5 or S_4 have length 4 or 5. Since $3 \leqslant \Omega(q \pm 1) \leqslant 4$, we see that the difference in this case is 1. If (iii) holds, Dickson's Theorem tells us that the maximal subgroups are as in case (ii) except with subgroups isomorphic to A_4 replacing those isomorphic to A_5 or S_4. Now we may calculate a chain difference of 1 just as in case (ii).

Now we show necessity. Suppose $G \cong L_2(q)$, $q > 3$ and G has chain difference 1.

First suppose q is even. The Sylow 2-subgroup of G is elementary abelian of order q and is a minimal normal subgroup of a Borel subgroup, its normalizer. Thus when $q = 2^n$, $n \geqslant 3$, a Borel subgroup already has chain difference greater than 1 by Corollary 2.2 because it has a minimal normal abelian subgroup of rank $n \geqslant 3$. Hence G also has chain difference greater than 1 by Lemma 1.3.

Next suppose q is odd, say $q = p^n$ where p is an odd prime. When $n \geqslant 3$, we see, exactly as in the preceding paragraph, that a Borel subgroup of G has chain difference greater than 1 and, hence, so does G. Thus, $n \leqslant 2$.

Consider the case where $n = 2$. By examining the possible last digit of the prime p, we see that either $p = 5$ or $5 \mid (p^4 - 1)$ (true for any odd number). If $p = 5$, $L_2(5^2)$ has chain difference greater than 1 (see the Atlas again). Thus, $p \neq 5$ and we have $5 \mid p^4 - 1$. That is, $5 \mid p^2 - 1$ or $5 \mid p^2 + 1$.

Assume $5 \mid p^2 + 1$, then 5 does not divide $p^2 - 1$ so neither $L_2(p)$ nor $PGL(2,p)$ has a subgroup isomorphic to A_5. Therefore, by Dickson's Theorem, $L_2(p^2)$ has a maximal subgroup isomorphic to A_5. There is a maximal chain containing this subgroup having length 4 (take one through A_4). Also by Dickson's Theorem $L_2(p^2)$ has a maximal subgroup which is dihedral of order $p^2 - 1$ or else $p = 3$, in which case $q = 9$ and we are finished. For $p > 3$, any maximal chain of the maximal dihedral subgroup has length $\Omega(p^2-1)+1$ since it is supersolvable. Suppose $\Omega(p^2-1)=3$, then $p^2-1=4r$ where r is a prime and so $p-1=2$, 4, r or $2r$. The first and third cases imply $p=3$ which is not so. The second case implies $p=5$ which we already eliminated. The fourth case implies $p+1=2$ which is absurd. Thus, $\Omega(p^2-1)>3$ and a maximal chain of $L_2(p^2)$ through a Borel subgroup has length $\Omega(p^2-1)+2>5$. Comparing this chain to the one of length 4 located earlier, we see that $L_2(p^2)$ has chain difference greater than 1 in case $5 \mid p^2+1$.

Assume $5 \mid p^2-1$, then $L_2(p)$ has a maximal subgroup isomorphic to A_5. The small cases $p=5$, 7, 11 are handled separately (again, using the Atlas or brute force) and only $p=5$ yields a group with chain difference 1. For $p>11$, $L_2(p)$ has dihedral maximal subgroups of orders $p+1$ and $p-1$. If either $\Omega(p+1)\leqslant 2$ or $\Omega(p-1)\leqslant 2$ then by looking at a maximal chain of $L_2(p)$ containing the corresponding dihedral subgroup and comparing with a maximal chain of length 5 containing the copy of A_5, we find $L_2(p)$, hence $L_2(p^2)$, has chain difference greater than 2. Thus, $\Omega(p\pm 1)\geqslant 3$. Now consider a maximal chain of $L_2(p^2)$ containing the Borel subgroup first passing through the Sylow p-subgroup. The length of this chain is $2+\Omega(p+1)+\Omega(p-1)\geqslant 8$. In $L_2(p)$, take a maximal chain of length 4 containing A_5, then extend it to a maximal chain of $L_2(p^2)$, by moving up to $PGL(2,p)$ and then to $L_2(p^2)$. This gives a maximal chain of length 6. Comparing the two maximal chains just constructed shows $L_2(p^2)$ has chain difference at least 2 in this case as well.

