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On Extended Block Induction

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1. INTRODUCTION

As generalizations of Brauer correspondence in the study of modular representation theory of finite groups, four different definitions of block induction have been proposed and used. The original Brauer block induction (cf. [Fe]), p-regular induction (cf. [B]), and extended induction (cf. [W]) are defined in terms of central characters, while Alperin–Burry induction (cf. [A]) is defined in terms of module-theoretic properties of block ideals. Although the four definitions are different, they are closely related. Among them, extended induction is the weakest. In this paper, we give some p-local characterizations of extended induction, study the behavior of principal blocks under extended induction, and examine whether properties of the other types of induction hold for extended induction. For instance, we show that the transitivity which holds along blocks under Brauer induction and p-regular induction does not hold under extended induction. In fact we give a class of infinitely many counterexamples for any prime p. (See Corollary 4.7.) Before we can precisely state and prove our results, we need to fix our notation and state some definitions.

Let G be a finite group. Let p be a prime number and let $(F, R, \mathbf{k})$ be a p-modular system. Assume that F and $\mathbf{k}$ are splitting fields for every subgroup of G. Let O be R or F and H a subgroup of G. The Brauer map Br_H^G (or simply Br_H): $OG \to OH$ is defined by $Br_H^G(x) = x$ if $x \in H$ and 0 if $x \in G - H$ for all $x \in G$ and is linearly extended to OG. If A is a G-algebra, let A^H denote the set of H-fixed points, and A_H^G the image of A^H under the trace map. Let G_p (resp. $G_{p'}$) denote the set of p- (resp. p'-) elements of G. For a set $S \subseteq G$, let FS denote the subspace spanned by S, and $(FS)^G$ the intersection of FS with the center ZFG of FG. If b is an

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OG-block, where "block" refers to block idempotent, then λ_b denotes the central character of FG associated with block b. If $\theta = \sum_{\phi \in \mathrm{Irr}(G)} a_\phi \phi$, with $a_\phi \in R$, define $\theta^b := \sum_{\phi \in \mathrm{Irr}(b)} a_\phi \phi$.

DEFINITION. Let $H \leq G$, b and $\hat{b}$ be OG, OH-blocks, respectively.

(1) $\tilde{b}^G$ *is defined and equal to* b (or $\tilde{b}$ *induces to* b in Brauer's sense) $\Leftrightarrow \lambda_{\tilde{b}} \circ (Br_H^G)_{ZFG} = \lambda_b$ (cf. [Fe]);

(2) $\tilde{b}^{\mathrm{reg}\,G}$ *is defined and equal to* b (or $\tilde{b}$ *p-regularly induces* to b) $\Leftrightarrow \lambda_{\tilde{b}} \circ (Br_H^G) ZFG_{p'} = (\lambda_b)_{ZFG p'}$ (cf. [B]);

(3) $\tilde{b}^{\mathrm{ext}\,G}$ *is defined and equal to* b (or $\tilde{b}$ *induces to* b in the *extended* sense) $\Leftrightarrow \lambda_{\tilde{b}} \circ (Br_H^G)(b') = \lambda_b(b')$, for all blocks b' of OG (cf. [W]);

(4) $\tilde{b}^{(G)}$ *is defined and equal to* b (or $\tilde{b}$ *induces to* b in Alperin–Burry's sense) $\Leftrightarrow$ there is only one OG-block b which covers $\tilde{b}$. (We say that an OG-block b *covers* $\tilde{b}$ if $\tilde{b}OH$ is isomorphic to a direct summand of $bOG_{H \times H}$ as $O(H \times H)$-modules.) (cf. [A]).

The four definitions are not equivalent, but if any two of the four types of block induction are defined, they are the same. In general we have

$$\tilde{b}^G = b \Rightarrow \tilde{b}^{\mathrm{reg}\,G} = b \Rightarrow \tilde{b}^{\mathrm{ext}\,G} = b$$

and

$$\tilde{b}^{(G)} = b \Rightarrow \tilde{b}^{\mathrm{ext}\,G} = b.$$

There are examples showing that it is not true in general to reverse any of the above arrows or get any further implications among these four definitions (cf. [B2]).

2. STRONG COVERING AND ITS CHARACTERIZATIONS

In order to investigate extended block induction more closely, we introduce the definition of *strong covering* of block which is indeed a strong case of covering of blocks. (See the remark following Proposition 2.2.) In this section, $\bar{x}$ denotes the canonical image in $RG/J(R)G$ of any $x \in RG$.

DEFINITION 2.1. Let G be a finite group and H a subgroup of G. Let $\tilde{b}$ be an OH-block and b a central idempotent of OG. Then we say b *strongly covers* $\tilde{b}$ if $\lambda_{\tilde{b}} \circ (Br_H^G)(b) \neq 0$.

By definition, we can see that $\tilde{b}$ induces to a block b of G in the extended sense if and only if b is the unique block which strongly covers $\tilde{b}$.

The following properties are useful when we study strong covering. (See also Prop. 1.4 in [W].)

Proposition 2.2. *Let H be a subgroup of G, let b and $\tilde{b}$ be OG, OH-blocks, resp. Then the following are equivalent.*

(1) *b strongly covers $\tilde{b}$;*

(2) *$\tilde{b}(Br_H^G)(b) \notin J(ZOH)$;*

(3) *let $f : \tilde{b}OH \to (bOG)_{H\times H}$ be the $O[H \times H]$-homomorphism defined by $f(x) = bx$ and let $g : (bOG)_{H\times H} \to \tilde{b}OH$ be defined by $g(y) = \tilde{b}Br_H^G(y)$; then $g \circ f$ is an automorphism of $\tilde{b}OH$;*

(4) *let $\theta \in \mathrm{Irr}(\tilde{b})$. Then $\theta^G(1)_p = \theta^b(1)_p$.*

Proof. The equivalence of (1), (2), and (3) is quite straightforward.

(1) ⇔ (4): By Lemma V.1.1 in [Fe], $\theta^G(1)\omega_\theta \circ Br_H^G = \theta^G$ on ZRG, where ω_θ is the central character of H corresponding to θ. So

$$\theta^G(1)\,\omega_\theta \circ Br_H^G(b) = \theta^G(b) = \theta^b(1).$$

As $\overline{\omega}_\theta = \lambda_{\tilde{b}}$ on ZFH, so $\lambda_{\tilde{b}} \circ Br_H^G(\bar{b}) = \overline{\omega_\theta \circ Br_H^G(b)}$. Hence $\lambda_{\tilde{b}} \circ Br_H^G(\bar{b}) \neq 0 \Leftrightarrow \theta^G(1)_p = \theta^b(1)_p$.

