Explicit methods for fractional differential equations and their stability properties

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1. Introduction

In the field of numerical treatment of differential equations of fractional order (FDEs) great attention has been recently dedicated to the development and the analysis of efficient and accurate methods, mostly of implicit type (e.g., see [2,4,8,10,14]).

As with ordinary differential equations (ODEs), there are several situations in which the use of implicit methods causes a huge amount of computation (e.g. with systems of very large dimension, in presence of strong nonlinearity or when the state–function is not well–known analytically and can be only evaluated at some points on a grid).

In order to deal with such difficulties, explicit methods for FDEs [9] and for partial differential equations of fractional order [11,18] have been introduced and some of their properties investigated.

The major drawback of explicit methods concerns their stability: for safely exploiting the advantages and potential of explicit schemes, stability properties have to be analyzed in depth and this is an area not widely investigated as yet.

The main aim of this paper is to study stability properties of some explicit methods for FDEs and introduce new schemes with better stability properties, so that numerical simulation can be carried out at reduced costs.

To this purpose we consider a FDE in the form

$$D^\beta y(t) = f(t, y(t))$$

where $f : [t_0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a sufficiently smooth function and $0 < \beta < 1$ is the fractional order. In the past some different definitions were introduced for the differential operator $D^\beta$ of fractional order $\beta$. From a theoretical point of view the most...
natural approach is the Riemann–Liouville definition \([16]\), with respect to the lower terminal \(t = t_0\), given by
\[
D^\beta y(t) \equiv \left. \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_{t_0}^{t} \frac{y(u)}{(t-u)^\beta} du \right|_{t_0}^t.
\]
(2)

The above definition is very important from an historical point of view since it allows the development of most of the theory of fractional derivatives. Nevertheless, in real applications this approach loses its usefulness since it enables the definition of initial conditions only as the limit value \(\lim_{t \to t_0} D^\beta y(t) = y_0\), without a clear physical meaning. For this reason we will refer to the alternative Caputo’s definition \([17]\)
\[
D^\beta y(t) \equiv \frac{C}{\Gamma(1-\beta)} \int_{t_0}^{t} \frac{y(u)}{(t-u)^\beta} du,
\]
which allows us to couple (1) with initial conditions in the traditional form \(y(t_0) = y_0\). By referring to the Caputo’s approach, problem (1) can be rewritten \([13]\) in form of the equivalent Volterra integral equation
\[
y(t) = y_0 + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} f(s, y(s)) ds.
\]
(3)

In the last decades dynamical systems modelled by FDEs have been investigated in several areas such as viscoelasticity, control theory, optics, finance, combustion, etc. (see \([17]\) and references therein). A well–known example is the propagation of a spherical flame initiated by a point source (derived by Joulin \([12]\) and successively studied in \([5,6]\)), which model is represented by means of the FDE
\[
D^{\beta-\gamma} y(t) = \log y(t) + \frac{E(t)(1-t)H(1-t)}{y(t)},
\]
where \(\gamma\) and \(E\) are positive constants and \(H(\theta)\) the Heaviside function.

This paper is organized as follows. In Section 2 a class of multistep methods for FDEs is introduced and order conditions are investigated. In Section 3 stability is investigated and a formula for the region of stability of methods under investigation is obtained, thanks to which in Section 4 the interval of stability of some existing explicit methods is studied. In Section 5 we derive new methods of first and second order with interval of stability larger than those of existing methods. Finally, in Section 6 results from previous sections are validated by means of some numerical experiments.

2. Fractional and \(\rho\)-fractional linear multistep methods

In order to solve numerically FDEs, in \([14]\) fractional linear multistep methods (FLMMs) were introduced and consistence, convergence and stability properties were investigated. As a main result an extension to the fractional case of the Dahlquist’s convergence theorem was proved. FLMMs are based on the approximation of the integral in (3) by means of the discrete convolution quadrature
\[
\frac{1}{\Gamma(\beta)} \int_{t_0}^{t_n} (t_n-s)^{\beta-1} f(s) ds \approx h^\beta \sum_{j=0}^{n} \omega_j z_{n-j} + h^\beta \sum_{j=0}^{n} w_{n,j} z_j,
\]
where weights \(\omega_j\) are the coefficients of \(\zeta^j\) in the expansion of a given generating function \(\omega(\zeta)\), generally depending on \(\beta\), and \(w_{n,j}\) are starting quadrature weights introduced in order to deal with the asymptotic behaviour of the true solution near the origin (since starting weights do not affect stability properties, which is the main subject of this paper, in the following we will not consider them).