So now we have $n=1$ or, in other words, $q=p$ is an odd prime. We need to show q satisfies the conditions in (ii) or (iii). As before we may assume $q>11$. If $5 \mid q^2-1$ or $16 \mid q^2-1$, $L_2(q)$ contains maximal subgroups isomorphic to A_5 or S_4 respectively (by Dickson's Theorem, of course). Either way, we obtain maximal chains of $L_2(q)$ of lengths 4 and 5. Comparing with the lengths of chains through a Borel and maximal dihedral subgroups gives $3\leqslant\Omega(p\pm 1)\leqslant 4$ since the chain difference is 1. If $5\cdot 16$ does not divide q^2-1, $L_2(q)$ contains a maximal subgroup isomorphic to A_4. A calculation like the one just outlined shows $2\leqslant\Omega(p\pm 1)\leqslant 3$ in this case.

THEOREM 3.3 *There are no nonabelian simple groups having chain difference equal to one except those found in Lemma* 3.1 *and Lemma* 3.2 *above.*

Proof. Having already analyzed the alternating groups in Lemma 3.1 and the family $L_2(q)$ in Lemma 3.2, we simply examine the remaining simple groups family by family.

The Family $A_n(q)\cong L_{n+1}(q)$, $n\geqslant 2$. First consider the case $n=2$. Suppose q is even. Then $L_3(q)$ contains $L_3(2)$ as a subgroup. $L_3(2)$

($\cong L_2(7)$) has chain difference greater than 1 by 3.2. Thus, $L_3(q)$ has chain difference greater than 1 by Lemma 1.3.

Now suppose $q = p^n$ where p is an odd prime. $L_3(q)$ contains $L_3(p)$ as a subgroup so it suffices to show $SL(3, p)$ has chain difference greater than 1, by Corollary 1.5 and Lemma 1.3. Consider the following subgroups of $SL(3, p)$:

$$T = \left\{ \begin{pmatrix} x & y & 0 \\ u & v & 0 \\ 0 & 0 & d \end{pmatrix} \in SL(3, p) : d = (xv - yu)^{-1} \right\}$$

and

$$M = \left\{ \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in SL(3, p) \right\}$$

M is elementary abelian of order p^2, $T \cong GL(2, p)$ and T acts irreducibly on M. (TM, in fact, is a maximal parabolic subgroup of $SL(3, p)$.) $GL(2, p)$ is not supersolvable so it is possible to find two maximal chains of different lengths. Call the lengths r and s where $r < s$. T is a maximal subgroup of TM (by the irreducible action of T on M) so by using the chain of length r in T, we obtain a maximal chain of TM of length $r + 1$. By first going through M, then using the chain of length s in T, we obtain a maximal chain of TM of length $s + 2$. The chain difference in TM then is at least $s + 2 - (r + 1) = s - r + 1 \geqslant 2$. Thus, the chain difference in $SL(3, p)$ is at least 2 as was to be shown.

For $n > 2$, $L_{n+1}(q)$ contains $L_3(q)$ as a subgroup and so, by the preceding, $L_{n+1}(q)$ has chain difference at least 2.

The Families $B_n(q)$, $n \geqslant 3$; $C_n(q)$, $n \geqslant 3$; $D_n(q)$, $n \geqslant 4$, $E_n(q)$, $n = 6, 7, 8$; $F_4(q)$, ${}^2D_n(q)$, $n \geqslant 4$, ${}^2E_6(q)$. The diagrams for the above families each contain a subdiagram of type A_2 (i.e. $0 - 0$). Thus, each group involves $A_2(q)$ which has chain difference greater than 1 as has just been shown. (For a discussion of this involvement, see [Go, pp. 77–78] or p. xv of the Atlas.)

The Family ${}^2A_2(q) \cong U_3(q)$. First suppose $q = p^n$, $n > 1$. A Borel subgroup of $U_3(q)$ is a semidirect product QK where Q is a normal Sylow p-subgroup of order q^3 and K is cyclic. Moreover, $Q/Z(Q)$ is elementary abelian of order q^2 and K acts irreducibly on it [O'N]. Thus, with $n > 1$, $Q/Z(Q)$ is a chief factor of QK of rank at least 4. By Corollary 2.2, KQ already has chain difference at least 3. Therefore, $cd(U_3(q)) \geqslant 3$ in this case.

Now suppose $q = p$, then p must be odd because $U_3(2)$ is solvable.

