The Invariant Imbedding Method for Transport Problems
II. Resolvent in Photon Diffusion Equation

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I. INTRODUCTION

In recent years, making use of functional equation techniques in various ways in connection with the principles of invariance due to Ambarzumian [1] and Chandrasekhar [15], the study of radiative transfer has exactly been carried out by Sobolev [21, 22], Miss Busbridge [10, 11], Horak and Lundquist [16], Stibbs [14, 25], Bellman and Kalaba [2, 3], King [17], Preisendörfer [19, 20], and Ueno [28, 29], respectively.

Let a particular physical process be given. Then, by imbedding this process within an appropriate class of processes the functional relationships existing among the various processes of the class will be found. In such a manner the reflected and transmitted fluxes and the probability distributions for these fluxes in neutron transport theory have been computed by Bellman, Kalaba, and Wing [4, 9].

Recently, allowing for the probability significance of the resolvent in the Milne first integral equation for the source function $J(\tau)$

$$J(\tau) = R(\tau) + \int_0^\infty K(\tau, \tau') B(\tau') d\tau',$$

(1.1)

where $B(\tau)$ is the distribution of the radiation sources acting in the medium and $K(\tau, \tau')$ is the resolvent, and proceeding in such a fashion that to the medium a layer of the infinitely small optical thickness $\Delta\tau$ is added, Sobolev [21] obtained

$$K(\tau + \Delta\tau, \tau' + \Delta\tau) = K(\tau, \tau') + K(\tau, 0) \Delta\tau K(0, \tau').$$

(1.2)

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Then, Eq. (4.8) of the Volterra type has been derived from equation (1.2) by Sobolev (cf. [21]).

When the resolvent is known and the arrangement of the radiation sources is given, following the usual procedure (cf. R. T.), the intensities in the outward and the inward directions at any level \( \tau \) in a semi-infinite medium is obtained by means of the integral transform of the source function \( J(\tau) \) as follows:

\[
I(\tau, + \mu) = \int_{\tau}^{\infty} J(t) \exp \left[ \frac{-(t - \tau)}{\mu} \right] \frac{dt}{\mu} \quad (0 < \mu \leq 1),
\]

\[
I(\tau, - \mu) = I(0, - \mu) \exp \left( \frac{-\tau}{\mu} \right) + \int_{0}^{\tau} J(t) \exp \left[ \frac{-(\tau - t)}{\mu} \right] \frac{dt}{\mu} \quad (0 < \mu \leq 1).
\]

In the case of the mathematical discussion of some physical process we must have recourse to the construction of simplified model of reality. In a statistical model the random process is described by a set of the probability distributions. The probabilistic theory of radiative transfer has the aim of determining the emergent intensity and the source function in the diffuse radiation field with the aid of the probability of the photon emergence. In the stationary case the optical depth parameter \( \tau \) is used in place of the time parameter in the usual stochastic process.

In our statistical model appropriate to radiative transfer and neutron diffusion in a plane-parallel medium, the probability distribution of emission \( \phi(\mu; \tau) \) is expressed in terms of a one-dimensional parameter \( \tau \) and a continuous random variable \( \mu(\tau) \) which denotes the cosine of the inclination to the outward normal, unless \( \mu \) is deterministically given. Mathematically, \( \phi(\mu; \tau) \ d\mu \) represents the probability of finding \( \mu \) in the range \( (\mu, \mu + d\mu) \) at level \( \tau \).

In the preceding paper [27], based on the stochastic model such that multiple scattering of a photon is of Markovian property, the exact solution of the equation of transfer in a semi-infinite atmosphere with isotropic scattering is provided by the reflectance integral transform (2.4) of any given distribution of the radiation sources acting in the medium. When the distribution is expressed in terms of an arbitrary polynomial of the optical depth, the exponential and the exponential integral, we can

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1 "R.T." will be used throughout paper to indicate reference to the book "Radiative Transfer" by S. Chandrasekhar (see [15]).
obtain explicitly the exact solution expressed in terms of the $H(\mu)$-function by means of the integral transformation. In more general case than the above it is required to compute the photon emergence probability in Eq. (3.4) with the aid of the iterative procedure. Mathematically, the photon diffusion equation (3.4) is equal to the auxiliary equation in the Ambarzumian technique (cf. Busbridge [12, 13]).