The main advantage of this approach is that methods for FDEs can be obtained as a generalization of classical multistep methods for ODEs. In fact, by considering the first and second characteristic polynomials \(\rho(\zeta)\) and \(\sigma(\zeta)\) of a generic multistep method, a generating function \(\omega(\zeta)\) for obtaining a FLMM with the same convergence properties of the underlying multistep method \((\rho, \sigma)\) is given by
\[
\omega(\zeta) = \left(\frac{\sigma(1/\zeta)}{\rho(1/\zeta)}\right)^\beta.
\]

For example, when \(\rho(\zeta) = 1 - \zeta\) and \(\sigma(\zeta) = 1\), the fractional backward differentiation formula (FBDF) of order 1 is obtained and weights \(\omega_j\) can be expressed as
\[
\omega_j = (-1)^j \binom{\beta}{j} = \frac{\Gamma(j-\beta)}{\Gamma(-\beta) \Gamma(j+1)}, \quad j = 0, 1, 2, \ldots,
\]
(4)
and can be recursively evaluated as
\[
\omega_0 = 1, \quad \omega_j = \left(1 - \frac{\beta + 1}{j}\right) \omega_{j-1}, \quad j = 1, 2, \ldots
\]
(5)
Unlike the above example, for most of the generating functions, a directly evaluable form for weights $\omega_j$ is not available and some more sophisticated and expensive techniques have to be used in order to determine weights in FLMMs.

As a consequence, even though generating functions for FBDFs of various order $p$, $1 \leq p \leq 6$, are known [14], only for the first order FBDF are weights known explicitly as coefficients (4) in the power series of the generating function $\omega(\xi) = (1 - \xi)^{-\beta}$. Moreover, by writing FLMMs in a more general formulation

$$\sum_{j=0}^{n} \alpha_j y_{n-j} = h^p \sum_{j=0}^{p} \gamma_j f(t_{n-j}, y_{n-j}),$$

weights of the fractional trapezoidal rule [14], based on the generating function $\omega(\xi) = \left(\frac{\xi}{1 + \xi}/1 - \xi\right)^{\beta}$, can be directly evaluated as $\alpha_j = \omega_j$ and $\gamma_j = (-1)\omega_j, j = 0, 1, \ldots$, where $\omega_j$ are the same introduced in (4). Even though this is a second order method with directly evaluable weights and good stability properties, the fractional trapezoidal rule has not been widely adopted in applications and we were only able to find a few examples of its application in the literature [1].

In order to reduce computational costs and, at the same time, make easier the development of new methods with directly evaluable weights, in [10] the following class of methods has been proposed

$$\sum_{j=0}^{n} \alpha_j y_{n-j} = h^p \sum_{j=0}^{p} \gamma_j f(t_{n-j}, y_{n-j}),$$

where $p$ is an integer fixed a priori. Since only the last previous $p$ values $f(t_{n-j}, y_{n-j}), j = 1, \ldots, p$, of the state–function $f(\cdot, \cdot)$ are used in the evaluation of each approximation $y_n$, methods (7) were named $p$-fractional linear multistep methods ($p$-FLMMs).

Several choices can be proposed for coefficients $\alpha_j$ and $\gamma_j$ and formula (7) allows generalization of some existing methods [9]. Among them, for $p = 1$ the backward fractional Euler method [14,17] obtained by choosing for each $n \geq 1$

$$\begin{cases}
\alpha_j = \omega_j, & j = 0, 1, \ldots, n - 1, \\
\alpha_n = -\sum_{j=0}^{n-1} \omega_j, \\
\gamma_0 = 1, & \gamma_1 = 0,
\end{cases}$$

where $\omega_j$ are those defined in (4), or the method of order $2 - \beta$ proposed by Diethelm [2] with

$$\begin{cases}
\alpha_0 = 1, \\
\alpha_j = (j + 1)^{1-\beta} - 2j^{1-\beta} + (j - 1)^{1-\beta}, & j = 1, 2, \ldots, n - 1, \\
\alpha_n = n^{1-\beta} - (n - 1)^{1-\beta}, \\
\gamma_0 = 1, & \gamma_1 = 0.
\end{cases}$$

In this paper we will restrict our attention to $p$-FLMMs in which weights $\alpha_j$ are selected as in the backward fractional Euler method and weights $\gamma_j$ are chosen in a suitable way in order to fulfill some consistency and stability conditions. Therefore, in the following we will consider $p$-FLMMs in the form

$$\sum_{j=0}^{n} \omega_j y_{n-j} = b_n y_0 + h^p \sum_{j=0}^{p} \gamma_j f(t_{n-j}, y_{n-j}),$$

where $b_n = \sum_{j=0}^{n-1} \omega_j$. Moreover, an explicit representation of $b_n$ is available [16] as

$$b_n = \frac{-\beta \Gamma(n - \beta)}{\Gamma(2 - \beta) \Gamma(n - 1)}, \quad n \geq 1.$$