Remark. The equivalence of (1) and (3) shows that strong covering implies general covering. So Alperin−Burry induction implies extended induction. The equivalence of (1) and (4) is useful when we compute examples.

H. Ellers and G. Hill have given a characterization of Alperin−Burry induction in terms of p-local subgroups, with a module-theoretic approach (cf. [EH]). Y. Fan has also done this for Alperin−Burry induction from a different point of view, with a ring-theoretic method (cf. [Fa, Theorem 1.1]). The following theorem, part of which is analogous to Fan's theorem, gives some p-local characterizations for extended induction.

Theorem 2.3. *Let $\tilde{b}$ be a block of FH with defect group D and b a central idempotent of FG. Denote $\tilde{B} := Br_{C_H(D)}^H(\tilde{b}) = \sum_i \tilde{b}_i$ and $B := Br_{C_G(D)}^G(b)$, where the $\tilde{b}_i$ are blocks in $FDC_H(D)$. (By Brauer's First Main Theorem, $\tilde{B}$ is a block of $FN_H(D)$.) Then the following are equivalent.*

(i) *b strongly covers $\tilde{b}$;*

(ii) *$Br_H^G(b)y = \tilde{b}$ for some $y \in \tilde{b}(FH)^H$;*

(iii) *$\tilde{B}Br_{C_H(D)}^{C_G(D)}(B)z = \tilde{B}$ for some $z \in FC_H(D)^{N_H(D)}$;*

(iv) *B in $N_G(D)$ strongly covers $\tilde{B}$ in $N_H(D)$;*

(v) *B in $DC_G(D)$ (or $C_G(D)$) strongly covers $\tilde{b}_i$, for all i, in $DC_H(D)$ (or $C_H(D)$ resp.);*

(vi) *B in $DC_G(D)$ (or $C_G(D)$) strongly covers $\tilde{b}_i$, for some i, in $DC_H(D)$ (or $C_H(D)$ resp.);*

(vii) *there exists $x \in FC_H(D)^{C_H(D)}$ such that $\tilde{B}Br_{C_H(D)}^{C_G(D)}(B)x = \tilde{B}$;*

(viii) *for any* i, *there is a* $z_i \in FC_H(D)^{C_H(D)}$ *such that* $\tilde{b}_i Br^{C_G(D)}_{C_H(D)}(B) z_i = \tilde{b}_i$;

(ix) *there is an* i, *such that* $\tilde{b}_i Br^{C_G(D)}_{C_H(D)}(B) z_i = \tilde{b}_i$ *for some* $z_i \in FC_H(D)^{C_H(D)}$;

(x) $\tau(B)$ *in* $DC_G(D)/D$ *strongly covers* $\tau(\tilde{b}_i)$ *in* $DC_H(D)/D$ *for any* i (*or some* i), *where* τ *is the homomorphism induced by the canonical map* $DC_G(D) \to DC_G(D)/D$.

Remark. By Theorem 2.3, in order to see if a block $\tilde{b}$ of H induces to a block b of G in the extended sense, we only need to know whether b is the unique block of G such that $Br_{C_G(D)}(b)$ satisfies any of those p-local conditions described in the theorem. So some problems on extended induction can be reduced to the p-local situation.

Proof. (i) ⇔ (ii) follows from Proposition 2.2 since $\tilde{b}(FH)^H$ is a commutative local ring.

(ii) ⇒ (iii): Let $\tilde{b} = Br_H(b)y$ for some $y \in \tilde{b}FH^H$. Then

$$\begin{aligned}
\tilde{B} &= Br_{C_H(D)}(\tilde{b}) \\
&= Br_{C_H(D)}(Br_H(b)y) \\
&= Br_{C_H(D)}(Br_H(by)) \\
&= Br_{C_H(D)}\Big(Br_{C_G(D)}(\tilde{b}by)\Big) \\
&= Br_{C_H(D)}\Big(Br_{C_G(D)}(\tilde{b})\,Br_{C_G(D)}(b)\,Br_{C_G(D)}(y)\Big) \\
&= Br_{C_H(D)}\Big(Br_{C_H(D)}(\tilde{b})\,Br_{C_G(D)}(b)\,Br_{C_H(D)}(y)\Big) \\
&= Br_{C_H(D)}(\tilde{b})\,Br_{C_H(D)}\big(Br_{C_G(D)}(b)\big)Br_{C_H(D)}(y) \\
&= \tilde{B}Br_{C_H(D)}(B)z,
\end{aligned}$$

where $z = Br_{C_H(D)}(y)$.

As $y \in \tilde{b}FH^H \subseteq FH^H_D$, $z \in FC_H(D)^{N_H(D)}_D \subseteq FC_H(D)^{N_H(D)}$. (See Lemma III.2.1 in [L].)

(iii) ⇒ (i): Assume (iii). Then $\tilde{B}z \in FC_H(D)^{N_H(D)}_D$ since $\tilde{B} = Br_{C_H(D)}(\tilde{b}) \in FC_H(D)^{N_H(D)}_D$.

Thus $\tilde{B}z = Br_{C_H(D)}(z_0)$ for some $z_0 \in (FH)^H_D$.

As $Br_{C_G(D)}$ is a ring homomorphism on $(FG)^D$ and $\tilde{B}$ is an idempotent in $FC_H(D)$,

$$\begin{aligned} Br_{C_G(D)}\big(b\tilde{b}z_0\big) &= Br_{C_G(D)}(b)\,Br_{C_H(D)}(\tilde{b})\,Br_{C_H(D)}(z_0) \\ &= B\tilde{B}z. \end{aligned}$$

Thus

$$\begin{aligned} Br_{C_H(D)}\big(b\tilde{b}z_0\big) &= Br_{C_H(D)}\Big(Br_{C_G(D)}\big(b\tilde{b}z_0\big)\Big) \\ &= Br_{C_H(D)}(B\tilde{B}z) \\ &= Br_{C_H(D)}(B)\,\tilde{B}z \\ &= \tilde{B}. \end{aligned}$$

Hence $\tilde{b}Br_H(b)z_0 = Br_H(b\tilde{b}z_0) \notin J(\tilde{b}FH^H)$ because $Br_{C_H(D)}(Br_H(b\tilde{b}z_0)) = Br_{C_H(D)}(b\tilde{b}z_0)) = \tilde{B}$ is an idempotent, and $Br_{C_H(D)}$ is a ring homomorphism on $(FH)^D$.

Therefore $\tilde{b}Br_H(b) \notin J(\tilde{b}FH^H)$.

(iii) $\Leftrightarrow$ (iv): Follows from (i) $\Leftrightarrow$ (iii).