Moreover, we may assume $p > 5$ by simply checking $U_3(3)$ and $U_3(5)$ in the Atlas.

For the moment, we will work in $SU_3(p)$. Consider $SU_3(p)$ acting on a 3-dimensional vector space V over $GF(p^2)$ where V has a nondegenerate Hermitian form which defines $SU_3(p)$. We may take a basis $\{w_1, w_2, w_3\}$ for V as in [Hu, Satz II.10.12]. Set $U = \langle w_2 \rangle$, a 1-dimensional nonisotropic subspace, and $U'(=U^{\perp}) = \langle w_1, w_3 \rangle$. U' contains a 1-dimensional isotropic subspace (e.g. $\langle w_1 \rangle$). By the theorem on page 373 of [Ki], the stabilizer of U in $SU_3(p)$, call it M, is a maximal subgroup of $SU_3(p)$. Furthermore, since M preserves the Hermitian form, a matrix in M has the form $\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$ with respect to the basis $\{w_1, w_3, w_2\}$ and it may be seen that

$$M = \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & (ad-bc)^{-1} \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GU_2(p) \right\} \cong GU_2(p).$$

In $GU_2(p)$, $GU_2(p)/SU_2(p)$ is cyclic of order $p+1$, $Z(SU_2(p)))$ has order 2 and $SU_2(p)/Z(SU_2(p)) \cong L_2(p)$. Let r be the length of a longest maximal chain of $L_2(p)$. By suitably refining the chain $1 < Z(SU_2(p)) < SU_2(p) < GU_2(p)$ to a maximal chain, we obtain a maximal chain of $GU_2(p)$ of length $1 + r + \Omega(p+1)$. Taking the image of this chain in M and recalling that M is a maximal subgroup of $SU_3(p)$ we have a maximal chain of $SU_3(P)$ of length $2 + r + \Omega(p+1)$. Lemma 1.4 implies there is a maximal chain of the same length having $Z(SU_3(p))$ as one of its terms. Factoring out that center, which has order $(p+1, 3)$, we obtain a maximal chain of $U_3(p)$ of length $2 + r + \Omega(p+1) - \Omega((p+1, 3))$.

Now we construct a shorter maximal chain of $U_3(p)$. The entries of $U_3(p)$ are in $GF(p^2)$. Let σ be a field automorphism of $GF(p^2)$ having order 2. We see that σ induces an automorphism of $U_3(p)$. Furthermore, by Theorem 2, p. 388 of [BGL], C, the centralizer in $U_3(p)$ of σ, is a maximal subgroup of $U_3(p)$. Furthermore $C \cong PSO_3(p) \cong PGL_2(p)$ (the first isomorphism is not hard to check by hand and the second is [Hu, II.10.11]).

Let s be the length of a shortest maximal chain in $L_2(p)$. Using the image of such a chain in C and recalling that $L_2(p)$ is maximal in $PGL_2(p) \cong C$ and C is maximal in $U_3(p)$, we construct a maximal chain of subgroups of $U_3(p)$ of length $s+2$.

From those two maximal chains and Lemma 1.2, we obtain

$$\begin{aligned} cd(U_3(p)) &\geqslant [2 + r + \Omega(p+1) - \Omega((p+1, 3))] - (s+2) \\ &= (r-s) + \Omega(p+1) - \Omega((p+1, 3)) \geqslant 1 + 2 - 1 = 2. \end{aligned}$$

The Family $^2A_n(q) \cong U_{n+1}(q)$, $n \geqslant 3$. By thinking of unitary matrices, we see $U_{n+1}(q)$, $n \geqslant 3$, involves $U_3(q)$ and $U_4(q)$. For $q=2$, $U_4(2)$ has chain difference greater than 1 from the Atlas. Otherwise, $U_3(q)$ has chain difference greater than 1 from the previous result. Lemma 1.3 thus implies the desired conclusion.

The Families $G_2(q)$, $^3D_4(q) = D_4^2(q)$, $PSp(4, q) \cong B_2(q) \cong C_2(q)$. For q even, $^3D_4(q)$ involves $L_2(q^3)$ (as may be seen from the diagram or from p. 376 of [Th] and thus it has chain difference greater than one. For the other groups, $q \geqslant 4$ (neither $B_2(2)$ nor $G_2(2)$ are simple). Now, $B_2(4)$ and $G_2(4)$ have chain difference greater than 1 from the Atlas and for $q > 4$, $B_2(q)$ and $G_2(q)$ each involve $L_2(q)$, which has been shown to have chain difference greater than 1.