In the present paper, from the probabilistic viewpoint, we shall show a new derivation of Eq. (4.8) determining the resolvent. With the aid of invariant imbedding technique, we derive the stochastic integro-differential equation governing the probability distribution of emission. Furthermore, establishing the photon diffusion equation from the probabilistic viewpoint and combining the above two equations, after some arguments, we obtain the equation determining the resolvent in the photon diffusion equation. Then the resolvent is expressed in terms of the auxiliary function $\Psi(\tau)$ depending only upon one level argument. Furthermore, an exact analytical expression of $\Psi(\tau)$ in terms of $H(\mu)$ was found by Minin [15]. Then, the Milne first integral equation in the theory of radiative transfer and of neutron diffusion can be solved without actually referring to the iterative procedure.

II. THE MILNE FIRST INTEGRAL EQUATION AND THE REFLECTANCE OPERATOR

Let the source function be $J(\tau)$ which satisfies the Milne first integral equation with the known distribution of the emitting-sources $B(\tau)$

$$\left(1 - \bar{\sigma} \lambda\right) \{J(\tau')\} = B(\tau), \quad (2.1)$$

where 1 is an identity operator, $\bar{\sigma}$ is the albedo for single scattering, and $\lambda$ is the Hopf operator

$$\lambda \{J(\tau')\} = \frac{1}{2} \int_0^\infty f(\tau') E_1(|\tau' - \tau|) \, d\tau'. \quad (2.2)$$

In Eq. (2.2) $E_1$ is the first exponential integral

$$E_1(\tau) = \int_0^1 \exp\left(-\frac{\tau}{\mu}\right) \frac{d\mu}{\mu}. \quad (2.3)$$

Then, based on the probabilistic method, the emergent intensity is written in the form (cf. Ueno [27])

$$I(0, \mu) = \Re_{\mu}\{B(\tau')\}, \quad (2.4)$$
where the reflectance operator $\mathcal{R}$ is given by

$$\mathcal{R}_\mu \{ g(\tau') \} = \int_0^\infty g(\tau') \rho(\mu; \tau') \frac{d\tau'}{\mu}. \tag{2.5}$$

In Eq. (2.5) $\rho(\mu; \tau)$ represents the probability that a photon produced at the optical depth $\tau$ will reappear in the radiation emerging from the medium through the surface $\tau = 0$ at an angle $\cos^{-1} \mu$ ($0 < \mu \leq 1$) to the normal after one or more scattering processes. A formula similar to Eq. (2.5) has independently been found by Sobolev [21].

Then, with the aid of the probability distribution $\rho(\mu; \tau)$ which is determined by Eq. (4.1), the required angular distribution of the emergent radiation is given by Eq. (2.4), provided that $B(\tau)$ is the known function of the optical depth.

III. The Stochastic Integro-Differential Equation

Consider a semi-infinite, plane-parallel medium of constant optical properties scattering radiation isotropically. For simplicity we assume that it is illuminated by a parallel beam of radiation of a constant net flux incident on the surface $\tau = 0$.

The probability that a photon will suffer absorption in an infinitesimal interval of optical depth $d\tau$ is taken to be $(d\tau/\mu) + O(d\tau)$, and the probability that it can pass through an interval $d\tau$ without absorption is $1 - (d\tau/\mu) + O(d\tau)$.

A photon incident on the surface $\tau = 0$ will be called a source photon. In order to obtain the stochastic equation characterizing the emission probability, we shall consider the behavior of a source photon at the surface.