(vi) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v): As $\lambda_{\tilde{b}_i} \circ Br_{C_H(D)} = \lambda_{\tilde{B}}$ on $ZFN_H(D)$, so

$$\begin{aligned} \lambda_{\tilde{B}} \circ Br_{N_H(D)}(B) &= \lambda_{\tilde{b}_i} \circ Br_{C_H(D)} \circ Br_{N_H(D)}(B) \\ &= \lambda_{\tilde{b}_i} \circ Br_{C_H(D)}(B). \end{aligned}$$

Thus B in $N_G(D)$ strongly covers $\tilde{B}$ in $N_H(D)$ if and only if B in $DC_G(D)$ strongly covers $\tilde{b}_i$ for all $i \Leftrightarrow$ for some i, B in $DC_G(D)$ strongly covers $\tilde{b}_i$.

(v) $\Leftrightarrow$ (viii) and (vi) $\Leftrightarrow$ (ix) follow from the equivalence of (i) and (ii).

(iii) $\rightarrow$ (vii): Clear

(vii) $\rightarrow$ (viii): Suppose $x \in FC_H(D)^{C_H(D)}$ such that $\tilde{B}Br^{C_G(D)}_{C_H(D)}(B)x = \tilde{B}$. Multiply both sides by $\tilde{b}_i$. We have

$$\tilde{b}_i Br^{C_G(D)}_{C_H(D)}(B)\,x = \tilde{b}_i \qquad \text{since } \tilde{b}_i\tilde{B} = \tilde{b}_i.$$

(x) $\Leftrightarrow$ (v) and (vi): τ induces a one-to-one correspondence between blocks of $DC_G(D)$ (or $DC_H(D)$) and blocks of $DC_G(D)/D$ (or $DC_H(D)/D$ resp.). So the $\tau(\tilde{b}_i)$ are blocks of $DC_H(D)/D$ and $\tau(B)$ is a central idempotent of $DC_G(D)/D$. Furthermore, for all i, we have

$$\begin{aligned}\lambda_{\tilde{b}_i} \circ Br_{DC_H(D)}(B) &= \lambda_{\tau(\tilde{b}_i)} \circ \tau \circ Br_{DC_H(D)}(B) \\ &= \lambda_{\tau(\tilde{b}_i)} \circ Br_{\overline{DC_H(D)}}(\tau(B)),\end{aligned}$$

where $\overline{DC_H(D)} = DC_H(D)/D$.

It is worth noticing that, in the case where the subgroup is normal, strong covering has a clear relationship to general covering.

PROPOSITION 2.4. *Let H be a normal subgroup of G. Let B and b be blocks of G and H, respectively. Then B strongly covers b if and only if B covers b and B is weakly regular with respect to H.*

Proof. $\Rightarrow$: If B strongly covers b, then B covers b.

Let $Br_H(B) = \sum_i a_i \hat{K}_i \in ZFG \cap ZFH$, where $\hat{K}_i = \sum_{k \in K_i} k$ with K_i running through all conjugacy classes of G in H. By Lemma V.3.3 and 3.4 in [Fe], we have

$$\begin{aligned}0 \neq \lambda_b(Br_H(B)) &= \sum_i a_i \lambda_b\left(\hat{K}_i\right) \\ &= \sum_i a_i \lambda_B\left(\hat{K}_i\right).\end{aligned}$$

So $a_i \lambda_B(\hat{K}_i) \neq 0$ for some i. Also the defect group of B is a defect group of K_i by Lemma IV.7.3 in [Fe]. Hence B is weakly regular with respect to H (see Lemma V.3.12 in [Fe]).

$\Leftarrow$: Let $\{B_j\}_j$ be the set of blocks of G covering b. Let $T(b)$ be the inertial group of b in G. Then $\sum_j B_j = \sum_g b^g$, where g runs through a set of coset representatives of $T(b)$ in G. So

$$\begin{aligned}\sum_j bBr_H(B_j) &= \sum_g bBr_H(b^g) \\ &= b\end{aligned}$$

since $b^g \in FH$ for all $g \in G$.

Thus $\lambda_b(Br_H(B_j)) \neq 0$ for some j. By the first part of the proposition, B_j is weakly regular with respect to H. Hence B_j and B have the same defects. Note that if $\hat{B} = \sum_{g \in G} \theta(g) g^{-1}$ is a block of RG, where θ is a virtual character of G, then $\theta(1)_p = p^{cd(\hat{B})}$ by a theorem of Broué (cf. [Br, Theorem 3.2.2].) Thus, by (2.6) and (2.7) in [B], $\lambda_b(Br_H(B)) \neq 0$.

Remark. The proposition above can also be proved using Lemma 2.5(i) and (iii) in [B2]. As a corollary, we can get a result which is a special case of Corollary 4 in [B2].

COROLLARY 2.5. *Let H be a normal subgroup of G, and let B, b be blocks of G, H, respectively. Then $b^{\text{reg}\,G} = B$ if and only if $b^{\text{ext}\,G} = B$.*

Proof. It follows immediately from Proposition 2.4 and Theorem 2 in [B].

3. PRINCIPAL BLOCKS UNDER EXTENDED INDUCTION

In this section we study the behavior of principal blocks under extended induction. It turns out that the question of whether the principal block of a group of G strongly covers the principal block of a group can be answered by considering somewhat restricted sections of G. Here a section of a group means a homomorphic image of a subgroup of the group. We will prove the following theorem.

THEOREM 3.1. *Let $H \leq G$. Suppose the principal block B_0 of G does not strongly cover the principal block b_0 of H. Then there is a section S of G with no non-trivial normal p-solvable subgroup, and a p'-subgroup T of S, such that the principal block $\tilde{B}_0$ of S does not strongly cover the principal block $\tilde{b}_0$ of T.*

Let us prove some lemmas first.