Consistency of $p$-FLMMs can be studied by introducing, associated to (7), the linear difference operator

$$\mathcal{L}_b[y(t), t, \beta] = \sum_{j=0}^{n} \alpha_j y(t - hj) - h^p \sum_{j=0}^{p} \gamma_j \nu_0 D^\beta y(t - hj),$$

where $y(t)$ is a sufficiently smooth function, and expanding at $t = t_n$ the true solution of (1) and its $\beta$-derivative in order to express $\mathcal{L}_b[y(t_n), t_n, \beta]$ as [10]

$$\mathcal{L}_b[y(t_n), t_n, \beta] = C_0(n, \beta)y(t_0) + \sum_{k=1}^{m} h^k C_k(n, \beta)y^{(k)}(t_0) + h^{m+1} R_{m+1},$$

where the remainder $R_{m+1}$ originates from the Taylor’s expansions of $y(t_n - jh)$ and $\nu_0 D^\beta y(t_n - jh)$ and coefficients $C_k(n, \beta)$ are given by

$$C_0(n, \beta) = \sum_{j=0}^{n} \alpha_j$$
and for $k = 1, 2, \ldots, m$

$$C_k(n, \beta) = \frac{1}{k!} \sum_{j=0}^{n} (n-j)^k \alpha_j - \frac{1}{\Gamma(k+1-\beta)} \sum_{j=0}^{p} y_j (n-j)^{k-\beta}. \quad (11)$$

It is easy to see that for method (8) it always holds that $C_0(n, \beta) \equiv 0$ for any $n$.

Thanks to the above expansion for $L_\lambda(y(t_n), t_n, \beta)$ the following characterization for the order of consistence for $p$-FLMMs has been given [9].

**Proposition 1.** A $p$-FLMM is of order $q$ when $C_k(n, \beta) = \Theta(n^q)$, as $n \to \infty$, with $q_k \leq k - \beta - q, k = 0, 1, \ldots$.

The following theorem, for the proof of which we refer to [10], allows us to establish an order condition for $p$-FLMMs (8).

**Theorem 2.** The $p$-FLMM (8) is consistent of order 1 when

$$\sum_{j=0}^{k} y_j = 1 \quad (12)$$

and it is consistent of order 2 when (12) holds together with

$$\sum_{j=0}^{p} y_j = \frac{\beta}{2} \quad (13)$$

By imposing conditions (12) and (13), in [10] the following 1-FLMM of order 2 has been proposed

$$\sum_{j=0}^{n-1} \omega_j y_{n-j} - b_j y_0 = h^{\beta} \left(1 - \frac{\beta}{2}\right) f(t_n, y_n) + h^{\beta} \frac{\beta}{2} f(t_{n-1}, y_{n-1}), \quad (14)$$

for which $A$–stability has also been proved. However, with respect to the fractional trapezoidal rule [14], which is $A$–stable and second order too, the proposed method has the advantage of reducing computational costs and the storage needs due to the limited number of addends in the right term. In [10] some numerical experiments have been done for testing performance of this method and comparing it with competitor methods.

**Remark 3.** From (5) observe that, by putting $\beta = 1$, it is $\omega_0 = 1, \omega_1 = -1, \omega_j = 0, j = 2, 3, \ldots$ and $b_n = 0, n \geq 2$. In this way methods for FDEs can be continued to the limiting case $\beta = 1$, corresponding to standard ODEs. In the following we will make use of this continuation in order to compare methods investigated in this paper with their counterparts for ODEs. We can see immediately that formula (14) returns to the trapezoidal rule when $\beta = 1$.

### 3. Analysis of stability

An important aspect in the numerical approximation of differential equations is related to the study of stability properties. To this purpose we introduce the linear test equation for FDEs

$$t_0 ^{D^\beta} y(t) = \lambda y(t), \quad y(t_0) = y_0, \quad \lambda \in \mathbb{C}, \quad 0 < \beta < 1. \quad (15)$$

A well–known result [17] states that the exact solution of (15) can be expressed in terms of the Mittag–Leffler function

$$E_\beta(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\beta k + 1)} \text{ as } y(t) = E_\beta(\lambda t - t_0) y_0.$$

When $\Re(\lambda) < 0$, the solution of (15) asymptotically tends to 0 as $t \to \infty$. A numerical method is said to be asymptotically stable when the same behaviour exists in the numerical solution. In order to investigate stability properties of $p$-FLMMs we first introduce the Z-transform and we recall some useful results [7].

**Definition 4.** Given a sequence $\{x_n\}_{n\in\mathbb{N}}$ with radius of convergence $R$, the function

$$Z(x_n) \equiv \bar{x}(z) = \sum_{n=0}^{\infty} x_n z^{-n}, \quad z \in \mathbb{C}, \quad |z| > R$$

called the Z-transform of $\{x_n\}_{n\in\mathbb{N}}$.