A source photon starting at $\tau = 0$ and traversing an optical depth interval $(0, \Delta\tau)$ without absorption will result in a reflected photon from the remaining medium whose surface is at the level $\Delta\tau$. The reflected photon is assumed to start at the level $\tau$ measured from the surface $\tau = \Delta\tau$. There is the eventuality that this photon may be absorbed in $(\Delta\tau, 0)$, or scattered forward in $(\Delta\tau, 0)$. Finally, a source photon absorbed in passing through $(0, \Delta\tau)$ may be re-emitted and will become a source photon for the remaining medium. Then it will be followed by reflection from the medium. It is easy to see that all other processes have probabilities of order of $\Delta\tau^2$ for small $\Delta\tau$. Hence, they can be neglected in the derivation of the stochastic equation. The enumeration of these foregoing possibilities will give rise to the emergence probability of a photon starting at the level $\tau + \Delta\tau$. 
This reasoning leads to the equation
\[
\rho(\mu; \tau + \Delta \tau) = \left(1 - \frac{\Delta \tau}{\mu}\right) \rho(\mu; \tau) \left(1 - \frac{\Delta \tau}{\mu}\right) + \frac{\Delta \tau}{2} \int_0^1 \rho(\mu'; \tau) \frac{d\mu'}{\mu'} \rho(\mu; \Delta \tau) + \frac{\Delta \tau}{\mu} \rho(\mu; \tau) + O(\Delta \tau). \tag{3.1}
\]

Passing to the limit as \(\Delta \tau \to 0\), we get the stochastic equation
\[
\frac{\partial \rho(\mu; \tau)}{\partial \tau} = -\frac{\rho(\mu; \tau)}{\mu} + \rho(\mu; 0) \Phi(\tau), \tag{3.2}
\]

where
\[
\Phi(\tau) = \frac{1}{2} \int_0^1 \rho(\mu'; \tau) \frac{d\mu'}{\mu'}. \tag{3.3}
\]

Furthermore, allowing for the probabilistic significance of \(\rho(\mu; \tau)\), we can formulate an alternative expression governing the photon emergence probability, i.e. the photon diffusion equation (cf. Ueno [26, 27]).

\[
\rho(\mu; \tau) = \mathcal{D} \exp \left(\frac{-\tau}{\mu}\right) + \mathcal{D} \mathcal{A}(\rho(\mu; \tau')), \tag{3.4}
\]

where the Hopf operator \(\mathcal{A}\) is given by Eq. (2.2).

In what follows we shall interpret probabilistically the linear integral equation (3.4). The probability distribution \(\rho(\mu; \tau)\) is composed of two components: the probability of the photon emergence without suffering any scattering process and that due to one or more scattering processes. The former is equal to the first exponential term on the right-hand side of equation (3.4). The latter is given by the Hopf operator which can be derived from the probabilistic consideration.

The probability that a photon produced at level \(\tau\) will be reabsorbed in the depth interval \((\tau', \tau + d\tau')\) in the direction \(\mu'\) is equivalent to \(\exp \left(- \frac{|\tau' - \tau|}{\mu'} \frac{d\tau'}{\mu'}\right)\). Hence the probability that the photon absorbed at level \(\tau\) will be reproduced at the depth from \(\tau'\) to \(\tau' + d\tau'\) is given by multiplying the above probability by \(\mathcal{D} d\mu'/2\) and then by integrating the product with respect to \(\mu'\) over \((0, 1)\). Multiplying the above integral by the probability \(\rho(\mu'; \tau')\) and integrating with respect to \(\tau'\) over \((0, \infty)\), we obtain the required probability of the photon
emergence from the medium through the surface \( \tau = 0 \) after successive scattering processes in the form

\[
\frac{\bar{\omega}}{2} \int_0^\infty \phi(\mu; \tau') d\tau' \left\{ \exp\left[ -\left( \frac{\tau' - \tau}{\mu'} \right) \right] d\mu'.
\]

which is equal to \( \bar{\omega}\Lambda_{\tau}\{\phi(\mu; \tau')\} \).