LEMMA 3.2. *Let $K \leq G$, K a p'-group. Let $\hat{B}_0 = \sum_{g \in G} a_g g$ be the principal block of RG, and let $\hat{b}_0$ be the principal block of RK. Let $B_0 = \hat{B}_0^*$ and $b_0 = \hat{b}_0^*$, where $*$ indicates images of elements of RG in FG under the natural map. Then B_0 strongly covers b_0 if and only if $(\sum_{k \in K} a_k)^* \neq 0$ in F.*

Proof. As K is a p'-group, $\hat{b}_0 = |K|^{-1} \sum_{k \in K} k$. So

$$\hat{b}_0 Br_K\left(\hat{B}_0\right) = \left(|K|^{-1} \sum_{k \in K} k\right)\left(\sum_{g \in K} a_g g\right)$$

$$= |K|^{-1} \sum_{g \in K} \left[a_g\left(\sum_{k \in K} k\right)\right]$$

$$= \left(\sum_{g \in K} a_g\right)\hat{b}_0.$$

Hence

$$\lambda_{b_0}(Br_K(B_0)) = \lambda_{b_0}\left(\left(\left(\sum_{k \in K} a_k\right)\hat{b}_0\right)^*\right)$$

$$= \left(\sum_{k \in K} a_k\right)^*.$$

LEMMA 3.3. *Let $K \leq G$, K a p'-group, and let B_0 and b_0 be the principal blocks of FG and FK, respectively. Then B_0 strongly covers b_0 if and only if $\sum_{g \in K} |(G_p g)_{p'}| \not\equiv 0 \pmod{p}$, where $(G_p g)_{p'} = \{xg : x \in G_p$ such that xg is a p'-element$\}$.*

Proof. Let $B_0 = \sum_{g \in G_{p'}} c_g g$, where $c_g \in F$. Then by Külshammer's formula (see [K]), where we read the cardinalities below modulo p,

$$\begin{aligned} c_g &= \left(|G_{p'}|\right)^{-1} \left|\{(u, s) \in G_p \times G_{p'} : us = g\}\right| \\ &= \left(|G_{p'}|\right)^{-1} \left|(G_p g)_{p'}\right|. \end{aligned}$$

Hence, by Lemma 3.2, B_0 strongly covers b_0 if and only if $\sum_{g \in K} c_g \neq 0$, if and only if $\sum_{g \in K} |(G_p g)_{p'}| \not\equiv 0 \pmod{p}$.

LEMMA 3.4. *Let K be a p'-subgroup of G. Let N be a p-subgroup of $Z(G)$. Then the principal block B_0 of G strongly covers the principal block b_0 of K if and only if the principal block $\overline{B}_0$ of G/N strongly covers the principal block $\overline{b}_0$ of KN/N.*

Proof. Fix $g \in K$. We claim $|(G_p g)_{p'}| = |(\overline{G}_p \bar{g})_{p'}|$, where $\bar{\ }$ indicates images under the canonical map $G \to G/N$.

Define map f as the restriction of the canonical map on $(G_p g)_{p'}$ into $(\overline{G}_p \bar{g})_{p'}$.

Let xg and yg be p'-elements with $x, y \in G_p$ and $\overline{xg} = \overline{yg}$. Then there is $n \in N$ such that $y = nx$. Since $n(xg) = yg$ is a p'-element, xg is a p'-element, and $n \in Z(G)$ is a p-element, we have $n = 1$. So $xg = yg$. Hence f is one-to-one.

Let $\bar{y} \in \overline{G}_p \bar{g}$ be a p'-element. Write $y = y_p y_{p'}$, where y_p and $y_{p'}$ are the p- and p'-part of y, respectively. Since $\bar{y}$ is a p'-element and $\bar{y}_p$ is the p-part of $\bar{y}$, $\bar{y}_p = 1$. Thus $\bar{y} = \bar{y}_{p'}$ with $y_{p'}$ a p'-element and $y_{p'} \in G_p g$. Hence f is also onto. Thus the claim is proved and the lemma then follows from Lemma 3.3.

LEMMA 3.5. *Let N be a normal p'-subgroup of G. Let τ be the natural map from RG to $R[G/N]$. Then for any block B of G with $N \subseteq \mathrm{Ker}(B)$, $\tau(B)$ is a block of G/N. This gives the one-to-one correspondence, given in* [Fe, *Lemma* V.4.3], *between the blocks B of G with $N \subseteq \mathrm{Ker}(B)$ and blocks $\overline{B}$ of G/N.*

Proof. Let B be a block of G with $N \subseteq \mathrm{Ker}(B)$. Then
$\mathrm{Irr}(B) = \mathrm{Irr}(\overline{B})$, where $\overline{B}$ is the block corresponding to B given in [Fe, Lemma V.4.3]. Now by Lemma IV.7.1 of [Fe],

$$\begin{aligned} B &= \sum_{g\in G}\left[\sum_{\chi\in \mathrm{Irr}(B)} (\chi(1)/|G|)\,\chi(g^{-1})\right]g \\ &= \sum_{g_i}\sum_{n\in N}\sum_{\chi\in \mathrm{Irr}(B)} (\chi(1)/|G|)\,\chi\left(g_i^{-1}n^{-1}\right)ng_i \\ &= \sum_{g_i}\sum_{n\in N}\sum_{\chi\in \mathrm{Irr}(B)} (\chi(1)/|G|)\,\chi\left(g_i^{-1}\right)ng_i, \end{aligned}$$

where g_i runs through a set of coset representatives of N in G.
Hence

$$\begin{aligned} \tau(B) &= \sum_{g_i}\sum_{\chi\in \mathrm{Irr}(B)} (\chi(1)/|G:N|)\,\chi\left(\bar{g}_i^{-1}\right)\bar{g}_i \\ &= \overline{B}. \end{aligned}$$

LEMMA 3.6. *Let N be a normal p'-subgroup of G and τ the natural map from FG to $F[G/N]$. Let $N \le K \le G$. Let B and b be blocks of G and K, respectively, with $N \subseteq \mathrm{Ker}(B)$ and $N \subseteq \mathrm{Ker}(b)$. Then B strongly covers b if and only if $\tau(B)$ in G/N strongly covers $\tau(b)$ in K/N.*

Proof. As $\lambda_b = \lambda_{\tau(b)} \circ \tau$ on ZFK,

$$\begin{aligned} \lambda_b Br_K(B) &= \lambda_{\tau(b)} \circ \tau \circ Br_K(B) \\ &= \lambda_{\tau(b)} \circ Br_{\overline{K}}(\tau(B)). \end{aligned}$$

Hence B strongly covers b if and only if $\tau(B)$ in G/N strongly covers $\tau(b)$ in K/N.

LEMMA 3.7. *Let N be a normal p'-subgroup of G, $K \le G$ a p'-group. Let B_0, b_0, and $\tilde{b}_0$ be the principal blocks of G, KN, and K, respectively. Then B_0 strongly covers b_0 if and only if B_0 strongly covers $\tilde{b}_0$.*

Proof. Let $\hat{B}_0 = \sum_{g\in G} a_g g$ be the principal block of RG. As K and KN are p'-groups, Lemma 3.2 implies:

$$B_0 \text{ strongly covers } b_0 \Leftrightarrow \left(\textstyle\sum_{g\in KN} a_g\right)^* \neq 0 \qquad (\text{in } F);$$

$$B_0 \text{ strongly covers } \tilde{b}_0 \Leftrightarrow \left(\textstyle\sum_{g\in K} a_g\right)^* \neq 0 \qquad (\text{in } F).$$

Let $m := |KN:K| = |N:N\cap K|$, and let $\{n_i : 1 \le i \le m\}$ be a set of coset representatives of $N \cap K$ in N. Then $KN = \{kn_i : 1 \le i \le m, k \in K\}$.