For q odd, the centralizer of an involution in each of these groups has a subgroup L_1L_2 of index 2 with $[L_1, L_2] = 1$, $|L_1 \cap L_2| = 2$ and $L_i \cong SL(2, q_i)$, q_i odd [Hr, p. 80]. $SL(2, q_i)$ is not supersolvable and so it has chain difference at least one. Thus, even a centralizer of an involution has chain difference at least 2 using Lemma 1.2.

The Families $^2B_2(2^r)$, $^2F_4(2^r)$, r *Odd*, $r > 1$. The groups $^2B_2(2^r)$, r odd, $r > 1$, are the Suzuki groups. Suzuki's paper [Su62, Theorem 9 of Sect. 15] gives the maximal subgroups. One of them is the Borel subgroup H of order $q^2(q-1)$ where H modulo its Sylow 2-subgroup is cyclic of order $q-1$. A maximal chain of $^2B_2(2^r)$ containing H and its Sylow 2-subgroup has length $2r + \Omega(q-1) + 1$. Another maximal subgroup is B_0 which is dihedral of order $2(q-1)$. A maximal chain containing B_0 has length $\Omega(q-1) + 2$. Thus, the chain difference is at least $2r - 1 > 2$.

Parrott [Pa, p. 343] shows $^2B_2(2^r)$ is involved in $^2F_4(2^r)$ and so the previous paragraph and Lemma 1.3 give $cd(^2F_4(2^r)) > 2$.

The Family $^2G_2(3^r)$, $r > 1$, r *Odd*. Centralizers of involutions in this family of Ree groups involve $L_2(3^r)$ [Wa, p. 332] and, thus, since r is odd and $r > 1$, the chain difference is greater than 1.

The Sporadic Simple Groups and the Tits Group $^2F_4(2)'$. Here one can simply glance at the Atlas and readily find a subgroup in each group that already has chain difference greater than 1. For the sporadic groups, it saves time to note that the table on p. 238 of the Atlas indicates the involvement of M_{11} in all the sporadic groups except J_1, M_{22}, J_2, J_3, *He*, *Ru*, and *Th*.

4. Nonsolvable Groups Having Chain Difference One

Lemma 4.1. *Suppose G/H is a finite simple group and the chief factors of G in H are cyclic, then $G/Z(G) = C/Z(G) \times H/Z(G)$, where $C = C_G(H)$, $C/Z(G) \cong G/H$.*

Proof. First we show $G/C_G(H)$ is solvable. Let $1 = H_k < H_{k-1} < \cdots < H_0 = H < G$ be a chief series of G and $A = \{\alpha \in \mathrm{Aut}(H): H_i^\alpha = H_i, 0 \leqslant i \leqslant k\}$. By hypothesis, each H_i/H_{i+1} is cyclic so $\mathrm{Aut}(H_i/H_{i+1})$ is abelian. Thus, $A/C_A(H_i/H_{i+1})$ is abelian and $A' \leqslant C_A(H_i/H_{i+1})$ for each i. By Hall's Theorem [Ha], A' is nilpotent. Thus, A is solvable. Since $G/C_G(H)$ embeds in A, it is also solvable.

Let $C = C_G(H)$; CH is normal in G. The simplicity G/H implies $CH = G$ or $C \leqslant H$. The latter is impossible since we have shown G/C is solvable. Thus, $CH = G$ and $G/C \cap H = C/C \cap H \times H/C \cap H$. It suffices to show $C \cap H = Z(G)$. Clearly, $C \cap H = Z(H) \leqslant Z(G)$. Conversely, $Z(G) \leqslant C$, obviously, and $Z(G) \leqslant H$ because G/H is simple.

Theorem 4.2. *Suppose G is a nonsolvable finite group with at least two noncyclic chief factors in any chief series, then $cd(G) \geqslant 2$.*

Before proving the theorem, we state and prove a corollary that along with Corollary 2.2 gives the result stated in the Introduction.