Equations (3.2) and (3.4) are equal to those given by Sobolev [21] and Miss Busbridge [12], respectively.

Making use of the iterative procedure, Eq. (3.4) becomes

\[
\phi(\mu; \tau) = \bar{\omega} \exp\left( \frac{\tau}{\mu} \right) + \bar{\omega} \sum_{i=1}^\infty P_i(\tau), \tag{3.5}
\]

where

\[
P_i(\tau) = \left( \frac{\bar{\omega}}{2} \right) \int_0^1 \cdots \int_0^1 \exp\left( -\frac{\tau^{(0)}}{\mu} \right) E_1(|\tau' - \tau|) \cdots E_1(|\tau^{(i)} - \tau^{(i-1)}|) d\tau' \cdots d\tau^{(i)}. \tag{3.6}
\]

In what follows, we shall show another derivation of the stochastic equation (3.2) in Markovian fashion.

In a manner similar to that used in the preceding paper [26], assuming the Markovian property of multiple scattering of a photon, we write the Chapman-Kolmogorov equation in the form

\[
\phi(\mu; \tau) = \int_0^1 \phi(\mu' \tau - \Delta \tau) \phi(\mu' \Delta \tau) d\mu', \tag{3.7}
\]

where

\[
\phi(\mu|\mu'; \Delta \tau) = R(\mu|\mu') \Delta \tau + \delta(\mu - \mu') \left\{ 1 - \frac{\Delta \tau}{\mu} \right\}. \tag{3.8}
\]

In Eq. (3.8) \( R(\mu|\mu') \) is given by

\[
R(\mu|\mu') = \frac{\phi(\mu; 0)}{2\mu'}, \tag{3.9}
\]

and \( \delta \) is the Dirac delta function.

In Eq. (3.9) \( R(\mu|\mu') \) can be interpreted as the conditional probability that if a change from the \( \mu' \) state occurs during \( (\tau - \Delta \tau, \tau) \), the change takes the state from \( \mu' \) to \( \mu \).

Then, letting \( \Delta \tau \to 0 \), we obtain the stochastic equation (3.2).
IV. THE RESOLVENT IN PHOTON DIFFUSION EQUATION

In a manner similar to that used in the formulation of Eq. (3.4), we get

\[ p(\mu; \tau) = \bar{\omega} \exp\left(\frac{-\tau}{\mu}\right) + \bar{\omega} \int_0^\infty K(\tau, \tau') \exp\left(\frac{-\tau'}{\mu}\right) d\tau', \quad (4.1) \]

where the \( K \)-function is the resolvent in the photon diffusion equation governing the emission probability. While the first term on the right-hand side of Eq. (4.1) represents the probability of the photon emergence without scattering, the second integral term is due to the successive scattering processes. In Eq. (4.1) the quantity \( K(t, t') \, dt \, dt' \) represents the probability that a photon absorbed in an interval \((\tau, \tau + dt)\) will subsequently be emitted in an interval \((\tau', \tau' + d\tau')\) after one or more scattering processes.

In what follows, we shall derive the equation of the resolvent from the stochastic equation (3.2).