For any $k \in K$, and $n \in N$, Lemma IV.4.12(ii) of [Fe] implies

$$\begin{aligned} a_{kn} &= |G|^{-1} \sum_{\chi \in \mathrm{Irr}(B_0)} \chi(1)\chi\big((kn)^{-1}\big) \\ &= |G|^{-1} \sum_{\chi \in \mathrm{Irr}(B_0)} \chi(1)\chi(k^{-1}) \\ &= a_k . \end{aligned}$$

Thus

$$\begin{aligned} \sum_{g \in KN} a_g &= \sum_{i=1}^{m} \sum_{k \in K} a_{kn_i} \\ &= \sum_{i=1}^{m} T \sum_{k \in K} a_k \\ &= m \sum_{k \in K} a_k . \end{aligned}$$

So $(\sum_{g \in KN} a_g)^* \neq 0 \Leftrightarrow (\sum_{g \in K} a_g)^* \neq 0$ since m is a p'-number.

Combining Lemma 3.6 and Lemma 3.7 we have

COROLLARY 3.8. *Let N be a normal p'-subgroup of G, $K \leq G$ a p'-group. Then the principal block B_0 of G strongly covers the principal block b_0 of K if and only if the principal block $\overline{B}_0$ of G/N strongly covers the principal block $\overline{b}_0$ of KN/N.*

LEMMA 3.9. *Let N be an abelian normal p-subgroup of G, and let K be a p'-subgroup of G. Let B_0, b_0, $\tilde{B}_0$, and $\tilde{b}_0$ be the principal blocks of $G, K, C_G(N)$, and $C_K(N)$, respectively. Then B_0 strongly covers b_0 if and only if $\tilde{B}_0$ strongly covers $\tilde{b}_0$.*

Proof. Fix $g \in K$ and $x \in G_p$.

Let $S_{(x,g)} := (Nxg)_{p'} = \{nxg : n \in N$ with nxg a p^*-element$\}$.

Then N acts on $S_{(x,g)}$ by conjugation. If $S_{(x,g)}$ is non-empty, let $nxg \in S_{(x,g)}$ and let $(nxg)^N$ be an orbit under the action. So $|N| = |(nxg)^N| \cdot |C_N(nxg)|$. Thus $|(nxg)^N| \equiv 0 \pmod p$ unless $C_N(nxg) = N$. Therefore $|S_{(x,g)}| \equiv 0 \pmod p$ unless $xg \in C_G(N)$.

Let $\{x_1 = 1, x_2, \ldots, x_t, y_{t+1}, \ldots, y_m\}$ be a set of coset representatives of N in G with $x_1, x_2, \ldots, x_t$ p-elements and $y_{t+1}, \ldots, y_m$ not p-elements. Note that for any $z \in Nx_i$, $\bar{z} = \bar{x}_i$ is a p-element; hence z is a p-element since N is a p-group. Therefore, $G_p = \bigcup_{1 \leq i \leq t} Nx_i$, so

$$\begin{aligned} (G_p g)_{p'} &= \textstyle\bigcup_{1 \leq i \leq t} (Nx_i g)_{p'} \\ &= \textstyle\bigcup_{1 \leq i \leq t} S_{(x_i, g)} . \end{aligned}$$

Thus $|(G_p g)_{p'}| \equiv \sum_{x_i g \in C_G(N)} |S_{(x_i,g)}| \pmod p$.

For any $x_i g \in C_G(N)$, if we let β be the canonical map from G to $G/C_G(N)$, then $\beta(x_i) = 1 = \beta(g)$ since $\beta(x_i)$ is a p-element while $\beta(g)$ is a p' element. Hence $x_i \in C_G(N)$ and $G \in C_G(N)$.

Also note that $C_G(N)_p = \bigcup_{1 \le i \le t,\, x_i \in C_G(N)} Nx_i$. Thus for $g \in K$, if g is not an element of $C_G(N) \cap K$, then $|(G_p g)_{p'}| \equiv 0 \pmod p$, and if $g \in C_G(N) \cap K$, then

$$\left|(G_p g)_{p'}\right| \equiv \sum_{1 \le i \le t,\, x_i \in C_G(N)} |S_{(x_i, g)}| \equiv \left|(C_G(N)_p g)_{p'}\right|.$$

Therefore $\sum_{G \in K} |(G_p g)_{p'}| \equiv \sum_{g \in C_G(N) \cap K} |(C_G(N)_p g)_{p'}|$. Thus the lemma follows by Lemma 3.3.

LEMMA 3.10. *Let $H \le G$. Suppose the principal block B_0 of G does not strongly cover the principal block b_0 of H. Then there is a section S of G with $O_p(S) = O_{p'}(S) = \langle 1 \rangle$, and a p'-subgroup T of S, such that the principal block $\tilde{B}_0$ of S does not strongly cover the principal block $\tilde{b}_0$ of T.*

Proof. Suppose not. Let $B_0(J)$ denote the principal block of an arbitrary group J. Then let G be of minimal order such that for some $H \le G$, $B_0(G)$ does not strongly cover $B_0(H)$, but for all sections S of G with $O_p(S) = O_{p'}(S) = \langle 1 \rangle$, $B_0(S)$ strongly covers $B_0(T)$ for all p'-subgroups T of S.

Let $D \in \mathrm{Syl}_p(H)$. Since $B_0(DC_G(D)/D)$ does not strongly cover $B_0(DC_H(D)/D)$ by Theorem 2.3, it follows that $D = \langle 1 \rangle$ and H is a p'-group by minimality of G. Since $B_0(G/O_{p'}(G))$ does not strongly cover $B_0(HO_{p'}(G)/O_{p'}(G))$ by Corollary 3.8, we have $O_{p'}(G) = \langle 1 \rangle$. If $O_p(G) \ne \langle 1 \rangle$ then G has an abelian normal p-subgroup $N \ne \langle 1 \rangle$ and $B_0(C_G(N))$ does not strongly cover $B_0(C_H(N))$ by Lemma 3.9. Hence $N \le Z(G)$. Then $B_0(G/N)$ does not strongly cover $B_0(HN/N)$ by Lemma 3.4. Therefore $N = \langle 1 \rangle$, a contradiction. Hence $O_p(G) = \langle 1 \rangle$. Now $S = G$ yields a contradiction which proves the lemma.