Corollary 4.3. *Suppose G is a nonsolvable finite group with $cd(G) = 1$, then G has exactly one noncyclic chief factor in any chief series and that factor is isomorphic to $L_2(q)$ where $q = 4$, 5, 9 or q is an odd prime with $3 \leqslant \Omega(q \pm 1) \leqslant 4$ when 5 or 16 divides $q^2 - 1$ and $2 \leqslant \Omega(q \pm 1) \leqslant 3$ when $5 \cdot 16$ does not divide $q^2 - 1$.*

Proof. That G has only one noncyclic chief factor in any chief series follows from the theorem. Nonsolvability implies such a chief factor is a direct product of nonabelian simple groups. If there is more than 1 direct factor in the chief factor we may apply Theorem 1.1 and Lemma 1.3 to see that $cd(G) \geqslant 2$ (take A/C as the chief factor, choose B with $C \leqslant B$, B normal in A and A/B simple then note that in the case of more than one direct factor in A/C, B is not supersolvable). Thus, the chief factor is a nonabelian simple group. By Lemma 1.3 again, it must be a simple group of chain difference one and, therefore, is one of the groups listed in the corollary by the results of the previous section.

Now we turn to the proof of the theorem.

Fix a chief series for G. By hypothesis, the series has at least two noncyclic chief factors. Suppose it has more than two such chief factors. Then there is a nontrivial normal subgroup L in the chief series with G/L

having a chief series with exactly two noncyclic factors. By induction, $cd(G/L) \geqslant 2$. Hence, $cd(G) \geqslant 2$. Thus, we may assume G has precisely two noncyclic chief factors. If either factor is the direct product of more than one nonabelian simple group, then the same argument as used in the proof of the corollary gives $cd(G) \geqslant 2$. Thus, we may assume G has precisely two noncyclic chief factors. If either factor is the direct product of more than one nonabelian simple group, then the same argument as used in the proof of the corollary gives $cd(G) \geqslant 2$. If both factors are nonabelian, then there is a factor A/B with both A and B nonsolvable so once again we employ Lemma 1.3 to give the desired result. In summary, we may henceforth assume our chief series for G has precisely two noncyclic factors: one simple and one elementary abelian.

Let K/H be the simple noncyclic chief factor and let N/L be the noncyclic elementary abelian chief factor. If $K \leqslant L$, then by Corollary 2.2 $cd(G/K) \geqslant 1$. Thus , $cd(G) \geqslant cd(G/K) + cd(K) \geqslant 2$. Hence we may assume $N \leqslant H$. If $L \neq 1$, we may use induction to obtain $cd(G) \geqslant cd(G/L) \geqslant 2$. Thus we may suppose $L = 1$ and N is a minimal normal subgroup of G.

We consider separately the cases $K < G$ and $K = G$.

Case 1. $K < G$. If K has a noncyclic chief factor other than K/H, then we are done by induction.

If all the chief factors of K contained in H are cyclic, then Lemma 4.1 shows there is a subgroup $C \leqslant K$ with $C/Z(K) \cong K/H$ and C is normal in G. Let S be the terminal member of the derived series of C. Since C is nonsolvable, $S \neq 1$ and S is not contained in $Z(K)$. Thus, $SZ(K)/Z(K)$ is a nontrivial normal subgroup of the simple group $C/Z(K)$, so $SZ(K) = C$ and $C/Z(K) \cong S/(S \cap Z(K))$. In as much as $S \cap Z(K) \leqslant Z(S)$ and $Z(S)$ is normal in S, we have $S \cap Z(K) = Z(S)$. Hence, by [Su82, 9.18(3), (5) and 9.14(2)], S is a homomorphic image of the representation group of $C/Z(K)$ with $Z(S)$ a homomorphic image of the center of the representation group.

We may assume $cd(C/Z(K)) = 1$, otherwise we are finished. From our examination of the simple groups in the previous section, $C/Z(K) \cong L_2(p)$, for some odd prime greater than 3, or $L_2(9)$. In either case the representation groups of the corresponding simple groups have cyclic centers [Hu, V.25.7]. Thus, $Z(S)$ is the image of a cyclic group and is therefore cyclic.

If $Z(S) \neq 1$, then any chief series of $G/Z(S)$ has two noncyclic chief factors and we are done by induction. If $Z(S) = 1$, then S is a minimal normal subgroup of G. By refining $1 < S < G$ to a chief series for G we see G/S is solvable and has one noncyclic chief factor and so $cd(G/S) \geqslant 1$. Thus, $cd(G) \geqslant cd(G/S) + cd(S) \geqslant 2$ as was to be shown.