Differentiating Eq. (4.1) with respect to \( \tau \), we have

\[ \frac{\partial p(\mu; \tau)}{\partial \tau} = -\frac{\bar{\omega}}{\mu} \exp\left(-\frac{\tau}{\mu}\right) + \bar{\omega} \int_0^\infty \frac{\partial K(\tau, \tau')}{\partial \tau} \exp\left(\frac{-\tau'}{\mu}\right) d\tau'. \quad (4.2) \]

On substituting from Eq. (4.1) into Eq. (3.2), we get

\[ \frac{\partial p(\mu; \tau)}{\partial \tau} = -\frac{\bar{\omega}}{\mu} \left[ \exp\left(\frac{-\tau}{\mu}\right) + \int_0^\infty K(\tau, \tau') \exp\left(\frac{-\tau'}{\mu}\right) d\tau' \right] \]

\[ + \bar{\omega} \left[ 1 + \int_0^\infty K(0, \tau') \exp\left(\frac{-\tau'}{\mu}\right) d\tau' \right] \Phi(\tau). \]

Then, combining Eq. (4.2) with Eq. (4.3) and assuming \( \Phi(\tau) \) equal to \( K(\tau, 0) \), we obtain

\[ \int_0^\infty F(\tau, \tau') \exp\left(\frac{-\tau'}{\mu}\right) d\tau' = 0, \quad (4.4) \]

where

\[ F(\tau, \tau') = \frac{\partial K(\tau, \tau')}{\partial \tau} + \frac{\partial K(\tau, \tau')}{\partial \tau'} - K(\tau, 0)K(0, \tau'). \quad (4.5) \]
Writing

\[ K(0, \tau') = \Psi(\tau'), \quad (4.6) \]

and assuming the symmetry of \( K \)-function with respect to \( \tau \) and \( \tau' \), from Eq. (4.4) we have

\[ \frac{\partial K(\tau, \tau')}{\partial \tau} + \frac{\partial K(\tau, \tau')}{\partial \tau'} = \Psi(\tau)\Psi(\tau'). \quad (4.7) \]

Equation (4.7) at \( \tau' > \tau \) yields

\[ K(\tau, \tau') = \Psi(\tau' - \tau) + \int_0^\tau \Psi(t)\Psi(t + \tau' - \tau) \, dt. \quad (4.8) \]

Equations (4.7) and (4.8) are equal to those given by Sobolev [23, 24].

V. THE PROOF OF IDENTITY \( \Phi(\tau) = \Psi(\tau) \)

Writing

\[ \phi(\mu; 0) = \omega H(\mu), \quad (5.1) \]

where \( H(\mu) \) is the \( H \)-function of Chandrasekhar (cf. [15]),

\[ H(\mu) = 1 + \frac{1}{2} \omega H(\mu) \int_0^1 \frac{H(\mu')}{\mu + \mu'} \, d\mu'. \quad (5.2) \]

and putting \( \tau = 0 \) in Eq. (5.1), we obtain

\[ \int_0^\infty \Psi(\tau') \exp \left( -\frac{\tau'}{\mu} \right) \, d\tau' = H(\mu) - 1. \quad (5.3) \]

On the other hand, an alternative expression of \( \phi(\mu; \tau) \) in Eq. (3.4) is (cf. Ueno [26, 27]),

\[ \phi(\mu; \tau) = \phi(\mu; 0) \exp \left( -\frac{\tau}{\mu} \right) \quad (5.4) \]

\[ + \frac{1}{2} \phi(\mu; 0) \int_0^\tau d\tau' \int_0^1 \phi(\mu'; \tau') \exp \left[ -\frac{(\tau - \tau')}{\mu} \right] \frac{d\mu'}{\mu'}. \]
In a manner similar to that used in the theory of random process of Markovian type, the probability distribution of emission can be accomplished either by simply remaining at the state \( \mu \) without suffering any scattering process in the depth interval \( \tau \) or by starting from the state \( \mu' \) at depth \( \tau' \), the depth of the last jump, jumping to a state \( \mu \) and remaining at the state through \( \tau - \tau' \) depth unit \( (0 \leq \tau' \leq \tau) \). Hence, multiplying the scattering component by \( \dot{p}(\mu; 0) \) and integrating successively with respect to \( \mu' \) over \((0, 1)\) and to \( \tau' \) over \((0, \tau)\), we get Eq. (5.4).