Proof of Theorem 3.1. The section S of G as in Lemma 3.10 has $O_{p'}(S) = \langle 1 \rangle = O_p(S)$. If S had a non-trivial p-solvable normal subgroup N, then at least one of $O_{p'}(S)$ and $O_p(S)$ would be non-trivial, with both characteristic in N and hence normal in S. This is a contradiction.

COROLLARY 3.11. *Let G be a p-solvable group, $H \le G$. Then the principal block B_0 of G strongly covers the principal block b_0 of H. Hence if $b_0^{\mathrm{ext}\, G}$ is defined then $b_0^{\mathrm{ext}\, G} = B_0$.*

Proof. It follows immediately from Theorem 3.1.

Remark. Corollary 3.11 can also be proved directly by using a theorem of P. Fong and W. Gaschütz (cf. Theorem X.1.5 in [Fe]). Also, we should notice that the corollary is not true without p-solvability of G (cf. [W, Example 2.10]). The following example shows that even in solvable groups, principal blocks may be induced in the extended sense by non-principal blocks of subgroups.

EXAMPLE 3.12. Let $k = GF(9)$. Let $k^{\#}$ be the multiplicative group of k with generator υ. Let $H = \langle(\begin{smallmatrix} \upsilon^2 & 0 \\ 0 & \upsilon^{-2} \end{smallmatrix})\rangle$, $S = \{(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}) : a \in k\}$.

Then $H \cong Z_4$ and $|S| = 9$. Then G is a solvable group of order 36. Let $p = 3$.

Let $G = S \cdot H \leq SL(2,9)$. Then G has 12 characters and two 3-blocks so that

$$\mathrm{Irr}(B_0) = \{\chi_1, \chi_3, \chi_5, \chi_7, \chi_9, \chi_{11}\},$$

$$\mathrm{Irr}(B_1) = \{\chi_2, \chi_4, \chi_6, \chi_8, \chi_{10}, \chi_{12}\},$$

where χ_1 is the trivial character, χ_2, χ_3, and χ_4 are other characters of degree 1, and $\chi_5, \chi_6, \dots, \chi_{12}$ are characters of degree 2.

Let $\beta \in \mathrm{Irr}(H)$ such that $\beta((\begin{smallmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{smallmatrix})) = 1$ and $\beta((\begin{smallmatrix} \upsilon^{\pm 2} & 0 \\ 0 & \upsilon^{\mp 2} \end{smallmatrix})) = -1$. Then β does not belong to the principal block of H, and $\beta \uparrow^G = \chi_3 + \chi_5 + \chi_7 + \chi_9 + \chi_{11}$. Let b_β be the block to which β belongs. Then, by Proposition 2.2, $b_\beta^{\mathrm{ext}\,G} = B_0$.

This gives an example where a non-principal block induces in the extended sense to the principal block of a solvable group. However, as we know, this can not happen for p-regular induction. Therefore extended induction is properly weaker than p-regular induction, even in solvable groups.

4. TRANSITIVITY

It is easy to see by definition that the transitivity of Brauer induction and p-regular induction holds along blocks. That is, if $T \leq H \leq G$ is a chain of three groups, and $\tilde{b}$, b, and B are blocks of T, H, and G, respectively, such that $\tilde{b}$ induces to b, and b induces to B in Brauer's sense (or p-regularly), then the induction of $\tilde{b}$ in G is defined and equals B. Ellers and Hill showed that Alperin−Burry induction does not have the transitivity property (cf. [EH]). However, general covering obviously has the transitivity property along blocks; therefore, if the induction of $\tilde{b}$ in G is defined then it must be B. Now we would like to know whether the transitivity property holds for extended block induction. Unfortunately, the answer is negative. We construct a class of infinitely many counterexamples to the transitivity of extended block induction.

Let p be a prime number. For any positive integer n, let $G_n = GL(n,p)$, $S_n = SL(n,p)$, and let V_n be the n-dimensional vector space of row vectors over the field of p elements. Let

$$P_n = \left\{ \begin{bmatrix} a & v \\ 0 & X \end{bmatrix} : X \in G_{n-1}, a = \det(X)^{-1}, v \in V_{n-1} \right\};$$

$$L_n = \left\{ \begin{bmatrix} a & 0 \\ 0 & X \end{bmatrix} : X \in G_{n-1}, a = \det(X)^{-1} \right\} \cong G_{n-1};$$

$$U_n = \left\{ \begin{bmatrix} 1 & v \\ 0 & I \end{bmatrix} : v \in V_{n-1} \right\}, \text{ where } I \text{ is the identity matrix in } S_{n-1};$$

$$Q_n = \left\{ \begin{bmatrix} 1 & v \\ 0 & X \end{bmatrix} : X \in S_{n-1}, v \in V_{n-1} \right\} \leq P_n.$$

Suppose S is a finite group with a BN-pair. Then S has parabolic subgroups P_J for all subsets J of the index set I of the generating involutions of the corresponding Weyl group. The Steinberg character St of S is an irreducible character of S (cf. p. 42 and p. 187 in [C]) defined as

$$\mathrm{St} = \textstyle\sum_{J \subseteq I} (-1)^{|J|} 1_{P_J} \uparrow^S.$$

For a fixed n, set $G := S_n$, $H := \left[\begin{smallmatrix} 1 & 0 \\ 0 & S_{n-1} \end{smallmatrix}\right] \cong S_{n-1}$. Let ϕ, ψ, and χ be the Steinberg characters of G, H, and L_n respectively. Since P_n is a parabolic subgroup of S_n and L_n is the Levi subgroup of P_n, Proposition 6.3.3 in [C] implies

LEMMA 4.1. $\phi_{P_n} = \chi \uparrow^{P_n}$.

LEMMA 4.2. $(\phi_{Q_n}, \phi_{Q_n})_{Q_n} = (\psi \uparrow^G, \phi)_G$.