Case 2. $K = G$. Recall we have a noncyclic elementary abelian subgroup N. Let $|N| = r^a$, $a > 1$, r a prime.

If N is not contained in $\Phi(G)$, then there exists a maximal subgroup M

not containing N. Thus, $G=NM$ and $N \cap M=1$. M is not solvable or else G would be solvable. Hence, $cd(M) \geqslant 1$. Take a shortest maximal chain of subgroups of M, say of length s, by adjoining G we get a maximal chain of subgroups of G of length $s+1$. Next build another maximal chain of G by taking one of length a in N and extending it by adjoining the preimage of a longest maximal chain in $G/N \cong M$. The length of this chain is $t+a$ where t is the length of a longest maximal chain in M. Comparing these two chains we see $cd(G) \geqslant (t+a)-(s+1)=cd(M)+(a-1) \geqslant 1+1$. Thus, we may now assume $N \leqslant \Phi(G)$. The simplicity of $K/H=G/H$ implies $\Phi(G) \leqslant H$.

If $\Phi(G)$ is not an r-group, then a chief series of $G/O_{r'}(\Phi(G))$ has two noncyclic chief factors (isomorphic to N and G/H) and, hence, we are finished by induction. Next we show $\Phi(G)=H$ by mimicking [Ko]. Suppose $\Phi(G)<H$. Let $R/\Phi(G)$ be a minimal normal subgroup of $G/\Phi(G)$ contained in $H/\Phi(G)$; $R/\Phi(G) \neq 1$ and so there is a maximal subgroup M of G not containing R. Thus, $RM=G$. Moreover, H is solvable and so $R/\Phi(G)$ is abelian which implies $R \leqslant F(G)=\bigcap C_G(C)$ where the intersection is taken over all chief factors C of G. Therefore, M acts on chief factors of G in the way G does. In particular, N is a minimal normal subgroup of M. M is not solvable and so it has two noncyclic chief factors: N and a nonabelian one. By induction, $cd(M) \leqslant 2$ and we are finished. Therefore, we may take $\Phi(G)=H$.

Now suppose $r=2$ or $r=3$. Let P be a Sylow p-subgroup of G where p is a prime divisor of $|G|$ different from 2 and 3. By Maschke's Theorem, N is a completely reducible P-module. The dimension of each irreducible P-submodule of N, except possibly one, is 1 and the exception can have dimension at most 2; for otherwise, $cd(PN) \geqslant 2$ by Corollary 2.2 and there is no more to show. The possible automorphism groups of the irreducible P-submodules are, therefore, 1, Z_2 and subgroups of $GL_2(2)$ or $GL_2(3)$, all $\{2,3\}$-groups. Thus, $P \leqslant C_G(N)$. Since $G/\Phi(G)=K/H$ is simple and P is not a subgroup of $\Phi(G)$, $P^G\Phi(G)=G$. Thus, $G=P^G \leqslant C_G(N)$, a contradiction because N is a noncyclic chief factor.

Finally, we consider the case $r \neq 2$ and $r \neq 3$. If the simple group $G/\Phi(G)$ has chain difference greater than 1, we are finished. Thus, by the previous section, we may assume $G/\Phi(G) \cong L_2(q)$ where q is odd so, by Dickson's Theorem, there is $B \leqslant G$ with $B/\Phi(G) \cong A_4$. Let A be an Hall $\{2,3\}$-subgroup of B, then $A \cong A_4$. By Maschke's Theorem again, N is a completely reducible A-module. Each irreducible A-submodule of N has dimension 1, for otherwise, since $cd(A)=2$, $cd(AN) \geqslant 2$ by Corollary 2.2. Therefore, each irreducible A-submodule has an abelian automorphism group which implies A' centralizes each submodule and, consequently, A' centralizes N. As before with P, it follows that $G=(A')^G \leqslant C_G(N)$, a contradiction. The proof of theorem is complete.

The converse of the Corollary is false. For example, from the Atlas, $PGL_2(29)$ has a "novel" maximal subgroup isomorphic to S_4. Comparing a short maximal chain of $PGL_2(29)$ containing that copy of S_4 with a long maximal chain containing A_5 and $L_2(29)$, we see $cd(PGL_2(29)) \geqslant 2$ even though the only noncyclic chief factor of $PGL_2(29)$ is $L_2(29)$.

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