On multiplying Eq. (5.4) by \( 1/(2\mu) \), integrating with respect to \( \mu \) over \((0, 1)\), and using Eq. (3.3), we get

\[
\Phi(\tau) = \phi(\tau) + \int_{0}^{\tau} \phi(\tau - \tau')\Phi(\tau') \, d\tau',
\]

where

\[
\phi(\tau) = \frac{1}{2} \int_{0}^{1} p(\mu; 0) \exp\left(\frac{-\tau}{\mu}\right) \frac{d\mu}{\mu}.
\]

Making use of the Laplace transform of Eq. (5.5) with respect to \( 1/\lambda \), and recalling Eq. (5.2), we obtain (after inverting the order of the integrations and integrating the second term by parts)

\[
\int_{0}^{\infty} \Phi(\tau) \exp\left(-\frac{\tau}{\mu}\right) \, d\tau = \frac{\tilde{\omega}}{2} \int_{0}^{\infty} \exp\left(-\frac{\tau}{\mu}\right) \, d\tau \int_{0}^{1} H(\mu') \exp\left(-\frac{\tau}{\mu'}\right) \frac{d\mu'}{\mu'}
\]

\[
+ \frac{\tilde{\omega}}{2} \int_{0}^{1} H(\mu') \frac{d\mu'}{\mu + \mu'} \int_{0}^{\infty} \exp\left(-\left(\frac{1}{\mu} + \frac{1}{\mu'}\right)\tau\right) \, d\tau \int_{0}^{\tau} \Phi(\tau') \exp\left(\frac{\tau'}{\mu'}\right) \, d\tau'
\]

\[
= \frac{\tilde{\omega}}{2} \mu \int_{0}^{1} H(\mu') \frac{d\mu'}{\mu + \mu'} + \frac{\tilde{\omega}}{2} \mu \int_{0}^{1} H(\mu') \frac{d\mu'}{\mu + \mu'} \int_{0}^{\infty} \Phi(\tau) \exp\left(-\frac{\tau}{\mu}\right) \, d\tau.
\]

Then

\[
\int_{0}^{\infty} \Phi(\tau) \exp\left(-\frac{\tau}{\mu}\right) \, d\tau = \left[1 - \frac{1}{2} \tilde{\omega} \mu \int_{0}^{1} H(\mu') \frac{d\mu'}{\mu + \mu'}\right]^{-1} - 1.
\]

Recalling Eq. (5.3), from Eq. (5.8) we have

\[
\Phi(\tau) = \Psi(\tau).
\]
It is of interest to mention that, on multiplying Eq. (3.4) by \(1/(2\mu)\) and integrating with respect to \(\mu\) over \((0, 1)\), we get

\[
\Phi(\tau) = \frac{1}{2} \tilde{w} E_1(\tau) + \frac{1}{2} \tilde{w} \int_0^\infty \Phi(\tau') E_1(|\tau' - \tau|) \, d\tau'.
\] (5.10)

The inversion of Laplace transform (5.8) leads to the determination of \(\Phi(\tau)\) (cf. Minin [16]). Then, the resolvent is given by Eq. (4.8).

Finally, we wish to express our cordial gratitude to Dr. R. Bellman for his kind advice and valuable suggestion on the subject of the present paper.

**Note added in proof:** While the proof of identity \(\Phi(\tau) = \Psi(\tau)\) in Section V is done with the aid of a stochastic type of integral equation (5.4), another straightforward proof is given by means of Eqs. (3.3), (5.2) and (5.3), allowing for the scattering functions \(S(\mu, \nu_0)\) defined by

\[
S(\mu, \nu_0) = \int_0^\infty f(\mu_0; \tau) e^{-\tau/\mu} \, d\tau = \frac{\tilde{w}_0^\nu \mu_0}{\mu + \mu_0} H(\mu) H(\nu_0).
\]

We are indebted to Dr. Ida W. Busbridge for the above concise derivation.

**References**