Proof. By Lemma 4.1 and Mackey decomposition, noting that $P_n = L_n Q_n$, $L_n \cap Q_n = H$, and $\chi_H = \psi$, we have

$$\begin{aligned} (\phi_{Q_n}, \phi_{Q_n})_{Q_n} &= \left(\chi \uparrow^{P_n} \downarrow_{Q_n}, \phi_{Q_n} \right)_{Q_n} \\ &= \left(\chi_H \uparrow^{Q_n}, \phi_{Q_n} \right)_{Q_n} \\ &= \left(\psi \uparrow^{Q_n}, \phi_{Q_n} \right)_{Q_n} \\ &= (\psi \uparrow^G, \phi)_G. \end{aligned}$$

LEMMA 4.3. *For $n > 2$, there are exactly two double cosets of (L_n, H) in P_n, and if $1 \neq g = \left(\begin{smallmatrix} 1 & v \\ 0 & I \end{smallmatrix}\right) \in U_n$, then with respect to a suitable basis of the underlying vector space of the natural module of G_n,*

$$L_n^g \cap H = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & u^T & Y \end{pmatrix} : Y \in S_{n-2}, u \in V_{n-2} \right\} \cong Q_{n-1}.$$

Proof. Since $P_n = L_n U_n$, we can choose double coset representatives of (L_n, H) in P_n to be in U_n. Now let

$$1 \neq g = \begin{pmatrix} 1 & v \\ 0 & I \end{pmatrix} \in U_n, \qquad l = \begin{pmatrix} a & 0 \\ 0 & X \end{pmatrix} \in L_n,$$

such that $l^g \in L^g \cap H$. Then

$$l^g = \begin{pmatrix} 1 & -v \\ 0 & I \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & I \end{pmatrix} = \begin{pmatrix} a & av - vX \\ 0 & X \end{pmatrix} \in H$$

forces $a = 1$, $X \in S_{n-1}$, and $vX = v$. Therefore under a suitable basis of the vector space, we have

$$L_n^g \cap H = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & u^T & Y \end{pmatrix} : Y \in S_{n-2}, u \in V_{n-2} \right\} \cong Q_{n-1}.$$

In particular, $|L_n gH| = |L_n||H|/|L^g \cap H| = |L_n|[H : Q_{n-1}]$ is independent of the element $g(\neq 1)$ in U_n.

Let k be the number of double cosets which are not L_n. Then

$$|P_n| = |L_n| + k|L_n|[H : Q_{n-1}]$$

and so

$$\begin{aligned} p^{n-1} &= [P : L_n] \\ &= 1 + k[H : Q_{n-1}] \\ &= 1 + k(p^{n-1} - 1). \end{aligned}$$

Hence $k = 1$, and the lemma is proved.

PROPOSITION 4.4. $(\psi \uparrow^G, \phi)_G = p + n - 2$ for $n \geq 2$.

Proof. We prove this by induction on n.

If $n = 2$, then $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \langle 1 \rangle$ and $\psi = \mathbf{1}_H$. Hence $\psi \uparrow^G$ is the character of the regular representation of G. Therefore

$$(\psi \uparrow^G, \phi)_G = \phi(1) = p.$$

Now for $n > 2$, by Frobenius reciprocity, Mackey decomposition, and Lemma 4.3,

$$\begin{aligned}(\psi\uparrow^G, \phi)_G &= \left(\psi, (\phi_{P_n})_H\right)_H \\ &= \left(\psi, \chi\uparrow^{P_n}\downarrow_H\right)_H \\ &= (\psi,\psi) + \left(\psi, \psi_{Q_{n-1}}\uparrow^H\right)_H \qquad (\text{since } \chi\downarrow_H = \psi) \\ &= 1 + \left(\psi_{Q_{n-1}}, \psi_{Q_{n-1}}\right)_{Q_{n-1}} \\ &= 1 + \left[p + (n-1) - 2\right] \\ &\qquad\qquad (\text{by Lemma 4.2 and the induction hypothesis}) \\ &= p + n - 2.\end{aligned}$$

Remark. E. Thoma had some similar results on Steinberg characters of $GL(n,q)$ and $GL(n-1,q)$, where q is a power of p (cf. [T]).

Let $GF(p^{n-1})^{\#} = \langle x\rangle$. Then by its multiplicative action on the vector space $GF(p^{n-1})$ over $GF(p)$, x can be regarded as an element of G_{n-1} of order $p^{n-1}-1$.

Let q be a Zsigmondy prime number with respect to $(p, n-1) \neq (2,6)$ or $(3,2)$. Namely, q is a divisor of $p^{n-1}-1$ but not a divisor of any $p^r - 1$ with $r < n-1$ (cf. Theorem 5.2.14 in [KL] for example).

Assume $n > 2$ for the rest of this section. Let $T = \langle z\rangle \in \mathrm{Syl}_q(\langle x\rangle)$. Then $T \leq S_{n-1}$ since $|G_{n-1}/S_{n-1}| = p - 1$ is not divisible by q. For any $t \in T^{\#}$,

$$N_{G_{n-1}}(\langle t\rangle) = N_{G_{n-1}}(\langle x\rangle) = \langle x\rangle \cdot Z$$

is a semidirect product of normal subgroup $\langle x\rangle$ and Z, where Z is the Galois group $\mathrm{Gal}(GF(p^{n-1})/GF(p))$ of order $n-1$. (See Theorem 7.3 [Hup].)

As $\langle x\rangle$ is transitive on the set of non-zero elements of V_{n-1}, $\langle x\rangle = C_{G_{n-1}}(x)$. Furthermore, for any $t \in T^{\#}$, as

$$\langle x\rangle = C_{G_{n-1}}(x) \leq C_{G_{n-1}}(t) \leq N_{G_{n-1}}(\langle t\rangle) = \langle x\rangle \cdot Z,$$

and q is not a divisor of $p^r - 1$ with $r < n-1$, the Main Theorem of Galois Theory implies that $\langle x\rangle = C_{G_{n-1}}(t)$. In particular, p is not a divisor of $|C_{G_{n-1}}(t)|$. Also, if we identify t with the element $\left(\begin{smallmatrix}1 & 0\\ 0 & t\end{smallmatrix}\right)$ in G_n, then we claim that

$$C_{G_n}(t) = \left\{\begin{pmatrix} a & 0 \\ 0 & Y\end{pmatrix} : a \in GF(p)^{\#}, Y \in C_{G_{n-1}}(t)\right\}.$$

In fact, if $X = \begin{pmatrix} a & c \\ b & Y \end{pmatrix} \in G_n$ such that $X\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}X$, then $\begin{pmatrix} a & ct \\ b & Yt \end{pmatrix} = \begin{pmatrix} a & c \\ tb & tY \end{pmatrix}$. Hence $tY = Yt$, $ct = c$, and $tb = b$.

Any $x^i \neq 1$ fixed no non-zero vectors by the definition of x. hence $c = 0$ and $b = 0$. Thus the claim holds. In particular, p is not a divisor of $|C_{G_n}(t)|$ for all $t \in T^{\#}$.

From the observation above, we can obtain the values of the Steinberg characters of S_n and H on elements of T. (We identify T as a subgroup of H.)

LEMMA 4.5. $\phi(t) = \psi(t) = (-1)^{n-2}$ for all $t \in T^{\#}$.