**Proposition 5.** Given any two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$, with respective Z-transforms $\bar{x}(z)$ and $\bar{y}(z)$, the following properties hold

1. $Z(\{x_n\}_{n} \ast \{y_n\}_{n}) = \bar{x}(z) \cdot \bar{y}(z)$
2. $Z(x_{n+1}) = z \bar{x}(z) - z \bar{x}_0$, $|z| > R$
3. $\lim_{n \to \infty} x_n = \lim_{n \to \infty} (z - 1) \bar{x}(z)$

where $\{x_n\}_{n} \ast \{y_n\}_{n}$ denotes the convolution $\sum_{j=0}^{n} x_j y_{n-j}$ and $R$ is the radius of convergence of $\{x_n\}_{n\in\mathbb{N}}$.

We are now able to give the following result concerning stability of $p$-FLMMs.
Proposition 6. Let \( 0 < \beta < 1 \), \( \gamma(\xi) = \sum_{j=0}^{\infty} \gamma_j \xi^j \) and denote with \( \{ y_n \} \) the numerical solution of (15) obtained by means of the \( p \)-FLMM (8). If
\[
\left( 1 - \frac{1}{z} \right)^\beta - h^\beta \lambda \left( \frac{1}{z} \right) \neq 0, \quad z \in \mathbb{C}, \quad |z| \geq 1
\]
then \( \lim_{n \to \infty} y_n = 0 \).

Proof. By applying method (8) to the test equation (15), and by adding and subtracting \( \omega_0 y_0 \), we are able to write
\[
\sum_{j=0}^{n} \omega_j y_{n-j} - b_{n+1} y_0 = h^\beta \lambda \sum_{j=0}^{n} \gamma_j y_{n-j},
\]
where we put \( \gamma_j = 0 \) for \( j > p \). Observe now that, thanks to (1) and (2) in Proposition 5, by putting \( b_0 = 0 \) we can write
\[
\tilde{\omega}(z)\tilde{y}(z) - b\tilde{\omega}(z) y_0 = h^\beta \lambda \tilde{\gamma}(z)\tilde{y}(z), \quad |z| > 1,
\]
where \( \tilde{\omega}(z) \), \( \tilde{y}(z) \) and \( b \) are Z-transforms of \( \{ \omega_n \} \), \( \{ y_n \} \) and \( \{ b_n \} \) respectively and, moreover, it is \( \tilde{\omega}(z) = \left( 1 - \frac{1}{z} \right)^\beta \), \( |z| \geq 1 \), \( \tilde{\gamma}(z) = \gamma \left( \frac{1}{z} \right) \) and \( b = \left( 1 - \frac{1}{z} \right)^{\beta - 1} \), \( |z| \geq 1, z \neq 1 \).

By observing the regions of convergence of \( \tilde{\omega}(z) \) and \( b(z) \), we note that equivalence (17) holds also for \( |z| \geq 1, z \neq 1 \). If we assume now (16), we have
\[
\tilde{y}(z) = \frac{z \tilde{b}(z)}{\tilde{\omega}(z) - h^\beta \lambda \tilde{\gamma}(z)} y_0, \quad |z| \geq 1, \quad z \neq 1
\]
and, thanks to part (3) in Proposition 5, we obtain
\[
\lim_{n \to \infty} y_n = \lim_{z \to 1} (z-1)\tilde{y}(z) = \lim_{z \to 1} \frac{(z-1)^\beta z^{2-\beta}}{\tilde{\omega}(z) - h^\beta \lambda \tilde{\gamma}(z)} y_0 = 0
\]
and the thesis can be continued to the case \( z = 1 \). \( \Box \)

By putting \( \xi = \frac{1}{z} \), hypothesis (16) can be reformulated by requiring that \( h^\beta \lambda \) lies in the set
\[
S = \mathbb{C} \setminus \left\{ \frac{(1-\xi)^\beta}{\gamma(\xi)} : |\xi| \leq 1 \right\}, \quad (18)
\]
coherently with the main result on stability regions proved in [15].

4. Explicit \( p \)-fractional linear multistep methods

In [9,10] some explicit \( p \)-FLMMs have been proposed and their order and stability properties have been investigated. Particularly, the explicit 1-FLMM of order 1
\[
\sum_{j=0}^{n-1} \omega_j y_{n-j} - b_n y_0 = h^\beta f(t_{n-1}, y_{n-1}) \quad (19)
\]
and the explicit 2-FLMM of order 2
\[
\sum_{j=0}^{n-1} \omega_j y_{n-j} - b_n y_0 = h^\beta \left[ \left( 2 - \frac{\beta}{2} \right) f(t_{n-1}, y_{n-1}) + \left( \frac{\beta}{2} - 1 \right) f(t_{n-2}, y_{n-2}) \right] \quad (20)
\]
have been studied. These methods are order–optimal in the sense that they are explicit \( p \)-FLMMs of order \( p \) with the minimum number of steps \( p \).

By means of (18) plots of stability regions can be drawn, as shown in Figs. 1 and 2, where the case \( \beta = 1 \) is obtained according Remark 3 (note that methods (19) and (20) correspond, for \( \beta = 1 \), to the explicit Euler and 2-step Adams–Bashforth methods respectively).

From a practical point of view, and in order to make a comparison of numerical methods, we restrict the analysis to test problems (15) in which \( \lambda \) is real and we investigate the point \( \alpha, -\infty < \alpha < 0 \), in which the boundary of the stability region \( S \) cuts the real axis. The interval \([-\alpha, 0]\) is usually called the interval of absolute stability.

In order to simplify the way in which intervals of stability are studied and completely move our analysis from the complex to the real field, we first see the following result.
**Lemma 7.** Let $0 < \beta < 1$ and denote with $g(\xi)$ the function

$$ g(\xi) = \frac{(1 - \xi)^{\beta}}{\gamma(\xi)}, \quad \text{with } \xi \in \mathbb{C} \text{ such that } |\xi| \leq 1, $$

where $\gamma(\xi) = \xi(1 - \gamma_1) + \gamma_1 \xi^2$ and $\gamma_1 \in [0, \frac{\beta}{2} - 1]$.

Then $\Re(\xi) = 0$ and $\Im(\xi) < 0$ (respectively $\Re(\xi) > 0$) iff $\Im(g(\xi)) = 0$ and $\Re(g(\xi)) < 0$ (respectively $\Re(g(\xi)) > 0$).

**Proof.** The first implication is trivial. For the converse, first consider $\Re(g(\xi)) < 0$ and by absurdity, assume the existence of $|\xi| \leq 1$ such that $\Im(g(\xi)) = 0$.

By writing $g(\xi) = (1 - \xi)^{\beta-1} \frac{1}{\gamma(\xi)}$, we are able to note that

$$ \arg(g(\xi)) = (\beta - 1) \arg(1 - \xi) - \arg \left( \frac{\gamma(\xi)}{1 - \xi} \right). $$

Moreover, since $|\xi| \leq 1$, it is $\arg(1 - \xi) \in [-\frac{\pi}{2}, 0)$, and as a consequence of $\Im(g(\xi)) = 0$ with $\Re(g(\xi)) < 0$, it is $\arg(g(\xi)) = \pi$.

Hence

$$ \arg \left( \frac{\gamma(\xi)}{1 - \xi} \right) = (\beta - 1) \arg(1 - \xi) - \pi \in \left[ -\pi, -\frac{1 + \beta}{2} \pi \right]. $$

Write now $\xi = x + iy$, $\rho = |\xi|$ and observe that $\Im \left( \frac{\gamma(\xi)}{1 - \xi} \right) = \left( \frac{1}{(1-x)^2 + y^2} - \gamma_1 \right) y > 0$ and hence $\arg \left( \frac{\gamma(\xi)}{1 - \xi} \right) \in [0, \pi)$, which contradicts (22).

In a similar way we can see that an analogous contradiction is obtained by assuming $\Im(\xi) < 0$ and hence from $\Im(g(\xi)) = 0$ it follows $\Im(\xi) = 0$.  

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**Fig. 1.** Plots of stability regions of the explicit 1-FLMM (19) of order 1.

**Fig. 2.** Plots of stability regions of the explicit 2-FLMM (20) of order 2.
As a consequence \( \arg(1 - \xi) = 0 \) and from (22) it follows \( \arg\left(\frac{\gamma(\xi)}{1 - \xi}\right) = -\pi \). Hence \( \Re\left(\frac{\gamma(\xi)}{1 - \xi}\right) = -\gamma x + \frac{x^2}{(1 - x)^2} < 0 \) from which, after some simple computation, we readily see that \( \Re(\xi) \leq 0 \).

The case \( \Re(\xi) > 0 \) is proved by observing that \( \arg(g(\xi)) = \beta \arg(1 - \xi) - \arg(\gamma(\xi)) \) and using arguments similar to those used for \( \Re(g(\xi)) < 0 \). \( \square \)

**Proposition 8.** The interval of stability of the 1-FLMM (19) of order 1 is given by \( -2^\beta < h^\beta \lambda < 0 \).

**Proof.** By means of Lemma 7, the interval of stability of the method can be studied by considering the stability function \( g(x) = (1 - x)^\beta \) as a real-valued function with real argument \( -1 \leq x \leq 1 \). To this purpose first note that \( g(x) \) has a pole in \( x = 0 \), with

\[
\lim_{x \to 0^-} g(x) = -\infty, \quad \lim_{x \to 0^+} g(x) = +\infty,
\]

and \( g(-1) = -2^\beta \) and \( g(1) = 0 \). Consider now the first derivative of \( g(x) \) with respect to \( x \)

\[
\frac{d}{dx}g(x) = \frac{-\beta(1-x)^{\beta-1} - (1-x)^\beta}{x^2} = \frac{-\beta + (\beta - 1)x}{x^2}
\]

which is always nonpositive as \( -1 \leq x \leq 1 \) and \( x \neq 0 \). Hence \( g(x) \) is a monotonically decreasing function both in \([-1, 0)\) and \((0, 1]\) and therefore \( g(x) \in (-\infty, -2^\beta] \) as \( x \in [-1, 0) \) and \( g(x) \in [0, +\infty) \) as \( x \in (0, 1] \) from which the thesis immediately follows. \( \square \)

**Proposition 9.** The interval of stability of the 2-FLMM (20) of order 2 is given by \( -\frac{2^\beta}{3^\beta} < h^\beta \lambda < 0 \).