Proof. Let $t \in T^{\#}$. As q is not a divisor of $p^r - 1$ for $r < n - 1$, S_{n-1} itself is the only Levi subgroup L_J of S_{n-1} which contains t. Therefore, by Theorem 6.4.7(iii) in [C], we have $\psi(t) = (-1)^{|J|}|C_G(t)|_p = (-1)^{n-2}$, since $|J| = n - 2$ when $L_J = S_{n-1}$.

Similarly, $\phi(t) = (-1)^{n-2}$.

Now let us consider the group $\tilde{G} := PSL(n, p)$. $\tilde{G}$ has exactly two p-blocks: the principal block B_0 and the Steinberg block B which contains the Steinberg character (cf. [D, Theorem 5] or Concluding Remarks in [Hum]).

As the center $Z(S_n)$ of S_n is contained in the kernel of the Steinberg character ϕ, ϕ is also the Steinberg character of $\tilde{G}$. Let n be chosen such that $n = kp + 2$, where k is a positive integer with $(k + 1, p - 1) = 1$. Then $(p - 1, n - 1) = 1$. Thus $S_{n-1} = PSL(n - 1, p)$ has exactly two blocks as well: the principal block b_0 and the Steinberg block b.

As $Z(S_n) \cap H = \langle 1 \rangle$, we can identify H as a subgroup of $\tilde{G}$. Thus we have a chain of three groups:

$$T < H < \tilde{G}.$$

Let $\tilde{b}$ be a non-principal p-block of T. Then we conclude the following.

THEOREM 4.6. *Let* $m = (n, p - 1) = |Z(S_n)|$. *Then with the notation and assumptions above, we have*

(i) $\tilde{b}^{\mathrm{ext}\, H} = b$;

(ii) $b^{\mathrm{ext}\, \tilde{G}} = B_0$; *and*

(iii) $\tilde{b}^{\mathrm{ext}\, \tilde{G}} = B$ *if* $m \equiv -1 \pmod{p}$ *or* $\tilde{b}^{\mathrm{ext}\, \tilde{G}}$ *is undefined if* $m \not\equiv -1 \pmod{p}$.

Proof. (i) $b = \sum_{g \in H} |S_{n-1}|^{-1} \psi(1)\psi(g^{-1})g$. Therefore

$$\begin{aligned} Br_T(\bar{b}) &= (-1)^{n-2}\big(|S_{N-1}|_{p'}\big)^{-1} \sum_{g \in T,\, g \neq 1} g \qquad \text{(by Lemma 4.5)} \\ &= (-1)^{n-2}/\big[(p^{n-2}-1)\dots(p^2-1)\big] \sum_{g \in T,\, g \neq 1} g \\ &= (-1)\big(|T|\tilde{b}_0 - 1\big), \end{aligned}$$

where $\tilde{b}_0 = |T|^{-1}\sum_{g \in T} g$ is the principal block of FT. Therefore $\lambda_{\tilde{b}}(Br_T(\bar{b})) = 1$ since $\lambda_{\tilde{b}}(\tilde{b}_0) = 0$. Also

$$\lambda_{\tilde{b}}\big(Br_T(\bar{b}_0)\big) = \lambda_{\tilde{b}}\big(Br_T(1-\bar{b})\big) = 1 - 1 = 0.$$

Hence $\tilde{b}^{\operatorname{ext} H} = b$ by definition of extended induction.

(iii) Similarly, by Lemma 4.5, we have

$$\begin{aligned} Br_T(\bar{B}) &= (-1)^{n-2}\big(|PSL(n,p)|_{p'}\big)^{-1} \sum_{g \in T,\, g \neq 1} g \\ &= (-1)^{n-2} m/\big[(p^{n-1}-1)\dots(p^2-1)\big] \sum_{g \in T,\, g \neq 1} g \\ &= m\big(|T|\tilde{b}_0 - 1\big). \end{aligned}$$

Therefore $\lambda_{\tilde{b}}(Br_T(\bar{B})) = -m \neq 0$ and

$$\begin{aligned} \lambda_{\tilde{b}}\big(Br_T(\bar{B}_0)\big) &= \lambda_{\tilde{b}}\big(Br_T(1-\bar{B})\big) \\ &= 1 + m \end{aligned}$$

which is equal to zero if $m \equiv -1 \pmod p$ and is non-zero if $m \not\equiv -1 \pmod p$. Hence $\tilde{b}^{\operatorname{ext} \tilde{G}} = B$ if $m \equiv -1 \pmod p$ and $\tilde{b}^{\operatorname{ext} \tilde{G}}$ is undefined if $m \not\equiv -1 \pmod p$ since there are exactly two blocks of $\tilde{G}$.

(ii) By Proposition 4.4, we have

$$\begin{aligned} \big(\psi \uparrow^{\tilde{G}}, \phi\big)_{\tilde{G}} &= (\psi, \phi_N)_H \\ &= p + n - 2. \end{aligned}$$

Note that $\psi(1)_p = |H|_p$ and $\phi(1)_p = |\tilde{G}|_p$. (See [C, Corollary 6.4.3] for example.) Thus

$$\begin{aligned} \psi \uparrow^{\tilde{G}}(1)_p &= [\tilde{G} : H]_p \psi(1)_p \\ &= [\tilde{G} : H]_p |H|_p \\ &= |\tilde{G}|_p \\ &= \phi(1)_p, \end{aligned}$$

and

$$\begin{aligned}\psi^{B}(1)_p &= (p+n-2)_p\phi(1)_p \\ &> \phi(1)_p \qquad (\text{since } p \text{ is a divisor of } n-2) \\ &= \psi\uparrow^{\tilde{G}}(1)_p.\end{aligned}$$

Hence B does not strongly cover b by Proposition 2.2.4. Therefore B_0 is the only block of $\tilde{G}$ strongly covering b, and so $b^{\text{ext}\,\tilde{G}} = B_0$.

COROLLARY 4.7. *Let p be any prime number. Let $n = kp + 2$ where k is a positive integer such that $k + 2$ is a multiple of $p - 1$. Then n satisfies the assumption in Theorem 4.6, and with the notation in Theorem 4.6, we have*

(i) $\tilde{b}^{\text{ext}\,H} = b$;
(ii) $b^{\text{ext}\,\tilde{G}} = B_0$; *and*
(iii) $\tilde{b}^{\text{ext}\,\tilde{G}} = B$.

Proof. Clearly $(k+1, p-1) = 1$ since $k+2$ is a multiple of $p-1$. Hence n satisfies the assumption in Theorem 4.6. As $n = kp + 2 = k(p-1) + k + 2$ is a multiple of $p - 1$,

$$\begin{aligned}m &= (p-1, n) \\ &= p-1 \\ &\equiv -1 \pmod p.\end{aligned}$$

Then the corollary follows from Theorem 4.6.

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