**Proof.** Thanks to Lemma 7 we restrict ourselves to investigating the real-valued function \( g(x) = \frac{2(1-x)^\beta}{x(4-\beta) + (\beta - 2)x} \) with real argument \( x \), which for \( x \in [-1, 1] \) has a unique pole in \( x = 0 \) and

\[
\lim_{x \to 0^-} g(x) = -\infty, \quad \lim_{x \to 0^+} g(x) = +\infty.
\]

Hence consider the first derivative of \( g(x) \) and observe that

\[
g'(x) = \frac{-2(1-x)^{\beta-1}((2-\beta)x^2 - (2-\beta)2x + (4-3\beta)x + (4-\beta))}{x^2((4-\beta) + (\beta - 2)x)^2} \leq 0
\]

for \( x \in [-1, 1], x \neq 0 \). Therefore, since \( g(1) = 0 \), it is \( g(x) \in [0, +\infty) \) for any \( x \in (0, 1] \). Moreover, since \( g(-1) = -\frac{2^\beta}{3^\beta} \), for \( x \in [-1, 0) \) we have \( g(x) \in (-\infty, -\frac{2^\beta}{3^\beta}] \) which concludes the thesis. \( \square \)

Propositions 8 and 9 allow us to determine bounds for \( h^\beta \lambda \) in order to the numerical solution of the linear test problem (15) asymptotically vanishes.

When problems with a low value of the fractional order \( \beta \) are dealt with, these bounds can involve severe restrictions on the step-size \( h \). In Tables 1 and 2 stability limits for \( h = \sqrt[3]{\alpha} \) have been evaluated and collected, where \( \alpha \) is the lower bound of the interval of stability.

### 5. Methods with improved intervals of stability

Results in Tables 1 and 2 indicate that explicit methods investigated in the previous section can be safely used with non-stiff, or moderately stiff problems, only for values of \( \beta \) quite close to 1.

In order to numerically solve problems with lower values of \( \beta \), new methods with a larger interval of stability must be developed.

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**Table 1**

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To this purpose we consider the family of generic 2-FLMMs of order 1 and, by imposing the first order condition \((12)\), it is easy to see that the function \(\gamma(\xi)\) can be written, in terms of a real parameter \(\phi\), as \(\gamma(\xi) = (1 - \phi) \xi + \phi \xi^2\).

Our aim is to detect a proper value for \(\phi\) in order to maximize the interval of stability. To this purpose we consider the real-valued function, as \(x \neq 0\),

\[
g(x, \phi) = \frac{(1 - x)^\beta}{T(x, \phi)}
\]

with real arguments \(x\) and \(\phi\). When \(\phi < \frac{1}{3}\), observe that for \(x > 0\) it is \(g(x, \phi) > 0\). Since for determining an interval of stability we are interested in negative values of \(g(x, \phi)\), we can restrict the analysis of the function \(g(x, \phi)\) to \(x \in [-1, 0)\).

**Proposition 10.** Let \(0 < \beta < 1\) and \(0 \leq \phi \leq \frac{2 - \beta}{2(3 - \beta)}\). The function \(g(x, \phi)\) defined in \((23)\) monotonically decreases with respect to \(x\) in \([-1, 0)\) with \(-\infty < g(x, \phi) \leq g(-1, \phi) = \frac{2\beta}{2(3 - \beta)}\) and the minimum value for \(g(-1, \phi)\), as \(\phi\) varies in \([0, \frac{2 - \beta}{2(3 - \beta)}]\), is reached at \(\phi = \frac{2 - \beta}{2(3 - \beta)}\).

**Proof.** As \(0 \leq \phi < \frac{1}{3}\) we note that \(g(x, \phi)\) has a unique pole at \(x = 0\) with \(\lim_{x \to 0^+} g(x, \phi) = -\infty\). Furthermore

\[
dx g(x, \phi) = - \frac{(1 - x)^{\beta - 1}}{(1 - \phi) x + \phi x^2} T(x, \phi),
\]

where \(T(x, \phi) = -\phi (2 - \beta) x^2 + ((3 - \beta) \phi + \beta - 1) x + (1 - \phi)\). For \(x \in [-1, 0)\) observe first that \(T(0, \phi) > 0\) and \(T(-1, \phi) = (2 - \beta) - 2\phi (3 - \beta)\). Hence for \(\phi \leq \frac{2 - \beta}{2(3 - \beta)}\) the polynomial \(T(x, \phi)\) is always nonnegative with respect to \(x \in [-1, 0)\) and then \(g(x, \phi)\) monotonically decreases and \(-\infty < g(x, \phi) \leq g(-1, \phi) = \frac{2\beta}{2(3 - \beta)}\).

Moreover, since \(\frac{2\beta}{2(3 - \beta)}\) decreases with respect to \(\phi\), when \(0 \leq \phi \leq \frac{2 - \beta}{2(3 - \beta)}\) its minimum value is reached at \(\phi = \frac{2 - \beta}{2(3 - \beta)}\) and \(g(-1, \phi) = \frac{2\beta}{2(3 - \beta)}\), which proves the thesis. \(\square\)

The above proposition suggests that, by selecting the real parameter \(\phi = \frac{2 - \beta}{2(3 - \beta)}\), an explicit 2-FLMM of order 1 with improved stability properties is obtained in the form

\[
\sum_{j=0}^{n-1} \omega_j y_{n-j} - b_n y_0 = h^\beta \left(\frac{4 - \beta}{2(3 - \beta)} f(t_{n-1}, y_{n-1}) + \frac{2 - \beta}{2(3 - \beta)} f(t_{n-2}, y_{n-2})\right).
\]

By again using \((18)\), the stability region of method \((24)\) is drawn, for some values of \(\beta\), in Fig. 3.

Since \(g(-1, \frac{2 - \beta}{2(3 - \beta)}) = -\frac{2\beta}{2(3 - \beta)}\), **Proposition 10** allows us to establish that \((-2\beta(3 - \beta), 0)\) is in the interval of absolute stability. In order to prove that the interval is exactly \((-\frac{2\beta}{2(3 - \beta)}, 0)\), we should prove that **Lemma 7** holds also for \(\gamma(x)\) as in method \((24)\). However, we graphically observed that the region of stability cuts the negative real axis just in \(-\frac{2\beta}{2(3 - \beta)}\).

Anyway we can note that the interval defined in **Proposition 10** is consistently greater than the interval of the rival explicit method \((19)\) of order 1 and hence we can conclude that method \((24)\) allows an enhancement in stability properties for methods of order 1.

This enhancement becomes more clearly observable when we fix a value for \(\lambda\) and we compare the bounds involved on the step-size \(h\) for stiff problems with low values of the fractional order \(\beta\), as shown in Fig. 4 where methods \((19)\) and \((20)\) are compared.

In a similar way we can consider the family of generic 3-FLMMs of order 2 for which, by imposing the first and second order conditions \((12)-(13)\), the function \(\gamma(\xi)\) can be written in terms of a real parameter \(\phi\) as

\[
\gamma(\xi) = \left(2 - \frac{\beta}{2} + \phi\right) \xi - \left(1 - \frac{\beta}{2} + 2\phi\right) \xi^2 + \phi \xi^3.
\]

As in the case of the previous method, we can see that for proper values of \(\phi\) the real-valued function
Fig. 3. Plots of stability regions of the explicit method (24) of order 1.

Fig. 4. Comparison of bounds on \( h \) for methods (19) and (24) of order 1.

\[
g(x, \phi) = \frac{(1 - x)^{\beta}}{(2 - \frac{\beta}{2} + \phi)x - (1 - \frac{\beta}{2} + 2\phi)x^2 + \phi x^3},
\]

for real arguments \( x \) and \( \phi \), is positive as \( x \in (0, 1] \) and hence we can restrict our investigation to \( x \in [-1, 0) \).

**Proposition 11.** Let \( 0 < \beta < 1 \) and \( -\frac{2 - \beta}{2} \leq \phi < 0 \). The function \( g(x, \phi) \) defined in (25) monotonically decreases with respect to \( x \in [-1, 0) \) with \( -\infty < g(x, \phi) \leq g(-1, \phi) = -\frac{2\beta}{3 - \beta + 4\phi} \) and the minimum value for \( g(-1, \phi) \), as \( \phi \) varies in \([-\frac{2 - \beta}{4}, 0]\), is reached at \( \phi = -\frac{2 - \beta}{4} \).

**Proof.** Note first, after some simple computation, that

\[
\frac{d}{d\xi} g(x, \phi) = -\frac{(1 - x)^{\beta - 1}}{(y(x))^2} T(x, \phi),
\]

where \( T(x, \phi) = \phi(1 - x(5 - \beta) + x^2(7 - 2\beta) - x^3(3 - \beta)) + \frac{4 - \beta}{2} - \frac{2 - \beta + 8}{2} x^2 + \frac{(2 - \beta)^2}{2} x^2 \). Moreover observe that \( T''(x) = -6(3 - \beta)\phi x - 2(7 - 2\beta)\phi + (2 - \beta)^2 \) and \( T''(x) < 0 \) as \( x < \frac{2(7 - 2\beta)\phi - (2 - \beta)^2}{6(3 - \beta)\phi} \). Since \( \frac{2(7 - 2\beta)\phi - (2 - \beta)^2}{6(3 - \beta)\phi} > 1 \) for any \( \phi \leq 1 - \frac{\beta}{2} \) we have that \( T''(x) < 0 \) for any \( x \in [-1, 0) \). By observing that \( T(-1) > 0 \), \( T(0) > 0 \) we can conclude that \( T(x) \) is positive in \([-1, 0)\) and hence \( g(x, \phi) \) is a monotonically decreasing function with respect to \( x \). Hence for any \( x \in [-1, 0) \) it is \( g(x, \phi) \leq g(-1, \phi) = -\frac{2\beta}{3 - \beta + 4\phi} \) and the minimum value is reached at \( \phi = -\frac{2 - \beta}{4} \), which proves the thesis. \( \square \)
Fig. 5. Plots of stability regions of the explicit method (26) of order 2.

Fig. 6. Comparison of bounds on $h$ for methods (20) and (26) of order 2.

By selecting $\phi$ as in Proposition 11, the following explicit 3-FLMM of order 2

$$
\sum_{j=0}^{n-1} \omega_j y_{n-j} - b_n y_0 = h^\beta \left( \frac{6 - \beta}{4} f(t_{n-1}, y_{n-1}) - \frac{2 - \beta}{4} f(t_{n-3}, y_{n-3}) \right)
$$

is obtained and its regions of stability are plotted in Fig. 5. Note that, since $g(-1, -\frac{2-\beta}{4}) = -2^\beta$, its interval of stability is given by $[-2^\beta, 0]$ which enhances, in a sensible way, the interval of stability of the 2-FLMM (20) of the same order.

Also in this case we observe that the bounds induced on $h$ are noticeably reduced as compared to method (20) and shown in Fig. 6.

6. Numerical experiments

In order to validate theoretical results from previous sections, we first consider the problem

$$
P^\beta y(t) = t^2 - y + \frac{2t^{2-\beta}}{\Gamma(3-\beta)}, \quad y(0) = 0,
$$

which true solution is $y(t) = \tilde{t}^2$. This equation has been considered in several papers [2,3,8] for verifying convergence results and making comparison among different methods.

In our tests, problem (27) has been integrated over the interval $[0, 1]$ with an increasing number $N$ of steps and errors $E(N) = y(t_N) - y_N$ have been evaluated. In Fig. 7 (for $\beta = 0.5$) and in Fig. 8 (for $\beta = 0.75$) we plot resulting errors $E(N)$ versus
Fig. 7. Errors in problem (27) for $\beta = 0.5$ at $t = 1.0$.

Fig. 8. Errors in problem (27) for $\beta = 0.75$ at $t = 1.0$.

<table>
<thead>
<tr>
<th>Method</th>
<th>Eq.</th>
<th>$I_\beta$</th>
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<tbody>
<tr>
<td>1-FLMM order 1</td>
<td>(19)</td>
<td>$-2^\beta$</td>
</tr>
<tr>
<td>2-FLMM order 1</td>
<td>(24)</td>
<td>$-2^\beta(3-\beta)$</td>
</tr>
<tr>
<td>2-FLMM order 2</td>
<td>(20)</td>
<td>$-2^{\frac{3}{2}-\beta}$</td>
</tr>
<tr>
<td>3-FLMM order 2</td>
<td>(26)</td>
<td>$-2^\beta$</td>
</tr>
</tbody>
</table>

Table 3
Lower boundary $I_\beta$ of intervals of stability for methods under investigation

Experimental results corroborate the convergence behaviour of first and second order for the methods under investigation and show that errors in the new methods (24) and (26) are only slightly greater than the errors in the standard methods (19) and (20).

Next test concerns with computational costs: we try to point out the saving in the number of floating-points operations caused by the use of methods with a larger interval of stability. To this purpose we integrate the linear test problem (15), on the interval $[0, 1]$, with a step-size $h$ chosen such that $h^2\lambda = 0.95I_\beta$, where with $I_\beta$ we indicate the lower boundary of the interval of stability of each method as evaluated in the previous Sections (their values are summarized in Table 3).

Executions have been performed for various values of $\beta$ and the number of floating-point operations has been evaluated by means of the built-in Matlab function `flops`. From results presented in Figs. 9 and 10 (for $\lambda = -10$ and $\lambda = -20$...
respectively) we observe that the feasibility of methods (24) and (26) of choosing the step-size $h$ in an interval larger than the ones corresponding to methods (19) and (20), allows a considerable reduction in the computational cost.

Therefore, even if an extra step is required, methods with larger intervals of stability seem to be more competitive with respect to standard methods.

References